IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received August 6, 2019, accepted August 24, 2019, date of publication September 2, 2019, date of current version October 2, 2019.

Digital Object Identifier 10.1109/ACCESS.2019.2939021

Finite-Time and Fixed-Time Consensus of Nonlinear Stochastic Multi-Agent Systems With ROUs and RONs via Impulsive Control

TAO CHEN, SHIGUO PENG, AND ZHENHUA ZHANG
Faculty of Automation, Guangdong University of Technology, Guangzhou 510006, China

Corresponding author: Shiguo Peng (sgpeng@gdut.edu.cn)

This work was supported by the National Natural Science Foundation of China under Grant 61374081 and Grant 61973092.

ABSTRACT This paper discusses the finite-time and the fixed-time consensus of nonlinear stochastic multi-agent systems (NSMSs) with randomly occurring uncertainties (ROUs) and randomly occurring nonlinearities (RONs) in a leader-following framework. Nonlinear control and impulsive pinning control protocols are designed to guarantee that follower agents realize consensus with the leader agent in finite time(fixed time). Based on the finite-time and fixed-time consensus theory, stochastic analysis technique, comparison system theory and algebra graph theory, some sufficient conditions are proposed to guarantee the finite-time and fixed-time consensus of systems. Then, the setting times of finite-time consensus and fixed-time consensus are estimated. Finally, two simulation examples are presented to illustrate the correctness of our conclusions.

INDEX TERMS Finite-time consensus, fixed-time consensus, stochastic multi-agent systems, impulsive control, ROUs, RONs.

I. INTRODUCTION

With the development of modern science and technology and the popularization of artificial intelligence, multi-agent system, as one kind of complex networks, has attracted considerable attention among the scholars and the experts [1]–[3]. Consensus problem is a basic research issue in multi-agent system, and it aims to design a distributed protocol to ensure that all controlled agents tend to a same value as the time goes by [4]–[6]. A consensus algorithm is an interaction criteria that an agent exchanges information between all of its neighbours in the complex system. Especially, a wonderful topic is the leader-following consensus, in which the leader is a solitary agent that leads other agents to reach an agreement in the whole network. The leader-following consensus protocol is designed to guarantee that states of all followers tend towards the leader as time goes on.

As is well known, classical consensus of multi-agent systems is asymptotical consensus, namely, the systems achieve consensus in an infinite time interval [7]. Unfortunately, due to the finiteness of the useful lifespan of equipment and the lifetime in humans, asymptotic consensus control is not the best control method. Therefore, people may hope to design a control method so that systems can reach an agreement as fast as possible [8]. Compared with asymptotic consensus, finite-time consensus greatly reduce the risk of theft and develop security of information when it is applied in secret communications. What's more, the finite-time control measure not only have the better robustness but it also has the property of external disturbance rejection [9]–[12]. Therefore, finite-time consensus also attracted scholars in various fields. In Ref. [12], Chen proposed a distributed algorithm for the finite-time consensus problem of stochastic multi-agent systems and at the meantime, it also optimized the convergence rate.

In the real engineering application environment, multi-agent systems are often affected by random disturbances. Such random disturbances such as information-transmitted random packet loss, random network congestion, random noise disturbances are described in [13]–[17]. This uncertainty may occur in a probabilistic way, which usually represents parameter excursions. Parameter uncertainties can seriously deteriorate the system performance and system failure. So far a large number of research results on stochastic

The associate editor coordinating the review of this article and approving it for publication was Zhan Bu.

systems have been published to the world. In Ref. [15], a new system with randomly occurring uncertainties (ROUs) and randomly occurring nonlinearities (RONs) has been proposed for the first time. In Ref. [18], the authors provided a leader-following consensus protocol for a class of multi-agent system with RONs and ROUs under the undirected and fixed topology. Not enough attention has been paid to research the non-linear stochastic multi-agent systems (NSMSs) with ROUs and RONs, as far as we know. Besides, finite-time consensus of NSMSs with ROUs and RONs obtained a few achievement.

Up till now, tremendous amount of methods have been used to solve the consensus problem of multi-agent systems, for instance, adaptive control, robust control, feedback control and so on [19]–[21]. Compared with these continuous-time control methods, impulsive control techniques is a definite advantage to control of the complex network, especially, the systems cannot be controlled by continuous-time control [22], [23]. In Ref. [23], the problem of finite-time stability for impulsive systems has been studied by using the impulsive control theory. Besides, only at the discrete instants did impulsive control use a fraction of control impulses, which not only reduced the energy loss but also reduced amount of control cost. However, it is impractical to add the control into each agent when the scale of systems is very huge. At this time, the pinning control technique is an effective way that decrease the amount of controlled nodes [24], [25]. Only on the designated nodes do we need to exert control. And this way greatly reduces the control pressure of large-scale systems. Particularly, the impulsive pinning control is one of most effective control methods. It not only has the merits of the impulsive control, but also has the advantages of pinning control [26], [27]. In this article, we will introduce impulsive pinning control for finite-time consensus problems of multi-agent systems.

Although our analysis can naturally estimate the setting time of the finite-time consensus, the setting time relies on the initial states of the system, which should be given beforehand. What the initial values are not known beforehand restricts the applications in practical situation. At present, authors proposed a new control method named fixed-time control which is an advanced finite-time control [28]. Unlike the finite-time approach, regardless of what initial value is chosen, the setting time for complex network to achieve agreement is invariant via fixed-time control. Since then, lots of results about fixed-time consensus of complex systems have been extensively published in the related journal [29]–[34]. In Ref. [29], in order to decrease communication costs, the authors proposed a novel protocol including quantized control and impulsive control to achieve fixed-time consensus of multi-agent systems. In Ref. [32], the authors designed a novel controller which can restrain the chattering phenomenon to guarantee that the aim systems with desynchronizing and synchronizing impulsive signal realized fixed-time synchronization with an isolated system. These highlight the important application of fixed-time consistency in multi-agent systems. However, it is worth noting that the study about finite-time and fixed-time consensus of NSMSs with RONs and ROUs is less covered.

Motivated by the above-mentioned considerations, this paper investigates the finite-time and fixed-time consensus of NSMSs with ROUs and RONs via impulsive control. In the following, the main theoretical contributions of this article can be summarized.

- Compared with previous work, two effective control protocols are both presented so that we can resolve the problem of finite-time and fixed-time consensus of NSMSs with RONs and ROUs.
- The high control cost of the target systems is reduced by using a pinning control strategy, which it only needs to control a small fraction of agents.
- With the help of the comparison system theory and average impulsive interval, the finite/fixed (dependent/independent on the initial states) settling time can be estimated, respectively.

The rest of this paper is organized as follows. Section II presents problem formulation and some preliminaries. Analytical arguments for the finite-time consensus are investigated in Section III.In Section IV, we propose novel control protocol for the fixed-time consensus of NSMSs. In Section V, two numerical simulation examples are provided to validate our results. At last, the conclusion is given in Section VI.

Notations: In this paper, unless otherwise specified, $\mathbb{N}^+$ denotes the positive integers, $\mathbb{R}$ represents the real numbers and $\mathbb{R}^n$ is the set of n-dimensional Euclidean space. I_n denotes the identity matrix with compatible dimensions. $\otimes$ represents the Kronecker product. $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ mean the smallest and the largest eigenvalue of matrix A, respectively. diag $\{\cdots\}$ stands for a diagonal matrix and *trace* $[A]$ stands for the trace of matrix A. $\left(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathcal{P}\right)$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ which satisfy the usual conditions: the filtration contains all P-null sets and is right continuous. $E(\cdot)$ stands for the exception operator with respect to some probability measure P. $\|\cdot\|$ stands for the Euclidean norm of a matrix. $|\cdot|$ denotes the absolute value of a number. $\mathbb{C}^{1,2}$ denotes the family of all nonnegative function $V(t,x)$ that are continuously once differential in t and twice in x. $Pr\{\alpha\}$ and $Pr\{\beta\}$ mean the occurrence probability of event α, β. For a continuous function $\vartheta : \mathbb{R} \to \mathbb{R}$, denote $\vartheta\left(t^-\right)| = \lim_{s\to 0^-} \vartheta(t+s)$, and the upper Dini derivative of $\vartheta(t)$ is represented by $D^+\vartheta(t) = \lim_{s\to 0^+} sup\,(\vartheta(t+s) - \vartheta(t))/s$.

