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Dynamic Sliding-Mode Control for T-S Fuzzy Singular Time-Delay Systems With H_∞ Performance

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ABSTRACT In this paper, we propose a novel sliding-mode control for Takagi-Sugeno (T-S) fuzzy singular system with time-delay and mismatched uncertainties and disturbances. In the majority of T-S fuzzy sliding-mode control ways, a restrictive assumption is required, that is every subsystem's input matrix is identical. In order to eliminate the restrictive assumption, we put forward a dynamic sliding-mode control method for T-S fuzzy singular time-delay systems. Additionally, the dynamic sliding-mode control law is designed to guarantee the reachability of the sliding surface in finite time interval. Stability of sliding motion is analyzed and the dynamic sliding-mode controller is parameterized in terms of the solutions of a set of linear matrix inequalities which facilitates design. In the end, three examples are shown to verify the merit and effectiveness of the proposed approaches are provided.

INDEX TERMS T-S fuzzy singular models, LMIs, Dynamic sliding-mode control, time-delay, H_∞ control.

I. INTRODUCTION

Since the 1980s, T-S fuzzy models have attracted great interest from the control community, because of their effectiveness in approximating the complex nonlinear systems [1]. A lot of efficient results have been proposed, such as observer-based output feedback control [2], non-quadratic membership-dependent Lyapunov functions [3], delay-dependent guaranteed cost control [4], robust H_∞ filtering [5], mismatched membership functions [6] and references therein.

Recently, singular systems have been drawn attention to more and more researchers, because singular systems are more effective to represent physical systems than the regular systems [7], [8]. It should be pointed out that singular systems need to be stable, regular and impulse free, it is different from the regular systems. For example, in [9], the authors pointed out that the paper [10] did not consider fully the impulse behavior, which is a vital feature of singular systems. A lot of good results have been addressed. For example, the admissibility, the paper [11] was the first to give the sufficient conditions for the T-S fuzzy singular systems. A fuzzy singular observer approach was proposed to solve the problem of fault estimation and fault-tolerant in [12]. The resilient estimation problem was researched in [13]. Uncertain T-S singular system was investigated in [14]. H_∞ fuzzy control was addressed in [15]. Generally, time-delays often occur in practical systems and in engineering problems involving rolling mills, transportation of signals, networked control systems, neural networks, and synchroniza tion between two chaotic systems [16]. Therefore, the issue of asymptotic stability and stabilization of singular time-delay system has been one of the hot topics in control research [13], [15].

As a popular method of robust control technique in the control community, the sliding-mode control (SMC) has a great deal of attractive merits, such as fast response, tracking ability and strong robustness. During the past decades, many results for different complex systems have been proposed, such as stochastic systems [17], markovian jump systems [18], networked control systems [19]. Based on SMC technique, a generalized regular form was first to introduce for singular systems in [20]. Researchers are desirable to establish the SMC approach to nonlinear systems, therefore, the SMC was extended to fuzzy systems in [21]. In the past two decades, a lot of great results have been achieved in T-S fuzzy mode with SMC techniques [22]–[27]. It was the first to discuss the fuzzy SMC in [22]. Uncertain fuzzy time-delay systems [23], [25]. T-S fuzzy singular systems with time-delay [24], [26]. The super-twisting algorithm was studied in [27].

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However, what worth mentioning is that almost all T-S fuzzy systems with SMC need to satisfy a restrictive assumption that every subsystem's input matrix is identical [23]–[27]. It is noted that many real models, such as the well-known inverted pendulum on a cart [28], do not satisfy this assumption. The constraint condition restricts those SMC techniques in [23]–[27] to be applied. However, so far, to the authors' best knowledge, in the current literature, few effective results on removing the constraint assumption for the T-S fuzzy singular systems with SMC [28]. In this paper, our proposed method can eliminate the restrictive assumption. That is the first motivation to study. On the other hand, most of the papers study T-S fuzzy systems via SMC with matched uncertainties and disturbances [24], [26]–[28]. However, our systems have the mismatched uncertainties and disturbances, so, the investigated case is much more general than those in [24], [26]–[28]. Especially, in the sense that the local input matrices are allowed to have unmatched uncertainties. In fact, the approaches in [24], [26]–[28] cannot be easily applied in this case. Our second motivation is to solve this problem.

