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A New Kriging-Based Learning Function for Reliability Analysis and Its Application to Fatigue Crack Reliability

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ABSTRACT In order to solve the problems in fatigue reliability analysis of mechanical components, a new learning function based on the Kriging model is proposed. In the process of fatigue reliability analysis, surrogate model is often used to fit the implicit performance function to avoid large numbers of calculations of fatigue crack samples. The existing learning functions ignore the information of probability density function (PDF). To overcome this defect and avoid unnecessary sampling in low PDF regions, a novel learning function takes into account the PDF and the local accuracy of Kriging model. The accuracy of the Kriging model is improved by adding samples step by step, and the new training samples are determined by the proposed learning function. The proposed method is verified by two examples from literatures. The results show that, compared with other surrogate models and learning functions, the proposed method has advantages in efficiency, convergence and accuracy. Finally, the proposed method is employed to calculate the fracture failure probability of cracked structures.

INDEX TERMS Structural reliability analysis, kriging model, probability density function, learning function, Monte Carlo, fatigue crack reliability.

I. INTRODUCTION

Recent years, the first-order reliability method (FORM), the second-order reliability method (SORM), Monte Carlo simulation (MCS) and surrogate models have been widely applied to structural reliability analysis [1]. However, based on Taylor expansion, FORM and SORM have been incapable of meeting the accuracy and application scope required by both theoretical analysis and engineerings. Estimating the structural failure probability by the failure rate of random samples, MC is of the greatest robustness [2]. When the order of magnitude of failure probability is small such as 10^{-5}, it is hard to perform MCS because too many calls to the time-consuming performance function are needed [3].

Surrogate models [4], including polynomial response surface [5], support vector machine (SVM) [6], artificial neural network model (ANN) [7] and the Kriging model [8], have been the most widely applied to structural reliability analysis. The basic idea of surrogate models is to selecte a small number of sample points in the input space, evaluate their performance function values, and approximate the target performance function by establishing an explicit mathematical model. Kriging is a high-efficiency interpolation model. It has been widely applied to reliability analysis, global optimization, sensitivity analysis and some other fields. Jones *et al.* [9] use the Kriging model to solve the problems in global optimization and propose the expected improvement function (EIF). Chen *et al.* [10] apply it to the fatigue reliability analysis of a Turbine Disc and optimize the parameters of correlation function by Particle Swarm Optimization (PSO). Pan *et al.* [11] introduce the Kriging model to sensitivity analysis, and the computational cost of sensitivity index is lessened significantly. Comparing with other surrogate models, the Kriging model has two advantages: (1) it is an interpolation model. (2) as a local accuracy measurement, the Kriging variance is available to construct efficient design of experiment (DoE) strategy.

According to the difference between structural reliability analysis and global optimization, Bichon *et al.* [12] propose the expected feasibility function (EFF), which makes

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the sample points distributed in the vicinity of the limit state surface. Echard *et al.* [13], [14] propose the structural reliability analysis methods AK-MCS [13] and AK-IS [14] by combining Kriging with MCS, Important Sampling (IS), and derive a new learning function U. However, the convergence criterion in AK-MCS and AK-IS is hard to satisfy. Tong *et al.* [15] combine the Kriging model with both subset simulation (SS) and IS, and propose a more reasonable convergence criterion for AK-SSIS. Yang *et al.* [16] and Lv *et al.* [17] develop learning functions of ERF and H, respectively. In addition, the PDF has also been considered in some literatures. Jiang *et al.* [18] propose a real-time estimation error-guided sampling method, and use the mean and variance of Kriging prediction to calculate the wrong-classification probability. Wang *et al.* [19] propose the error-based stopping criterion (ESC) to address existing stopping criteria, which takes PDF into account. Jiang *et al.* [20] propose a failure-pursuing sampling framework that could employ various surrogate models or active learning strategies. In their proposed method, a part of parameters may vary with problems.

In engineering, fatigue fracture failure is one of the most common failure forms of mechanical parts under cyclic loading. Fatigue crack growth is a slow accumulation process [21]. The randomness of crack growth makes it difficult for prediction models to get accurate prediction results [22]. Therefore, the randomness of crack growth has been drawn more and more attention. Scholars often introduce restrictive assumptions to simplify the complexity of prediction models. Macias *et al.* [23] apply the concept of probabilistic fracture mechanics, combining finite element analysis (FEA) with FORM, The reliability life of plate with central crack is studied by quadratic response surface method. Cai *et al.* [24] propose an extended model based on the S-N method to consider the frequency effect. They use heteroscedasticity method to predict fatigue life and study effects of increasing loading frequency and stress amplitude on fatigue life. Mohamed *et al.* [25] propose a reliability method for crack propagation under thermal loading, which can be coupled with any finite element software for reliability evaluation. However, for the actual structure with complex geometry, irregular load, imperfect boundary conditions, and random defect shape and direction, many influencing factors are ignored, so the prediction life model presented is not accurate enough [26], [27]. Above all, the existing methods are hardly to predict fatigue life accurately and efficiently.

To overcome the shortcomings described above, this paper proposes a new learning function, which considers both PDF and the misclassification probability of point. Misclassification probability means the probability that a Kriging model wrongly predicts the sign of the performance function at a point. It guarantees that the point selected by the learning function lies in the area of interest, and avoids the waste of sample points caused by sampling in unimportant areas. Combining with Kriging and MCS, the learning function proposed can adaptively and efficiently enhance the Kriging model for fatigue crack reliability analysis.

The rest of this paper is organized as follows. Section II introduces the Kriging and Monte Carlo simulation. Section III describes the proposed learning function and the failure probability algorithm combining Kriging and MCS. In Section IV, the proposed method is verified by two examples from literatures. And then the proposed method is employed to calculate the fracture failure probability of cracked structures. Section V is the conclusion.

