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Best Proximity Point Results for γ-Controlled Proximal Contraction

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ABSTRACT In this article, we introduce the notion of weak P_γ-property and γ-controlled proximal contraction in the setting of b-metric spaces and prove best proximity results for such mappings. By restricting these results, we get some new results to study the existence of best proximity points and fixed points of mappings.

INDEX TERMS Fixed points, best proximity points, γ-controlled proximal contraction, weak P_γ-property.

I. INTRODUCTION AND PRELIMINARIES

Let (X, d_s) be a metric space. A mapping $\Upsilon : L \subset X \to K \subset X$ has a fixed point $\beta \in L$, if $\beta = \Upsilon\beta$, that is, $d_s(\beta, \Upsilon\beta) = 0$. When $d_s(\beta, \Upsilon\beta) > 0$ for all $\beta \in L$. Then one can discuss and find a point $\beta \in L$ for which $d_s(\beta, \Upsilon\beta)$ has least value. Finding of such point is the base of best proximity theory. The literature of best proximity and fixed point is very rich and we have many significant results some of them are given in [1]–[17]. We could see from the literature that rich branches of best proximity theory are based on the concepts of P-property/Weak P-property, approximately compactness and uniformly convex Banach space. Whereas Almeida *et. al.* [1] showed that some best proximity point results proved by using the concept of Weak P-property can be obtained from their associated fixed point results. In this paper, we modify and generalized the concept of Weak P-property to overcome the finding of Almeida *et. al.* [1] for best proximity point results. By using our generalized concept of Weak P-property almost all existing results of best proximity point could be further extended and the finding of Almeida *et. al.* [1] are not applicable.

It is not false to say that the most classical result of this theory was given by Fan [3].

Theorem 1 ([3]): Let L be a nonempty convex and compact subset of normed linear space X and $\Upsilon : L \to X$ be a continuous function. Then there exists $\beta \in L$ such that

$$\|\beta - \Upsilon\beta\| = \inf_{a\in L}\{\|\Upsilon\beta - a\|\}.$$

Abkar and Gbeleh [5] gave best proximity result for nonself multivalued mappings satisfying P-property. Kiran *et al*. [7] generalized the result of [5] by giving the concept of controlled proximal contraction. Jleli and Samet [9] gave the notion of α-proximal admissible and α-ψ-proximal contractive type mappings and proved the corresponding best proximity point theorems. These notions and results have been extended to multivalued nonself mappings by Ali *et al*. [10] and Choudhurya *et al*. [11], independently. These results also generalized the result of [5].

The purpose of this paper is to introduce some results in the setting of b-metric spaces which generalize the results given in the above articles.

Throughout this section (X, d_s) be a metric space, K and L are nonempty subsets of X. The following notations and definitions are used in this article.

$$d_s(\beta, L) = \inf\{d_s(\beta, l) : l \in L\}$$
$$dist(K, L) = \inf\{d_s(k, l) : k \in K,\ l \in L\}$$
$$K_0 = \{k \in K : d_s(k, l) = dist(K, L) \text{ for some } l \in L\}$$
$$L_0 = \{l \in L : d_s(k, l) = dist(K, L) \text{ for some } k \in K\}$$

and

$$B(\beta_0, r) = \{\beta \in X : d_s(\beta_0, \beta) \le r\}.$$

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$CL(\widehat{L})$ is used to represent the collection of all nonempty closed subsets of $\widehat{L}$. For every $K, L \in CL(\widehat{L})$, let

$$H_s(K,L) = \begin{cases} \max\{\sup_{\beta\in K} d_s(\beta, L), \sup_{\zeta\in L} d_s(\zeta, K)\} \\ \qquad\qquad \text{if maximum exists;} \\ \infty \qquad\qquad \text{otherwise.} \end{cases}$$

Such a map H_s is known as generalized Hausdorff metric induced by d_s.

Definition 2 ([6]): A pair (K, L) has a weak P-property, if $K_0 \neq \emptyset$, for any $\beta_1, \beta_2 \in K$ and $\zeta_1, \zeta_2 \in L$ with $d_s(\beta_1, \zeta_1) = dist(K, L) = d_s(\beta_2, \zeta_2)$, we have $d_s(\beta_1, \beta_2) \leq d_s(\zeta_1, \zeta_2)$.

