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Flip and Neimark-Sacker Bifurcations of a Discrete Time Predator-Pre Model

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ABSTRACT In this paper, we investigate the system undergoes flip and Neimark Sacker bifurcation in the interior of R^2_+ by using the center manifold theorem and bifurcation theory. The dynamics of this discrete time predator-pre model is investigated in the closed first quadrant of R^2_+.

INDEX TERMS Flip and Neimark sacker bifurcation, time predator-pre model, manifold.

I. INTRODUCTION

The dynamics of biological system is usually described by Lotka voltera system. Lotka Voltera system is also known as predator pre system and is a pair of two differential equations of first order in which two species interact one as prey and other as predator. It was first time suggest by Lotka in 1920 [21] and then by Voltera 1926 [26] and further Holling investigate it and extend this to density depended predator pre model [15]. Afterwards the continuous system has been used to deals with specific interactions, that includes only limit cycles [8], [13], [17], [20], [24]. Further, the study of predator-prey system has become the main research of interest among mathematics due to its universal existence and importance and it played a vital role in the study of mathematical ecology [18], [19], [28], [29]. Recently it has been observed that discrete dynamical system has much richer set of properties than those of continuous system [1]–[3], [12].

Bifurcation was first time introduced bu Henri Poincare in 1885. First time, the word bifurcation was introduced in the paper [4], that deals with the topological or qualitative structure of a solution of a class of differential equations and a class of integral curves of vector fields and is also used to study the mathematical behavior of dynamical systems. Bifurcation occurs when a small change to bifurcation parameter makes a sudden change to qualitative or topological behavior of a system. Bifurcation occurs in both continuous and discrete time dynamical systems [14]. Further, Henri introduced a different types of stationary points and classified them.

Bifurcation theory is used to study the laser dynamics [27] and a lot of theoretical examples that can't be dealt experimentally, for example coupled quantum wells [10] and kicked top [25]. Bifurcation theory is also used to make connection between quantum systems to the dynamics of their classical analogous in atomic systems [5], [9], [23], resonant tunneling diodes [22] and molecular systems [7]. The main reason for the connection between quantum systems and the bifurcation in the classical equation of motion is that at bifurcation, the signature of classical orbits becomes larger, that was dealt by Martin Gutzwiller in his classical work on quantum chaos [16]. Many kinds of bifurcations such as Neimak Sacker bifurcation, Hopf bifurcation, period doubling bifurcation, cusp bifurcation and tangent bifurcation have been discussed in [11].

Flip bifurcation that is also known as period doubling bifurcation occurs when a small changes in bifurcations parameters leads to a new system that bifurcate in twice the period as that of the original system and it takes as many iterations as before for the numerical values visited by the system to repeat themselves [30]–[33]. For more details about this bifurcation, we recommend [34]–[37] and references therein.

Definition 1 (Flip Bifurcation): Let $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ *be a one-parameter family of* C^3 *maps satisfying*

$$f(0,0) = 0$$
$$\left[\frac{\partial f}{\partial x}\right]_{\alpha=0,x=0} = -1$$
$$\left[\frac{\partial^2 f}{\partial x^3}\right]_{\alpha=0,x=0} < 0$$
$$\left[\frac{\partial^3 f}{\partial x^3}\right]_{\alpha=0,x=0} < 0$$

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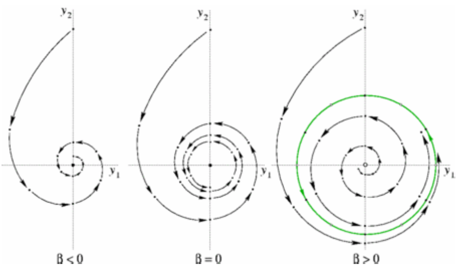

FIGURE 1. Supercritical Neimark-Sacker bifurcation in the plane.