II. KRIGING AND MONTE CARLO SIMULATION

For the input vector $\boldsymbol{X}$ of a structure, its dimension is set to be M and joint probability density function is $f(\boldsymbol{x})$. The performance function is set to be $G(\boldsymbol{x})$, by which the input space is divided into two parts: safety domain $S_s = \{\boldsymbol{x}|G(\boldsymbol{x}) > 0, x \in R^M\}$ and failure domain $S_f = \{\boldsymbol{x}|G(\boldsymbol{x}) \leq 0, \boldsymbol{x} \in R^M\}$. Then, the failure probability can be expressed as

$$P_f = \int \cdots \int_{G(\boldsymbol{x})\leq 0} f(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \tag{1}$$

If the sample set is given as

$$\Omega = \{(\boldsymbol{x}_i, y_i), i = 1, 2, \ldots, N\}$$

In the theory of Kriging, $G(\boldsymbol{x})$ is set to be expressed as

$$\begin{aligned} G(\boldsymbol{x}) &= \sum_{h=1}^{p} \beta_h g_h(\boldsymbol{x}) + z(\boldsymbol{x}) \\ &= \boldsymbol{g}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{\beta} + z(\boldsymbol{x}) \end{aligned} \tag{2}$$

where $g_h(\boldsymbol{x})(h = 1, 2, \ldots, p < N)$ is polynomial. Nguyen *et al.* [8] has studied the influence that $g_h(\boldsymbol{x})$ imposes on the accuracy of Kriging. In this paper, the degree of $g_h(\boldsymbol{x})$ is 1, and $\beta_h(h = 1, 2, \ldots, p)$ is the coefficient of $g_h(\boldsymbol{x})$. $z(\boldsymbol{x})$ is zero-mean Gaussian process. The covariance of $z(\boldsymbol{x}_i)$ and $z(\boldsymbol{x}_j)$ is

$$\mathrm{Cov}[z(\boldsymbol{x}_i), z(\boldsymbol{x}_i)] = \sigma^2 R(\boldsymbol{x}_i, \boldsymbol{x}_j; \boldsymbol{\theta}) \tag{3}$$

where σ^2 is the variance of $z(\boldsymbol{x})$. $R(\boldsymbol{x}_i, \boldsymbol{x}_j; \boldsymbol{\theta})$ is used to describe the correlation coefficient for $z(\boldsymbol{x}_i)$ and $z(\boldsymbol{x}_j)$. And $\boldsymbol{\theta}$ is a parameter for the correlation function $R(\boldsymbol{x}_i, \boldsymbol{x}_j; \boldsymbol{\theta})$. Gaussian correlation function is one of the most widely used correlation functions, which is expressed as

$$R(\boldsymbol{x}_i, \boldsymbol{x}_j; \boldsymbol{\theta}) = \prod_{m=1}^{M} \exp\left[-\theta_m (x_i^m - x_j^m)^2\right] \tag{4}$$

where x_i^m refers to the mth element of vector $\boldsymbol{x}_i$.

At point $\boldsymbol{x}$, the Kriging linear combination predictor of the structural performance function can be described as

$$\hat{G}(\boldsymbol{x}) = \boldsymbol{c}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{Y} \tag{5}$$

where $\boldsymbol{c}(\boldsymbol{x}) = [c_1(\boldsymbol{x}), \ldots, c_N(\boldsymbol{x})]^{\mathrm{T}}$ is the $\boldsymbol{x}$-related vector, and $\boldsymbol{Y} = [y_1, \ldots, y_N]^{\mathrm{T}}$. The unbiased minimum variance estimator of $G(\boldsymbol{x})$ is

$$\mu_G(\boldsymbol{x}) = \hat{G}(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x})\hat{\boldsymbol{\beta}} + \boldsymbol{r}(\boldsymbol{x})^{\mathrm{T}}\boldsymbol{\gamma} \tag{6}$$

$$\sigma_G^2(\boldsymbol{x}) = \hat{\sigma}^2 \begin{pmatrix} 1 + \boldsymbol{u}^{\mathrm{T}}(\boldsymbol{x})(\boldsymbol{G}^{\mathrm{T}}\boldsymbol{R}^{-1}\boldsymbol{F})^{-1}\boldsymbol{u}(\boldsymbol{x}) \\ -\boldsymbol{r}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{R}^{-1}\boldsymbol{r}(\boldsymbol{x}) \end{pmatrix} \tag{7}$$

$$\hat{\sigma}^2 = \frac{1}{M}(\boldsymbol{Y} - \boldsymbol{G}^{\mathrm{T}}\boldsymbol{\beta})^{\mathrm{T}}\boldsymbol{R}^{-1}(\boldsymbol{Y} - \boldsymbol{G}^{\mathrm{T}}\boldsymbol{\beta})$$
$$\hat{\boldsymbol{\beta}} = (\boldsymbol{G}^{\mathrm{T}}\boldsymbol{R}^{-1}\boldsymbol{G})^{-1}\boldsymbol{G}^{\mathrm{T}}\boldsymbol{R}^{-1}\boldsymbol{Y}$$
$$\boldsymbol{G} = [\boldsymbol{g}(\boldsymbol{x}_1), \boldsymbol{g}(\boldsymbol{x}_2), \ldots, \boldsymbol{g}(\boldsymbol{x}_M)]^{\mathrm{T}}$$
$$\boldsymbol{R} = \left(R(\boldsymbol{x}_i, \boldsymbol{x}_j; \boldsymbol{\theta})\right)_{N\times N}$$
$$\boldsymbol{\gamma} = \boldsymbol{R}^{-1}(\boldsymbol{Y} - \boldsymbol{G}\hat{\boldsymbol{\beta}})$$
$$\boldsymbol{r}(\boldsymbol{x}) = [R(\boldsymbol{x}_1, \boldsymbol{x}; \boldsymbol{\theta}), \ldots, R(\boldsymbol{x}_N, \boldsymbol{x}; \boldsymbol{\theta})]$$
$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{G}^{\mathrm{T}}\boldsymbol{R}^{-1}\boldsymbol{r}(\boldsymbol{x}) - \boldsymbol{g}(\boldsymbol{x})$$

Eqs. (6) and (7) are elaborately derived in Ref. [12].