II. PRELIMINARIES

A. GRAPH THEORY

Graph theory, as a practical tool, is a great option to express the join conditions among agents. In this subsection,for some basic notations of algebraic graph theory, we give a brief introduction. A multi-agent system can be represented by an undirected graph $G = (\mathbb{V}, \mathbb{E}, A)$. Without multiple edges and self-loops, the N agents is consisted of a set of nodes $\mathbb{V} = \{v_1, v_2, \cdots, v_N\}$ and the edge set $\mathbb{E} \subseteq \mathbb{V} \times \mathbb{V}$ stands for the communication transmission among nodes. The weighted adjacency matrix is denoted by $A = (a_{ij}) \in \mathbb{R}^{N\times N}$. An edge of undirected graph G is denoted by (i, j), representing there is bidirectional information flow between agent i and agent$j$. The elements of A are denoted as follows: if the edge $(i, j) \in \mathbb{E}$, then $a_{ij} = a_{ji} > 0$, otherwise $a_{ij} = 0$, and the diagonal elements of A are zeros, that is $a_{ii} = 0$. $L = [l_{ij}] \in \mathbb{R}^{N\times N}$ is stood for the Laplacian matrix, which $l_{ij} = -a_{ij}$ when $i \neq j$ and $l_{ii} = \sum_{j\in\mathbb{N}_i}^{N} a_{ij}$. In the graph, the leader is represented by s_0 which could send the information to the following agents. An undirected graph $\bar{G}$ stands for the leader-following communication topology. We use a matrix $H = L + C$ to describe the structure of $\bar{G}$, where $C = \mathrm{diag}\{c_1, c_2, \cdots, c_N\}$ with $c_i = 1$ if (s_0, i) is an edge of $\bar{G}$ and with $c_i = 0$ otherwise. $H = (h_{ij})_{N\times N}$ can be written as

$$h_{ij} = \begin{cases} l_{ii} + c_i, & i = j, \\ l_{ij}, & i \neq j. \end{cases}$$

B. PROBLEM STATEMENT

In this paper, a class of NSMSs with ROUs and RONs is considered. The dynamics of follower agents can be described as follows:

$$\begin{cases} ds_i(t) = [A(t)s_i(t) + \beta(t)f(t, s_i(t)) + u_i(t)]\,dt \\ \qquad\qquad + \sigma(t, s_i(t))\,dw(t), \\ s_i(t) = \phi_i(t), t \leq 0, \end{cases} \tag{1}$$

where $s_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}^m$ are the state vector and the control protocol of the ith agent to be designed later, respectively, $i = 1, 2, \cdots, N$. $f(\cdot) : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ stands for a continuous nonlinear function of agent i. $\sigma : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ is the noisy intensity function. $w(t)$ is a scalar Brownian motion defined on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathcal{P})$, which satisfies $E\{dw(t)\} = 0$ and $E\{[dw(t)]^2\} = dt$; $\phi_i(t)$ is the continuous initial condition of ith agent. $A(t)$ is a weight matrix and can be further described as $A(t) = A + \alpha(t)\Delta A(t)$, $\Delta A(t) = MF(t)Q$. A, M and Q are approximatively dimensional known constant matrices and $F(t)$ stands for the non-linear time-varying function satisfied that

$$F(t)^T F(t) \leq I. \tag{2}$$

The terms $\alpha(t)\Delta A(t)$ represents the phenomena of ROUs, and $\beta(t)f(\cdot,\cdot)$ stands for the phenomena of RONs. Random variables $\alpha(t), \beta(t)$ are both Bernoulli distributing white sequences which the values are either one or zero. Naturally, they are assumed as follows:

$$\begin{cases} Pr\{\alpha(t) = 1\} = \alpha, & Pr\{\alpha(t) = 0\} = 1 - \alpha, \\ Pr\{\beta(t) = 1\} = \beta, & Pr\{\beta(t) = 0\} = 1 - \beta, \end{cases} \tag{3}$$

where $\alpha, \beta \in [0, 1]$ are known constants. Further, it is assumed that the stochastic variables $\alpha(t)$, $\beta(t)$ and $w(t)$ are reciprocally independent. From (3), we can get that $E\{\alpha(t) - \alpha\} = 0, E\{\beta(t) - \beta\} = 0$.

Let $s_0 \in \mathbb{R}^n$ be the state of the leader agent. The dynamic of s_0 is described as follows:

$$\begin{cases} ds_0(t) = [A(t)s_0(t) + \beta(t)f(t, s_0(t))] \\ \qquad\qquad + \sigma(t, s_0(t))\,dw(t), \\ s_0(t) = \psi(t), t \leq 0, \end{cases} \tag{4}$$

where $\psi(t)$ is the continuous initial state of leader s_0.

Before staring the main results, some assumptions, lemmas and definitions are introduced in the following.

Definition 1 [35]: Suppose that there exists a positive constant T_a and a positive integer N_0 such that

$$\frac{\tilde{t} - t}{T_a} - N_0 \leq N_\varsigma(t, \tilde{t}) \leq N_0 + \frac{\tilde{t} - t}{T_a} \tag{5}$$

for any $\tilde{t} > t \geq 0$, where $N_\varsigma(t, \tilde{t})$ stands for the number of impulsive times of the impulsive sequence $\varsigma = \{t_1, t_2, \cdots\}$ on the interval $(t, \tilde{t})$. Then, it is said that the impulsive sequence $\varsigma = \{t_1, t_2, \cdots\}$ have an average impulsive interval T_a.

Definition 2: System (1) is said to achieve finite-time leader-following consensus in probability, if there exists a constant $T > 0$ which depends on the initial condition vector value $s(0)$ such that

$$P\left\{\lim_{t\to T_{s(0)}} s_i(t) - s_0(t) = 0\right\} = 1, \quad \forall i \in N.$$

Definition 3: The multi-agent (1) is said to achieve fixed-time leader-following consensus in probability, if the finite-time leader-following consensus is solved and an estimated settling-time $T_{s(0)}$ is further uniformly bounded by a fixed positive constant.

Assumption 1: The non-linear function $f(\cdot)$, $\sigma(\cdot)$ are Lipschitz continuous, there exist known constant matrices J and Σ such that

$$\|f(t, x_1) - f(t, x_2)\| \leq \|J(x_1 - x_2)\|,$$
$$\|\sigma(t, x_1) - f(t, x_2)\| \leq \|\Sigma(x_1 - x_2)\|,$$

for all $x_1, x_2 \in \mathbb{R}^n$.

Lemma 1 [36]: For any $x, y \in \mathbb{R}^n$ and $\varepsilon > 0$, then we obtain that

$$x^T y + y^T x \leq \varepsilon x^T x + \varepsilon^{-1} y^T y.$$

Lemma 2 [29]: Let $s_1, s_2, \cdots, s_N \geq 0$, $0 < p < 1, q > 1$, the following two inequalities hold

$$\sum_{i=1}^{N} s_i^p \geq \left(\sum_{i=1}^{N} s_i\right)^p, \quad \sum_{i=1}^{N} s_i^q \geq N^{1-q}\left(\sum_{i=1}^{N} s_i\right)^q.$$

Lemma 3 [22]: A continuous Lipschitz function $V(t)$ is assumed that satisfies

$$\dot{V}(t) \leq -aV^b(t), \quad \forall t \geq t_0, \; V(t_0) \geq 0,$$

where $a > 0$, $0 < b < 1$ are constants, then $V(t)$ satisfies

$$V^{1-b}(t) \leq V^{1-b}(t_0) - a(1 - b)(t - t_0), \quad t_0 \leq t \leq T,$$

and $V(t) \equiv 0, \forall t \geq T$, with the settling time T given by

$$T = t_0 + \frac{V^{1-b}(t_0)}{a(1-b)}.$$

Lemma 4 [22]: Suppose that the continuous and non-negative function $V(t)$ satisfies conditions as follows:

$$\begin{cases} \dot{V}(t) \leq -aV^b(t), & t \neq t_k, \\ V(t_k^+) \leq \xi_k V(t_k^-), & t = t_k, \end{cases}$$

where $a > 0$, $0 < b < 1$, $0 < \xi_k < 1$, $k = 1, 2, \cdots, n$, then following inequality holds:

$$V^{1-b}(t) \leq V^{1-b}(t_0) - a(1-b)(t - t_0), \quad t_0 \leq t \leq T,$$

where T is a constant which stands for the settling time.

