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Robust H_∞ Interval Observer for Linear Systems With a Controllable Convergence Rate: A Parametric Method

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ABSTRACT In this paper, the design method of a robust interval observer for linear systems with time-varying disturbances is proposed. First, the H_∞-gain performance is established by constructing the transfer function from disturbances to error dynamic systems of the interval observers. Second, the design problem about a robust interval observer, equivalent to the eigenstructure assignment of the observer error systems under the above form and the idea of the eigenstructure decomposition, is solved. Finally, in view of this situation where there does not exist an observation gain ensuring the cooperativity of the error systems, a novel parametric approach to design an interval observer with a controlled convergence rate and the robustness with respect to disturbances is proposed by a linear transformation and the solutions to a type of generalized Sylvester equations. Besides, the correctness and efficiency of the obtained results are illustrated by numerical examples and an actual physical system about the longitudinal motion of a Charlie Aircraft.

INDEX TERMS H_∞-performance, interval observer, parametric method, time-varying disturbances.

I. INTRODUCTION

Aiming at this widespread problem where some physical states are quite difficult to be directly measured in the actual control engineering, the research on the state reconstruction problem has been intensively concerned by numerous researchers. And there have been dozens of available and effective results [1]–[3] since the concept of a Luenberger-like observer was introduced by D. G. Luenberger in 1966 [4]. Moreover, with continuous improvement of the control quality requirements, the parts, which are initially ignored by researchers for a simple design, are gradually taken into consideration in the system design, such as non-linear items [5], uncertainties [6], or delays [7], etc.

As the model-relied observer, its performance will be inevitably challenged by uncertain factors. Therefore, in order to achieve an accurate estimation of the system states, necessary to make a deep study on the observer design for the systems with disturbances. Immediately, many control strategies are introduced, such as Adaptive Control [8], [9], Sliding Mode Control [10], Disturbance-Decoupled Method [11], [12] and Lyapunov Stability Analysis [13], etc. And further, according to their ways to deal with the model uncertainties, the above methods can come down to a ''Deterministic Method'' – in other words, their main goals are completely to eliminate the impact of uncertainties on a system. However, as pointed out in [14] by M. Kline, the uncertainty problem cannot be solved with the deterministic methods thoroughly.

Similar to the traditional observer design, the construction of the interval observers contains the input and output of a system. Interestingly, the partial information of the system uncertainties is also involved in the design process as its feature. Finally, the observation for the states of a system is well achieved by a pair of dynamic systems, which surround the estimated states tightly with the upper and lower bounds. Therefore, because of its great breakthrough in structure and the unique treatment of uncertainties, the interval observers become one of the research hotspots in observer theory recently. After introduced by J. L. Gouzé in [15] and wildly applied to the biological positive systems [16], [17], the interval observers attract the considerable interest of researchers.

In [18], M. Moisan et al. developed the relevant design methods of the interval observers from the positive

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systems to the more general systems, which leads to a wide discussion about the construction of system cooperativity. Then, a time-invariant change of coordinates was applied to design a full-order interval observer for nonlinear systems in [19]. Besides, the design thought of a reduced-order interval observer was firstly proposed for the time-delay systems by Efimov *et al.* [20]. Recently, the functional observer was successfully pioneered into the interval observer design theory in [21], where the definition, the sufficient existence conditions and the effective design method about the functional interval observers were successively proposed. Meanwhile, in view of the limitations of the simple linear time-invariant model in describing dynamics, the systems with the additional characteristics were deeply considered in the development process of the interval observers, such as time-delay [22], time-varying [23], [24], switching [25]–[27] or fuzzy [28].

Regrettably, the current research results about the interval observers mainly focus on the design problem, namely how to construct a cooperative error system, and correspondingly, the improvement of performance about the interval observer itself is always ignored. Although there is better inclusiveness for the system uncertainties as to its feature and advantage, the truth is, a more accurate estimation of the states is the most fundamental requirement for an observer. And the robustness of the observers was firstly proposed by Doyle and Stein [30], that is the sensitivity of the observers to uncertainties. Many useful methods are put forward on the premise of fully considering the anti-interference ability in observer design [31]–[35]. Applying an L_1/L_2 framework, the robustness and estimation accuracy concerning the model uncertainties were analyzed in [33]. Moreover, in terms of the tractable finite-dimensional linear programs, an optimal L_∞-to-L_∞ interval observer was designed in [34]. Recently, the H_∞ and $\mathcal{D}$-stability performance were all considered in the design process of an unknown input interval observer in [35].

Comparing with the above methods, the contributions of this paper is that

1. A simpler form of H_∞-gain, is constructed in this paper, avoiding to a calculation of the LMIs.

2. The design of the interval observers with the robustness to the disturbances and the designed poles in a certain area is equivalent to the solution to a type of Sylvester equations and the simple selection of the given parameters.

3. The existence of an observer gain, namely error system cooperativity, is guaranteed in the design process of a robust interval observer.

The paper is organized as follows. The preliminaries and problem statement are given in Section II. And Section III presents the main results about parametric design methods of the robust interval observers. Finally, numerical examples and an actual physical system are provided to verify the correctness of the proposed results in Section IV.

The corresponding notations are introduced as a clearer explanation for the derivation and proof in this paper.

1. The $\mathbb{R}^n$, $\mathbb{R}^{n\times m}$, $\mathbb{R}_+$, $\mathbb{R}^{n\times m}[s]$, $\mathbb{C}^-$ and $\mathbb{C}^{n\times m}$ define the set of all real vectors of dimension n, the set of all real vectors of dimension $n\times m$, the set of all positive real numbers, the set of all polynomial matrices of dimension $n\times m$ with real coefficients, the left-half complex plane and the set of all complex matrices of dimension $n\times m$ respectively. The I_n denotes the identity matrix of order n, and the Ω^- is an area in the left-hand of s-plane.

2. The $\lambda_i(A)$, eig(A), Re(A) and $|A|$ represent the ith eigenvalue of matrix A, the set of all eigenvalues of matrix A, the real part of matrix A, and the matrix of the absolute values of all elements of matrix A respectively. The deg$(\cdot)$ denotes the degree n of a polynomial matrix $P_0+P_1s+\cdots+P_ns^n$. The diag$(d_1,d_2,\cdots,d_n)$ represents the diagonal matrix with diagonal elements $d_i, i=1,2,\cdots,n$. Also, the max$(A,0)$ means that each element of the matrix A compares with zero and take a larger value.

3. The relations of vectors or matrices mean element-wise, i.e. if $x_1=[a_{11},a_{12}]\le x_2=[a_{21},a_{22}]$, it represents $a_{11}\le a_{21}$ and $a_{12}\le a_{22}$.

II. PRELIMINARIES AND PROBLEM STATEMENT

Let us consider the following system with disturbances as

$$\begin{cases}\dot{x}(t)=Ax(t)+Bu(t)+Ff(t),\\ y(t)=Cx(t),\end{cases}\tag{1}$$

where the state $x(t)$ is a n-dimensional vector, the system input $u(t)$ is a p-dimensional vector, and the system output $y(t)$ is a m-dimensional vector with the known constant matrices A, B, C and F of appropriate dimensions. Meanwhile, the time-varying disturbances $f(t)$ are bounded by the known upper and lower bounds $\overline{f}(t), \underline{f}(t)$.