Combining Eqs. (1) and (6), the failure probability of this structure is estimated to be

$$\hat{P}_f = \int_{\hat{G}(\boldsymbol{x})\le 0} f(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \tag{8}$$

When $\boldsymbol{X}$ has a greater number of dimensions, it is rather hard or even impossible to solve equations by numerical integration. MCS is applied to obtain the failure probability by estimating the failure rate of the randomly selected samples. It is achievable to get a sufficiently precise $\hat{P}_f$ when the random sample N_{MC} is of a large enough size. With an acceptable calculation amount, MCS is the most robust method. MCS is employed in this paper to approximate $\hat{P}_f$.

$$\hat{P}_f \approx \frac{1}{N_{\mathrm{MC}}}\sum_{i=1}^{N_{\mathrm{MC}}} I(\hat{G}(\boldsymbol{x}_{\mathrm{MC},i}) \le 0) \tag{9}$$

where N_{MC} refers to the number of random sampling and $\boldsymbol{x}_{\mathrm{MC},i}(i = 1, 2, \ldots, N_{\mathrm{MC}})$ is the independently and identically distributed random sequence from $f(\boldsymbol{x})$. $I(\cdot)$ is a failure indicator function.

$$I(\cdot) = \begin{cases} 0 & > 0 \\ 1 & \le 0 \end{cases}$$

The coefficient of variation of $\hat{P}_f$ is

$$\delta_{\mathrm{MC}} = \frac{\sqrt{var(\hat{P}_f)}}{\hat{P}_f} = \sqrt{\frac{1-\hat{P}_f}{N_{\mathrm{MC}}\hat{P}_f}} \tag{10}$$

Let $\delta_{\mathrm{MC}} < [\delta]$, then

$$N_{\mathrm{MC}}\hat{P}_f > (1-\hat{P}_f)/[\delta]^2 \approx 1/[\delta]^2 \tag{11}$$

III. THE NEW LEARNING FUNCTION

A. THE LEARNING FUNCTION TAKING PDF INTO CONSIDERATION

According to Eqs. (8) and (9), the accuracy of $\hat{P}_f$ is directly influenced by the sign of $\hat{G}(\boldsymbol{x}_{\mathrm{MC},i})$. When the signs $\hat{G}(\boldsymbol{x}_{\mathrm{MC},i})$ and $G(\boldsymbol{x}_{\mathrm{MC},i})$ are opposite, the accuracy of $\hat{P}_f$ is disturbed. According to the Kriging theory, $G(\boldsymbol{x})$, the true value of the structural performance function at point $\boldsymbol{x}$, is normally distributed.

$$G(\boldsymbol{x}) \sim N\left(\mu_G(\boldsymbol{x}), \sigma_G^2(\boldsymbol{x})\right)$$

Therefore, the probability that $\hat{G}(\boldsymbol{x}_{\mathrm{MC},i})$has a wrong sign is

$$P_{\mathrm{wrong}}(\boldsymbol{x}) = \Phi\left(-\left|u_G(\boldsymbol{x})/\sigma_G(\boldsymbol{x})\right|\right) \tag{12}$$

In structural reliability analysis, Eq. (12) can be used to measure the local accuracy of Kriging model. A large value of P_{wrong} means the estimated performance function at point $\boldsymbol{x}$ has great indeterminacy about its sign. It is necessary to perform the true structural performance function to enhance the local accuracy of the Kriging model. In addition, according to Eq. (8), a larger value of PDF at point $\boldsymbol{x}$ means that $\boldsymbol{x}$ holds a larger weight in P_f and has more random samples in its neighbourhood when $\hat{P}_f$ is calculated with MCS. The only consideration of the local accuracy of a surrogate model will result in some problems: (1) the surrogate model has an insufficient accuracy at point $\boldsymbol{x}$. But $f(\boldsymbol{x})$ is so small that the contribution that $\boldsymbol{x}$ makes to P_f can be ignored. (2) in the input space there may be a point where Kriging local accuracy is higher but the large PDF make the point influence the accuracy of $\hat{P}_f$ more significantly than the "optimum" point of Eq.(12). Therefore, it is necessary to consider both the local accuracy of Kriging model and PDF in P_f while selecting points to refresh Kriging model. This paper proposes the learning function (L_f) which taking into account both local model accuracy and PDF.

$$\begin{aligned} L_f(\boldsymbol{x}) &= P_{\mathrm{wrong}}(\boldsymbol{x}) \cdot f(\boldsymbol{x}) \\ &= \Phi\left(-\left|u_G(\boldsymbol{x})/\sigma_G(\boldsymbol{x})\right|\right) \cdot f(\boldsymbol{x}) \end{aligned} \tag{13}$$

B. THE FAILURE PROBABILITY ALGORITHM COMBINING KRIGING AND MCS

Combining Kriging with MCS, a new method of structural reliability analysis is developed in this paper, which is similar to AK-MCS [13] and AK-SSIS [28]. This method employs the learning function described by Eq. (13) as the criterion to select new sample points and enhance the Kriging model. And the convergence criterion presented in Ref. [15] is adopted here. The procedure of the new method is summarized as follows:

Step 1: Generate N_{MC} independently and identically distributed M-dimension random vectors whose probability density function is $f(\boldsymbol{x})$. $\boldsymbol{S} = [\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{N_{\mathrm{MC}}}]^{\mathrm{T}}$.

Step 2: Use LHS (Latin Hypercube Sampling) [26] to generate N_0 random sample points. Ref. [13] shows that the number of initial sample point N_0 should be selected as few as possible. According to previous experience, N_0 should be larger than the dimension of the input vectors. Compute values of performance function at N_0 initial points, and construct the sample set Ω of the Kriging model.

Step 3: Build the Kriging model $\hat{G}(\boldsymbol{x})$ according to the existing sample set $\Omega$and Eqs. (2)-(7).