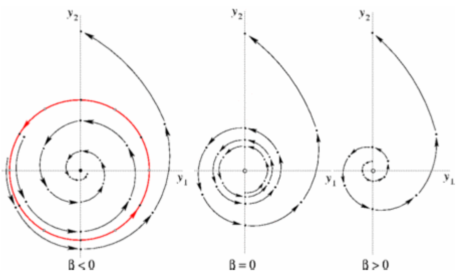

FIGURE 2. Subcritical Neimark-Sacker bifurcation in the plane.

Then there are intervals $(\alpha_1, 0)$, $(0, \alpha_2)$, *and* $\epsilon > 0$ *such that*

1. If $\alpha \in (0, \alpha_2)$, *then* $f_\alpha(x)$ *has one stable orbit of period two for* $x \in (-\epsilon, \epsilon)$ *and one unstable fixed point and*

2. If $\alpha \in (\alpha_1, 0)$, *then* $f_\alpha(x)$ *has a unique stable fixed point for* $x \in (-\epsilon, \epsilon)$.

This type of bifurcation is known as a flip bifurcation.

Neimark sacker bifurcation occur when a fixed point chnages stability by a pair of complex eigen values with unit modulus in a discrete time dynamical system. Neimark sacker bifurcation can be subcritical or supercritical in an unstable or stable closed invariant curve.

Definition 2 (Neimark Sacker): Consider a map $x \longrightarrow f(x, \alpha)$, $x \in R^n$ *depending on a parameter* $\alpha \in R$, *where f is smooth.*

Suppose that for all sufficiently small $|\alpha|$ *the system has a family of fixed points* $x_0(\alpha)$. *Further assume that its Jacobian matrix* $A(\alpha) = f_x(x^0(\alpha), \alpha)$ *has one pair of complex eigenvalues* $\lambda_{1,2}(\alpha) = r(\alpha)e^{\pm i\theta(a)}$ *on the unit circle when* $\alpha = 0$, *i.e.,* $r(0) = 1$ *and* $0 < \theta(0) < p$. *Then, generically, as a passes through* $\alpha = 0$, *the fixed point changes stability and a unique closed invariant curve bifurcates from it. This bifurcation is characterized by a single bifurcation condition* $|\lambda_{1,2}| = 1$ *(has codimension one) and appears generically in one-parameter families of smooth maps. (see Figures 1, 2).*

We will discus the following discrete time dynamical system.

$$\frac{dx}{dt} = rx(1-x) - \frac{cxy}{\alpha + x}$$
$$\frac{dy}{dt} = -dy + \frac{cxy}{\alpha + x} \tag{1}$$

The system describes the dynamics of a simple predator-prey ecosysytem. Here x, y are (scaled) population numbers, and r, c, d and α are parameters characterizes the behaviour of isolated populations and their interaction. Let us consider α as a central parameter and assume $c > d$. Where y, x are the predator and prey densities.

Apply the forward Euler's scheme we get the following system:

$$x \longrightarrow x + \lambda\left[xr(1-x) - \frac{cxy}{\alpha + x}\right]$$
$$y \longrightarrow y + \lambda\left[dy + \frac{cxy}{\alpha + x}\right] \tag{2}$$

where λ is the step size. we are intended to investigate this system as a discrete time dynamical system by using the bifurcation and center manifold theory in the interior of $\mathbb{R}^2_+$

II. EXISTENCE AND UNIQUENESS OF FIXED POINT

Fixed points of above system (2) can be obtained by solving the following system of equations simultaneously

$$x = x + \lambda\left[xr(1-x) - \frac{cxy}{\alpha + x}\right]$$
$$y = y + \lambda\left[dy + \frac{cxy}{\alpha + x}\right]$$

Lemma 1: The model (2) *has two fixed points one is* $(0, 0)$ *and other is* $(\bar{x}, \bar{y})$ *and later is obtained by solving the following system*

$$r(1-\bar{x}) = \frac{\bar{y}}{\alpha + \bar{x}}$$
$$d = \frac{c\bar{x}}{\alpha + \bar{x}}$$