Definition 1: A matrix $A\in\mathbb{R}^{n\times m}$ can be represented as

$$A=A^+-A^-,\tag{2}$$

where $A^+=\max(A,0)$ and $A^-=\max(-A,0)$, and then there exists $x(t)$ with $\underline{x}(t)\le x(t)\le\overline{x}(t)$, satisfying the following equation

$$A^+\underline{x}(t)-A^-\overline{x}(t)\le Ax(t)\le A^+\overline{x}(t)-A^-\underline{x}(t).$$

Definition 2: A square matrix $A\in\mathbb{R}^{n\times n}$ is a Metzler and Hurwitz matrix if and only if its all non-diagonal elements are non-negative and its eigenvalues lie in the left-hand of s-plane, namely for any Metzler and Hurwitz matrix $A=(a_{ij})$, there exists

$$a_{ij}\ge 0,\quad \lambda_i(A)\in\mathbb{C}^-,\ (1\le i\ne j\le n).$$

Lemma 1 [21]: The following system is structured by a Metzler and Hurwitz matrix $A\in\mathbb{R}^{n\times n}$ and an uniformly bounded vector $f_+(t)$

$$\dot{x}(t)=Ax(t)+f_+(t), f_+(t):\mathbb{R}_+\to\mathbb{R}^n_+.\tag{3}$$

where the initial state is non-negative, namely $x(t_0)\ge 0$, then all the solutions of (3) are non-negative and uniformly bounded.

From [15], we give the following lemma for the interval observers of the system (1).

Lemma 2: The observation of the system (1) *can be carried out with the design of a Luenberger-like interval observer satisfying* $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$ *as*

$$\begin{cases} \dot{\overline{x}}(t) = A\overline{x}(t) + Bu(t) + L(y(t) - C\overline{x}(t)) + \overline{\phi}(t), \\ \dot{\underline{x}}(t) = A\underline{x}(t) + Bu(t) + L(y(t) - C\underline{x}(t)) + \underline{\phi}(t), \end{cases} \tag{4}$$

where

$$\begin{cases} \overline{\phi}(t) = F^+\overline{f}(t) - F^-\underline{f}(t), \\ \underline{\phi}(t) = F^+\underline{f}(t) - F^-\overline{f}(t). \end{cases}$$

if all the following conditions are established

1. *There exists the observer gain matrix* $L \in \mathbb{R}^{n\times m}$ *making the matrix* $A - LC$ *be a Metzler and Hurwitz matrix;*
2. *The initial condition of the system* (1) *satisfies*

$$\underline{x}(t_0) \leq x(t_0) \leq \overline{x}(t_0).$$

The proof of Lemma 2 is given in [15].

Lemma 3: Given the observable system (1) *and the proposed interval observer* (4)*, the transfer function of the error observation systems from* $\overline{x}(t) - \underline{x}(t)$ *to* $\overline{f}(t) - \underline{f}(t)$ *can be obtained as*

$$T_{e\phi}(s) = (\lambda I_n - (A - LC))^{-1}|F|. \tag{5}$$

Proof: Denote $e(t) = \overline{x}(t) - \underline{x}(t)$, then we have

$$\begin{aligned} \dot{e}(t) &= \dot{\overline{x}}(t) - \dot{\underline{x}}(t) \\ &= A\overline{x}(t) + Bu(t) + L(y(t) - C\overline{x}(t)) + \overline{\phi}(t) \\ &\quad - A\underline{x}(t) - Bu(t) - L(y(t) - C\underline{x}(t)) - \underline{\phi}(t) \\ &= (A - LC)e(t) + F^+\overline{f}(t) - F^-\underline{f}(t) \\ &\quad - F^+\underline{f}(t) + F^-\overline{f}(t) \\ &= (A - LC)e(t) + |F|(\overline{f}(t) - \underline{f}(t)). \end{aligned} \tag{6}$$

The transfer function from disturbances $\phi(t) = \overline{f}(t) - \underline{f}(t)$ to states $e(t)$ in the error dynamic system (6) is deduced as

$$T_{e\phi}(s) = (\lambda I_n - (A - LC))^{-1}|F|.$$

□

Consider the following type of generalized Sylvester equations in [37]

$$\sum_{i=0}^{\varphi} \mathcal{A}_i \mathcal{V} \mathcal{F}^i = \sum_{i=0}^{\varphi} \mathcal{B}_i \mathcal{W} \mathcal{F}^i, \tag{7}$$

where

1. $\mathcal{A}_i \in \mathbb{R}^{n\times q}$, $\mathcal{B}_i \in \mathbb{R}^{n\times r}$ and $\mathcal{F} \in \mathbb{R}^{p\times p}$ are the parameter matrices;
2. $\mathcal{V} \in \mathbb{C}^{q\times p}$, $\mathcal{W} \in \mathbb{C}^{r\times p}$ are the matrices to be determined.

and the polynomial matrices associated with the generalized Sylvester equation (7) are

$$\begin{cases} \mathcal{A}(s) = \sum_{i=0}^{\varphi} \mathcal{A}_i s^i, \\ \mathcal{B}(s) = \sum_{i=0}^{\varphi} \mathcal{B}_i s^i. \end{cases} \tag{8}$$

Definition 3 $\mathcal{F}$*-Left Coprime [37]: Let* $\mathcal{A}(s) \in \mathbb{R}^{n\times q}[s]$ *and* $\mathcal{B}(s) \in \mathbb{R}^{n\times r}[s]$, $q + r > n$ *be given as in* (8)*, and* $\mathcal{F} \in \mathbb{C}^{p\times p}$ *be an arbitrary matrix. Then* $\mathcal{A}(s)$ *and* $\mathcal{B}(s)$ *are said to be* $\mathcal{F}$*-left coprime if*

$$\text{rank}\begin{bmatrix} \mathcal{A}(s) & \mathcal{B}(s) \end{bmatrix} = n, \quad s \in \text{eig}(\mathcal{F}). \tag{9}$$

Lemma 4 [38]: Consider the given transfer function

$$G(\lambda) = C(\lambda I_n - A)^{-1}B \in H_\infty,$$

then $\|G(\lambda)\|_\infty < \gamma$*, if and only if all eigenvalues of following Hamilton matrix*

$$H = \begin{bmatrix} A & BB^{\mathrm{T}}/\gamma^2 \\ -C^{\mathrm{T}}C & -A^{\mathrm{T}} \end{bmatrix},$$

are not on the imaginary axis.

With all these elements in mind, we can state the considered observation problem:

Problem 1: For the system (1)*, design a robust interval observer as the form of* (4)*, namely find the observation gain L to make the following conditions hold on*

1. *The matrix* $A - LC$ *is a Metzler and Hurwitz matrix;*
2. *The pre-designated* H_∞ *is bounded, namely* $\|T_{e\phi}\|_\infty < \gamma$*;*
3. *The eigenvalues of error system are constrained to lie in a prescribed region.*

III. MAIN RESULTS

A. ROBUST INTERVAL OBSERVER DESIGN

According to Definition 3, there exists the following right coprime factorization (RCF)

$$\mathcal{A}(s)N(s) - \mathcal{B}(s)D(s) = 0, \tag{10}$$

where $N(s) \in \mathbb{R}^{q\times\beta_0}[s]$ and $D(s) \in \mathbb{R}^{r\times\beta_0}[s]$, $\beta_0 = q+r-n$, are a pair of polynomial matrices.

Denote $D(s) = [d_{ij}(s)]_{r\times\beta_0}$, $N(s) = [n_{ij}(s)]_{q\times\beta_0}$ and

$$\begin{aligned} \omega_1 &= \max\{\deg(d_{ij}(s)), i = 1, 2, \ldots, r, j = 1, 2, \ldots, \beta_0\}, \\ \omega_2 &= \max\{\deg(n_{ij}(s)), i = 1, 2, \ldots, q, j = 1, 2, \ldots, \beta_0\}, \\ \omega &= \max\{\omega_1, \omega_2\}, \end{aligned}$$

then the $N(s)$ and $D(s)$ can be represented in the following forms:

$$\begin{cases} N(s) = \sum_{k=0}^{\omega} N_k s^k, & N_k \in \mathbb{R}^{q\times\beta_0}, \\ D(s) = \sum_{k=0}^{\omega} D_k s^k, & D_k \in \mathbb{R}^{r\times\beta_0}. \end{cases} \tag{11}$$