Step 4: Use the Kriging model established in Step 3 to estimate the performance function and Kriging variances of the points in $\boldsymbol{S}$. And estimate the failure probability according to Eq.(9). The number of failure samples in $\boldsymbol{S}$ is

$$N_{\mathrm{fail}} = N_{\mathrm{MC}}\hat{P}_f$$

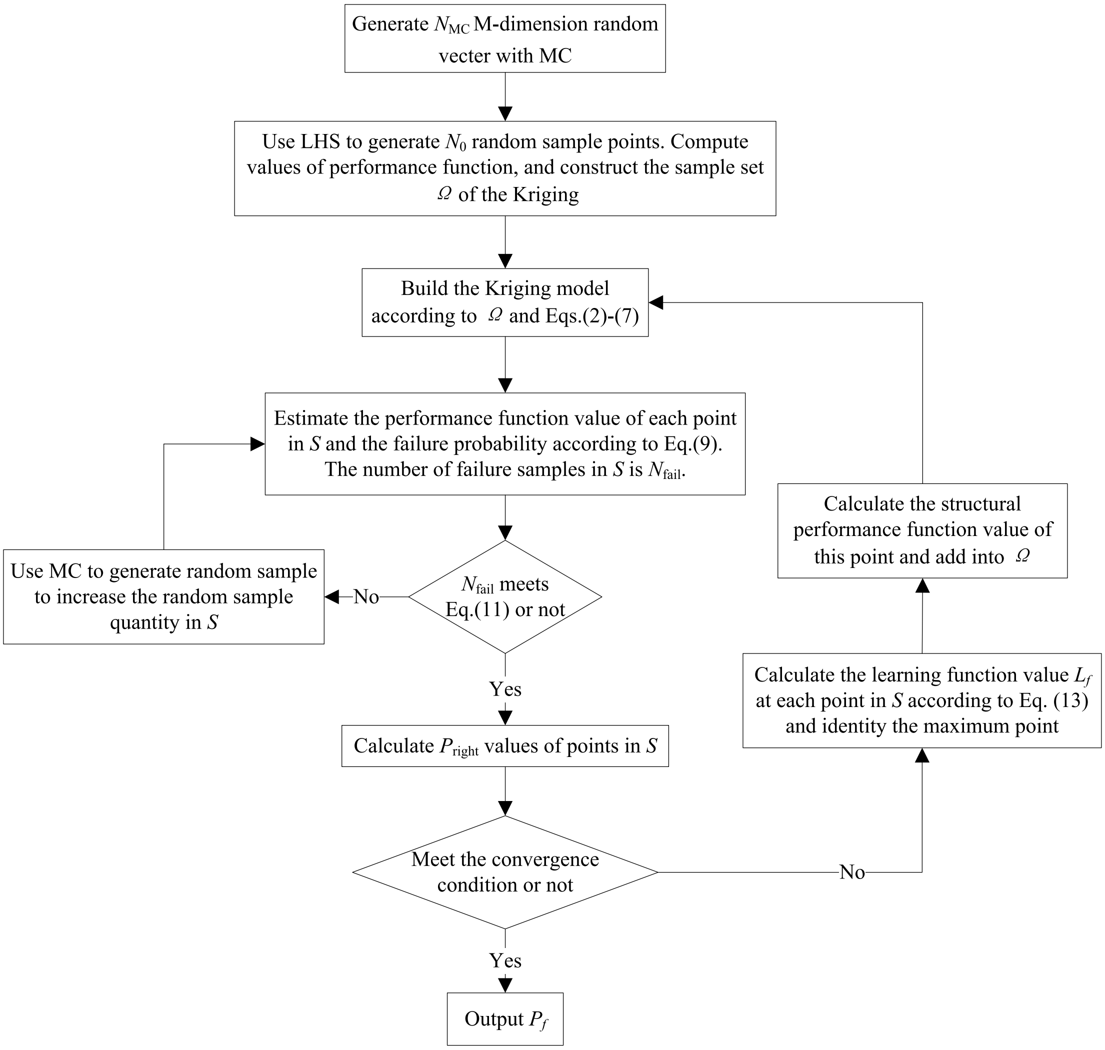

FIGURE 1. Procedure of the proposed algorithm for estimating failure probability.

If N_{fail} can make Eq. (11) true, turn to Step 5. Otherwise, generate more random samples for $\boldsymbol{S}$ until N_{fail} satisfies Eq. (11), and then move to Step 5.

Step 5: Use the convergence criterion proposed in Ref. [15] to judge whether to converge or not. The details are described as follows.

Let $P_{right}(\boldsymbol{x}) = \Phi\left(|u_G(\boldsymbol{x})/\sigma_G(\boldsymbol{x})|\right)$

Calculate $P_{right}(\boldsymbol{x})$ values of points in $\boldsymbol{S}$ and rearrange them in order from largest to smallest, so as to get $S' = [x'_1, \ldots, x'_{N_{MC}}]^T$. The P_{right} value at x'_i is represented by $P_{right,i}$. If there is any i ($i = 1, \ldots, N_{MC}$) that makes Eq. (14) true, the procedure is ended with $\hat{P}_f$ as the structural failure probability; otherwise, move to Step 6. See Ref. [15] for the detailed derivation process.

$$i \cdot P_{right,i} \geq P_{right,limit} \cdot N_{MC} \tag{14}$$

where $P_{right,limit}$ is a fixed positive number which is less than 1, and $P_{right,limit} = 0.9772$ in this paper.

Step 6: Identify the point in $\boldsymbol{S}$ which maximizes L_f and update Ω. The learning function value L_f at each point in $\boldsymbol{S}$ is calculated according to Eq. (13), so as to identify the point maximizing L_f. Then, the true value of performance function will be calculated at the maximum point and added into Ω. Finally turn to Step 3.

The procedure is also shown as Fig. 1.

IV. NUMERICAL APPLICATIONS

To demonstrate the efficiency of the learning function L_f and structural reliability analysis proposed in this paper, in this section the proposed method will be used to study two examples from literatures and compared with other relative methods. Then, two cracked structures are analyzed.