III. LOCAL STABILITY

The Jacobian matrix J of (2) at any fixed point (x, y) can be obtained as $J = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ where,

$$a_{11} = 1 + \lambda\left[r - 2rx - \frac{cy\alpha}{\alpha + x}\right], \quad a_{12} = -\frac{c\lambda x}{\alpha + x},$$
$$a_{21} = \frac{c\lambda\alpha y}{(\alpha + x)^2}, \quad a_{22} = 1 - \lambda\left[d + \frac{cx}{\alpha + x}\right]$$

Characteristic equation of J is

$$T^2 + fT + g = 0$$

where $f = -(a_{11} + a_{22})$, $g = a_{11}a_{22} - a_{12}a_{21}$.

The following lemma is discussed the stability of $(\bar{x}, \bar{y})$.

Lemma 2: Let $F(T) = T^2 + fT + g$. *Suppose that* $F(1) > 0$ *and* T_1, T_2 *are roots of* $F(T) = 0$. *Then*

a. $|T_1| < 1$ *and* $|T_2| < 1$ *iff* $F(-1) > 0$ *and* $q < 1$
b. $|T_1| < 1$ *and* $|T_2| > 1$(*or* $|T_1| > 1$ *and* $|T_2| < 1$) *iff* $F(-1) < 0$
c. $T_1 = -1$ *and* $|T_2| \neq 1$ *iff* $F(-1) = 0$ *and* $p \neq 0, 2$
d. T_1, T_2 *are complex and* $|T_1| = 1$ *and* $|T| = 1$ *iff* $p^2 - 4q < 0$ *and* $q = 1$.

Further, we recall the topological interpretation for a fixed point $(\bar{x}, \bar{y})$ is as follows.

a. If $|T_1| < 1$ and $|T_2| < 1$ then $(\bar{x}, \bar{y})$ is a sink and is locally asymptotically stable.
b. If $|T_1| > 1$ and $|T_2| > 1$ then $(\bar{x}, \bar{y})$ is called a source and is locally asymptotically unstable.
c. If $|T_1| > 1$ and $|T_2| < 1$ (or $|T_1| < 1$ and $|T_2| > 1$) then $(\bar{x}, \bar{y})$ is saddle.
d. If either $|T_1| = 1$ or $|T_2| = 1$ then $(\bar{x}, \bar{y})$ is called non-hyperbolic.

The characteristic equation of J at $(\bar{x}, \bar{y})$ is

$$T^2 - fT + g = 0$$

where,

$$f = 2 + MT \quad \text{and } g = 1 + MT + NT^2$$

where,

$$M = r - 2r\bar{x} - \frac{c\alpha\bar{y}}{(\alpha+\bar{x})^2} \quad \text{and } N = \frac{\alpha c^2 \bar{x}\bar{y}}{(\alpha+\bar{x})^3}.$$

Then the characteristic polynomial will become

$$F(T) = T^2 - (2 + M\lambda)T + (1 + M\lambda + N\lambda^2) = 0$$

We need the following proposition

Proposition 1: 1 *: It is defined as sink if one of the following properties is holds*

i) $-2\sqrt{N} \le M < 0$ *and* $0 < \lambda < -\frac{M}{N}$

ii) $M < -2\sqrt{N}$ *and* $0 < \lambda < \frac{-M-\sqrt{M^2-4N}}{N}$

2 *: It is defined as source if one of the following two conditions is holds*

i) $-2\sqrt{N} \le M < 0$ *and* $\lambda > \frac{-M}{N}$

ii) $M < -2\sqrt{N}$ *and* $\lambda > \frac{-M+\sqrt{M^2-4N}}{N}$

iii) $M \ge 0$

3 *: It is defined as saddle if the following property holds* $M < -2\sqrt{N}$ *and* $\frac{-M-\sqrt{M^2-4N}}{N} < \lambda < \frac{-M+\sqrt{M^2-4N}}{N}$