Assumption 1: The matrix $A - LC$ *is considered as a non-defective matrix because of the robustness with respect to the parameter perturbations, which means its Jordan normal form can be represented as*

$$\Lambda(\lambda_l) = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n), \tag{12}$$

where $\lambda_l, l = 1, 2, \cdots, n$*, is the eigenvalues of* $A - LC$*.*

Assumption 2: The $N(s)$ and $D(s)$, in the forms of (11), are right coprime factorization satisfying (10), namely polynomial matrices $\mathcal{A}(s)$ and $\mathcal{B}(s)$ are $\mathcal{F}$-left coprime, where

$$\begin{cases} \mathcal{A}(s) = sI_n - A^{\mathrm{T}} \\ \mathcal{B}(s) = -C^{\mathrm{T}} \end{cases}$$

Theorem 1: Under Assumption 1 and 2, suppose the $\lambda_l, l = 1, 2, \cdots, n$, as a set of the self-conjugated complex values with the negative real part, if there exist the optional matrix $Z = \{z_{ij}\}_{i=1,2,\cdots,\beta_0, j=1,2,\cdots,n}$, and the λ_l, $l = 1, 2, \cdots, n$, satisfying the following conditions

1. *The matrix $V^{-1}\Lambda V$ is a Metzler matrix;*
2. *All eigenvalues of the following Hamilton matrix*

$$H(\lambda_l, z_{ij}) = \begin{bmatrix} \Lambda & V^{-1}|F||F|^{\mathrm{T}}(V^{-1})^{\mathrm{T}}/\gamma^2 \\ -V^{\mathrm{T}}V & -\Lambda^{\mathrm{T}} \end{bmatrix}, \quad l = 1, 2, \cdots, n, \ i = 1, 2, \cdots, \beta_0, \quad j = 1, 2, \cdots, n, \tag{13}$$

are not on the imaginary axis;

3. *$\lambda_l, l = 1, 2, \cdots, n$, lie in a prescribed region,*

where $V(\lambda_l, z_{ij}) = \sum_{k=0}^{\omega} \Lambda^k Z^{\mathrm{T}} N_k^{\mathrm{T}}$ is the right eigenvectors matrix, $\Lambda(\lambda_l)$ is in the form of (12) and $W(\lambda_l, z_{ij}) = \sum_{k=0}^{\omega} \Lambda^k Z^{\mathrm{T}} D_k^{\mathrm{T}}$ is an equivalent matrix, then a robust interval observer is designed as the dynamic system (4) with the observation gain $L = V^{-1}W$.

Proof: Under a assumption of non-existing disturbance, the error dynamic system is obtained as

$$\dot{e}(t) = (A - LC)e(t).$$

According to Assumption 1, there must exist the nonsingular matrix $V \in \mathbb{C}^{n\times n}$ such that

$$V(A - LC) = \Lambda V.$$

Let

$$W = VL,$$

we have

$$VA - WC = \Lambda V. \tag{14}$$

Further, from (7), (8), and (14), we have

$$\begin{cases} \mathcal{A}(s) = sI_n - A^{\mathrm{T}}, \\ \mathcal{B}(s) = -C^{\mathrm{T}}. \end{cases}$$

Next, utilizing Assumption 2 and results in [36], the parametric forms of the observation gain L and the right eigenvectors matrix V are obtained as

$$\begin{cases} V = \sum_{k=0}^{\omega} \Lambda^k Z^{\mathrm{T}} N_k^{\mathrm{T}}, W = \sum_{k=0}^{\omega} \Lambda^k Z^{\mathrm{T}} D_k^{\mathrm{T}}, \\ L = V^{-1}W, \end{cases} \tag{15}$$

where $Z = \{z_{ij}\}_{i=1,2,\cdots,\beta_0, j=1,2,\cdots,n}$, is an arbitrary parameter matrix, and the error system matrix $A - LC$ will be re-expressed as $V^{-1}\Lambda V$.

Next, deduced from Lemma 3, the transfer function (5) is transformed into

$$\begin{aligned} T_{e\phi}(s) &= (\lambda I_n - (A - LC))^{-1}(F^+ + F^-) \\ &= V^{-1}(\lambda I_n - \Lambda)^{-1} V|F|. \end{aligned} \tag{16}$$

According to Lemma 4 and equation (16), the pre-designated H_∞ bounded, namely

$$\|T_{e\phi}(s)\|_\infty = \|(\lambda I_n - (A - LC))^{-1}|F|\|_\infty < \gamma,$$

is equivalent to that all eigenvalues of the Hamilton matrix

$$H(\lambda_l, z_{ij}) = \begin{bmatrix} \Lambda & V|F||F|^{\mathrm{T}}(V)^{\mathrm{T}}/\gamma^2 \\ -(V^{-1})^{\mathrm{T}}V^{-1} & -\Lambda^{\mathrm{T}} \end{bmatrix}, \quad l = 1, 2, \cdots, n, i = 1, 2, \cdots, \beta_0, \quad j = 1, 2, \cdots, n,$$

are not on the imaginary axis.

From Hamilton matrix $H(\lambda_l, z_{ij})$, $\Lambda(\lambda_l)$ and $V(\lambda_l, z_{ij})$, easy to find all above matrices are the matrix functions with respect to parameters λ_l and z_{ij}. Therefore, the conditions can be satisfied by choosing the parameters λ_l and z_{ij} appropriately. Finally, the observation gain L can be calculated by (15) under the selected parameters λ_l and z_{ij}. And the proof is completed. □

Remark 1: From Theorem 1, under the eigenstructure decomposition, the design problem about a robust interval observer is equivalent to the problem of eigenstructure assignment. Further, based on the solution to a type of the Sylvester equations, the conditions of designing a robust interval observer are transformed into the parametric forms, related to eigenvalues $\Lambda(\lambda_l)$ and eigenvectors matrix $V(\lambda_l, z_{ij}), l = 1, 2, \cdots, n, i = 1, 2, \cdots, \beta_0, j = 1, 2, \cdots, n$. Thereby, the robustness to model uncertainties and the convergence rate of an interval observer can be controlled under the selected parameters, which simplifies the design difficulties and makes a clear process of design.

However, the robust interval observers, designed by Theorem 1, are obtained under the strong assumption, of which there exists the observation gain L ensuring the cooperativity and stability of the error systems simultaneously. But cooperativity, as a rather specific feature, is difficult to be satisfied in practice, then under some changes of coordinates, a robust interval observer is developed to overcome the difficulties as follows.

Firstly, using a non-singular matrix transformation $z(t) = Tx(t)$, the system (1) is transformed into

$$\begin{cases} \dot{z}(t) = \bar{A}z(t) + \bar{B}u(t) + TFf(t), \\ y(t) = \bar{C}(t)z(t), \end{cases} \tag{17}$$

where $T \in \mathbb{R}^{n\times n}$, $\bar{A} = TAT^{-1}$, $\bar{B} = TB$ and $\bar{C} = CT^{-1}$. And after the cooperative condition, namely $\bar{A} - L\bar{C}$ being a Metzler, is satisfied, the corresponding interval observer with $\underline{z}(t_0) \le z(t_0) \le \bar{z}(t_0)$ is designed as

$$\begin{cases} \dot{\bar{z}}(t) = \bar{A}\bar{z}(t) + \bar{B}u(t) + L(y(t) - \bar{C}\bar{z}(t)) + \bar{\phi}(t), \\ \dot{\underline{z}}(t) = \bar{A}\underline{z}(t) + \bar{B}u(t) + L(y(t) - \bar{C}\underline{z}(t)) + \underline{\phi}(t), \end{cases} \tag{18}$$

where

$$\begin{cases} \overline{\phi}(t) = (TF)^{+}\overline{f}(t) - (TF)^{-}\underline{f}(t), \\ \underline{\phi}(t) = (TF)^{+}\underline{f}(t) - (TF)^{-}\overline{f}(t). \end{cases} \tag{19}$$