A. EXAMPLE 1

To elaborate the point selection and convergence processes of the proposed method, a series system with two input variables in Ref. [13] is studied in this section. Its performance

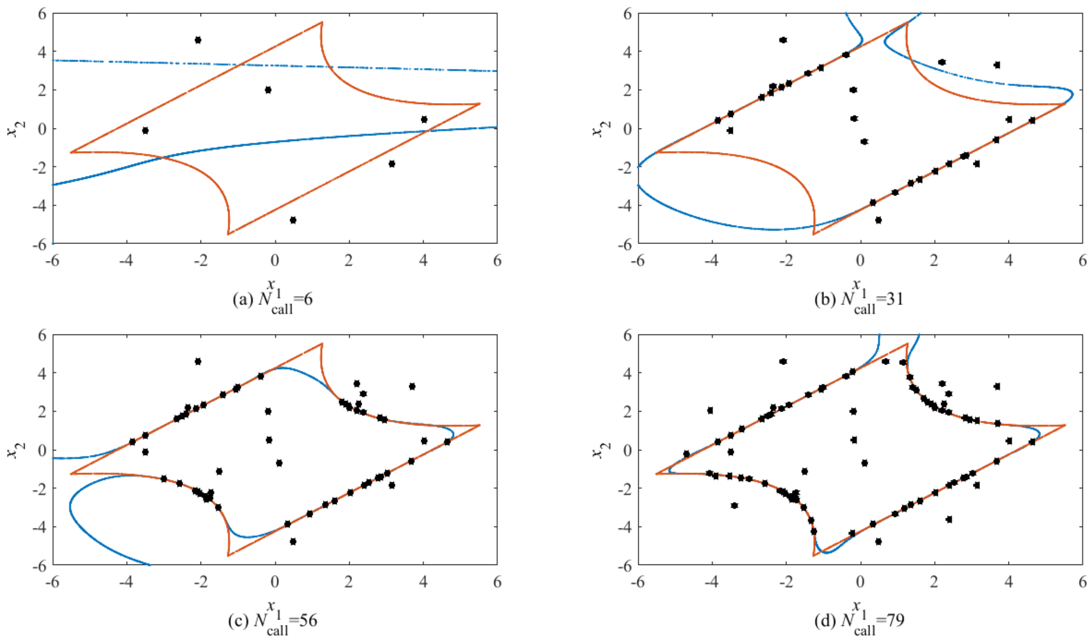

FIGURE 2. The convergence procedure of predicted limit state for example 1.

function is

$$G(\boldsymbol{x}) = \min \left\{ \begin{array}{l} 3 + \frac{(x_1 - x_2)^2}{10} - \frac{x_1 + x_2}{\sqrt{2}}; \\ 3 + \frac{(x_1 - x_2)^2}{10} + \frac{x_1 + x_2}{\sqrt{2}}; \\ (x_1 - x_2) + 3\sqrt{2}; \\ (x_2 - x_1) + 3\sqrt{2}; \end{array} \right\}$$

$\boldsymbol{x} = [x_1, x_2]^{\mathrm{T}}$ is subjected to 2-dimension standardized normal distribution. The correlation coefficient of x_1 and x_2 is 0, which means they are mutually independent. The limit state surface of this structural system ($G(\boldsymbol{x}) = 0$) is shown as Fig. 2(a).

To keep consistent with Ref. [13], let $[\delta] = 0.03$. And according to Eq. (11), the number of failure samples in MCS needs to satisfy

$$N_{\mathrm{fail}} = N_{\mathrm{MC}} \hat{P}_f \geq 1112 \tag{15}$$

According to the procedure shown in Fig. 1, This section sets $N = 10^6$. $N_0 = 6$ initial points are generated in $[-5, 5]^2$ with LHS, their performance function values are calculated to construct the original Kriging sample set Ω. The next is to gradually add new point one by one into Ω and supplement $\boldsymbol{S}$ when necessary to satisfy Eq. (15). It is not converged until the element quantity of Ω reaches 79. Fig. 2(d) shows the sample points of Ω and $G(\boldsymbol{x}) = 0$ when the convergence criterion Eq. (14) is satisfied. As showed by Fig. 2, all the points selected according to the proposed learning function L_f are scattered in the vicinity of $G(\boldsymbol{x}) = 0$. Therefore, the proposed method can help to estimate the failure probability more accurately with less times of evaluations of the performance function.

TABLE 1. Comparison of results for example 1.

Method	N_{call}	$\hat{P}_f$ (10^{-3})	δ(%)	ε(%)
Monte Carlo	10^6	4.416	1.5	-
AK-MCS+U	126	4.42	-	<0.1
AK-MCS+EFF	124	4.42	-	<0.1
The proposed method	6+73	4.51	2.1	2.13

Fig. 2 compares $\hat{G}(\boldsymbol{x}) = 0$ with $G(\boldsymbol{x}) = 0$ during the procedure of reliability analysis. Table 1 shows the comparison between the proposed method and some other methods, where N_{call} refers to the times that the true performance function is called when each corresponding method is converged, and ε refers to the relative error compared with MCS result, and it is calculated as Eq. (16).

$$\varepsilon = \frac{\left| \hat{P}_f - P_{f,\mathrm{MC}} \right|}{P_{f,\mathrm{MC}}} \tag{16}$$

In these sub figures, the lines in blue and red are $\hat{G}(\boldsymbol{x}) = 0$ and $G(\boldsymbol{x}) = 0$, respectively. The numbers of calls to the performance function in the four subfigures are 6, 31, 56 and 79 ($N_0 = 6$). From Fig. 2, it can be seen that when N_{call} is close to 56, the Kriging model has already been able to fit $G(\boldsymbol{x}) = 0$ well. And the accuracy of the corresponding $\hat{P}_f$ has already been acceptable. To meet the employed convergence criterion, more points are needed to further enhance the accuracy of $\hat{G}(\boldsymbol{x}) = 0$ in the "unimportant area" of small $f_X(\boldsymbol{x})$. Selected by the proposed learning function L_f, the sample points are well-distributed in the vicinity of $G(\boldsymbol{x}) = 0$. According to Ref. [13], points selected

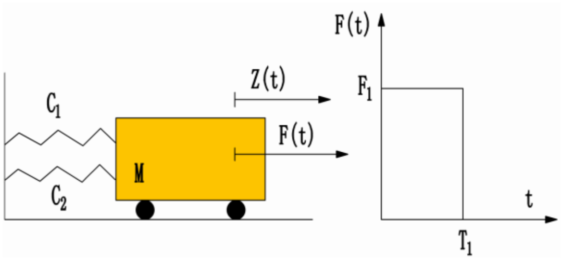

FIGURE 3. The non-linear undamped single degree of freedom system.