Motivated by the fact, this paper researches a dynamic sliding-mode control (DSMC) strategy for a class of T-S fuzzy singular time-delay systems subject to mismatched norm-bounded uncertainties and disturbances. A remarkable characteristic of the DSMC strategy is that the singular derivative-term matrix and the system state and the state of controller are taken in to account in the sliding surface function. Owing to mismatched disturbances cannot be removed, the H_∞ control technique can decrease the influence of the mismatched disturbance. The dynamic sliding-mode control for T-S fuzzy singular time-delay systems with the mismatched disturbances is shown in Fig.1. The design parameter matrices defining the sliding variable are obtained by solving LMIs. In finite time, the desired system states can asymptotically converge to equilibrium point via the user-defined DSMC technique. The contributions of this paper are as follows

1. A novel dynamic sliding-mode controller is designed to stabilize the T-S fuzzy singular time-delay systems.
2. Eliminate the restrictive assumption that the input matrix B for all the subsystems is the same.
3. Solve the problem that is the systems with mismatched uncertainties and disturbances, especially, the local input matrices are allowed to have unmatched uncertainties.

The remainder of this paper is organized as follows. Section 2 formulates the T-S fuzzy singular time-delay systems and preliminaries. Section 3 focuses on the DSMC strategy design, analysis of sliding motion and H_∞ control. In section 4, to verify the effectiveness of proposed DSMC technique, three simulation results are shown. Finally, Section 5 summarizes the paper.

Notations: Throughout this paper, the n-dimensional Euclidean space is represented by R^n, the set of all $m \times n$ matrices is denoted by $\mathrm{R}^{m\times n}$. $\mathbf{I}_n$ and $0_{m\times n}$ denote the $n \times n$ identity matrix and $m \times n$ zero matrix, respectively. The superscript "T" and "-1" denote matrix transposition and inverse, respectively. $\|\cdot\|$ represents the Euclidean norm and the induced norm for vectors and matrices, respectively. $\tilde{X} > 0$ ($\tilde{X} < 0$) means that $\tilde{X}$ is a positive (negative) definite matrix. The star $\star$ is used as a term that is induced by symmetric position and $sym(\tilde{X})$ is defined as $\tilde{X} + \tilde{X}^T$. The notation $L_2[0, T]$ denotes the space of square-integrable vector functions, i.e., $\omega(t) : [0, T] \rightarrow \mathrm{R}^P \in L_2[0, \mathcal{T}]$, if $\int_0^{\mathcal{T}} \omega^T(t)\omega(t) < \infty$.

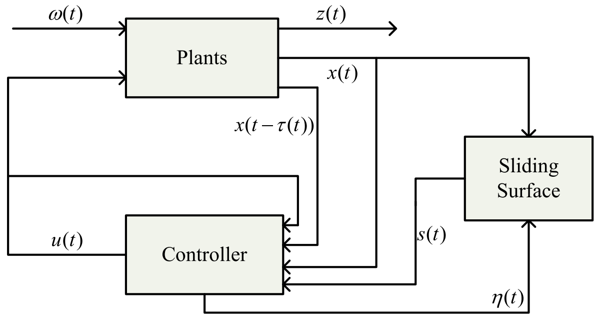

FIGURE 1. The block diagram of dynamic sliding-mode control strategy.

II. SYSTEM DESCRIPTION AND PRELIMINARIES

In this paper, we consider a T-S fuzzy singular time-delay systems with r plant rules that can be represented by the following.

Plant Rule i: IF ϑ_1 is υ_{i1} and ϑ_2 is υ_{i2} and $\ldots$ ϑ_g is υ_{ig}, *THEN*

$$\begin{aligned} E\dot{x}(t) &= [A_i + \Delta A_i]\,x(t) + [A_{\tau i} + \Delta A_{\tau i}]\,x(t-\tau(t)) \\ &\quad + [B_i + \Delta B_i]\,u(t) + [H_i + \Delta H_i]\,\omega(t) \\ z(t) &= C_i x(t) + D_i x(t-\tau(t)) + C_{id}u(t) \\ x(t) &= \phi(t), \quad t \in [-\tau_M, 0], \quad i = 1, 2, \cdots, r. \end{aligned} \tag{1}$$