Abkar and Gabeleh in [4] showed that every nonempty, bounded, closed and convex pair in a uniformly convex Banach space X satisfies the above definition with an equality sign.

Definition 3 ([5]): An element $\beta^* \in K$ is called best proximity point of a multivalued nonself map Υ, if $d_s(\beta^*, \Upsilon\beta^*) = dist(K, L)$.

Ali *et al*. [10] extended the concept of Jleli and Samet [9] for multivalued mappings in the following way:

Definition 4 ([10]): A multivalued map $\Upsilon : K \to 2^L \setminus \emptyset$ is γ-proximal admissible if there is a function $\gamma : K \times K \to [0, \infty)$ such that

$$\begin{cases} \gamma(\beta_1, \beta_2) \geq 1 \\ d_s(u_1, \zeta_1) = dist(K, L) \\ d_s(u_2, \zeta_2) = dist(K, L) \end{cases} \Rightarrow \gamma(u_1, u_2) \geq 1$$

where $\beta_1, \beta_2, u_1, u_2 \in K$ and $\zeta_1 \in \Upsilon\beta_1, \zeta_2 \in \Upsilon\beta_2$.

Czerwik [2] stated the following generalization of metric space.

Definition 5: A mapping $d_s : X \times X \to [0, \infty)$ is known as b-metric on a nonempty set X, if for every $k, l, m \in X$, we have a real number $s \geq 1$ holding the following axioms:

(i) $d_s(k, l) = 0$ if and only if $k = l$;
(ii) $d_s(k, l) = d_s(l, k)$;
(iii) $d_s(k, m) \leq s[d_s(k, l) + d_s(l, m)]$.

Then (X, d_s, s) is said to be a b-metric space.

The following famous lemma of the existing literature will be used in our main results.

Lemma 6: Let (X, d_s, s) be a b-metric space, $L \in CL(X)$ and $p > 1$. Then for each $\beta \in X$, there exists $l \in L$ such that

$$d_s(\beta, l) \leq p d_s(\beta, L). \tag{1}$$

II. MAIN RESULT

Here, Ω_s denotes the collection of functions $\chi : [0, \infty) \to [0, \infty)$ having the following properties:

(i) χ is nondecreasing function;
(ii) $\chi(at) = a\chi(t)$ for all $a, t \geq 0$;
(iii) $\sum_{n=0}^{\infty} s^{2n}\chi^n(t) < \infty$, where $s \geq 1$;
(iv) $\chi^0(t) = t$.

Through out this section: We consider K and L are nonempty subsets of a b-metric space (X, d_s, s), $\beta_0 \in K_0$ and $B(\beta_0, r)$ is a closed ball in X.

Definition 7: A mapping $\Upsilon : K \to CL(L)$ is γ-controlled proximal contraction on $B(\beta_0, r)$, if for each $\beta, \zeta \in B(\beta_0, r) \cap K$ with $\gamma(\beta, \zeta) \geq 1$, we have

$$H_s(\Upsilon\beta, \Upsilon\zeta) \leq \chi(d_s(\beta, \zeta)) \tag{2}$$

where, $\chi \in \Omega_s$ and $\gamma : K \times K \to [0, \infty)$.

The following definition is a generalization of the [15, Definition 2.1]

Definition 8: A pair (K, L) of nonempty subsets of (X, d_s, s) has a weak P_γ-property, if for any $\beta_1, \beta_2 \in K_0$, $K_0 \neq \emptyset$, and $\zeta_1, \zeta_2 \in L$, we have

$$\begin{cases} \gamma(\beta_1, \beta_2) \geq 1 \\ d_s(\beta_1, \zeta_1) = dist(K, L) \\ d_s(\beta_2, \zeta_2) = dist(K, L) \end{cases} \Rightarrow d_s(\beta_1, \beta_2) \leq d_s(\zeta_1, \zeta_2)$$

where $\gamma : K \times K \to [0, \infty)$ is a function.