Theorem 2: Under Assumption 2, if there exist the Metzler and Hurwitz matrix $M = \{m_{lj}\}_{l=1,2,\cdots,n,j=1,2,\cdots,n}$, and an arbitrary parameter matrix $Z = \{z_{ij}\}_{i=1,2,\cdots,\beta_0,j=1,2,\cdots,n}$, making all eigenvalues of the Hamilton matrix

$$H(m_{lj}, z_{ij}) = \begin{bmatrix} M & |TF||TF|^{\mathrm{T}}/\gamma^2 \\ -I_n & -M^{\mathrm{T}} \end{bmatrix}, \quad l = 1, 2, \cdots, n, i = 1, 2, \cdots, \beta_0, \quad j = 1, 2, \cdots, n,$$

are not on the imaginary axis, then a robust interval observer is designed as

$$\begin{cases} \overline{x}(t) = (T^{-1})^{+}\overline{z}(t) - (T^{-1})^{-}\underline{z}(t), \\ \underline{x}(t) = (T^{-1})^{+}\underline{z}(t) - (T^{-1})^{-}\overline{z}(t), \end{cases} \tag{20}$$

where $\overline{z}(t)$ and $\underline{z}(t)$ are the states of the systems as (18) with the transformation matrix $T(m_{lj}, z_{ij}) = \sum_{k=0}^{\omega} M^k Z^{\mathrm{T}} N_k^{\mathrm{T}}$ and the observation gain $L(m_{lj}, z_{ij}) = \sum_{k=0}^{\omega} M^k Z^{\mathrm{T}} D_k^{\mathrm{T}}$ respectively.

Proof: Denote $M = \{m_{lj}\}_{l=1,2,\cdots,n,j=1,2,\cdots,n}$, as a Metzler and Hurwitz matrix, then the design issue of ensuring the $\bar{A} - L\bar{C}$ be a Metzler and Hurwitz matrix is transformed into the solution to the following Sylvester equation:

$$TAT^{-1} - LCT^{-1} = M. \tag{21}$$

After the simple matrix transformation and transposition, we obtain the standard form of the Sylvester matrix equation as

$$A^{\mathrm{T}}T^{\mathrm{T}} - C^{\mathrm{T}}L^{\mathrm{T}} = T^{\mathrm{T}}M^{\mathrm{T}}. \tag{22}$$

Further, from (7), (8), and (22), we obtain

$$\begin{cases} \mathcal{A}(s) = sI_n - A^{\mathrm{T}}, \\ \mathcal{B}(s) = -C^{\mathrm{T}}. \end{cases}$$

Based on Assumption 2, where the $\mathcal{A}(s)$ and $\mathcal{B}(s)$ satisfying the $\mathcal{F}$-left coprime, therefore, the parametric forms of transformation matrix T and observation matrix L are necessarily obtained according to the solution to a type of generalized Sylvester equations in [36] as

$$\begin{cases} T = \sum_{k=0}^{\omega} M^k Z^{\mathrm{T}} N_k^{\mathrm{T}}, \\ L = \sum_{k=0}^{\omega} M^k Z^{\mathrm{T}} D_k^{\mathrm{T}}, \end{cases} \tag{23}$$

where $Z = \{z_{ij}\}_{i=1,2,\cdots,\beta_0,j=1,2,\cdots,n}$, is an arbitrary parameter matrix. By Definition 3 and the systems (18), the transfer function of the error dynamic systems is obtained as

$$T_{\bar{e}\phi}(s) = (\lambda I_n - M)^{-1}|TF|.$$

Under Lemma 4, the parametric form of transformation matrix T as (23) and the equation (21), the following statements are equivalent:

1. the dynamic systems as (18) is an interval observer with a robustness to uncertainties $f(t)$;
2. The pre-designated H_∞ is bounded, namely
$$\|T_{\bar{e}\phi}(s)\|_\infty = \|(\lambda I_n - M)^{-1}|TF|\|_\infty < \gamma;$$
3. All eigenvalues of the Hamilton matrix
$$H(m_{lj}, z_{ij}) = \begin{bmatrix} M & |TF||TF|^{\mathrm{T}}/\gamma^2 \\ -I_n & -M^{\mathrm{T}} \end{bmatrix}, \tag{24}$$
are not on the imaginary axis.

Similar to find that the Hamilton matrix $H(m_{lj}, z_{ij})$ as (24) is the matrix function with respect to m_{lj} and z_{ij}, $l = 1, 2, \cdots, n, i = 1, 2, \cdots, \beta_0, j = 1, 2, \cdots, n$. Therefore, the conditions can be satisfied by choosing the parameters m_{lj} and z_{ij} appropriately. Finally, the transformation matrix T and the observation gain L can be calculated by (23) under the selected parameters m_{lj} and z_{ij}.

Furthermore, applying the inverse transformation $x(t) = T^{-1}z(t)$, we have

$$\underline{x}(t) \le x(t) = T^{-1}z(t) \le \overline{x}(t).$$

By Definition 1, the robust interval observer is given as

$$\begin{cases} \underline{x}(t) = (T^{-1})^{+}\underline{z}(t) - (T^{-1})^{-}\overline{z}(t), \\ \overline{x}(t) = (T^{-1})^{+}\overline{z}(t) - (T^{-1})^{-}\underline{z}(t). \end{cases}$$

where $\overline{z}(t)$ and $\underline{z}(t)$ are the states of the dynamic systems as (18) with the transformation matrix and the observation gain as (23). The proof is completed. □

Remark 2: Form Theorem 2, the error dynamic system of (18) is obtained as

$$\dot{e}_z(t) = (\bar{A} - L\bar{C})e_z(t) + \overline{\phi}(t) - \underline{\phi}(t) = Me_z(t) + \Phi(t).$$

It is obvious that the convergence rate is decided by M and because of the selected M under the satisfied condition (24), we learn that the output interval of the interval observer for (20) can converge to a constant with a controllable rate.

Denote M as a following special diagonal form

$$M = \mathrm{diag}(\zeta_1, \zeta_2, \cdots, \zeta_n), \tag{25}$$

easy to find that $\zeta_l, l = 1, 2, \cdots, n$, are eigenvalues of M, namely, the eigenvalues of the error systems. Then, according to Theorem 2, the corollary about the design method of the interval observer with designed eigenvalues and robust to the disturbance is obtained as

Corollary 1: Under Assumption 2, if there exist the matrix M in the form of (25) and an arbitrary parameter matrix $Z = \{z_{ij}\}_{i=1,2,\cdots,\beta_0,j=1,2,\cdots,n}$, making all eigenvalues of the Hamilton matrix

$$H(\zeta_l, z_{ij}) = \begin{bmatrix} M & |TF||TF|^{\mathrm{T}}/\gamma^2 \\ -I_n & -M^{\mathrm{T}} \end{bmatrix}, \quad l = 1, 2, \cdots, n, \ i = 1, 2, \cdots, \beta_0, \quad j = 1, 2, \cdots, n$$

are not on the imaginary axis, then the robust interval observer is designed with $\zeta_l \in \Omega^-, l = 1, 2, \cdots, n$ as

$$\begin{cases} \overline{x}(t) = (T^{-1})^+\overline{z}(t) - (T^{-1})^-\underline{z}(t), \\ \underline{x}(t) = (T^{-1})^+\underline{z}(t) - (T^{-1})^-\overline{z}(t), \end{cases}$$

where $\overline{z}(t)$ and $\underline{z}(t)$ are the states of the systems as (18) with the transformation matrix $T(\zeta_l, z_{ij}) = \sum_{k=0}^{\omega} M^k Z^{\mathrm{T}} N_k^{\mathrm{T}}$ and the observation gain $L(\zeta_l, z_{ij}) = \sum_{k=0}^{\omega} M^k Z^{\mathrm{T}} D_k^{\mathrm{T}}$.

Proof: *Because the special form of M itself is the Metzler and Hurwitz matrix with $\zeta_l < 0, l = 1, 2, \cdots, n$, according to Theorem 2, the results can be verified by direct deduction. The proof is completed.* □

Remark 3: In Theorem 1, 2 and Corollary 1, the transfer functions are constructed in terms of the difference between upper and lower bounds. Therefore, the robustness focuses on the effect of the interval between upper and lower bounds from the disturbances in the system, namely the thickness of the interval. So the robust interval observers own the better performance of encircling the observed state and further, achieves the better estimation.