4 *: It is non-hyperbolic if one of the following two conditions is satisfied*

i) $M < -2\sqrt{N}$ *and* $\lambda = \frac{-M\pm\sqrt{M^2-4N}}{N}$ *and* $\lambda \ne \frac{-2}{M}, \frac{4}{M}$

ii) $-2\sqrt{N} < M < 0$ *and* $\lambda = \frac{-M}{N}$

IV. FLIP BIFURCATION

Define Ω_{11} and Ω_{12} as

$$\Omega_{11} = \Big\{(r, c, d, \alpha, \lambda) : \lambda = \frac{-M - \sqrt{M^2-4N}}{N}, \; M < -2\sqrt{N}, r, c, d, \alpha, \lambda > 0\Big\}$$

$$\Omega_{12} = \Big\{(r, c, d, \alpha, \lambda) : \lambda = \frac{-M + \sqrt{M^2-4N}}{N}, \; M < -2\sqrt{N}, r, c, d, \alpha, \lambda > 0\Big\}$$

Flip bifurcation undergoes in a small neighborhood of Ω_{11} or Ω_{12}.

Consider parameters $(r, c, d, \alpha, \lambda_1) \in \Omega_{11}$, s.t.

$$\begin{cases} x \longrightarrow x + \lambda_1 \left[xr(1-x) - \frac{cxy}{\alpha+x}\right] \\ y \longrightarrow y + \lambda_1 \left[dy + \frac{cxy}{\alpha+x}\right] \end{cases} \tag{3}$$

The equation (3) has a unique positive fixed point $(\bar{x}, \bar{y})$.

Since $(r, c, d, \alpha, \lambda_1) \in \Omega_{11}$, $\lambda_1 = \frac{-M-\sqrt{M^2-4N}}{N}$. Choosing $\lambda^\star$ as a bifurcation parameter and a perturbation of (3) as

$$\begin{cases} x \longrightarrow x + (\lambda_1+\lambda^*) \left[rx(1-x) - \frac{cxy}{\alpha+x}\right] \\ y \longrightarrow y + (\lambda_1+\lambda^*) \left[dy + \frac{cxy}{\alpha+x}\right] \end{cases} \tag{4}$$

where $|\lambda^\star| \ll 1$.

Let $u = x - \bar{x}, v = y - \bar{y}$. Then the transformation of the fixed point $(\bar{x}, \bar{y})$ into the origin can be obtained by the following map (4)

$$\begin{pmatrix} u \\ v \end{pmatrix} \longrightarrow \begin{pmatrix} a_1u + a_2v + a_3uv + a_4u^2 + a_5uv\lambda^* + a_6u^2\lambda^* \\ +a_7u^3 + a_8u\lambda^* + O((|u| + |v| + |\lambda^*|)^4) \\ \\ b_1u + b_2v + b_3uv + b_4u^2 + b_5uv\lambda^* + b_6u^2\lambda^* \\ +b_7u^3 + b_8u\lambda^* + O((|u| + |v| + |\lambda^*|)^4) \end{pmatrix} \tag{5}$$

where

$$a_1 = 1 + \lambda\left[r - 2r\bar{x} - \frac{\alpha c\bar{y}}{(\alpha+\bar{x})^2}\right], \quad a_2 = \frac{-\lambda c\bar{x}}{\alpha+\bar{x}},$$

$$a_3 = \frac{-\alpha c\lambda}{(\alpha+\bar{x})^2}, \quad a_4 = \lambda\left[-2\,r + \frac{2\alpha c\bar{y}}{(\alpha+\bar{x})^3}\right]$$

$$a_5 = \frac{-\alpha c}{(\alpha+\bar{x})^2}, \quad a_6 = -2r + \frac{2\alpha c\lambda\bar{y}}{(\alpha+\bar{x})^3},$$

$$a_7 = \frac{-6\alpha c\lambda\bar{y}}{(\alpha+\bar{x})^4}, \quad a_8 = r\lambda - 2r\bar{x} - \frac{\alpha c\bar{y}}{(\alpha+\bar{x})^2}$$