Example 9: Let $X = \mathbb{R}^3$ and $d_s((\beta_1, \beta_2, \beta_3), (\zeta_1, \zeta_2, \zeta_3)) = \sum_{i=1}^{3} |\beta_i - \zeta_i|$. Take the sets $K = \{(0, 0, \beta) : \beta \in [3, 4]\} \cup \{(1, 0, 0)\}$ and $L = \{(1, 0, \zeta) : \zeta \in [3, 4]\} \cup \{(1, 0, 1)\}$. Define $\gamma : K \times K \to [0, \infty)$ by

$$\gamma((a, 0, \beta), (b, 0, \zeta)) = \begin{cases} 1, & \text{if } \beta, \zeta \in [3, 4] \text{ and } a = b = 0 \\ 0, & \text{otherwise.} \end{cases}$$

For the above defined γ, K, and L, one can easily verify that (K, L) has a weak P_γ-property. But by taking $\beta_1 = (1, 0, 0)$, $\beta_2 = (0, 0, 3) \in K_0$ and $\zeta_1 = (1, 0, 1)$, $\zeta_2 = (1, 0, 3) \in L$, we have $d_s(\beta_1, \zeta_1) = 1 = dist(K, L)$ and $d_s(\beta_2, \zeta_2) = 1 = dist(K, L)$; and $d_s(\beta_1, \beta_2) = 4$ and $d_s(\zeta_1, \zeta_2) = 2$. That is, $d_s(\beta_1, \beta_2) > d_s(\zeta_1, \zeta_2)$. Hence, weak P-property does not hold for (K, L).

In the following results, we take (X, d_s, s) as a complete and continuous b-metric space, and K, L are nonempty subsets of X. The following hypotheses may also be used in our results.

(T-i) For each $\beta \in K_0$, we have $\Upsilon\beta \subseteq L_0$ and the pair (K, L) satisfies weak P_γ-property.
(T-ii) Υ is γ-proximal admissible.
(T-iii) Υ is γ-controlled proximal contraction on the closed ball $B(\beta_0, r)$, for some $\beta_0 \in K_0$ and $r > 0$, and $\sum_{n=0}^{\infty} s^{2n+2}\chi^n(d_s(\beta_0, \Upsilon\beta_0) + dist(K, L)) < r$. Further, for $\beta_0 \in K_0$, there exist $\zeta_0 \in \Upsilon\beta_0$ and $\beta_1 \in K_0$ such that $d_s(\beta_1, \zeta_0) = dist(K, L)$ and $\gamma(\beta_0, \beta_1) \geq 1$.
(T-iv) Υ is continuous.
(T-v) for each sequence $\{\beta_n\}$ in K with $\gamma(\beta_n, \beta_{n+1}) \geq 1 \;\forall\, n \in \mathbb{N}$ and $\beta_n \to \beta \in K$, we have $\gamma(\beta_n, \beta) \geq 1 \;\forall\, n \in \mathbb{N}$.

where $\Upsilon : K \to CL(L)$ and $\gamma : K \times K \to [0, \infty)$.

Now, we present the first result of this article.

Theorem 10: Let (X, d_s, s) with $s > 1$, let K_0 be nonempty and $\Upsilon : K \to CL(L)$ be a mapping which satisfies the hypotheses: (T-i)-(T-iv). Then Υ has a best proximity point in $B(\beta_0, r) \cap K_0$.

Proof: From (T-iii), there are $\beta_0, \beta_1 \in K_0$ and $\zeta_0 \in \Upsilon\beta_0$ such that

$$d_s(\beta_1, \zeta_0) = dist(K, L) \ \ and \ \ \gamma(\beta_0, \beta_1) \geq 1. \tag{3}$$

By triangle inequality, (T-iii) and (3), we calculate

$$\begin{aligned} d_s(\beta_0, \beta_1) &\leq s[d_s(\beta_0, \Upsilon\beta_0) + d_s(\Upsilon\beta_0, \beta_1)] \\ &\leq s[d_s(\beta_0, \Upsilon\beta_0) + d_s(\zeta_0, \beta_1)] \\ &= s\Lambda < r \end{aligned} \tag{4}$$

where $\Lambda = d_s(\beta_0, \Upsilon\beta_0) + dist(K, L)$. This implies $\beta_1 \in B(\beta_0, r) \cap K$, since $\beta_1 \in K_0 \subseteq K$. From (2), we get

$$d_s(\zeta_0, \Upsilon\beta_1) \leq H_s(\Upsilon\beta_0, \Upsilon\beta_1) \leq \chi(d_s(\beta_0, \beta_1)). \tag{5}$$