B. GENERAL ALGORITHM

Based on Theorem 1,2 and Corollary 1, a general algorithm is proposed to design the robust interval observers for linear systems with bounded disturbances, namely the prescribed region.

Step 1: According to the stability and performance requirements of the closed-loop systems, determine the configuration area of the desired closed-loop eigenvalues, namely the constrains about Λ (or M). Go to Step 2.

Step 2: For the specific systems, obtain its Sylvester matrix equation as (14), and the polynomial matrices associated with the generalized Sylvester equation are

$$\begin{cases} \mathcal{A}(s) = sI_n - A^{\mathrm{T}} \\ \mathcal{B}(s) = -C^{\mathrm{T}} \end{cases}$$

Then check if $\mathcal{A}(s), \mathcal{B}(s)$ satisfies the $\mathcal{F}$-left coprime, if yes, go to Step 3, if not, the Assumption 2 is not satisfied, then the proposed methods are all not valid. Stop (No solution).

Step 3: According to the generalized RCF as (10), we have a pair of polynomial matrices $(N(s), D(s))$. Thereby, the N_i and $D_i, i = 0, 1, \cdots \omega$, are obtained. Check if there exists L making the $A - LC$ be a non-defective matrix (Assumption 1) or not, if yes, go to Step 4, if not, go to Step 5.

Step 4: By (15), we obtain the parametric right eigenvector matrix V and the observation matrix L with free parameters Z and constrained matrix Λ. Further to check if we can assign matrices Z and Λ to ensure the following conditions hold:

B1. The matrix $V^{-1}\Lambda V$ is a Metzler matrix;
B2. All eigenvalues of the Hamilton matrix (13) are not on the imaginary axis;
B3. $\lambda_l, l = 1, 2, \cdots, n$, lie in a prescribed region (constrains in Step 1).

If yes, the robust interval observer is designed by Theorem 1, if not, the Theorem 1 is not valid, and go to Step 5.

Step 5: Denote a Metzler and Hurwitz matrix M and a non-singular matrix T, then a transformed system is obtained as (17). Go to Step 6.

Step 6: The parametric transformation matrix T and the observation matrix L is calculate as (23) with free parameters Z and constrained matrix M. Then check if we can assign matrices Z and M to ensure the following conditions hold:

C1. All eigenvalues of the Hamilton matrix (24) are not on the imaginary axis;
C2. The matrix M satisfy the constrains in Step 1.

If yes, the robust interval observer is designed by Theorem 2, if not, the Theorem 2 is not valid. Stop (No solution).

Remark 4: From the design algorithm, find that the premise of Theorem 1 is that there exists L making the A − LC being a Metzler and non-defective matrix (Assumption 1 in Step 3 and B1 in Step 4). And it means a low method applicability. Therefore, under some changes of coordinates, the parametric design method is advanced to ensure the cooperativity of the error systems and to relax the restrictions in Assumption 1. Moreover, comparing with their proof procedures, easy to deduce that Theorem 2 has more degrees of freedom than Theorem 1, which means that the condition C2 is easier to be satisfied than the condition B2 by selecting parameters. However, how to guarantee that the condition B2 (or C2) holds is still an open problem. Currently, there are some research results, for example, the sufficient conditions that the eigenvalues of complex Hamiltonian matrices are the real or the pure imaginary number are proposed in [39]. Because of the rich degrees of freedom in Theorem 2 and the limitations of the current methods about the Hamiltonian matrices, the cut-and-try method is adopted in this paper. Furthermore, the empirical requirements of this method for designers makes the research on the eigenvalues of Hamiltonian matrices be the key point in our further work.

IV. NUMERICAL EXAMPLES

A. EIGENSTRUCTURE ASSIGNMENT BASED METHOD

Consider a linear system (1) with bounded disturbance $-\mu \leq f(t) \leq \mu$ as

$$A = \begin{bmatrix} -8 & 0 \\ 0 & -9 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; F = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$
$$C = \begin{bmatrix} -1 & -1 \end{bmatrix},$$

and $u = \sin(t)$. And because of

$$\mathrm{rank}\begin{bmatrix} \mathcal{A}(s) & \mathcal{B}(s) \end{bmatrix} = \mathrm{rank}\begin{bmatrix} s+8 & 0 & 1 \\ 0 & s+9 & 1 \end{bmatrix} = 2,$$

Assumption 2 is satisfied, further, from the equation (10), the RCF matrices $N(s)$ and $D(s)$ can be chosen as

$$N(s) = \begin{bmatrix} s+9 \\ s+8 \end{bmatrix}, \quad D(s) = s^2 + 17s + 72.$$

The parametric matrices V and W are calculated with $Z = \begin{bmatrix} z_{11} & z_{12} \end{bmatrix}$ and $\Lambda = \mathrm{diag}(s_1, s_2)$ as

$$V = \begin{bmatrix} s_1 z_{11} + 9z_{11} & s_1 z_{11} + 8z_{11} \\ s_2 z_{12} + 9z_{12} & s_2 z_{12} + 8z_{12} \end{bmatrix},$$
$$W = \begin{bmatrix} z_{11}s_1^2 + 17z_{11}s_1 + 72z_{11} \\ z_{12}s_2^2 + 17z_{12}s_2 + 72z_{12} \end{bmatrix}.$$

Require $\gamma = 0.6$, namely $\|H_\infty\| < 0.6$ and by assigning the variables as $z_{11} = -5$, $z_{12} = -4$, $s_1 = -8$ and $s_2 = -7$, we have

$$\begin{cases} V^{-1}\Lambda V = \begin{bmatrix} -7 & 0 \\ 2 & -8 \end{bmatrix}, \\ \mathrm{Re}(\mathrm{eig}(H)) = \{-5.5400, 5.5400, -8.2716, 8.2716\}. \end{cases}$$

Accordingly, the conditions of Theorem 1 are all satisfied, a robust interval observer under $\|H_\infty\| < \gamma$ is designed with $L = V^{-1}W = \begin{bmatrix} 0 & 2 \end{bmatrix}^{\mathrm{T}}$as

$$\begin{cases} \dot{\overline{x}}(t) = \begin{bmatrix} -8 & 0 \\ 0 & -9 \end{bmatrix}\overline{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix}u(t) \\ \qquad + \begin{bmatrix} 0 \\ 2 \end{bmatrix}(y(t) - \begin{bmatrix} -1 & -1 \end{bmatrix}\overline{x}(t)) + \overline{\phi}(t), \\ \dot{\underline{x}}(t) = \begin{bmatrix} -8 & 0 \\ 0 & -9 \end{bmatrix}\underline{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix}u(t) \\ \qquad + \begin{bmatrix} 0 \\ 2 \end{bmatrix}(y(t) - \begin{bmatrix} -1 & -1 \end{bmatrix}\underline{x}(t)) + \underline{\phi}(t), \end{cases}$$

where

$$\begin{cases} \overline{\phi}(t) = \begin{bmatrix} \mu & 2\mu \end{bmatrix}^{\mathrm{T}}, \\ \underline{\phi}(t) = \begin{bmatrix} -\mu & -2\mu \end{bmatrix}^{\mathrm{T}}. \end{cases} \tag{26}$$

B. LINEAR TRANSFORMATION BASED METHOD

Let us consider the another system with bounded disturbance $-\mu \leq f(t) \leq \mu$ as

$$A = \begin{bmatrix} -8 & 4 \\ -4 & -7 \end{bmatrix}, \quad B = \begin{bmatrix} 10 \\ 0 \end{bmatrix}, \quad F = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$
$$C = \begin{bmatrix} 0 & 1 \end{bmatrix},$$

and $u = \sin(t)$.