TABLE 2. The variable distribution parameters for example 2.

Variable	C_1	C_2	M	R	T_1	F_1
Mean	1	0.1	1	0.5	1	0.45
Standard deviation	0.1	0.01	0.05	0.05	0.2	0.075

according to U and EFF are obviously too concentrated and consequently result in a "waste" of sample points.

B. EXAMPLE 2

A nonlinear undamped single degree of freedom system, as presented in Fig. 3, is studied in this section. This system has been used several literatures Refs. [13]–[15], [28] to demonstrate the accuracy of reliability analysis method. It has 6 input variables, and its performance function is

$$g\left(C_1, C_2, M, R, T_1, F_1\right) = 3R - \left| \frac{2F_1}{M\omega_0^2} \sin\left(\frac{\omega_0 T_1}{2}\right) \right|$$

where $\omega_0 = \sqrt{(C_1 + C_2)/M}$ and multiple random variables $\boldsymbol{x} = [C_1, C_2, M, R, T_1, F_1]^{\mathrm{T}}$. The variables of $\boldsymbol{x}$ are mutually independent and normally distributed.

Their distribution parameters are listed in Table 2.

Let $[\delta] = 0.03$ and $N = 10^5$. According to the method developed in Sec. III.B, the original Kriging model is set to have $N_0 = 10$ sample points and 10^5 random sample points are generated; the next is to call to the performance function at each initial points and iterativley enhance the Kriging model until it is converged. To demonstrate the stability of the proposed method, it is randomly repeated in MATLAB for five times, the results are respectively compared with those acquired from other methods like AK-MCS+U, AK-MCS+EFF, IS+RS and IS+ANN. Relative results and accuracy are list in Table 3.

According to the comparison shown in Table 3, the proposed method is able to estimate the target failure probability of the system with less number of evaluations of the true performance function. The results of random repeat are not greatly different from each other, which indicates the proposed method is stable. Fig. 4 presents how the estimated failure probability $\hat{P}_f$ from each repeat changes along with N_{call}.

TABLE 3. Comparison of results for example 2.

Method	N_{call}	$\hat{P}_f$ (10^{-2})	δ(%)	ε(%)
Monte Carlo	7×10^4	2.844	2.21	-
AK-MCS+U	57	2.845	1.85	<0.1
AK-MCS+EFF	55	2.855	1.80	0.39
IS+RS	109	2.5	-	0.05
IS+ANN	68	3.1	-	0.09
The proposed method	10+33	2.88	1.86	1.01
	10+37	2.87	1.87	1.01
	10+36	2.87	1.87	1.01
	10+34	2.88	1.86	1.01
	10+41	2.87	1.87	1.01

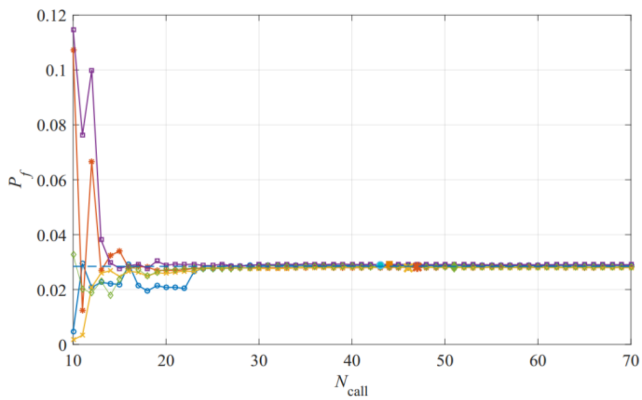

FIGURE 4. Graphs of $\hat{P}_f$ with different N_{call} for example 2.

As shown in Fig. 4, it can be seen that all the estimated failure probability values tend rapidly to the target value. When $N_{\mathrm{call}} \approx 25$, estimated failure probabilities tend to converge.

C. EXAMPLE 3

A three point bending test is analyzed in this section. This example has been used by Refs. [30], [31]. The structure is shown in Fig. 5.

In this application, an initial crack is set in the middle of the bottom side, and stress P is exerted to the middle of the top side. Elastic material of the beam is set with Young's modulus $E = 210$ GPa and Poisson's ratio $\nu = 0.2$.

Its performance function is G, as shown at the bottom of this page, where the beam high W_{v} and the initial crack length a are the random variables, and the beam length S_{v}, the structural thickness B_{v} and the material toughness K_{c} are the deterministic parameters. All the parameters are summarized in Table 4.

$$G = \frac{K_C 2\left(1 + 2\left(a/W_{\mathrm{v}}\right)\right)\left(1 - \left(a/W_{\mathrm{v}}\right)\right)^{3/2} B_{\mathrm{v}} W_{\mathrm{v}}^{3/2}}{S_{\mathrm{v}} 3\left(a/W_{\mathrm{v}}\right)^{1/2}\left[1.99 - \left(a/W_{\mathrm{v}}\right)\left(1 - \left(a/W_{\mathrm{v}}\right)\right)\left(2.15 - 3.93\left(a/W_{\mathrm{v}}\right) + 2.7\left(a/W_{\mathrm{v}}\right)^2\right)\right]} - P_{\mathrm{Applied}}$$

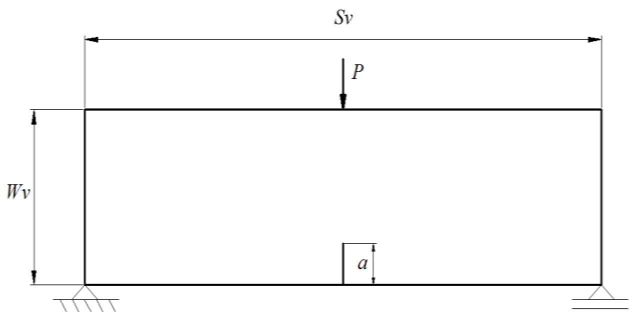

FIGURE 5. The three point bending test.