As $s > 1$, by Lemma 6, there exists $\zeta_1 \in \Upsilon\beta_1$ such that

$$d_s(\zeta_0, \zeta_1) \leq sd_s(\zeta_0, \Upsilon\beta_1) \leq s\chi(d_s(\beta_0, \beta_1)). \tag{6}$$

As $\zeta_1 \in \Upsilon\beta_1 \subseteq L_0$ then we get $\beta_2 \in K_0$ such that

$$d_s(\beta_2, \zeta_1) = dist(K, L). \tag{7}$$

It is given that Υ is γ-proximal admissible, then (3) and (7) yield, $\gamma(\beta_1, \beta_2) \geq 1$. By hypothesis (T-i), from $\gamma(\beta_1, \beta_2) \geq 1$, (3) and (7), we get

$$d_s(\beta_1, \beta_2) \leq d_s(\zeta_0, \zeta_1). \tag{8}$$

From (6) and (8), we have

$$d_s(\beta_1, \beta_2) \leq s\chi(d_s(\beta_0, \beta_1)). \tag{9}$$

By applying χ in (9), we have

$$\chi(d_s(\beta_1, \beta_2)) \leq s\chi^2(d_s(\beta_0, \beta_1)). \tag{10}$$

The triangle inequality, (4) and (9), yield

$$\begin{aligned} d_s(\beta_0, \beta_2) &\leq sd_s(\beta_0, \beta_1) + s^2 d_s(\beta_1, \beta_2) \\ &\leq sd_s(\beta_0, \beta_1) + s^3\chi(d_s(\beta_0, \beta_1)) \\ &\leq s^2\Lambda + s^4\chi(\Lambda) < r. \end{aligned}$$

This inequality and the fact $\beta_2 \in K_0 \subseteq K$ implies that $\beta_2 \in B(\beta_0, r) \cap K$. Since $\gamma(\beta_1, \beta_2) \geq 1$ and $\beta_1, \beta_2 \in B(\beta_0, r) \cap K$, then from (2), we get

$$d_s(\zeta_1, \Upsilon\beta_2) \leq H_s(\Upsilon\beta_1, \Upsilon\beta_2) \leq \chi(d_s(\beta_1, \beta_2)). \tag{11}$$

Lemma 6 ensures there is $\zeta_2 \in \Upsilon\beta_2$ such that $d_s(\zeta_1, \zeta_2) \leq sd_s(\zeta_1, \Upsilon\beta_2)$. Thus by (11), we get

$$d_s(\zeta_1, \zeta_2) \leq s\chi(d_s(\beta_1, \beta_2)). \tag{12}$$

As $\zeta_2 \in \Upsilon\beta_2 \subseteq L_0$, there is $\beta_3 \in K_0$ such that

$$d_s(\beta_3, \zeta_2) = dist(K, L). \tag{13}$$

As $\gamma(\beta_1, \beta_2) \geq 1$ then by using hypothesis (T-ii) we get $\gamma(\beta_2, \beta_3) \geq 1$, since (7) and (13) hold. From hypothesis (T-i), by using the facts of (7), (13) and $\gamma(\beta_2, \beta_3) \geq 1$, we get

$$d_s(\beta_2, \beta_3) \leq d_s(\zeta_1, \zeta_2). \tag{14}$$

From (14), (12) and (10), we get

$$d_s(\beta_2, \beta_3) \leq s\chi(d_s(\beta_1, \beta_2)) \leq s^2\chi^2(d_s(\beta_0, \beta_1)). \tag{15}$$

The triangle inequality, (9) and (15), yield

$$\begin{aligned} d_s(\beta_0, \beta_3) &\leq sd_s(\beta_0, \beta_1) + s^2 d_s(\beta_1, \beta_2) + s^3 d_s(\beta_2, \beta_3) \\ &\leq sd_s(\beta_0, \beta_1) + s^3\chi(d_s(\beta_0, \beta_1)) \\ &\quad + s^5\chi^2(d_s(\beta_0, \beta_1)) \\ &\leq s^2\Lambda + s^4\chi(\Lambda) + s^6\chi^2(\Lambda) < r. \end{aligned}$$

This inequality and the fact $\beta_3 \in K_0 \subseteq K$ implies that $\beta_3 \in B(\beta_0, r) \cap K$. Proceeding in the same way, we get $\{\beta_n\} \subseteq K_0$ with $\beta_n \in B(\beta_0, r) \cap K$ and $\{\zeta_n\} \subseteq L_0$ with $\zeta_n \in \Upsilon\beta_n$ such that

$$\gamma(\beta_{n-1}, \beta_n) \geq 1 \text{ and } d_s(\beta_n, \zeta_{n-1}) = dist(K, L) \ \forall\, n \in \mathbb{N}. \tag{16}$$