$$b_1 = \frac{\alpha c\lambda\bar{y}}{(\alpha+\bar{x})^2}, \quad b_2 = 1 + d\lambda + \frac{c\lambda\bar{x}}{\alpha+\bar{x}}, \quad b_3 = \frac{\alpha c\lambda}{(\alpha+\bar{x})^2},$$

$$b_4 = \frac{-2\alpha c\bar{y}\lambda}{(\alpha+\bar{x})^3}, \quad b_5 = \frac{\alpha c}{(\alpha+\bar{x})^2}, \quad b_6 = \frac{-2\alpha c\bar{y}}{(\alpha+\bar{x})^3},$$

$$b_7 = \frac{6\alpha c\bar{y}\lambda}{(\alpha+\bar{x})^4}, \quad b_8 = \frac{\alpha c\bar{y}}{(\alpha+\bar{x})^2} \quad \text{and } \lambda = \lambda_1.$$

We construct an invertible matrix

$$T = \begin{pmatrix} a_2 & a_2 \\ -1-a_1 & t-a_1 \end{pmatrix} \tag{6}$$

and use the translation

$$\begin{pmatrix} u \\ v \end{pmatrix} = T \begin{pmatrix} \mu \\ \nu \end{pmatrix} \tag{7}$$

for the equation (5). Then the map (5) becomes

$$\begin{pmatrix} \mu \\ \nu \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 0 \\ 0 & t \end{pmatrix}\begin{pmatrix} \mu \\ \nu \end{pmatrix} + \begin{pmatrix} f(\mu, \nu, \lambda^*) \\ g(\mu, \nu, \lambda^*) \end{pmatrix} \tag{8}$$

where

$$f(\mu,\nu,\lambda^*) = \frac{a_3(t-a_1)-a_2b_3}{a_2(t+1)}uv + \frac{a_4(t-a_1)-a_2b_4}{a_2(t+1)}u^2 + \frac{a_5(t-a_1)-a_2b_5}{a_2(t+1)}uv\lambda^* + \frac{a_6(t-a_1)-a_2b_6}{a_2(t+1)}u^2\lambda^* + \frac{a_7(t-a_1)-a_2b_7}{a_2(t+1)}u^3 + O((|u|+|v|+|\lambda^*|)^4)$$

$$g(x,y,\lambda^*) = \frac{a_3(1+a_1)+a_2b_3}{a_2(t+1)}uv + \frac{a_4(1+a_1)+a_2b_4}{a_2(t+1)}u^2 + \frac{a_5(1+a_1)+a_2b_5}{a_2(t+1)}uv\lambda^* + \frac{a_6(1+a_1)+a_2b_6}{a_2(t+1)}u^2\lambda^* + \frac{a_7(1+a_1)+a_2b_7}{a_2(t+1)}u^3 + O((|u|+|v|+|\lambda^*|)^4)$$

and

$$u = a_2(\mu+\nu), \quad v = -(1+a_1)\mu + (t-a_1)\nu$$
$$u^2 = a_2^2(\mu^2+2\mu\nu+\nu^2)$$
$$uv = -a_2(1+a_1)\mu^2 + a_2(t-a_1)\mu\nu - a_2(1+a_1)\mu\nu + a_2(t-a_1)\nu^2$$
$$u^3 = a_2^3(\mu^3+3\mu^2\nu+3\mu\nu^2+\nu^3)$$

We observe that the center manifold $W^C(0,0,0)$ of (8) in the neighborhood of $\lambda^* = 0$ at the fixed point (0,0) can be approximated as follows

$$W^c(0,0,0) = \{(\mu,\nu,\lambda^*) \in \mathbb{R}^3 : \nu = a_1\mu^2 + a_2\mu\lambda^* + a_3\lambda^{*2} + O((|\mu|+|\lambda^*|)^3)\} \quad (9)$$

where $O((|\mu|+|\lambda^*|)^3)$ is a function with order atleast 3 in the variables and