where i denotes that the ith fuzzy inference rule, υ_{ij} is the fuzzy set, r is the number of *IF-THEN* rules, and $\vartheta_g(t)$ is the premise variables; $x(t) \in \mathrm{R}^n$ is the state vector, $z(t) \in \mathrm{R}^k$ is the controlled output vector, $u(t) \in \mathrm{R}^m$ is the control input vector, $\omega(t) \in \mathrm{R}^p$ is the external disturbance which belongs to $L_2[0, \mathcal{T})$. $E \in \mathrm{R}^{n\times n}$, $A_i \in \mathrm{R}^{n\times n}$, $A_{\tau i} \in \mathrm{R}^{n\times n}$, $B_i \in \mathrm{R}^{n\times m}$, $C_i \in \mathrm{R}^{k\times n}$, $C_{id} \in \mathrm{R}^{k\times m}$, $D_i \in \mathrm{R}^{k\times n}$ and $H_i \in \mathrm{R}^{n\times p}$ are known real constant matrices, and $rank(E) = q \leq n$, $\tau(t)$ is a time varying delay and satisfies

$$0 \leq \tau(t) \leq \tau_M < \infty, \quad \dot{\tau}(t) \leq \tau_d < 1 \tag{2}$$

and $\phi(t)$ is an initial function on $[-\tau_M, 0]$.

There is an assumption as follows, which will be used in this paper:

Assumption 1: The parameter uncertainties ΔA_i, $\Delta A_{\tau i}$, ΔB_i and ΔH_i are norm-bounded, which satisfy

$$\|[\Delta A_i, \Delta B_i]\| \leq \varepsilon_{ai}, \|\Delta A_{\tau i}\| \leq \varepsilon_{bi}, \Delta H_i \Delta H_i^T \leq \varepsilon_h^2 \mathbf{I}_n \tag{3}$$

where ε_{ai}, ε_{bi}, and ε_h are known real positive constants, $i = 1, 2, \cdots, r$.

Based on the center-average defuzzifier, product inference and the singleton fuzzifier, the overall T-S fuzzy singular systems can be inferred as

$$\begin{aligned} E\dot{x}(t) &= \sum_{i=1}^{r} \psi_i(\vartheta(t)) \{[A_i + \Delta A_i] x(t) + [B_i + \Delta B_i] u(t) \\ &\quad + [A_{\tau i} + \Delta A_{\tau i}] x(t-\tau(t)) + [H_i + \Delta H_i]\omega(t)\} \\ z(t) &= \sum_{i=1}^{r} \psi_i(\vartheta(t)) \{C_i x(t) + D_i x(t-\tau(t)) + C_{id} u(t)\} \\ x(t) &= \phi(t), \quad t \in [-\tau_M, 0]. \end{aligned} \tag{4}$$

where

$$\psi_i(\vartheta(t)) = \frac{\theta_i(\vartheta(t))}{\sum_{i=1}^{r} \theta_i(\vartheta(t))}, \quad \theta_i(\vartheta(t)) = \prod_{j=1}^{g} \upsilon_{ij}(\vartheta(t))$$

with $\upsilon_{ij}(\vartheta(t))$ represents the degree of membership function of $\vartheta_j(t)$ in υ_{ij}. Note that:

$$\psi_i(\vartheta(t)) \geq 0, \quad \sum_{i=1}^{r} \psi_i(\vartheta(t)) = 1, \quad i = 1, 2, \cdots, r$$

By adding a term $0 \times u(t-\tau(t))$, $t \in [-\tau_M, 0]$ in (4), we can obtain the equivalent system as follows:

$$\begin{aligned} E\dot{x}(t) &= \sum_{\hat{i}=1}^{r} \psi_i(\vartheta(t)) \{[A_i + \Delta A_i] x(t) + [B_i + \Delta B_i] u(t) \\ &\quad + [A_{\tau i} + \Delta A_{\tau i}] x(t-\tau(t)) + \bar{H}_i \omega(t)\} \\ z(t) &= \sum_{i=1}^{r} \psi_i(\vartheta(t)) \Big\{ \bar{C}_i \left[x^T(t), u^T(t)\right]^T \\ &\quad + \bar{D}_i \left[x^T(t-\tau(t)), u^T(t-\tau(t))\right]^T \Big\} \\ x(t) &= \phi(t), \quad t \in [-\tau_M, 0] \end{aligned} \tag{5}$$

where $\bar{C}_i = [C_i, C_{id}]$, $\bar{D}_i = [D_i, 0_{k\times m}]$, and $\bar{H}_i = [H_i + \Delta H_i]$.