Moreover,

$$d_s(\beta_n, \beta_{n+1}) \leq d_s(\zeta_{n-1}, \zeta_n) \leq s^n\chi^n(d_s(\beta_0, \beta_1)) \ \forall\, n \in \mathbb{N}.$$

For $n, m \in \mathbb{N}$, $n > m$, we get

$$\begin{aligned} d_s(\beta_n, \beta_m) &\leq \sum_{j=n}^{m-1} s^j d_s(\beta_j, \beta_{j+1}) \\ &\leq \sum_{j=n}^{m-1} s^{2j}\chi^j(d_s(\beta_0, \beta_1)) \\ &< \sum_{j=n}^{\infty} s^{2j}\chi^j(d_s(\beta_0, \beta_1)) < \infty. \end{aligned}$$

This proves that $\{\beta_n\}$ is Cauchy in $B(\beta_0, r) \cap K \subseteq K$. Similarly one can also prove that $\{\zeta_n\}$ is Cauchy in L. Since K and L are closed in the complete space X and $B(\beta_0, r) \cap K$ is closed in K. Thus, we get $\beta^* \in B(\beta_0, r) \cap K$ and $\zeta^* \in L$ such that $\beta_n \to \beta^*$ and $\zeta_n \to \zeta^*$. By the continuity of d and (16), we get $d_s(\beta^*, \zeta^*) = dist(K, L)$ as $n \to \infty$. Clearly, $\zeta^* \in \Upsilon\beta^*$, since, Υ is continuous. Thus, $dist(K, L) \leq d_s(\beta^*, \Upsilon\beta^*) \leq d_s(\beta^*, \zeta^*) = dist(K, L)$. Hence, β^* is a best proximity point of Υ. □

Theorem 11: Let (X, d_s, s) with $s > 1$, let K_0 be nonempty and $\Upsilon : K \to CL(L)$ be a mapping which satisfies the hypotheses: (T-i)-(T-iii) and (T-v). Then Υ has a best proximity point in $B(\beta_0, r) \cap K_0$.

Proof: Following the proof of the last theorem, we have $\{\beta_n\}$ as Cauchy in $B(\beta_0, r) \cap K \subseteq K$ and $\{\zeta_n\}$ as Cauchy in L satisfying

$$\gamma(\beta_{n-1}, \beta_n) \geq 1 \text{ and } d_s(\beta_n, \zeta_{n-1}) = dist(K, L) \quad \forall n \in \mathbb{N}, \tag{17}$$

and

$$d_s(\beta_n, \beta_{n+1}) \leq d_s(\zeta_{n-1}, \zeta_n) \leq s^n\chi^n(d_s(\beta_0, \beta_1)) \quad \forall n \in \mathbb{N}.$$

Further, $\beta_n \to \beta^*$ and $\zeta_n \to \zeta^*$. By using (17) and the continuity of d, we get $d_s(\beta^*, \zeta^*) = dist(K, L)$ as $n \to \infty$.

Since $\beta_n, \beta^* \in B(\beta_0, r) \cap K$ and $\gamma(\beta_n, \beta^*) \geq 1$. Thus from (2), we get

$$H_s(\Upsilon\beta_n, \Upsilon\beta^*) \leq \chi(d_s(\beta_n, \beta^*)) \text{ for each } n \in \mathbb{N}.$$

When n tends to infinity in the last inequality, we get $\Upsilon\beta_n \to \Upsilon\beta^*$. As $\zeta_n \in \Upsilon\beta_n$, $\zeta_n \to \zeta^*$ and $\Upsilon\beta_n \to \Upsilon\beta^*$. Then, $\zeta^* \in \Upsilon\beta^*$. Thus, $dist(K, L) \leq d_s(\beta^*, \Upsilon\beta^*) \leq d_s(\beta^*, \zeta^*) = dist(K, L)$. Hence, β^* is a best proximity point of Υ. □

For $s = 1$ we have the following result, which can be proved on the same lines as the proof of Theorem 10 and 11 done.