TABLE 4. The variable distribution parameters and deterministic for example 3.

Vatiable	Distribution	Mean	Standard deviation
K_c (kN/m3/2)	Deterministic	500.0	-
$P_{Applied}$ (kN)	Deterministic	140.0	-
B_v (m)	Deterministic	1.0	-
S_v (m)	Deterministic	5.0	-
W_v (m)	Normal	1.25	0.10
a (m)	Normal	0.1	0.01

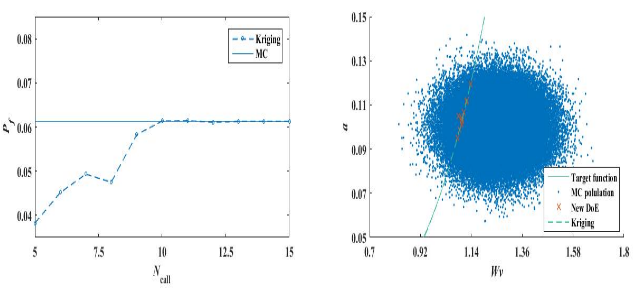

FIGURE 6. Convergence history of the proposed method.

To further illustrate the accuracy and engineering practicability of the proposed method, two of the above-mentioned variables are set to be random variable. According to the proposed method, the convergence and result of this application are shown in Fig. 6.

From the left subfigure, even with a small number of initial sample points, the proposed method can converge after several iterations. From the right subfigure, the added sample points are generally distributed around the target limit state surface, and the model can fit the limit state function quite well in the area of interest. The proposed method can be used to estimate the failure probability of fatigue crack growth with favorable accuracy and engineering practicability.

D. EXAMPLE 4

This section employs the proposed method to calculate the failure probability of a cracked structure [32]. The flat plate with park-through crack and two tensile loads apply to its top and bottom sides is presented in Fig. 7.

In this application, an part-through crack is set in the middle of the flat plate, and load P applied its top and bottom edges. The crack depth a, the crack length $2c$, the structural thickness h, the applied load P and the material toughness K_{IC} are the uncertain variables. The Young's modulus E and Poisson's ratio v are the deterministic parameters. All the parameters are summarized in Table 5.

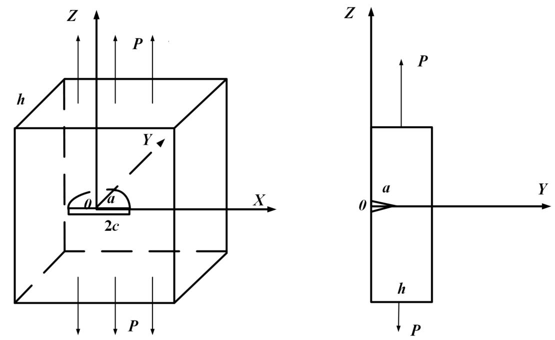

FIGURE 7. Geometry of the structure.

TABLE 5. The variable distribution parameters and deterministic for example 4.

Variable	Distribution	Mean	CoV
E（MPa）	Deterministic	2.1×10^5	-
v	Deterministic	0.3	-
a（mm）	Normal	6	0.1
$2c$（mm）	Normal	24	0.1
h（mm）	Normal	15	0.1
P（MPa）	Normal	150	0.1
K_{IC}（MPa*m-1/2）	Normal	30	0.1

The finite element model (FEM) software is used to model the park-through crack structure, the FEM of the structure is shown in Fig. 8. The initial mesh is composed by of 46406 elements and 197181 nodes, and Fig. 9 shows the contour result of the stress when the random variables are set to be the corresponding mean values.

The 20 samples generated from Latin hypercube sampling and stress intensity factors at crack tip are obtained by FEM software. The Kriging prediction of maximum stress intensity factor based on the initial conditions of these samples can be described as

$$\max\{K_{I1}, K_{I2}, ..., K_{In}\} = Y(P, a, c)$$

According to Ref. [29], the structure is failure if stress intensity factor is greater than plane strain fracture toughness at point of part-through crack tip. So the performance function of this problem can be described as

$$G(P, a, c) = K_{IC} - \max\{K_{I1}, K_{I2}, ..., K_{In}\}$$

where $K_{Ii}(i = 1, 2, \ldots, n)$ is stress intensity factor of each node at the crack tip, and the stress intensity factor of each node is calculated by the FEM software. $G(P, a, c) > 0$ represents that the structure is safe.

According to the procedure presented in Sec. III.B, the failure probability $P_f = 1.46 \times 10^{-3}$ is obtained after 32 iterations. 52(20 + 32) times of the structure model are required to calculate the failure probability using the proposed method. Calculating the structure structural model with