In the following, to simplify the calculation, the variables ψ_i, x, u, z, $x_{\tau(t)}$, $u_{\tau(t)}$, ω and ϕ are used to denote the $\psi_i(\vartheta(t)), x(t), u(t), z(t), x(t-\tau(t)), u(t-\tau(t)), \omega(t)$ and $\phi(t)$, respectively.

Next, a definition and two lemmas will be employed throughout this paper.

Definition 1 [9], [24]:

1) The system (5) is said to be regular if $det(sE - \sum_{i=1}^{r} \psi_i A_i) \not\equiv 0$.
2) The system (5) is said to be impulse-free if $deg(det(sE - \sum_{i=1}^{r} \psi_i A)) = rank(E)$.
3) The system (5) is said to be stable if exists a scalar $\Psi(\varepsilon) > 0$ such that, for any compatible initial conditions ϕ satisfying $sup_{-\tau_M \leq t \leq 0} \|\phi\| < \Psi(\varepsilon)$, $\|x\| < \varepsilon$ ($\forall \varepsilon > 0$ and $\forall t \geq 0$). Furthermore, $x \to 0$, $t \to \infty$.
4) The system (5) is said to be admissible if it is regular, impulse-free and stable.

Lemma 1 [17]: Let $\tilde{X}, \tilde{Y}, \tilde{\Delta}$ are matrices of appropriate dimensions with $\tilde{\Delta}^T \tilde{\Delta} \leq \mathbf{I}_n$ then

$$\tilde{X}\tilde{\Delta}\tilde{Y} + \tilde{Y}^T \tilde{\Delta}^T \tilde{X}^T \leq \varepsilon \tilde{X}\tilde{X}^T + \varepsilon^{-1} \tilde{Y}^T \tilde{Y} \tag{6}$$

Lemma 2 [29]: Consider a singular system as follows

$$E\dot{x} = Ax + Bu \tag{7}$$

where $x \in \mathrm{R}^n$ is the state vector, $u \in \mathrm{R}^m$ is the input vector, E, A, B are constant matrices of appropriate dimensions, $rank(E) = r < n$. we assume the system (7) is regular and impulse-free. If the system (7) is stable, the following are equivalent.

1. there exists a matrix $P \in \mathbf{R}^{n\times n}$ which satisfies
$$PE^T = EP^T, \quad PA^T + AP^T < 0 \tag{8}$$
2. there exists a positive definite matrix $Z \in \mathrm{R}^{n\times n}$ and a matrix $S \in \mathrm{R}^{(n-r)\times(n-r)}$ which satisfy
$$(ZE^T + VSU^T)^T A^T + A(ZE^T + VSU^T) < 0 \tag{9}$$
where matrices $V, U \in \mathrm{R}^{n\times(n-r)}$ are full column rank and $EV = 0$, $E^T U = 0$.

III. DESIGN OF DYNAMIC SLIDING-MODE CONTROL

In this section, we focus on the DSMC design and the admissibility of the sliding motion. Hence, the section will be divided into two parts. In the first part, the sliding surface and dynamic sliding-mode controller (DSM controller) are designed. In the second part, after the system turns into sliding motion, the admissibility of the systems will be analyzed.

A. DESIGN OF SLIDING SURFACE AND SLIDING-MODE CONTROLLER

To obtain DSMC, we give a novel sliding surface for the system (5) defined as:

$$s(t) = G_x Ex + G_u \eta = \bar{G}\bar{E}\bar{x} = \tilde{G}\bar{x} = 0 \tag{10}$$

where $\eta \in \mathrm{R}^m$ is the state of the controller, $G_x \in \mathrm{R}^{m\times n}$, $G_u \in \mathrm{R}^{m\times m}$, $\bar{G} = [G_x, G_u]$, $\tilde{G} = [G_x E, G_u]$, $\bar{E} = diag[E, \mathbf{I}_m]$, $\bar{E} \in \mathrm{R}^{(n+m)\times(n+m)}$, $rank(\bar{E}) = q + m \leq n + m$, $\tilde{G} = \bar{G}\bar{E}$, and $\bar{x} = \left[x^T, \eta^T\right]^T$. G_u is designed to be nonsingular.