Theorem 12: Let K and L be nonempty closed subsets of a complete metric space (X, d_s) and K_0 be nonempty. Let $\gamma : K \times K \to [0, \infty)$ and $\Upsilon : K \to CL(L)$ be mappings such that for each $\beta, \zeta \in B(\beta_0, r) \cap K$, for some $\beta_0 \in K_0$ and $r > 0$, with $\gamma(\beta, \zeta) \geq 1$ we have

$$H_s(\Upsilon\beta, \Upsilon\zeta) \leq \chi(d_s(\beta, \zeta))$$

with a strict inequality, if $\beta \neq \zeta$. Where, $\chi \in \Omega_1$ and $\sum_{n=0}^{\infty} \chi^n(d_s(\beta_0, \Upsilon\beta_0) + dist(K, L)) < r$. Further, for $\beta_0 \in K_0$, there exist $\zeta_0 \in \Upsilon\beta_0$ and $\beta_1 \in K_0$ such that $d_s(\beta_1, \zeta_0) = dist(K, L)$ and $\gamma(\beta_0, \beta_1) \geq 1$. Moreover, the hypotheses: (T-i), (T-ii), (T-iv) or (T-v) are also hold. Then Υ has a best proximity point in $B(\beta_0, r) \cap K_0$.

Example 13: Let $X = \mathbb{R}^2$ and $d_s((\beta_1, \zeta_1), (\beta_2, \zeta_2)) = |\beta_1 - \beta_2| + |\zeta_1 - \zeta_2|$ be a metric on X. Take $K = \{(1, \beta) : \beta \in \mathbb{R}\}$ and $L = \{(0, \beta) : \beta \in \mathbb{R}\}$. Define $\Upsilon : K \to CL(L)$ by

$$\Upsilon(1, \beta) = \begin{cases} \{(0, \beta)\}, & \beta \leq 0 \\ \{(0, 0), (0, \beta/8)\}, & 0 < \beta \leq 4 \\ \{(0, b) : b \geq \beta\}, & \beta > 4 \end{cases}$$

and $\gamma : K \times K \to [0, \infty)$ by

$$\gamma((1, \beta), (1, \zeta)) = \begin{cases} 1, & \beta, \zeta \in [0, 4] \\ 0, & \text{otherwise.} \end{cases}$$

One can see that Υ is γ-controlled proximal contraction on closed ball $B(\beta_0 = (1, 0.4), r = 4)$ with $\chi(t) = \frac{1}{4}t$, and $\sum_{n=0}^{\infty} \chi^n(d_s(\beta_0, \Upsilon\beta_0) + dist(K, L)) < 4$. Also, note that $K_0 = K$, $L_0 = L$; for each $\beta \in K_0$ we have $\Upsilon\beta \subseteq L_0$ and the pair (K, L) satisfies the weak P_γ-property. For $\beta_0 = (1, 0.4) \in K_0$, we have $\zeta_1 = (0, \frac{0.4}{8}) \in \Upsilon\beta_0$ in L_0 and $\beta_1 = (1, \frac{0.4}{8}) \in K_0$ such that $d_s(\beta_1, \zeta_1) = dist(K, L)$ and $\gamma(\beta_0, \beta_1) = 1$. If $\beta_0, \beta_1 \in \{(1, \beta) : 0 \leq \beta \leq 4\}$, then $\Upsilon\beta_0, \Upsilon\beta_1 \subseteq \{(0, \frac{\beta}{8}) : 0 \leq \beta \leq 4\}$. Take $\zeta_1 \in \Upsilon\beta_0$, $\zeta_2 \in \Upsilon\beta_1$ and $u_1, u_2 \in K$ such that $d_s(u_1, \zeta_1) = dist(K, L)$ and $d_s(u_2, \zeta_2) = dist(K, L)$. Then we have $u_1, u_2 \in \{(1, \beta) : 0 \leq \beta \leq \frac{1}{2}\}$. Hence Υ is an γ-proximal admissible. Also, for each sequence $\{\beta_n\}$ in K with $\gamma(\beta_n, \beta_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\beta_n \to \beta \in K$, by definition of γ, we have $\gamma(\beta_n, \beta) \geq 1$ for all $n \in \mathbb{N}$. Therefore, all the conditions of Theorem 12 hold and Υ has a best proximity point.