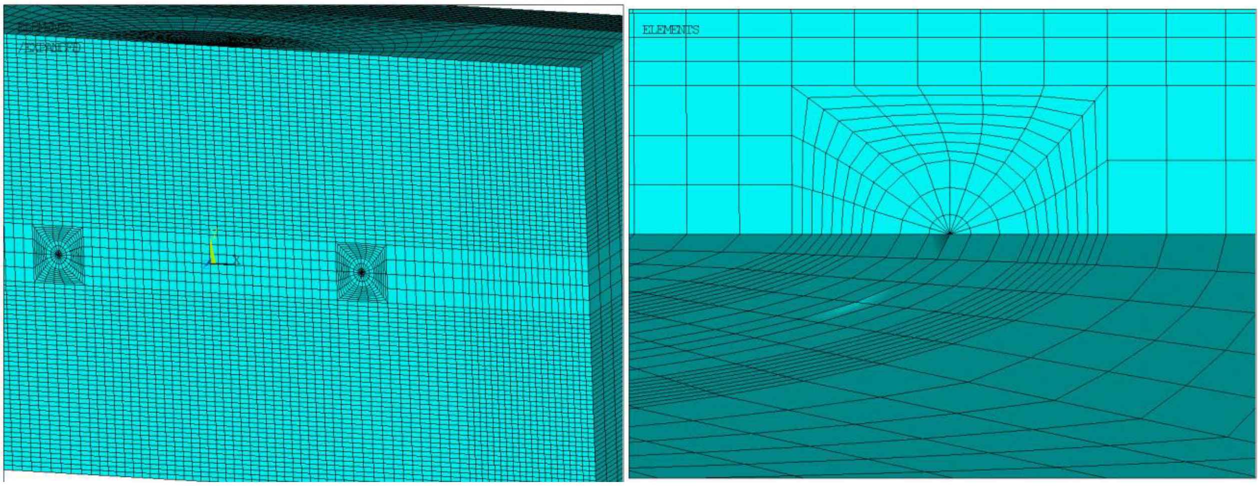

FIGURE 8. The finite elements mesh.

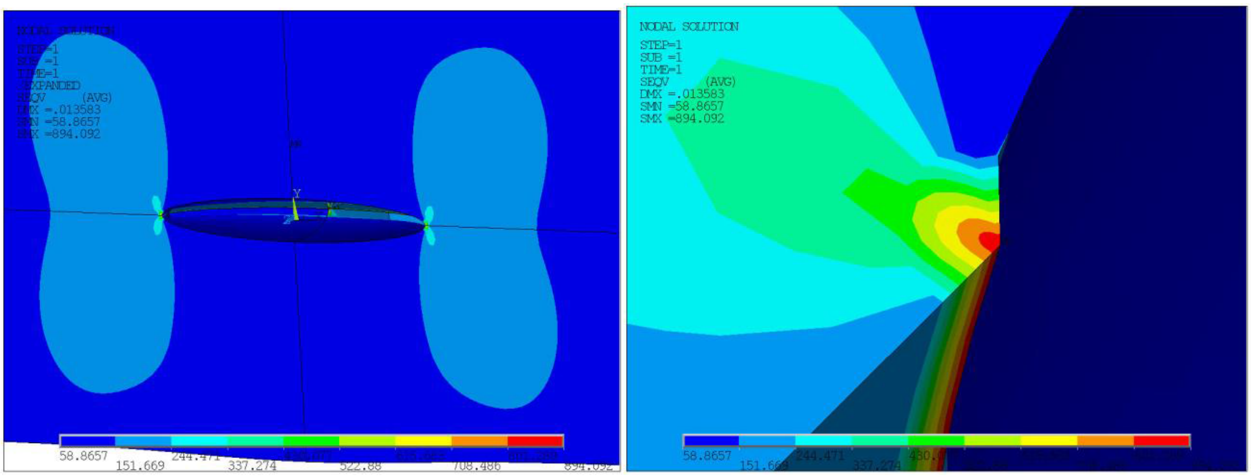

FIGURE 9. Stress contour of the structure and the crack tip.

crude IS combined with FEM software would require at least 11 days (1000 × 932s, note that it takes 932 seconds per a calculation evaluation of the structure model); however, the computational time can be reduced to 13.5 hours (52 × 932 + 52 × 1.58s) by the proposed method. For this structure, both MCS and IS are too time-consuming.

E. ANALYSIS OF RESULTS

According to Example 1 and 2, compared with other methods, the method proposed in this paper is able to both remarkably reduce the evaluation times of structural performance function in reliability analysis and guarantee the accurate failure probability estimation. performance functions in engineering are always implicit and time-consuming. This kind of implicit functions has to be solved with, for example, limit element numerical model. It may take hours or even longer time to run a numerical model only once. However, the proposed method can not only make sure about calculation accuracy, but also maximize efficiency. Compared with the existing learning functions, the learning function L_f (Eq. (13)) proposed focuses on the influence that the Kriging model accuracy at the different points in input space imposes on the accuracy of failure probability estimation, considers both the local accuracy of Kriging model and PDF $f(\boldsymbol{x})$, and guarantees that the selected samples are located in the important areas (Fig. 2). Besides, through the reliability analysis of cracked structures in Example 3 and Example 4, the proposed model is an useful tool for engineering problems with time-consuming model. By calling to target performance function as few times as possible, the proposed method can reduce the computational time required by reliability analysis of cracked structures, and obtain more accurate results.

V. CONCLUSION

By analyzing the shortcomings of the existing learning functions in the Kriging-based structural reliability analysis, this paper proposes a new learning function which integrates the local accuracy of Kriging model and PDF and applies it to reliability analysis method. According to studies of four examples, it shows that: (1) the proposed learning function can avoid the unnecessary sampling in the unimportant area during reliability analysis. (2) Comparing with other methods, the proposed method can efficiently enhance the Kriging model, and reduce the number of evaluations of structural performance function. (3) In the proposed reliability analysis method, there is no special hypotheses about the nonlinearity of structural performance function or its explicit-implicit feature, so both linear and nonlinear situations are included in the performance functions of the examples. Results show that the proposed method can predict the reliability and failure probability of engineering structures with time-consuming model. (4) The proposed method is available to calculate the fracture failure probability of crack structures. The proposed method can be used to solve the reliability analysis of other practical engineering problems, especially for implicit complex problems, so it has certain engineering application value.

The method proposed in this paper combines the Kriging model with MCS. When failure probability of a structure is small, N_{MC} needs to be very large. For example, N_{MC} will be above 10^7 if P_f is about 10^{-4}. In above situation, the computational procedure consumes a large amount of computer memory, and each iteration may take even longer than an hour. Combining the proposed learning function with IS and SS can further increase the computational efficiency.

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