Remark 1: Two equivalent sliding surface functions are given in (10), $s(t) = \bar{G}\bar{E}\bar{x}$ and $s(t) = \tilde{G}\bar{x}$, respectively. In the proof of Theorem 1, we choose the first form. In the proof of Theorem 2, the second form is considered.

Since so, we can design a DSM controller for the system (5),

$$\begin{aligned} \dot{\eta} &= -\sum_{i=1}^{r} \psi_i \Big\{ G\left[A_i x + A_{\tau i} x_{\tau(t)} + B_i u\right] \\ &\quad + (\alpha + \xi(t)) G_u^{-1} sgn(s(t)) \Big\} \\ u &= \eta \end{aligned} \tag{11}$$

where

$$\xi(t) = \varepsilon_{ai} \|\bar{x}\| + \varepsilon_{bi} \left\|\bar{x}_{\tau(t)}\right\| + \left(\|H_i\| + \varepsilon_h^2\right) \rho(t)$$

$G = G_u^{-1}G_x$, ε_{ai}, ε_{bi} and ε_h are defined in (3), α is a known positive constant, $\rho(t)$ is the known uniform upper bound of ω.

Remark 2: It is noted that the sliding surface defined in (10) is dependent on both the system state vector and the state of the controller [30], and the sliding-mode controller is in the form fuzzy dynamic-state feedback control. These features are similar to dynamic controller. This is also the reason why it is called the DSMC approach.

Denote: $s(t) = s$, $L_1 = \left[\mathbf{I}_n, 0_{n\times m}\right]^T$, $L_2 = [0_{m\times n}, \mathbf{I}_m]^T$, $\bar{A}_i = [A_i, B_i]$, $\bar{A}_{\tau i} = [A_{\tau i}, 0_{n\times m}]$, $\Delta\bar{A}_i = [\Delta A_i, \Delta B_i]$ and $\Delta\bar{A}_{\tau i} = [\Delta A_{\tau i}, \mathbf{0}_{n\times m}]$. Then, substitute (11) into (5), we can obtain the closed-loop system to be represented by a compact form as follows:

$$\begin{aligned}
\bar{E}\dot{\bar{x}} &= \sum_{i=1}^{r}\psi_i\left\{\left[(L_1 - L_2G)\bar{A}_i + L_1\Delta\bar{A}_i\right]\bar{x}\right.\\
&\quad + \left[(L_1 - L_2G)\bar{A}_{\tau i} + L_1\Delta\bar{A}_{\tau i}\right]\bar{x}_{\tau(t)}\\
&\quad \left. + L_1\bar{H}_i\omega - L_2(\alpha + \xi(t))G_u^{-1}sgn(s)\right\}\\
z &= \sum_{i=1}^{r}\psi_i\left\{\bar{C}_i\bar{x} + \bar{D}_i\bar{x}_{\tau(t)}\right\}\\
\bar{x} &= \left[\phi^T, 0\right]^T, \quad t\in[-\tau_M, 0]
\end{aligned} \tag{12}$$

Theorem 1: For the system (12), in finite time, the system states can reach onto the sliding surface (10) by the DSM controller (11).