III. FINITE-TIME LEADER-FOLLOWING CONSENSUS

In this section, the finite-time leader-following consensus criteria for NSMSs with ROUs and RONs will be investigated. Let $e_i(t) = s_i(t) - s_0(t)$. To achieve the followers to track synchronization for the leader within finite time, a control algorithm of ith agent is designed as follows:

$$\begin{aligned} u_i(t) = &-\rho e_i(t) - \eta sign(e_i(t)) |e_i(t)|^\gamma \\ &+ b_k[\sum_{j\in N_i} a_{ij}(e_i(t) - e_j(t)) + c_i e_i(t)] \sum_{k=1}^{\infty} \delta(t - t_k), \\ &k \in \mathbb{N}^+, \quad i = 1, 2, \cdots, N, \end{aligned} \tag{6}$$

where $\rho, \eta > 0$ are control parameters, γ is a variable constant satisfying $0 < \gamma < 1, \gamma \in \mathbb{R}$, b_k represents the control gains, $\delta(t)$ is the Dirac delta function satisfying $\delta(t) = 0$ for $t \neq 0$. $\mathbb{Z} = \{t_k\}$ is a strict increased sequence which satisfies $t_{k-1} < t_k$ and $\lim_{t_k} \to +\infty$ as $k \to +\infty$, $sign(e_i(t))$ denotes the signum function satisfying $sign(e_i(t)) = \mathrm{diag}(sign(e_{i1}(t)), sign(e_{i2}(t)), \cdots, sign(e_{in}(t)))$, which is defined as follows:

$$sign(x) = \begin{cases} -1, & if\ x < 0, \\ 0, & if\ x = 0, \\ 1, & if\ x > 0. \end{cases}$$

Remark 1: Compared with system (1) in [17], we add $A(t)$ in system (1). At the same time, the RONs is considered in it. Also, different from the continuous-time control method in [17], both nonlinear controller and impulsive controller are used for finite-time leader-following consensus of (1) in this paper. Besides, compared with the continuous-time control, the impulsive control accelerates the convergence of the multi-agent systems.

Remark 2: Different from the control protocol (4) in [7], we propose the control method to realize leader-following consensus in finite time in this paper. Although the energy cost of the finite-time control method may be much more expensive than traditional technique, the pinning control technique is proposed to reduce the cost as far as possible.

Thus, our control objective is to use control strategy (6) to make the trajectory of system (1) consistent with that of system (4). And the tracking error dynamics systems can be described by

$$\begin{cases} de_i(t) = [A(t)e_i(t) + \beta(t)F(t, e_i(t)) \\ \qquad - \rho e_i(t) - \eta sign(e_i(t))|e_i(t)|^\gamma]dt \\ \qquad + \tilde{\sigma}(t, e_i(t))dw(t), t \neq t_k, \\ \Delta e_i(t_k) = b_k[\sum_{j\in N_i} a_{ij}(e_i(t_k^-) - e_j(t_k^-)) \\ \qquad + c_i e_i(t_k^-)], t = t_k, \\ e_i(t) = \phi_i(t) - \psi(t), \end{cases} \tag{7}$$

where $i = 1, 2, \cdots, N$, $F(t, e_i(t)) = f(t, s_i(t)) - f(t, s_0(t))$ and $\tilde{\sigma}(t, e_i(t)) = \sigma(t, s_i(t)) - \sigma(t, s_0(t))$. Yet the general, $s_i(t), s_0(t), e_i(t)$ are assumed to be right-hand continuous at $t = t_k$, for instance, $s_i(t_k) = s_i(t_k^+)$, $s_0(t_k) = s_0(t_k^+)$ and $e_i(t_k) = e_i(t_k^+)$. $\Delta e_i(t_k) = e_i(t_k^+) - e_i(t_k^-)$.

Let $e(t) = [e_1^T(t), e_2^T(t), \cdots, e_N^T]^T$, $\bar{F}(t, e(t)) = [F^T(t, e_1(t)), F^T(t, e_2(t)), \cdots, F^T(t, e_N(t))]^T$, $\bar{\sigma}(t, e(t)) = [\tilde{\sigma}^T(t, e_1(t)), \tilde{\sigma}^T(t, e_2(t)), \cdots, \tilde{\sigma}^T(t, e_N(t))]^T$, $\Phi(t) = (\phi_1^T(t), \cdots, \phi_N^T(t))^T$, $\Psi(t) = (\psi^T(t), \cdots, \psi^T(t))^T$, $\varphi(t) = \Phi(t) - \Psi(t)$.

On the basis of the Kronecker product technology, system (7) can be rewritten in a compact form as follows:

$$\begin{cases} de(t) = [(I_N \otimes A(t))e(t) + \beta(t)\bar{F}(t, e(t)) \\ \qquad - \rho e(t) - \eta sign(e(t))|e(t)|^\gamma]dt \\ \qquad + \bar{\sigma}(t, e(t))dw(t), t \neq t_k, \\ \Delta e(t_k) = b_k((H \otimes I_N)e(t_k^-)), t = t_k, \\ e(t) = \varphi(t), \end{cases} \tag{8}$$

In the following, we present a theoretical result to guarantee that the follower system (1) and the leader system (4) with ROUs, RONs and stochastic disturbances can reach finite-time consensus via the control protocol (6). And then, we give a following result.

Theorem 1: Suppose that Assumption 1 hold. Suppose that positive constants $\varepsilon_1, \rho, \eta$, and $0 < \gamma < 1$ satisfying

$$\lambda_{\max}(\Pi_1) + \lambda_{\max}(\Pi_2) + 2\beta\|J\| - 2\rho < 0, \tag{9}$$

$$\theta < 1, \tag{10}$$

where $\Pi_1 = I_N \otimes (A^T + A + \varepsilon_1\alpha^2 MM^T + \varepsilon_1^{-1}Q^T Q)$, $\Pi_2 = I_N \otimes (\Sigma^T\Sigma)$, $\theta = sup_{k\in\mathbb{N}^+}\lambda_{\max}(\Theta_k)$, $\Theta_k = (b_k(H \otimes I_n) + I_{Nn})^T(b_k(H \otimes I_n) + I_{Nn})$.
then the NSMSs (1) and (4) with the controller (6) can realize the finite-time consensus in a finite time T

$$T \leq \frac{V(0)^{1-\frac{1+\gamma}{2}}}{2\eta\left(1 - \frac{1+\gamma}{2}\right)}. \tag{11}$$

Proof: Define the following Lyapunov function candidate

$$V(t, e(t)) = e^T(t)e(t). \tag{12}$$

Similar as previous papers, let the notation $\mathcal{L}$ stand for the Kolmogorov operator of Itô stochastic system. Let function

$V(t,s) \in \mathbb{C}^{1,2}$, on the basis of Itô rule, the equation is obtained along with the evolution of random system $ds(t) = f(t,s(t))dt + g(t,s(t))dw(t)$ as follows:

$$dV(t,s(t)) = \mathcal{L}V(t,s(t))dt + V_s(t,s(t))g(t,s(t))dw(t),$$

where $\mathcal{L}V(t,s) = V_t(t,s) + V_s(t,s)f(t,s)$ $+\frac{1}{2}Tr\left[g^T(t,s)V_{ss}(t,s)g(t,s)\right]$.

According to Itô differential formula, the derivative of (12) with respect to (8) can be given as follows:

$$dV(t,e(t)) = \mathcal{L}V(t,e(t))dt + 2e^T(t)\bar{\sigma}(t,e(t))dw(t), \tag{13}$$

$$\begin{aligned}\mathcal{L}V(t,e(t)) = 2e^T(t)&[(I_N \otimes A(t))e(t) \\ &+ \beta(t)\bar{F}(t,e(t)) - \rho e(t) \\ &- \eta sign(e(t))|e(t)|^{\gamma}] \\ &+ trace[\bar{\sigma}^T(t,e(t))\bar{\sigma}(t,e(t))].\end{aligned} \tag{14}$$

On the basis of Lemma 1, it can be obtained that

$$\begin{aligned}&2e_i^T(t)(A(t))e_i(t) \\ &= 2e_i^T(t)(A + \alpha(t)MF(t)Q)e_i(t) \\ &= e_i^T(t)\left(A^T + A\right)e_i(t) + 2\alpha e_i^T(t)MF(t) \\ &\quad \times Qe_i(t) + 2(\alpha(t) - \alpha)e_i^T(t)MF(t)Qe_i(t) \\ &\leq e_i^T(t)\left[A^T + A + \varepsilon_1\alpha^2 MM^T + \varepsilon_1^{-1}Q^TQ\right] \\ &\quad \times e_i(t) + 2(\alpha(t) - \alpha)e_i^T(t)MF(t)Qe_i(t).\end{aligned} \tag{15}$$

Based on the Kronecker product, we can rewrite (13) as follows:

$$\begin{aligned}2e^T(t)(I_N \otimes A(t))e(t) \leq e^T(t)(I_N \otimes (&A^T + A \\ &+ \varepsilon_1\alpha^2 MM^T + \varepsilon_1^{-1}Q^TQ)) \\ &\times e(t) + 2(\alpha(t) - \alpha)e^T(t) \\ &\times MF(t)Qe(t),\end{aligned} \tag{16}$$

and on the basis of Assumption 1, one can obtain

$$\begin{aligned}2e^T(t)\beta(t)\bar{F}(t,e(t)) = 2e^T(t)\beta\bar{F}(t,e(t)) &+ 2(\beta(t) \\ &- \beta)e^T(t)\bar{F}(t,e(t)) \\ \leq 2\beta\|J\|e^T(t)e(t) &+ 2(\beta(t) \\ &- \beta)e^T(t)\bar{F}(t,e(t)).\end{aligned} \tag{17}$$