Suppose $L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$ and then, we have

$$A - LC = \begin{bmatrix} -8 & 4 \\ -4 & -7 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} -8 & 4 - l_1 \\ -4 & -l_2 - 7 \end{bmatrix}.$$

It is obvious that there does not exist the matrix L make the $A - LC$ be a Metzler matrix, namely the condition B2 is not satisfied. Then Theorem 1 is invalid, and Theorem 2 is applied to constructed a robust interval observer for the system.

Due to

$$\mathrm{rank}\begin{bmatrix} \mathcal{A}(s) & \mathcal{B}(s) \end{bmatrix} = \mathrm{rank}\begin{bmatrix} s+8 & 4 & 0 \\ -4 & s+7 & -1 \end{bmatrix} = 2,$$

the RCF matrices $N(s)$ and $D(s)$ can be chosen as

$$N(s) = \begin{bmatrix} 1 \\ -\frac{s}{4} - 2 \end{bmatrix}, \quad D(s) = \frac{s^2}{4} + \frac{15s}{4} + 18.$$

Denote

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}, \quad Z = \begin{bmatrix} z_{11} & z_{12} \end{bmatrix},$$

then we obtain

$$\begin{cases} T = \begin{bmatrix} z_{11} & -2z_{11} - \frac{m_{11}z_{11}}{4} - \frac{m_{12}z_{12}}{4} \\ z_{12} & -2z_{12} - \frac{m_{21}z_{11}}{4} - \frac{m_{22}z_{12}}{4} \end{bmatrix}, \\ L = \begin{bmatrix} 18z_{11} + \frac{\alpha_1 + \alpha_2}{4} \\ 18z_{12} + \frac{\alpha_3 + \alpha_4}{4} \end{bmatrix}. \end{cases}$$

where

$$\begin{aligned} \alpha_1 &= z_{11}(m_{11}^2 + m_{12}m_{21}) + 15m_{11} \\ \alpha_2 &= 15m_{12} + z_{12}(m_{11}m_{12} + m_{12}m_{22}) \\ \alpha_3 &= z_{12}(m_{22}^2 + m_{12}m_{21}) + 15m_{22} \\ \alpha_4 &= 15m_{21} + z_{11}(m_{11}m_{21} + m_{21}m_{22}) \end{aligned}$$

Firstly, not consider a H_∞-performance and a general interval observer is designed by assigning the variables

$$M = \begin{bmatrix} -1 & 3 \\ \frac{1}{3} & -2 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & -1 \end{bmatrix},$$

as

$$\begin{cases} \overline{x}(t) = \begin{bmatrix} 3.4 & 2.4 \\ 2.4 & 2.4 \end{bmatrix}\overline{z}(t) - 0_{2\times 2}\underline{z}(t), \\ \underline{x}(t) = \begin{bmatrix} 3.4 & 2.4 \\ 2.4 & 2.4 \end{bmatrix}\underline{z}(t) - 0_{2\times 2}\overline{z}(t), \end{cases} \tag{27}$$

where

$$\begin{cases} \dot{\overline{z}}(t) = \begin{bmatrix} 12.8 & 16.8 \\ -25.4667 & -27.8 \end{bmatrix}\overline{z}(t) + \begin{bmatrix} 10 \\ -10 \end{bmatrix}u(t) \\ \qquad + \begin{bmatrix} 5.75 \\ -10.75 \end{bmatrix}(y(t) - \begin{bmatrix} 2.4 & 2.4 \end{bmatrix}\overline{z}(t)) + \overline{\phi}(t), \\ \dot{\underline{z}}(t) = \begin{bmatrix} 12.8 & 16.8 \\ -25.4667 & -27.8 \end{bmatrix}\underline{z}(t) + \begin{bmatrix} 10 \\ -10 \end{bmatrix}u(t) \\ \qquad + \begin{bmatrix} 5.75 \\ -10.75 \end{bmatrix}(y(t) - \begin{bmatrix} 2.4 & 2.4 \end{bmatrix}\underline{z}(t)) + \underline{\phi}(t), \end{cases}$$

and

$$\begin{cases} \overline{\phi}(t) = \begin{bmatrix} \mu & \frac{7}{12}\mu \end{bmatrix}^{\mathrm{T}}, \\ \underline{\phi}(t) = \begin{bmatrix} -\mu & -\frac{7}{12}\mu \end{bmatrix}^{\mathrm{T}}. \end{cases} \tag{28}$$

Secondly, the H_∞-performance is involved under $\gamma = 1$ by choosing

$$M = \begin{bmatrix} -3 & 0 \\ 0 & -4 \end{bmatrix}, \quad Z = \begin{bmatrix} 2 & -3 \end{bmatrix},$$

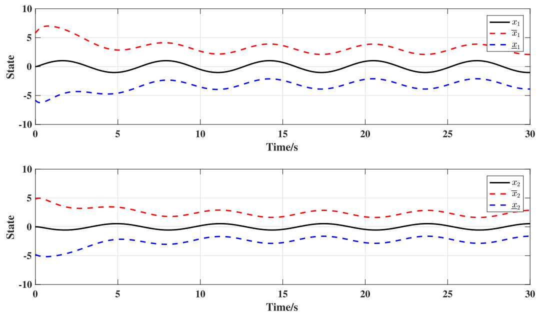

FIGURE 1. Interval observation of the state $x(t)$ in system with uncertainties under $\mu = 0.2$ without considering the H_∞-gain. $\overline{x}(t)$ and $\underline{x}(t)$ represent the upper and lower bounds in (27) respectively.

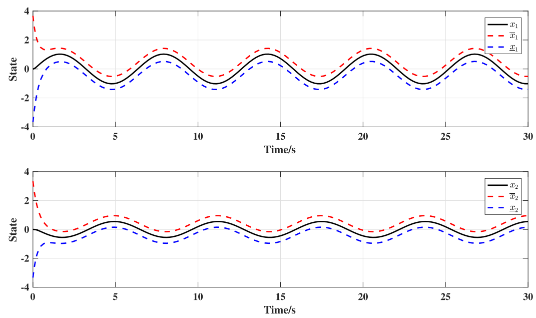

FIGURE 2. Interval observation of the state $x(t)$ in system with uncertainties under $\mu = 0.2$ under considering the H_∞-gain, namely $\gamma = 1$. $\overline{x}(t)$ and $\underline{x}(t)$ represent the upper and lower bounds in (29) respectively.

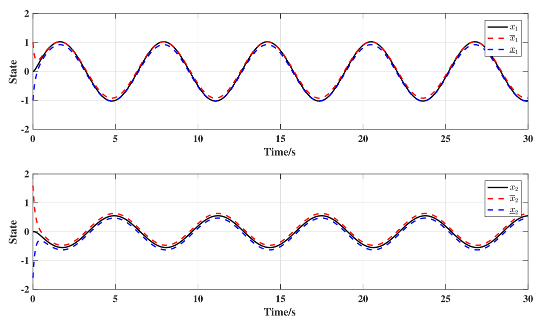

FIGURE 3. Interval observation of the state $x(t)$ in system with uncertainties under $\mu = 0.2$ under considering the H_∞-gain, namely $\gamma = 0.4$. $\overline{x}(t)$ and $\underline{x}(t)$ represent the upper and lower bounds in (31) respectively.

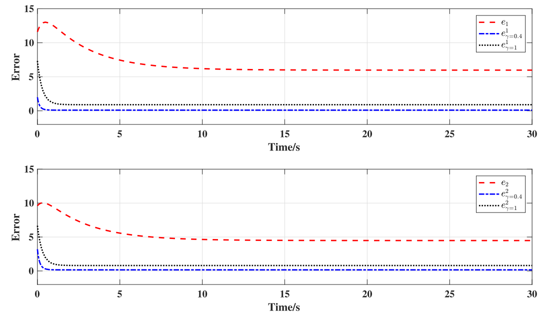

FIGURE 4. Interval error $e(t)$, $e_{\gamma=1}(t)$ and $e_{\gamma=0.4}(t)$ of interval observer in (27), (29) and (31) under the time-varying disturbance $f(t)$.