$$a_1 = \frac{a_{12}[a_{13}(1+a_{11})+a_{12}a_{23}]+(1+a_{11})[a_{25}-a_{14}-a_{12}a_{24}]}{1-t_2^2}$$

$$a_2 = \frac{(1+a_{11})[b_2(1+a_{11})+a_{12}c_2]-a_{12}[b_1(1+a_{11})+a_{12}c_1]}{a_{12}(1+t_2^2)}$$

$$a_3 = 0$$

Therefore, we consider a restriction of map (8) to the center manifold

$$\widetilde{F} : \mu \longrightarrow -\mu + h_1\mu^2 + h_2\mu^3 + h_3\mu\lambda^* + h_4\mu^2\lambda^* + O((|\mu|+\lambda^*)^3) \quad (10)$$

where

$$h_1 = \frac{a_2[a_4(t-a_1)-a_2b_4]-(1+a_1)[a_3(t-a_1)-a_2b_3]}{t+1}$$
$$h_2 = \frac{a_2^2[a_1(t-a_1)-a_2b_1]}{t+1}, \quad h_3 = \frac{a_8(t-a_1)-a_2b_8}{t+1}$$
$$h_4 = \frac{a_2[a_6(t-a_1)-a_2b_6]-(1+a_1)[a_5(t-a_1)-a_2b_5]}{t+1}$$

In order for map (10) to undergo a Flip bifurcation, α_1 and α_2 should be nonzero.

$$\alpha_1 = \left(\frac{\partial^2 F}{\partial\tilde{x}\partial\lambda^*} + \frac{1}{2}\frac{\partial F}{\partial\lambda^*}\frac{\partial^2 F}{\partial\tilde{x}^2}\right) \quad \text{at } (0,0).$$

and

$$\alpha_2 = \left(\frac{1}{6}\frac{\partial^3 F}{\partial\tilde{x}^3} + \left(\frac{1}{2}\frac{\partial^2 F}{\partial\tilde{x}^2}\right)^2\right)$$

Theorem 1: When $\alpha_2 \neq 0$ and parameter λ varies in a small neighbourhood of λ_1^* then the system (2) undergoes a flip bifurcation. Furthermore, if $\alpha_2 > 0$(resp. $\alpha_2 < 0$) then the orbits of period-2 that bifurcate from $(\bar{x},\bar{y})$ are stable(resp. unstable)

V. NEIMARK-SACKER BIFURCATION

Define

$$\Omega_2 = \left\{(r,c,d,\alpha,\lambda) : \lambda = \frac{-G}{H}, -2\sqrt{H} < G < 0, r,c,d,\alpha,\lambda > 0\right\}$$

and we will investigate the Neimark-Sacker bifurcation of system (2) if parameters varry in a small neighborhood of Ω_2.

Taking parameters $(r,c,d,\alpha,\lambda_2) \in \Omega_2$, then the given system becomes

$$\begin{cases} x \longrightarrow x + \lambda_2\left[xr(1-x) - \frac{cxy}{\alpha+x}\right] \\ y \longrightarrow y + \lambda_2\left[dy + \frac{cxy}{\alpha+x}\right] \end{cases} \quad (11)$$

This map has only one positive fixed point $(\bar{x},\bar{y})$.

Since $(r,c,d,\alpha,\lambda_2) \in \Omega_2$, $\lambda_2 = -\frac{M}{N}$.

Consider $\bar{\lambda}^*$ as a bifurcation parameter and consider a perturbation of (11) as follows

$$\begin{cases} x \longrightarrow x + (\lambda_2+\bar{\lambda}^*)\left[xr(1-x) - \frac{cxy}{\alpha+x}\right] \\ y \longrightarrow y + (\delta_2+\bar{\delta}^*)\left[dy + \frac{cxy}{\alpha+x}\right] \end{cases} \quad (12)$$

where $|\bar{\lambda}^*| \ll 1$, which is small perturbation parameter.

$$\begin{pmatrix} u \\ v \end{pmatrix} \longrightarrow \begin{pmatrix} a_1u+a_2v+a_3uv+a_4u^2+a_7u^3+O((|u|+|v|)^4) \\ b_1u+b_2v+b_3uv+b_4u^2+b_7u^3+O((|u|+|v|)^4) \end{pmatrix} \quad (13)$$

where a_1, a_2, a_3, a_4, a_7 and b_1, b_2, b_3, b_4, b_7 are given as above.