Note that, one can check in this example for assumed ball and γ function the [7, Theorem 3] and [10, Theorem 15] are not applicable.

Following result is obtained by take $\gamma(\beta, \zeta) = 1$ for each $\beta, \zeta \in K$ in Theorem 10 and 11.

Theorem 14: Let (X, d_s, s) with $s > 1$, let K_0 be nonempty and $\Upsilon : K \to CL(L)$ be a mapping satisfying (T-i) and the following inequality

$$H_s(\Upsilon\beta, \Upsilon\zeta) \leq \chi(d_s(\beta, \zeta))$$

for all $\beta, \zeta \in B(\beta_0, r) \cap K$, for some $\beta_0 \in K_0$, with $\sum_{n=0}^{\infty} s^{2n+2}\chi^n(d_s(\beta_0, \Upsilon\beta_0) + dist(K, L)) < r$ and $\chi \in \Omega_s$. Then Υ has a best proximity point in $B(\beta_0, r) \cap K_0$.

In case $\Upsilon : K \to L$, we get the following result, obtained by our main results:

Corollary 15: Let (X, d_s, s) with $s > 1$ and let K_0 be nonempty. Let $\gamma : K \times K \to [0, \infty)$ and $\Upsilon : K \to L$ be mappings such that

$$d_s(\Upsilon\beta, \Upsilon\zeta) \leq \chi(d_s(\beta, \zeta)),$$

for each $\beta, \zeta \in B(\beta_0, r) \cap K$, for some $\beta_0 \in K_0$, with $\gamma(\beta, \zeta) \geq 1$, where, $\chi \in \Omega_s$ and $\sum_{n=0}^{\infty} s^{2n+2}\chi^n(d_s(\beta_0, \Upsilon\beta_0) + dist(K, L)) < r$. Also assume, for $\beta_0 \in K_0$, there are $\Upsilon\beta_0 \in L_0$ and $\beta_1 \in K_0$ satisfying $d_s(\beta_1, \Upsilon\beta_0) = dist(K, L)$ and $\gamma(\beta_0, \beta_1) \geq 1$. Moreover, the hypotheses: (T-i), (T-ii), (T-iv) or (T-v) are also hold. Then Υ has a best proximity point in $B(\beta_0, r) \cap K_0$.

By taking $K = L = X$, we have the following fixed point theorem. Note that this is a new result in b-metric spaces, as far as we know.

Corollary 16: Let (K, d_s, s) be a complete and continuous b-metric space with $s > 1$. Let $\gamma : K \times K \to [0, \infty)$ and $\Upsilon : K \to CL(K)$ be mappings satisfying the following hypotheses:

(i) Υ is γ-admissible, that is, for each $\beta, \zeta \in K$ with $\gamma(\beta, \zeta) \geq 1$, we have $\inf_{a \in \Upsilon\beta, b \in \Upsilon\zeta} \gamma(a, b) \geq 1$;
(ii) Υ is γ-controlled contraction on the closed ball $B(\beta_0, r)$, for some $\beta_0 \in K_0$ and $r > 0$, that is, for each $\beta, \zeta \in B(\beta_0, r)$ with $\gamma(\beta, \zeta) \geq 1$, we get

$$H_s(\Upsilon\beta, \Upsilon\zeta) \leq \chi(d_s(\beta, \zeta))$$

where, $\chi \in \Omega_s$ and $\sum_{n=0}^{\infty} s^{2n+2}\chi^n(d_s(\beta_0, \Upsilon\beta_0)) < r$. Further, for $\beta_0 \in K$, there exists $\beta_1 \in \Upsilon\beta_0$ such that $\gamma(\beta_0, \beta_1) \geq 1$;
(iii) Υ is continuous, or, for each $\{\beta_n\}$ in K with $\gamma(\beta_n, \beta_{n+1}) \geq 1\ \forall n \in \mathbb{N}$ and $\beta_n \to \beta \in K$, we have $\gamma(\beta_n, \beta) \geq 1\ \forall n \in \mathbb{N}$.

Then Υ has a fixed point in $B(\beta_0, r)$.

III. CONCLUSION

This article provides a tool to study the existence of best proximity point of the nonself mappings satisfying certain conditions, like γ-controlled proximal contraction and weak P_γ-property. Further the notion of weak P_γ-property generalizes the notion of weak P-property and removes all those limitations which may occur due to the use of weak P-property.

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