Next, based on Assumption 1, we have

$$\begin{aligned}&trace\left[\bar{\sigma}^T(t,e(t))\bar{\sigma}(t,e(t))\right] \\ &\qquad \leq e^T(t)\left(I_N \otimes \left(\Sigma^T\Sigma\right)\right)e(t),\end{aligned} \tag{18}$$

combining (16)-(18) and condition (9) , we can further get

$$\begin{aligned}\mathcal{L}V(t) \leq 2(\alpha(t) - \alpha)&e^T(t)(I_N \otimes (MF(t)Q))e(t) \\ &- 2(\beta(t) - \beta)e^T(t)\bar{F}(t,e(t)) \\ &- 2\eta\left(e^T(t)e(t)\right)^{\frac{1+\gamma}{2}}.\end{aligned} \tag{19}$$

Since $E\mathcal{L}V(t,e(t))$ is continuous in $t \in (t_{k-1},t_k]$, it can be obtained as follows

$$D^+EV(t,e(t)) = E\mathcal{L}V(t,e(t)), \quad t \in (t_{k-1},t_k], \ k \in \mathbb{N}^+. \tag{20}$$

Then, (19) can be yielded by using the mathematical expectation as follows:

$$E\mathcal{L}V(t,e(t)) \leq -2\eta(E[V(t,e(t))])^{\frac{1+\gamma}{2}}. \tag{21}$$

Obviously, according to Lemma 3, we can get that

$$\begin{aligned}V^{1-\frac{1+\gamma}{2}}(t) \leq V^{1-\frac{1+\gamma}{2}}(t_0) - 2\eta\left(1 - \frac{1+\gamma}{2}\right)& \\ \times (t - t_0), t \in (t_{k-1},t_k], \quad k \in \mathbb{N}^+&.\end{aligned} \tag{22}$$

In addition, when $t = t_k$, on can obtain

$$\begin{aligned}V\left(t_k^+, e\left(t_k^+\right)\right) &= e^T\left(t_k^+\right)e\left(t_k^+\right) \\ &= \left((b_k(H \otimes I_n) + I_{Nn})e\left(t_k^-\right)\right)^T \\ &\quad \times \left((b_k(H \otimes I_n) + I_{Nn})e\left(t_k^-\right)\right) \\ &\leq \theta V\left(t_k^-, e\left(t_k^-\right)\right),\end{aligned} \tag{23}$$

where $\theta < 1$. Use the mathematical expectation, on has that

$$E\left\{V\left(t_k^+\right)\right\} \leq \theta EV\left(t_k^-\right). \tag{24}$$

Based on Lemma 4, we can obtain that

$$\begin{aligned}V^{1-\frac{1+\gamma}{2}}(t) \leq V^{1-\frac{1+\gamma}{2}}(t_0) - 2\eta\left(1 - \frac{1+\gamma}{2}\right)& \\ \times (t - t_0), \quad t_0 \leq t \leq T&.\end{aligned} \tag{25}$$

Easily, we can find that $V(t)$ will be reach in the finite time, which indicates that error system will be zero. And we can estimate the finite time as follows:

$$T \leq \frac{V(0)^{1-\frac{1+\gamma}{2}}}{2\eta\left(1 - \frac{1+\gamma}{2}\right)}. \tag{26}$$

Therefore, by the above analysis, based on Lemma 3 and Lemma 4, the NSMSs (1) can achieve consensus with system (4) in the finite time T by using the proposed control protocol.

The proof is completed.

Remark 3: It is noted that the setting time in (26) relies on the initial values. However, we may not give the estimation of the setting time while the initial values are hard to get. Moreover, if the initial state enlarges, the setting time may also become too big. Thus, we propose a new class of control protocol in next section which the estimated setting time is independent of the initial state.

IV. FIXED-TIME LEADER-FOLLOWING CONSENSUS

In this part, the fixed-time leader-following consensus criteria for NSMSs with ROUs and RONs will be investigated. Let $e_i(t) = s_i(t) - s_0(t)$ stand for the consistency error between the follower s_i and the leader s_0. To achieve the followers to track synchronization for the leader within fixed time, we design a control algorithm of ith agent as follows:

$$\begin{aligned} u_i(t) = &-\kappa_1 e_i(t) - \kappa_2 sign(e_i(t))|e_i(t)|^{\iota} \\ &- \kappa_3 sign(e_i(t))|e_i(t)|^{d} + b_k[\sum_{j\in N_i} a_{ij}(e_i(t) \\ &- e_j(t)) + c_i e_i(t)]\sum_{k=1}^{\infty}\delta(t-t_k), \\ &k\in\mathbb{N}^+, \quad i=1,2,\cdots,N. \end{aligned} \tag{27}$$

where $\kappa_1, \kappa_2, \kappa_3 > 0$ are control parameters, ι, d are variable constants satisfying $0 < \iota < 1, d > 1, \iota, d \in \mathbb{R}$

Remark 4: Compared with the control protocol (6) in Theorem 1, the control protocol (27) adds an extra term $-\kappa_3 sign(e_i(t))|e_i(t)|^d$, $\kappa_3 > 0, d > 1$, which can realize fixed-time consensus. Without the term $-\kappa_3 sign(e_i(t))|e_i(t)|^d$, the control protocol (27) returns to the control protocol which can only realize finite-time consensus.

Remark 5: Although the problem for the fixed-time consensus of multi-agent systems have been studied in [33], [34], the fixed-time consensus of NSMSs via impulsive control protocol is not considered. In this article, we have proposed a theory of the fixed-time consensus of NSMSs by using impulsive control, which is of great significant from a theoretical and practical viewpoint.

Therefore, our control objective is to use control strategy (27) to make the trajectory of system (4) to be tracked by the system (1). And the error dynamics can be expressed as follows:

$$\begin{cases} de_i(t) = [A(t)e_i(t) + \beta(t)F(t,e_i(t)) \\ \qquad - \kappa_1 e_i(t) - \kappa_2 sign(e_i(t))|e_i(t)|^{\iota} \\ \qquad - \kappa_3 sign(e_i(t))|e_i(t)|^{d}]dt \\ \qquad + \tilde{\sigma}(t,e_i(t))dw(t), t\neq t_k, \\ \Delta e_i(t_k) = b_k[\sum_{j\in N_i} a_{ij}\left(e_i(t_k^-) - e_j(t_k^-)\right) \\ \qquad + c_i e_i(t_k^-)], t = t_k, \\ e_i(t) = \phi_i(t) - \psi(t). \end{cases} \tag{28}$$

Similarly, on the basis of Kronecker product, we can rewrite Eq.(28) in the following:

$$\begin{cases} de(t) = [(I_N\otimes A(t))e(t) + \beta(t)\bar{F}(t,e(t)) \\ \qquad - \kappa_1 e(t) - \kappa_2 sign(e(t))|e(t)|^{\iota} \\ \qquad - \kappa_3 sign(e(t))|e(t)|^{d}]dt \\ \qquad + \bar{\sigma}(t,e(t))dw(t), t\neq t_k, \\ \Delta e(t_k) = b_k\left((H\otimes I_N)e(t_k^-)\right), t=t_k, \\ e(t) = \varphi(t). \end{cases} \tag{29}$$

Next, we present a theoretical result to guarantee that the follower system (1) and the leader system (4) with ROUs, RONs and stochastic disturbances can reach finite-time consensus via control protocol (27). And then, we give a following result.