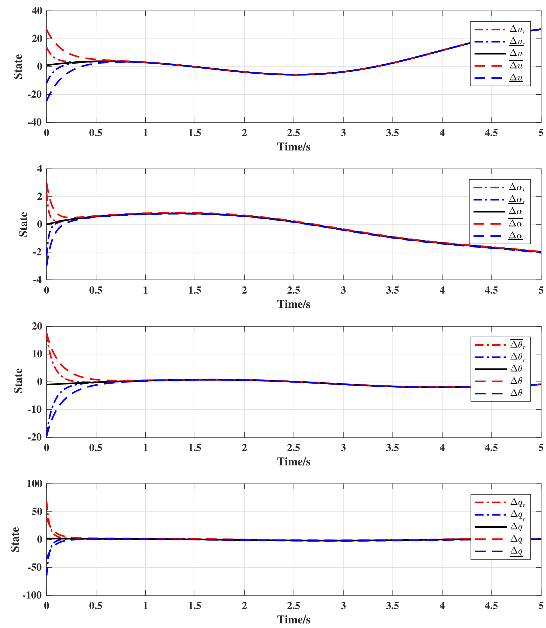

FIGURE 5. Interval observation of the state $x(t)$ in the longitudinal motion of a Charlie Aircraft with uncertainties $f(t)$ as (33). $\overline{x}(t)$ and $\underline{x}(t)$ represent the upper and lower bounds in (34). $\overline{x}_r(t)$ and $\underline{x}_r(t)$ represent the upper and lower bounds in (35) under considering the H_∞-gain, namely $\gamma = 0.6$.

then according to Theorem 2, the eigenvalues are obtained as

$$\mathrm{Re}(\mathrm{eig}(H)) = \{1.5405, -1.5405, 3.3729, -3.3729\}.$$

and a robust interval observer is constructed as

$$\begin{cases} \underline{x}(t) = 0_{2\times 2}\underline{z}(t) - \begin{bmatrix} 2 & \frac{5}{3} \\ 2 & \frac{4}{3} \end{bmatrix} \overline{z}(t), \\ \overline{x}(t) = 0_{2\times 2}\overline{z}(t) - \begin{bmatrix} 2 & \frac{5}{3} \\ 2 & \frac{4}{3} \end{bmatrix} \underline{z}(t), \end{cases} \tag{29}$$

where

$$\begin{cases} \dot{\overline{z}}(t) = \begin{bmatrix} -39 & -24 \\ 42 & 24 \end{bmatrix} \overline{z}(t) + \begin{bmatrix} 20 \\ -30 \end{bmatrix} u(t) \\ \qquad + \begin{bmatrix} 18 \\ -21 \end{bmatrix} (y(t) - \begin{bmatrix} -2 & -\frac{4}{3} \end{bmatrix} \overline{z}(t)) + \overline{\phi}(t), \\ \dot{\underline{z}}(t) = \begin{bmatrix} -39 & -24 \\ 42 & 24 \end{bmatrix} \underline{z}(t) + \begin{bmatrix} 20 \\ -30 \end{bmatrix} u(t) \\ \qquad + \begin{bmatrix} 18 \\ -21 \end{bmatrix} (y(t) - \begin{bmatrix} -2 & -\frac{4}{3} \end{bmatrix} \underline{z}(t)) + \underline{\phi}(t), \end{cases}$$

and

$$\begin{cases} \overline{\phi}(t) = \begin{bmatrix} \frac{3}{2}\mu & 3\mu \end{bmatrix}^{\mathrm{T}}, \\ \underline{\phi}(t) = \begin{bmatrix} -\frac{3}{2}\mu & -3\mu \end{bmatrix}^{\mathrm{T}}. \end{cases} \tag{30}$$

Comparing with the observer (29), a better robust interval observer with $\gamma = 0.4$ is constructed under the following selected parameters

$$M = \begin{bmatrix} -5 & 0 \\ 0 & -10 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & -1 \end{bmatrix},$$

then according to Theorem 2, we have

$$\mathrm{Re}(\mathrm{eig}(H)) = \{2.8328, 8.2551, -2.8328, -8.2551\}.$$

Then the robust interval observer is constructed as

$$\begin{cases} \underline{x}(t) = \begin{bmatrix} \frac{2}{5} & 0 \\ 0 & 0 \end{bmatrix} \underline{z}(t) - \begin{bmatrix} 0 & \frac{3}{5} \\ \frac{4}{5} & \frac{4}{5} \end{bmatrix} \overline{z}(t), \\ \overline{x}(t) = \begin{bmatrix} \frac{2}{5} & 0 \\ 0 & 0 \end{bmatrix} \overline{z}(t) - \begin{bmatrix} \text{[3pt]}0 & \frac{3}{5} \\ \frac{4}{5} & \frac{4}{5} \end{bmatrix} \underline{z}(t), \end{cases} \tag{31}$$

1. General interval observer:

$$\begin{cases} \overline{x}(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 3.5 & 0 & 2.5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overline{z}(t) - \begin{bmatrix} 4.905 & 0 & 4.905 & 0 \\ 0 & 0.1 & 0 & 0.1 \\ 0 & 0.1 & 0 & 0.1 \\ 0 & 1.901 & 0 & 0.901 \end{bmatrix} \underline{z}(t), \\ \underline{x}(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 3.5 & 0 & 2.5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underline{z}(t) - \begin{bmatrix} 4.905 & 0 & 4.905 & 0 \\ 0 & 0.1 & 0 & 0.1 \\ 0 & 0.1 & 0 & 0.1 \\ 0 & 1.901 & 0 & 0.901 \end{bmatrix} \overline{z}(t), \end{cases} \tag{34}$$

where

$$\begin{cases} \dot{\overline{z}}(t) = \begin{bmatrix} -18.1103 & 0.4454 & -13.1103 & 0.4454 \\ 5.9757 & -18.4275 & 5.9757 & -8.4275 \\ 25.1033 & -0.6451 & 18.1033 & -0.6451 \\ -12.2541 & 36.8975 & -12.2541 & 16.8975 \end{bmatrix} \overline{z}(t) + \begin{bmatrix} 0.04 \\ 12.1396 \\ -0.04 \\ -11.7396 \end{bmatrix} u(t) \\ \quad + \begin{bmatrix} 2.6729 & -4.4539 \\ -1.2183 & 84.2746 \\ -5.1179 & 6.4514 \\ 2.4983 & -368.9746 \end{bmatrix} \left(y(t) - \begin{bmatrix} -4.905 & 0 & -4.905 & 0 \\ 0 & -0.1 & 0 & -0.1 \end{bmatrix} \overline{z}(t)\right) + \begin{bmatrix} 0.008 \\ 2.4279 \\ 0.008 \\ 2.3479 \end{bmatrix}, \\ \dot{\underline{z}}(t) = \begin{bmatrix} -18.1103 & 0.4454 & -13.1103 & 0.4454 \\ 5.9757 & -18.4275 & 5.9757 & -8.4275 \\ 25.1033 & -0.6451 & 18.1033 & -0.6451 \\ -12.2541 & 36.8975 & -12.2541 & 16.8975 \end{bmatrix} \underline{z}(t) + \begin{bmatrix} 0.04 \\ 12.1396 \\ -0.04 \\ -11.7396 \end{bmatrix} u(t) \\ \quad + \begin{bmatrix} 2.6729 & -4.4539 \\ -1.2183 & 84.2746 \\ -5.1179 & 6.4514 \\ 2.4983 & -368.9746 \end{bmatrix} \left(y(t) - \begin{bmatrix} -4.905 & 0 & -4.905 & 0 \\ 0 & -0.1 & 0 & -0.1 \end{bmatrix} \underline{z}(t)\right) + \begin{bmatrix} -0.008 \\ -2.4279 \\ -0.008 \\ -2.3479 \end{bmatrix}. \end{cases}$$