Note that the characteristic equation of the map (13) at point (u,v) is given by

$$T^2 + f(\bar{\lambda}^*)T + g(\bar{\lambda}^*) = 0 \quad (14)$$

where

$$f(\bar{\lambda}^*) = -2 - M(\lambda_2+\bar{\lambda}^*)$$
$$g(\bar{\lambda}^*) = 1 + M(\lambda_2+\bar{\lambda}^*) + N(\lambda_2+\bar{\lambda}^*)$$

Since the parameters $(r, c, d, \alpha, \lambda_2) \in \Omega_2$. Therefore the eigenvalues of $(0, 0)$ are a pair of conjugate complex numbers $T, \overline{T}$ with $|T| = |\overline{T}| = 1$, where

$$T, \overline{T} = -\frac{f(\overline{\lambda}^*)}{2} \pm \frac{i}{2}\sqrt{4g(\overline{\lambda}^*) - f^2(\overline{\lambda}^*)}$$

$$T, \overline{T} = 1 + \frac{M(\lambda_2 + \overline{\lambda}^*)}{2} \pm \frac{i(\lambda_2 + \overline{\lambda}^*)}{2}\sqrt{4N - M^2}$$

We have $|T| = \sqrt{(q(\overline{\lambda}^*))}$, $l = \frac{d|T|}{d\overline{\lambda}^*} = \frac{-M}{2} > 0$

In addition it is required that when $\overline{\lambda}^*$, $T^m, \overline{T}^m \neq 1 (m = 1, 2, 3, 4)$ which is equivalent to $f(0) \neq -2, 0, 1, 2$. Note that $(r, c, d, \alpha, \lambda_2) \in \Omega_2$. Thus the eigen values $T, \overline{T}$ of the fixed point $(0, 0)$ do not lie in the intersection of the unit circle with the coordinate axes when $\overline{\lambda}^* = 0$ if the following conditions are hold.

$$M^2 \neq 2N, 3N \tag{15}$$

Next we find the normal form of (13) at $\overline{\lambda}^* = 0$.

Let $\overline{\lambda}^* = 0$, $\rho = 1 + \frac{M\lambda_2}{2}$, $\omega = \frac{\lambda_2}{2}\sqrt{4N - M^2}$.

$$T = \begin{pmatrix} a_2 & 0 \\ \rho - a_1 & -\omega \end{pmatrix}$$

and consider the translation

$$\begin{pmatrix} u \\ v \end{pmatrix} = T \begin{pmatrix} \mu \\ \nu \end{pmatrix} \tag{16}$$

for the equation (13); then the map (13) becomes

$$\begin{pmatrix} \mu \\ \nu \end{pmatrix} \longrightarrow \begin{pmatrix} \rho & -\omega \\ \omega & \rho \end{pmatrix} \begin{pmatrix} \mu \\ \nu \end{pmatrix} + \begin{pmatrix} f(\mu, \nu) \\ g(\mu, \nu) \end{pmatrix} \tag{17}$$

where

$$f(\mu, \nu) = \frac{a_3}{a_2}uv + \frac{a_4}{a_2}u^2 + \frac{a_7}{a_2}u^3 + O((|\mu| + |\nu|)^4)$$

$$g(\mu, \nu) = -\frac{a_3(a_1 - \rho) + a_2 b_3}{a_2\omega}uv - \frac{a_4(a_1 - \rho) + a_2 b_4}{a_2\omega}u^2 - \frac{a_7(a_1 - \rho) + a_7 b_3}{a_2\omega}u^3 + O((|\mu| + |\nu|)^4)$$