Theorem 2: Suppose that Assumption 1 hold. Suppose that positive constants $\varepsilon_2, \kappa_1, \kappa_2, \kappa_3$, and $0 < \iota < 1, d > 1$ satisfying

$$\lambda_{\max}(\Pi_1) + \lambda_{\max}(\Pi_2) + 2\beta\|J\| - 2\kappa_1 < 0, \tag{30}$$

where $\Pi_1 = I_N \otimes \left(A^T + A + \varepsilon_2\alpha^2 MM^T + \varepsilon_2^{-1}Q^T Q\right)$, $\Pi_2 = I_N \otimes \left(\Sigma^T\Sigma\right)$, $\theta = sup_{k\in\mathbb{N}^+}\lambda_{\max}(\Theta_k)$, $\Theta_k = (b_k(H\otimes I_n) + I_{Nn})^T(b_k(H\otimes I_n) + I_{Nn})$,

then NSMSs (1) and (4) with the controller (27) can reach consensus in the fixed time. Moreover, the setting time can be estimated at T.

$$\begin{aligned} T = T_1 + T_2 = &\frac{T_a}{(1-m)\ln\theta}\ln\left(1 - \frac{\theta^{N_0(1-m)}\ln\theta}{2T_a\bar{\kappa}_3}\right) \\ &+ \frac{T_a}{(1-\omega)\ln\theta}\ln\left[\frac{2T_a\kappa_2}{2T_a\kappa_2\theta^{2N_0(1-\omega)} - \ln\theta}\right] \\ &+ 2T_a N_0 \end{aligned}$$

when $0 < \theta < 1$ and

$$T = T_1 + T_2 = \frac{1}{2\bar{\kappa}_3(m-1)} + \frac{1}{2\kappa_2(1-\omega)}$$

when $\theta = 1$, where $m = \frac{1+d}{2} > 1$ and $0 < \omega = \frac{1+\iota}{2} < 1$, $\bar{\kappa}_3 = \kappa_3(Nn)^{\frac{1-d}{2}}$.

Proof: Similar to the proof of Theorem 1, the same Lyapunov candidate is considered as follows:

$$V(t,e(t)) = e^T(t)e(t). \tag{31}$$

Based on the Itô differential formula and using Lemma 2, we can obtain the derivative of (31) with respect to (29) in the following:

$$\begin{aligned} \mathcal{L}V(t,e(t)) \le &2(\alpha(t)-\alpha)e^T(t)(I_N\otimes(MF(t)Q))e(t) \\ &- 2(\beta(t)-\beta)e^T(t)\bar{F}(t,e(t)) \\ &- 2\kappa_2(V(t,e(t)))^{\frac{1+\iota}{2}} - 2\bar{\kappa}_3(V(t,e(t)))^{\frac{1+d}{2}}, \end{aligned} \tag{32}$$

where $\bar{\kappa}_3 = \kappa_3(Nn)^{\frac{1-d}{2}}$.

Then, by taking the mathematical expectation on both sides of (13) with (32), one has

$$\begin{aligned} E\mathcal{L}V(t,e(t)) \le &-2\kappa_2(E[V(t,e(t))])^{\frac{1+\iota}{2}} \\ &- 2\bar{\kappa}_3(E[V(t,e(t))])^{\frac{1+d}{2}} \end{aligned} \tag{33}$$

On the other hand, when $t = t_k$, one can obtain

$$\begin{aligned} V(t_k^+, e(t_k^+)) &= e^T(t_k^+)e(t_k^+) \\ &= \left((b_k(H\otimes I_n) + I_{Nn})e(t_k^-)\right)^T \\ &\quad\times\left((b_k(H\otimes I_n) + I_{Nn})e(t_k^-)\right) \\ &\le \theta V(t_k^-, e(t_k^-)), \end{aligned} \tag{34}$$

Take the mathematical expectation, then we can obtain

$$E\left\{V(t_k^+)\right\} \le \theta EV(t_k^-). \tag{35}$$

On the basis of the above analysis, we can consider the system for comparison purpose as follows:

$$\begin{cases} \dot{v}(t) = \begin{cases} -2\kappa_2 v^{\frac{1+\iota}{2}}(t), & 0 < v(t) < 1, t \neq t_k \\ -2\bar{\kappa}_3 v^{\frac{1+d}{2}}(t), & v(t) \geq 1, t \neq t_k \\ 0, v(t) = 0, & t \neq t_k \end{cases} \\ v(t_k) = \theta v(t_k^-), \\ v(0) = v_0. \end{cases} \quad (36)$$

Comparing (33) and (35) with (36), we can find that $0 \leq V(t) \leq v(t)$. As a result, if there exists $T > 0$ such as $v(t) \equiv 0$ for $t > T$, then $V(t) \equiv 0$ for $t > T$. Therefore, in order to demonstrate the fixed-time consensus of the error system (29), it is only necessary to demonstrate the corresponding problem of the zero solution of system (36).

Next, the system (36) will be considered in two cases such that $0 < \theta < 1$ and $\theta = 1$.

Case 1: $0 < \theta < 1$

Suppose $w(t) = v^{1-m}(t)$ when $v(t) \geq 1$. It can be seen from Eq. (36) that $w(t) \to 1$ when $v(t) \to 1$ and $w(t) \to 0$ when $v(t) \to +\infty$, then it is obtained that

$$\begin{cases} \dot{w}(t) = 2\bar{\kappa}_3(m-1), 0 \leq w(t) \leq 1, t \neq t_k \\ w(t_k) = \bar{\theta} w(t_k^-), t = t_k \\ w(0) = w_0 = v_0^{1-m}, \end{cases} \quad (37)$$

where $\bar{\theta} = \theta^{1-m}$ implies $\bar{\theta} \in [1, +\infty)$. It can be deduced from Eq. (37) that

$$w(t) = \bar{\theta}^{N_\varsigma(0,t)} w(0) + 2\bar{\kappa}_3(m-1)\int_0^t \bar{\theta}^{N_\varsigma(s,t)} ds \quad (38)$$

Since $w(0) = \bar{\theta}^{N_\varsigma(0,0)} w(0) = w_0 < 1$, $\lim_{t\to+\infty} w(t) = \infty$ and $w(t)$ is monotonously increasing on $[0, +\infty)$, there exists a positive number T_1 such that $\lim_{t\to T_1} w(t) = 1$ and $0 < w(t) < 1$ for $0 < t < T_1$.

Therefore, based on Eq. (38), one obtain that

$$\bar{\theta}^{N_\varsigma(0,t)} w(0) + 2\bar{\kappa}_3(m-1)\int_0^t \bar{\theta}^{N_\varsigma(s,t)} ds = 1, \quad (39)$$

which implies

$$2\bar{\kappa}_3(m-1)\int_0^t \bar{\theta}^{N_\varsigma(s,t)} ds \leq 1. \quad (40)$$

By simple computation from (5) and (40), we can get

$$t \leq \frac{T_a}{(1-m)\ln\theta} \ln\left(1 - \frac{\theta^{N_0(1-m)}\ln\theta}{2T_a\bar{\kappa}_3}\right). \quad (41)$$

Let

$$T_1 = \frac{T_a}{(1-m)\ln\theta} \ln\left(1 - \frac{\theta^{N_0(1-m)}\ln\theta}{2T_a\bar{\kappa}_3}\right).$$

In other word, one can obtain $w(t) \to 1$ when $t \to T_1$.

Suppose $w(t) = v^{1-\omega}(t)$, when $0 < v(t) \leq 1$,. It can be seen from Eq. (36) that $w(t) \to 1$ when $v(t) \to 1$ and $w(t) \to 0$ when $v(t) \to 0$, then one obtains

$$\begin{cases} \dot{w}(t) = -2\kappa_2(1-\omega), 0 < w(t) \leq 1, t \neq t_k, t \geq T_1 \\ w(t_k) = \tilde{\theta} w(t_k^-), t = t_k, t \geq T_1 \\ w(T_1) = v(T_1) = 1. \end{cases} \quad (42)$$

where $\tilde{\theta} = \theta^{1-\omega}$ implies $\tilde{\theta} \in (0, 1)$. According to Eq. (42), we can deduce that

$$w(t) = \tilde{\theta}^{N_\varsigma(T_1,t)} w(T_1) - 2\kappa_2(1-\omega)\int_{T_1}^t \tilde{\theta}^{N_\varsigma(s,t)} ds. \quad (43)$$

On the basis of Definition 1, it can be obtained that

$$w(t) \leq \tilde{\theta}^{\frac{t-T_1}{T_a} - N_0} - 2\kappa_2(1-\omega)\int_{T_1}^t \tilde{\theta}^{\frac{t-s}{T_a} + N_0} ds. \quad (44)$$

Further, let the right-hand side of (44) becomes zero and we can get

$$t - T_1 = \frac{T_a}{(1-\omega)\ln\theta} \ln\left[\frac{2T_a\kappa_2}{2T_a\kappa_2\theta^{2N_0(1-\omega)} - \ln\theta}\right] + 2T_a N_0. \quad (45)$$

Let

$$T_2 = \frac{T_a}{(1-\omega)\ln\theta} \ln\left[\frac{2T_a\kappa_2}{2T_a\kappa_2\theta^{2N_0(1-\omega)} - \ln\theta}\right] + 2T_a N_0.$$

In other word, we need time T_2 for the sake of $w(t) \to 0$ after $w(t) \to 1$ at time T_1.