2. Robust interval observer:

$$\begin{cases} \overline{x}_r(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overline{z}(t) - \begin{bmatrix} 1.962 & 0 & 1.9620 & 0 \\ 0 & 0.0303 & 0 & 0.0303 \\ 0 & 0.0303 & 0 & 0.0303 \\ 0 & 1.6064 & 0 & 0.6064 \end{bmatrix} \underline{z}(t), \\ \underline{x}_r(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underline{z}(t) - \begin{bmatrix} 1.962 & 0 & 1.9620 & 0 \\ 0 & 0.0303 & 0 & 0.0303 \\ 0 & 0.0303 & 0 & 0.0303 \\ 0 & 1.6064 & 0 & 0.6064 \end{bmatrix} \overline{z}(t), \end{cases} \tag{35}$$

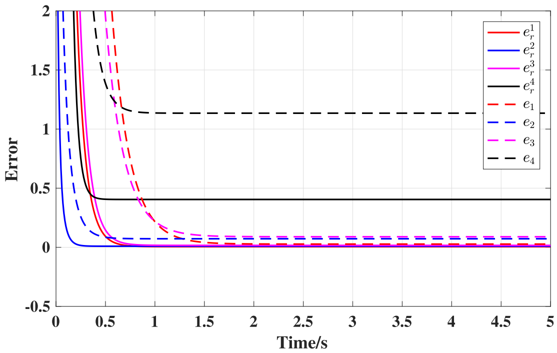

FIGURE 6. Interval error $e(t)$, $e_r(t)$ of interval observer in (34), (35) under the time-varying disturbance $f(t)$.

where

$$\begin{cases} \dot{\overline{z}}(t) = \begin{bmatrix} -9.4 & -4.4 \\ 4.4 & -5.6 \end{bmatrix} \overline{z}(t) + \begin{bmatrix} 10 \\ -10 \end{bmatrix} u(t) \\ \qquad + \begin{bmatrix} \frac{11}{2} \\ -\frac{11}{2} \end{bmatrix} (y(t) - \begin{bmatrix} -0.8 & -0.8 \end{bmatrix} \overline{z}(t)) + \overline{\phi}(t), \\ \dot{\underline{z}}(t) = \begin{bmatrix} -9.4 & -4.4 \\ 4.4 & -5.6 \end{bmatrix} \underline{z}(t) + \begin{bmatrix} 10 \\ -10 \end{bmatrix} u(t) \\ \qquad + \begin{bmatrix} \frac{11}{2} \\ -\frac{11}{2} \end{bmatrix} (y(t) - \begin{bmatrix} -0.8 & -0.8 \end{bmatrix} \underline{z}(t)) + \underline{\phi}(t), \end{cases}$$

and

$$\begin{cases} \overline{\phi}(t) = \begin{bmatrix} \frac{5}{4}\mu & \frac{5}{2}\mu \end{bmatrix}^{\mathrm{T}}, \\ \underline{\phi}(t) = \begin{bmatrix} -\frac{5}{4}\mu & -\frac{5}{2}\mu \end{bmatrix}^{\mathrm{T}}. \end{cases} \tag{32}$$

Based on the above calculations, the simulation results are shown in Figure 1-4. The interval observers based on a parametric design method can achieve the interval observation of the states in Figures 1-3, besides, the robust interval observers (29) and (31) possess a thinner thickness of the interval length in Figure 2-3, and meanwhile, because of the different matrix M, the convergence rate of the robust interval observers is faster. Therefore, the simulations explain the advantages of a parametric method to design a H_∞ performance-based the robust interval observer than general interval observer.

C. LONGITUDINAL MOTION OF A CHARLIE AIRCRAFT

Consider the dynamic system (1) with bounded disturbances $-0.2 \leq f(t) \leq 0.2$, associated to the longitudinal motion of a Charlie Aircraft in [40] as

$$x = \begin{bmatrix} \Delta u & \Delta\alpha & \Delta\theta & \Delta q \end{bmatrix}^{\mathrm{T}}, \quad u = \delta_e$$

$$A = \begin{bmatrix} -0.007 & 0.012 & -9.81 & 0 \\ -0.128 & -0.54 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0.065 & 0.96 & 0 & -0.99 \end{bmatrix},$$

$$B = F = \begin{bmatrix} 0 \\ -0.04 \\ 0 \\ -12.5 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \tag{33}$$

where u, α, θ and q represent the aircraft longitudinal velocity, the aircraft attack angle, the aircraft pitch angle and the aircraft pitch angular rate respectively. And δ_e is the elevator deflection, and Δ is associated with the perturbation of the variables from the nominal values. The input signal of the system can be chosen as any random signal. In simulation, we considered that the input vector of the system is $u(t) = -0.2\sin(t)\cos(t)$. Then according to Corollary 1, we design two interval observers, one of which is a robust

where

$$\begin{cases} \dot{\overline{z}}(t) = \begin{bmatrix} -30.2371 & 0.2863 & -20.2371 & 0.2863 \\ 5.1528 & -33.3771 & 5.1528 & -12.3771 \\ 45.2301 & -0.4376 & 30.2301 & -0.4376 \\ -13.4402 & 85.8471 & -13.4402 & 31.8471 \end{bmatrix} \overline{z}(t) + \begin{bmatrix} 0.04 \\ 11.6996 \\ -0.04 \\ -10.3796 \end{bmatrix} u(t) \\ \qquad + \begin{bmatrix} 10.3 & -9.4 \\ -2.6 & 408.4 \\ -23.1 & 14.4 \\ 6.9 & -2833 \end{bmatrix} (y(t) - \begin{bmatrix} -1.962 & 0 & -1.962 & 0 \\ 0 & -0.0303 & 0 & -0.0303 \end{bmatrix} \overline{z}(t)) + \begin{bmatrix} 0.008 \\ 2.3399 \\ 0.008 \\ 2.0759 \end{bmatrix} \\ \dot{\underline{z}}(t) = \begin{bmatrix} -30.2371 & 0.2863 & -20.2371 & 0.2863 \\ 5.1528 & -33.3771 & 5.1528 & -12.3771 \\ 45.2301 & -0.4376 & 30.2301 & -0.4376 \\ -13.4402 & 85.8471 & -13.4402 & 31.8471 \end{bmatrix} \underline{z}(t) + \begin{bmatrix} 0.04 \\ 11.6996 \\ -0.04 \\ -10.3796 \end{bmatrix} u(t) \\ \qquad + \begin{bmatrix} 10.3 & -9.4 \\ -2.6 & 408.4 \\ -23.1 & 14.4 \\ 6.9 & -2833 \end{bmatrix} (y(t) - \begin{bmatrix} -1.962 & 0 & -1.962 & 0 \\ 0 & -0.0303 & 0 & -0.0303 \end{bmatrix} \underline{z}(t)) + \begin{bmatrix} -0.008 \\ -2.3399 \\ -0.008 \\ -2.0759 \end{bmatrix} \end{cases}$$

H_∞ interval observer. The specific forms are represented as the dynamics (34) and (35), as shown at the bottom of the 9th page, and the simulation results are shown in Figure 5-6. As shown in Figure 6, the robust H_∞ interval observer (35) has a narrower interval and faster convergence rate than (34). The applicability and effectiveness of the design method are further explained in this paper.

V. CONCLUSION

In this paper, the design problem of an interval observer with the robustness to the bounded disturbances for a linear system is discussed. Utilizing the eigenstructure decomposition, the change of coordinates and the solution to a type of Sylvester equations, the parametric forms of closed-loop system, transformation matrix T and observation gain L are all obtained, and further, the upper bound constraint for the H_∞-gain of the transfer function from the disturbances to the states of the error dynamic system is transformed into the conditions decided by the designed parameters (13) or (24), needing not LMIs. By the parametric H_∞-gain performance, two effective methods to design a robust interval observer is proposed, one of which is to solve the construction problem about the error system cooperativity, and has a controllable convergence rate.

Note that the parametric conditions of the robust interval observers are built as (13) or (24), but how to select the parameters effectively is still a problem worthy of discussion. Fortunately, the rich degree of freedom greatly improves the possibility of the robust interval observers existence.

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