$$u = a_2\mu, \quad v = (\rho - a_1)\mu - \omega\nu, \quad u^2 = a_2^2\mu^2, \quad u^3 = a_2^3\mu^3,$$
$$uv = a_2(\rho - a_1)\mu^2 - a_2\omega\mu\nu$$

and

$$f_{\mu\mu} = 2a_3(\rho - a_1) + 2a_2a_4, \quad f_{\mu\nu} = -a_3\omega,$$
$$f_{\nu\nu} = 0, \quad f_{\mu\mu\mu} = 6a_2a_1, \; f_{\mu\mu\nu} = f_{\mu\nu\nu} = f_{\nu\nu\nu} = 0$$
$$g_{\mu\mu} = \frac{2}{\omega}\{a_3(\rho - a_1) - a_2b_3 + a_2(a_4(\rho - a_1) - a_2b_4)\}$$
$$g_{\mu\nu} = a_2b_3 + a_3(a_1 - \rho), \quad g_{\nu\nu} = 0,$$
$$g_{\mu\mu\mu} = \frac{6a_2^2}{\omega}(a_7(\rho - a_1) - a_7b_3), \quad g_{\mu\mu\nu} = g_{\mu\nu\nu} = g_{\nu\nu\nu}.$$

The map (17) will undergo a Neimark-Sacker bifurcation at $\overline{\lambda}^* = 0$, $a \neq 0$, where

$$a = \left[-Re\left(\frac{(1-2T)\overline{T}^2}{1-T}\xi_{20}\xi_{11}\right) - \frac{1}{2}|\xi_{11}|^2 - |\xi_{02}|^2 + Re(\overline{T}\xi_{21})\right]$$

where

$$\xi_{20} = \frac{1}{8}\left[(f_{\mu\mu} - f_{\nu\nu} + 2g_{\mu\nu}) + i(g_{\mu\mu} - g_{\nu\nu} - 2f_{\mu\nu})\right]$$
$$\xi_{11} = \frac{1}{4}\left[(f_{\mu\mu} + f_{\nu\nu}) + i(g_{\mu\mu} + g_{\mu\mu})\right]$$
$$\xi_{02} = \frac{1}{8}\left[(f_{\mu\mu} - f_{\nu\nu} - 2g_{\mu\nu}) + i(g_{\mu\mu} - g_{\nu\nu} + 2f_{\mu\nu})\right]$$
$$\xi_{21} = \frac{1}{16}\Big[(f_{\mu\mu\mu} + f_{\mu\nu\nu} + g_{\mu\mu\nu} + g_{\nu\nu\nu}) + i(g_{\mu\mu\mu} + g_{\mu\nu\nu} - f_{\mu\mu\nu} + f_{\nu\nu\nu})\Big]$$

Theorem 2: If the (15) *hold and* $a \neq 0$ *then the map* (2) *undergoes a Neimark-Sacker bifurcation at the equilibrium point* $(\overline{x}, \overline{y})$ *when the parameter* λ *varies in a very small neighborhood of* λ_2*. Furthermore, if* $a > 0$*(resp.* $a < 0$*), then a repelling (resp. attracting) invariant closed curve bifurcates from the equilibrium point for* $\lambda < \lambda_2$*(resp.* $\lambda > \lambda_2$*).*

VI. CONCLUSION

Bifurcation theory is the mathematical study of changes in the qualitative or topological structure of a given family, such as the integral curves of a family of vector fields, and the solutions of a family of differential equations. Most commonly applied to the mathematical study of dynamical systems, a bifurcation occurs when a small smooth change made to the parameter values (the bifurcation parameters) of a system causes a sudden 'qualitative' or topological change in its behavior. Bifurcations occur in both continuous systems (described by ODEs, DDEs or PDEs) and discrete systems (described by maps). In this paper, we investigated the system undergoes flip and Neimark Sacker bifurcation in the interior of R_+^2 by using the center manifold theorem and bifurcation theory. The dynamics of this discrete time predator-pre model is investigated in the closed first quadrant of R_+^2.

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