On the basis of the previous discussions, when $0 < \theta < 1$, one can see that the setting time is

$$T = T_1 + T_2 = \frac{T_a}{(1-m)\ln\theta} \ln\left(1 - \frac{\theta^{N_0(1-m)}\ln\theta}{2T_a\bar{\kappa}_3}\right) + \frac{T_a}{(1-\omega)\ln\theta} \ln\left[\frac{2T_a\kappa_2}{2T_a\kappa_2\theta^{2N_0(1-\omega)} - \ln\theta}\right] + 2T_a N_0. \quad (46)$$

That being said, $v(t) \equiv 0$ when $t \geq T$.

Case 2: $\theta = 1$

We use a similar analytic method used Eq. (39) and Eq.(43). The results are obtained as follows:

$$T_1 = \frac{1}{2\bar{\kappa}_3(m-1)}, \quad T_2 = \frac{1}{2\kappa_2(1-\omega)}.$$

Therefore, when $\theta = 1$, we can estimate the setting time as follows:

$$T = T_1 + T_2 = \frac{1}{2\bar{\kappa}_3(m-1)} + \frac{1}{2\kappa_2(1-\omega)}.$$

The proof is thus completed.

Remark 6: When $\theta > 1$, it follows that $\tilde{\theta} = \theta^{1-\omega} > 1$ and the fixed-time consensus of NSMSs (1) can not be guaranteed. When $\tilde{\theta} > 1$, the setting time T_2 which make $w(t)$ of system (42) tend to zero can not be due to be estimated.

V. NUMERICAL EXAMPLES

In this section, we provide two simulation examples to illustrate the feasibility and effectiveness of the proposed control method. The consensus of multi-agent Chua's circuit systems is considered. Using the similar example as that given in [7], the Chua's oscillator is expressed in the following:

$$\begin{cases} \dot{s}_{i1}(t) = -p_1 s_{i1}(t) + p_1 s_{i2}(t) - p_1 g(s_{i1}(t)), \\ \dot{s}_{i2}(t) = s_{i1}(t) - s_{i2}(t) + s_{i3}(t), \\ \dot{s}_{i3}(t) = -p_2 s_{i2}(t), \end{cases} \tag{47}$$

where $s_i(t)$ denotes the state of ith agent, $g(s_{i1}(t)) = \epsilon_2 s_{i1}(t) + 0.5(\epsilon_1 - \epsilon_2)(|s_{i1}(t)+1| - |s_{i1}(t)-1|)$,$\epsilon_1 < \epsilon_2 < 0$ are known constants. Let $p_1 = 8.92, p_2 = 16.223, \epsilon_1 = -2.16$ and $\epsilon_2 = -0.885$. The chaotic behaviour is shown by (47). According to (1) and (47), we obtain that

$$A = \begin{bmatrix} -p_1(1+\epsilon_2) & p_1 & 0 \\ 1 & -1 & 1 \\ 0 & -p_2 & 0 \end{bmatrix},$$

$$f(t, s_i(t)) = \begin{bmatrix} -0.5p_1(\epsilon_1 - \epsilon_2)(|s_{i1}(t)+1| - |s_{i1}(t)-1|) \\ 0 \\ 0 \end{bmatrix}.$$

Let $F(t) = \mathrm{diag}\{-\sin(t), cons(t), \sin(t)\}$, $M = \mathrm{diag}\{0.3, -0.5, 0.4\}$, $Q = \mathrm{diag}\{0.2, 0.3, -0.5\}$, $\sigma(t, s_i(t)) = [\sqrt{0.2}\sin(t)s_{i1}(t), \sqrt{0.4}\sin(t)s_{i2}(t), \sqrt{0.6}\sin(t)s_{i3}(t)]^T$, $i = 1, 2, 3, 4$. Easily, we can see that $J = |\epsilon_1 p_1| I_n$, $\Sigma = \mathrm{diag}\{0.4, 0.6, 0.8\}$.

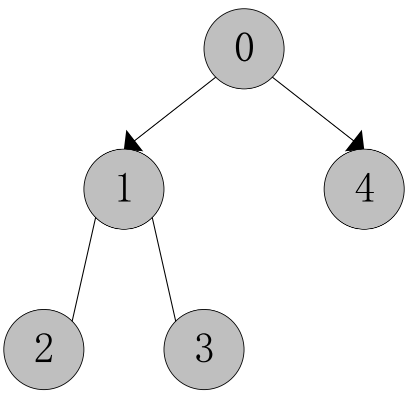


FIGURE 1. **The communication topology $\bar{G}$ of a multi-agent system.**

We consider a team of 5 agents which the interaction topology $\bar{G}$ is indicated by undirected graph presented in FIGURE1. Obviously, the leader is expressed as 0th agent and the rest of agents denote the followers.

From $\bar{G}$, the Laplacian matrix L and pinning gain matrix C are obtained as follows:

$$L = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & 1 & \\ & & & 0 \end{bmatrix},$$

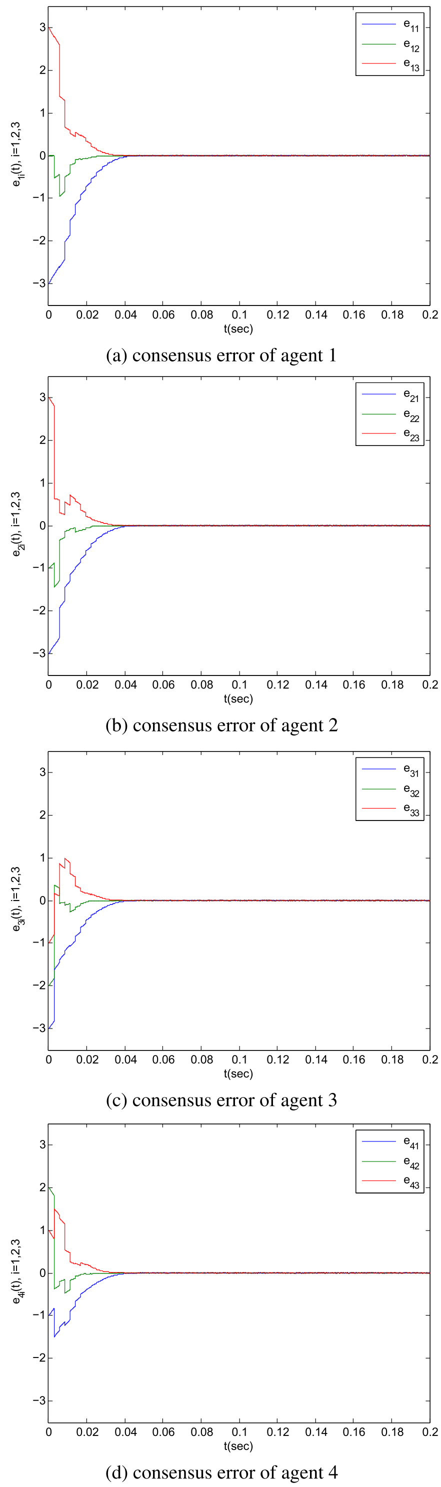
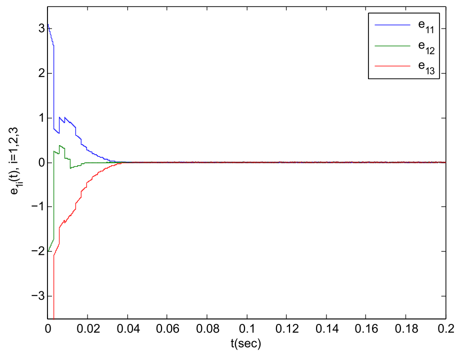


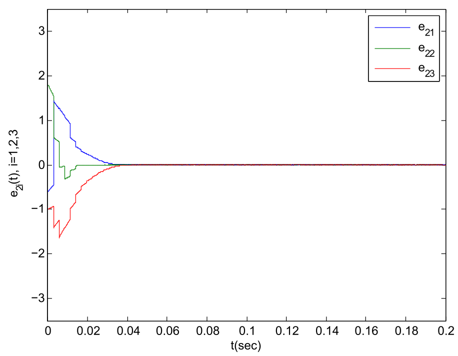
FIGURE 2. **Finite-time consensus of NSMSs via protocol (6) with initial states $s_1 = [2.1\ 0\ -1.82]^T$, $s_2 = [-1.6\ 3.8\ 2]^T$, $s_3 = [2.12\ 2\ 1]^T$, $s_4 = [-0.15\ 1\ 0]^T$, $s_0 = [-1\ 2\ 3]^T$.**

thus

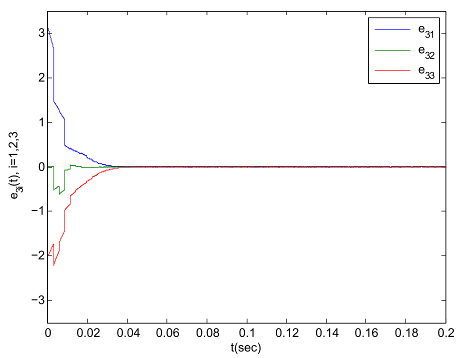
$$H = L + C = \begin{bmatrix} 3 & -1 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}.$$



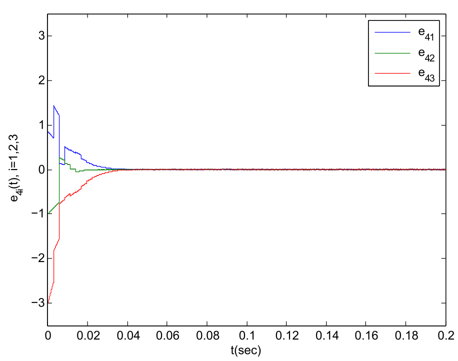
(a) consensus error of agent 1



(b) consensus error of agent 2

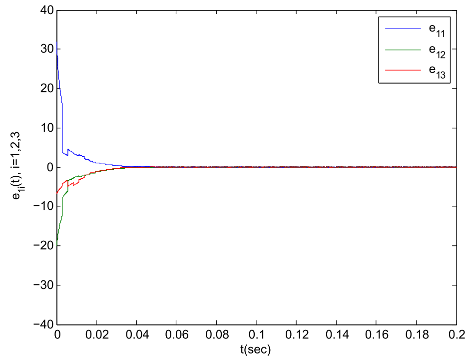


(c) consensus error of agent 3

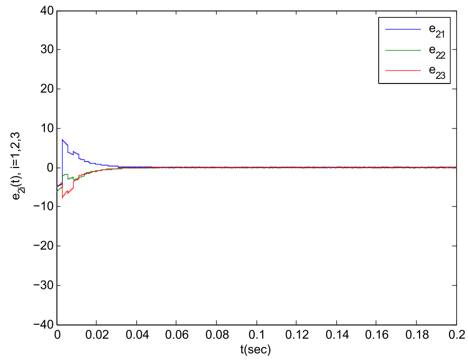


(d) consensus error of agent 4

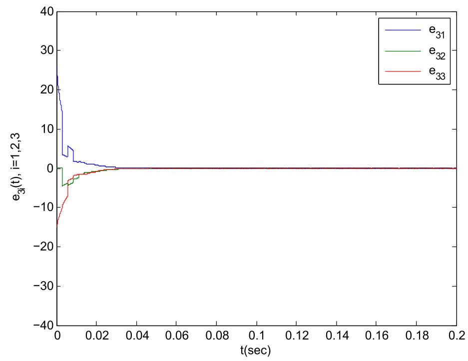
FIGURE 3. **Fixed-time consensus of NSMSs via protocol (27) with initial states $s_1 = [2.1\ 0\ -1.82]^T$, $s_2 = [-1.6\ 3.8\ 2]^T$, $s_3 = [2.12\ 2\ 1]^T$, $s_4 = [-0.15\ 1\ 0]^T$, $s_0 = [-1\ 2\ 3]^T$.**



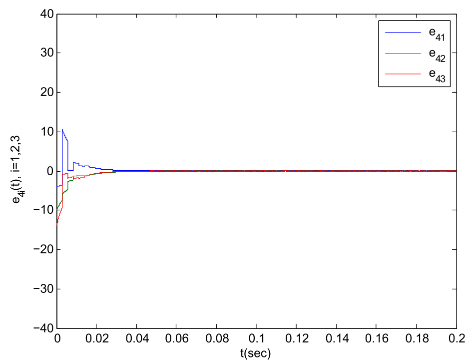
(a) consensus error of agent 1



(b) consensus error of agent 2



(c) consensus error of agent 3



(d) consensus error of agent 4

FIGURE 4. **Fixed-time consensus of NSMSs via protocol (27) with initial states $s_1 = [21\ 0\ 18.2]^T$, $s_2 = [-16\ 13.8\ 20]^T$, $s_3 = [15\ 20\ 10]^T$, $s_4 = [-15\ 10\ 11]^T$, $s_0 = [-11\ 20\ 25]^T$.**

Example 1 (Finite-Time Leader-Following Consensus of NSMSs With ROUs and RONs): Let $b_k = -0.6$, step-length is 0.0001. By simple calculation, it can be obtained that $\theta = 0.9032$. This value satisfies the condition (10) of Theorem 1. The Bernoulli-distributed stochastic variable $\alpha(t)$, $\beta(t)$ are assumed to satisfy (3) with $\alpha = 0.5$, $\beta = 0.5$. The initial states are taken as $s_1 = [2.1\ 0\ -1.82]^T$, $s_2 = [-1.6\ 3.8\ 2]^T$, $s_3 = [2.12\ 2\ 1]^T$, $s_4 = [-0.15\ 1\ 0]^T$, and the leader evolves from $s_0 = [-1\ 2\ 3]^T$. Besides, the condition (9) in Theorem 1 implies that $\rho > 4.9$ should be satisfied. Take $\rho = 5$, $\eta = 40$, $\gamma = 0.5, \varepsilon_1 = 1$. The numerical results for the proposed controller (6) is presented in FIGURE2. According to (11), the multi-agent system can achieve consensus within the settling time $T = 0.141$. In FIGURE2, under protocol (6), the setting time is approximate $t = 0.05$. This result proves the effectiveness and feasibility of Theorem 1.

Example 2 (Fixed-Time Leader-Following Consensus of NSMSs With ROUs and RONs): Set $\kappa_1 = 5, \kappa_2 = 40, \iota = 0.5, \kappa_3 = 10, d = 2$ in (27). Meantime $T_a = 0.028$ and $N_0 = 1$ characterize the impulsive time sequence. Other parameters are the same as Example 1. And the initial values are the same as those in Example 1. In FIGURE3, the settling time is about $t = 0.045$ under protocol (27). Then, change the initial states to $s_1 = [21\ 0\ 18.2]^T$, $s_2 = [-16\ 13.8\ 20]^T$, $s_3 = [15\ 20\ 10]^T$, $s_4 = [-15\ 10\ 11]^T$, and the leader evolves from $s_0 = [-11\ 20\ 25]^T$. In FIGURE4, under protocol (27) the setting time is approximate $t = 0.038$. According to Theorem 2, the multi-agent system can achieve consensus in settling time $T = 0.101$. The settling time for both cases is smaller than T and thereby demonstrating the effectiveness of Theorem 2.

Remark 7: According to the numerical simulations, obviously, the convergence rate under our control protocol is much faster than the results in [7]. Furthermore, numerical simulations demonstrate that the estimated setting time of fixed-time consensus is independent on the initial states.

VI. CONCLUSION

The finite-time and fixed-time consensus problems of NSMSs with RONs and ROUs are investigated in this paper. A new class of protocols proposed integrates nonlinear control and impulsive control. By using finite-time consensus method, the follower agents can achieve consensus with the leader in finite time. Furthermore, the control protocol has been improved and a fixed-time protocol has been proposed. This protocol allows followers to reach the agreement with the leader in finite time under arbitrary initial states. Compared with the finite-time consensus, the estimated setting time of the fixed-time consensus is shorter. Two simulation results have been presented to show the effectiveness of the new designed protocols. In the future, we will further research finite-time consensus of discrete dynamic systems with communication delays and Markovian jumping topology.

REFERENCES

[1] Z. H. Guan, Z. W. Liu, G. Feng, and M. Jian, "Impulsive consensus algorithms for second-order multi-agent networks with sampled information," *Automatica*, vol. 48, no. 7, pp. 1397–1404, 2012.

[2] T. Li and J.-F. Zhang, "Consensus conditions of multi-agent systems with time-varying topologies and stochastic communication noises," *IEEE Trans. Autom. Control*, vol. 55, no. 9, pp. 2043–2057, Sep. 2010.

[3] L. Cheng, Z. G. Hou, M. Tan, and X. Wang, "Necessary and sufficient conditions for consensus of double-integrator multi-agent systems with measurement noises," *IEEE Trans. Autom. Control*, vol. 56, no. 8, pp. 1958–1963, Aug. 2011.

[4] G. Miao, S. Xu, and Y. Zou, "Consentability for high-order multi-agent systems under noise environment and time delays," *J. Franklin Inst.*, vol. 350, no. 2, pp. 244–257, 2013.

[5] Y. Cao and W. Ren, "Finite-time consensus for multi-agent networks with unknown inherent nonlinear dynamics," *Automatica*, vol. 50, no. 10, pp. 2648–2656, 2014.

[6] M. Defoort, G. Demesure, Z. Zuo, and A. Polyakov, "Fixed-time stabilisation and consensus of non-holonomic systems," *IET Control Theory Appl.*, vol. 10, no. 18, pp. 2497–2505, Sep. 2016.

[7] Y. Xu, S. Peng, and A. Guo, "Leader-following consensus of nonlinear multi-agent systems with randomly occurring uncertainties and stochastic disturbances under impulsive control input," *Int. J. Syst. Sci.*, vol. 48, no. 15, pp. 3138–3146, 2017.

[8] X. Yang and J. Lu, "Finite-time synchronization of coupled networks with Markovian topology and impulsive effects," *IEEE Trans. Autom. Control*, vol. 61, no. 8, pp. 2256–2261, Aug. 2016.

[9] X. Yang and J. Cao, "Finite-time stochastic synchronization of complex networks," *Appl. Math. Model.*, vol. 34, no. 11, pp. 3631–3641, 2010.

[10] M. P. Aghababa, S. Khanmohammadi, and G. Alizadeh, "Finite-time synchronization of two different chaotic systems with unknown parameters via sliding mode technique," *Appl. Math. Model.*, vol. 35, no. 6, pp. 3080–3091, 2011.

[11] L. Wang and F. Xiao, "Finite-time consensus problems for networks of dynamic agents," *IEEE Trans. Autom. Control*, vol. 55, no. 4, pp. 950–955, Apr. 2010.

[12] X. Chen, J. Li, Z. Wu, and J. Yu, "Finite-time consensus protocol for stochastic multi-agent systems," *IET Control Theory Appl.*, vol. 13, no. 6, pp. 755–762, 2019.

[13] J. Hu, Z. Wang, S. Liu, and H. Gao, "A variance-constrained approach to recursive state estimation for time-varying complex networks with missing measurements," *Automatica*, vol. 64, pp. 155–162, Feb. 2016.

[14] J. Hu, Z. Wang, B. Shen, and H. Gao, "Quantised recursive filtering for a class of nonlinear systems with multiplicative noises and missing measurements," *Int. J. Control*, vol. 86, no. 4, pp. 650–663, 2013.

[15] J. Hu, Z. Wang, H. Gao, and L. K. Stergioulas, "Robust sliding mode control for discrete stochastic systems with mixed time delays, randomly occurring uncertainties, and randomly occurring nonlinearities," *IEEE Trans. Ind. Electron.*, vol. 59, no. 7, pp. 3008–3015, Jul. 2012.

[16] S. Yue, J. Hu, D. Chen, D. Ji, and F. Liu, "Recursive approach to networked fault estimation with packet dropouts and randomly occurring uncertainties," *Neurocomputing*, vol. 214, pp. 340–349, Nov. 2016.

[17] X. Liu, D. W. C. Ho, S. Qiang, and W. Xu, "Finite/fixed-time pinning synchronization of complex networks with stochastic disturbances," *IEEE Trans. Cybern.*, vol. 49, no. 6, pp. 2398–2403, Jun. 2018.

[18] M. Hu, L. Guo, A. Hu, and Y. Yang, "Leader-following consensus of linear multi-agent systems with randomly occurring nonlinearities and uncertainties and stochastic disturbances," *Neurocomputing*, vol. 149, pp. 884–890, Feb. 2015.

[19] A. Hu and Z. Xu, "Stochastic linear generalized synchronization of chaotic systems via robust control," *Phys. Lett. A*, vol. 372, no. 21, pp. 3814–3818, 2008.

[20] X. Liu, G. Ma, X. Jiang, and H. Xi, "$H\infty$ stochastic synchronization for master–slave semi-Markovian switching system via sliding mode control," *Complexity*, vol. 21, no. 6, pp. 430–441, 2016.

[21] Z. Tang, H. P. Ju, T. H. Lee, and J. Feng, "Random adaptive control for cluster synchronization of complex networks with distinct communities," *Int. J. Adapt. Control Signal Process.*, vol. 30, no. 3, pp. 534–549, 2016.

[22] H. Ren, F. Deng, and Y. Peng, "Finite time synchronization of markovian jumping stochastic complex dynamical systems with mix delays via hybrid control strategy," *Neurocomputing*, vol. 272, pp. 683–693, Jan. 2018.

[23] X. Li, D. W. Ho, and J. Cao, "Finite-time stability and settling-time estimation of nonlinear impulsive systems," *Automatica*, vol. 99, pp. 361–368, Jan. 2019.

[24] Y. Sun, W. Li, H. Shi, D. Zhao, and S. Azaele, "Finite-time and fixed-time consensus of multiagent networks with pinning control and noise perturbation," *SIAM J. Appl. Math.*, vol. 79, no. 1, pp. 111–130, 2019.

[25] X. Lu, X. Zhang, and Q. Liu, "Finite-time synchronization of nonlinear complex dynamical networks on time scales via pinning impulsive control," *Neurocomputing*, vol. 275, pp. 2104–2110, Jan. 2017.

[26] H. Ren, Y. Peng, F. Deng, and C. Zhang, "Impulsive pinning control algorithm of stochastic multi-agent systems with unbounded distributed delays," *Nonlinear Dyn.*, vol. 92, no. 4, pp. 1453–1467, 2018.

[27] H. Ren, F. Deng, Y. Peng, B. Zhang, and C. Zhang, "Exponential consensus of non-linear stochastic multi-agent systems with ROUs and RONs via impulsive pinning control," *IET Control Theory Appl.*, vol. 11, no. 2, pp. 225–236, 2016.
[28] A. Polyakov, "Nonlinear feedback design for fixed-time stabilization of linear control systems," *IEEE Trans. Autom. Control*, vol. 57, no. 8, pp. 2106–2110, Aug. 2012.
[29] Z. Xu, C. Li, and Y. Han, "Leader-following fixed-time quantized consensus of multi-agent systems via impulsive control," *J. Franklin Inst.*, vol. 356, no. 1, pp. 441–456, Jan. 2019.
[30] H. Li, C. Li, T. Huang, and D. Ouyang, "Fixed-time stability and stabilization of impulsive dynamical systems," *J. Franklin Inst.*, vol. 354, no. 18, pp. 8626–8644, 2017.
[31] J. Hu, G. Sui, X. Lv, and X. Li, "Fixed-time control of delayed neural networks with impulsive perturbations," *Nonlinear Anal., Modelling Control*, vol. 23, no. 6, pp. 904–920, 2018.
[32] X. Yang, J. Lam, D. W. C. Ho, and Z. Feng, "Fixed-time synchronization of complex networks with impulsive effects via nonchattering control," *IEEE Trans. Autom. Control*, vol. 62, no. 11, pp. 5511–5521, Nov. 2017.
[33] H. Hong, W. Yu, G. Wen, and X. Yu, "Distributed robust fixed-time consensus for nonlinear and disturbed multiagent systems," *IEEE Trans. Syst., Man, Cybern. A, Syst.*, vol. 47, no. 7, pp. 1464–1473, Jul. 2017.
[34] J. Ni, L. Liu, C. Liu, X. Hu, and S. Li, "Fast fixed-time nonsingular terminal sliding mode control and its application to chaos suppression in power system," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 64, no. 2, pp. 151–155, Feb. 2017.
[35] J. Lu, D. W. C. Ho, and J. Cao, "A unified synchronization criterion for impulsive dynamical networks," *Automatica*, vol. 46, no. 7, pp. 1215–1221, Jul. 2010.
[36] Z. Wang, S. Lauria, J. Fang, and X. Liu, "Exponential stability of uncertain stochastic neural networks with mixed time-delays," *Chaos Solitons Fractals*, vol. 32, no. 1, pp. 62–72, 2007.

TAO CHEN was born in Guangdong, China. He received the B.S. degree from the Zhongkai University of Agriculture and Engineering, Guangzhou, China, in 2013, and the M.S. degree from the Guangdong University of Technology, Guangzhou, in 2017, where he is currently pursuing the Ph.D. degree with the Faculty of Automation. His research interests include complex system control and stochastic systems.

SHIGUO PENG was born in Hunan, in 1967. He received the B.S. degree from Xiangtan University, Xiangtan, in 1989, and the M.S. and Ph.D. degrees from Zhongshan University, Guangzhou, China, in 1992 and 1995, respectively. He is currently a Professor with the School of Automation, Guangdong University of Technology, Guangzhou. His research interests include nonlinear analysis, robust control, and stochastic systems.

ZHENHUA ZHANG received the B.S. degree from West Anhui University, Lu'an, China, in 2015, and the M.S. degree from the Guangdong University of Technology, Guangzhou, China, in 2018, where he is currently pursuing the Ph.D. degree with the School of Automation. His current research interests include impulsive control theory, event-triggered control theory, and consensus of stochastic multi-agent systems.