Proof: Choose the Lyapunov function candidate as $V_s(t) = s^T s$, one has

$$\begin{aligned}
\dot{V}_s(t) &= 2s^T\bar{G}\bar{E}\dot{\bar{x}}\\
&= 2\sum_{i=1}^{r}\psi_i s^T\bar{G}\left\{\left[(L_1 - L_2G)\bar{A}_i + L_1\Delta\bar{A}_i\right]\bar{x}\right.\\
&\quad + \left[(L_1 - L_2G)\bar{A}_{\tau i} + L_1\Delta\bar{A}_{\tau i}\right]\bar{x}_{\tau(t)}\\
&\quad \left. + L_1\bar{H}_i\omega - L_2(\alpha + \xi(t))G_u^{-1}sgn(s)\right\}
\end{aligned} \tag{13}$$

In the fact that $\bar{G}(L_1 - L_2G) = 0$, we obtain

$$\begin{aligned}
\dot{V}_s(t) &= 2\sum_{i=1}^{r}\psi_i s^T\bar{G}\left\{L_1\Delta\bar{A}_i\bar{x} + L_1\Delta\bar{A}_{\tau i}\bar{x}_{\tau(t)}\right.\\
&\quad \left. + L_1\bar{H}_i\omega - L_2(\alpha + \xi(t))G_u^{-1}sgn(s)\right\}\\
&= 2\sum_{i=1}^{r}\psi_i s^T G_x\left[\Delta\bar{A}_i\bar{x} + \Delta\bar{A}_{\tau i}\bar{x}_{\tau(t)}\right.\\
&\quad \left. + \bar{H}_i\omega - 2\xi(t)\|s\|\right] - 2\alpha\|s\|\\
&\le 2\sum_{i=1}^{r}\psi_i\left\{\left(\|G_x\|\left[\varepsilon_{ai}\|\bar{x}\| + \varepsilon_{bi}\left\|\bar{x}_{\tau(t)}\right\|\right.\right.\right.\\
&\quad \left.\left.\left. + \left(\|H_i\| + \varepsilon_h^2\right)\rho(t)\right] - \xi(t)\right)\|s\|\right\} - 2\alpha\|s\|\\
&= -2\alpha\|s\|\\
&= -2\alpha\sqrt{V_s(t)}
\end{aligned} \tag{14}$$

Hence, in finite time, the system states can arrive at the sliding surface (10). So, the proof is end.

B. ANALYSIS OF THE SLIDING MOTION

We have already proved that the DSM controller guarantees the reachability of the system states in finite time in Theorem 1. In this subsection, after the system turns into sliding motion, we will analyze the admissibility of the sliding motion.

In finite time, since the system states can reach onto the sliding surface (10), the system (12) becomes

$$\begin{aligned}
\bar{E}\dot{\bar{x}} &= \sum_{i=1}^{r}\psi_i\left\{\left[(L_1 - L_2G)\bar{A}_i + L_1\Delta\bar{A}_i\right]\bar{x} + [(L_1\right.\\
&\quad \left. - L_2G)\bar{A}_{\tau i} + L_1\Delta\bar{A}_{\tau i}\right]\bar{x}_{\tau(t)} + L_1\bar{H}\omega\}\\
z &= \sum_{i=1}^{r}\psi_i\left\{\bar{C}_i\bar{x} + \bar{D}_i\bar{x}_{\tau(t)}\right\}\\
\bar{x} &= \left[\phi^T, 0\right]^T, \quad t\in[-\tau_M, 0]
\end{aligned} \tag{15}$$

In the following, we will give a sufficient condition such that the system (15) is robustly admissible with norm-bounded parameters uncertainties and mismatched disturbances.

Theorem 2: Given a positive scalar γ, τ_d. The system (15) is robustly admissible, if there exist nonsingular matrices X, $Q > 0$ and two sets of matrices W_{1i}, $W_{2i}(i = 1, 2, \cdots, r)$ with appropriate dimensions, a set of positive scalars $\epsilon_i(i = 1, 2, \cdots, r)$ such that the following LMIs are satisfied:

$$X^T\bar{E}^T = \bar{E}X \ge 0 \tag{16}$$

$$\Omega_i = \begin{bmatrix}
\Omega_{11i} & \Omega_{12i} & X^T\bar{C}_i^T & \varepsilon_{ai}X^T & 0\\
\star & \Omega_{22i} & X^T\bar{D}_i^T & 0 & \varepsilon_{bi}X^T\\
\star & \star & -\mathbf{I}_k & 0 & 0\\
\star & \star & \star & -\epsilon_i\mathbf{I}_{n+m} & 0\\
\star & \star & \star & \star & -\epsilon_i\mathbf{I}_{n+m}
\end{bmatrix} < 0 \tag{17}$$

where

$$\begin{aligned}
\Omega_{11i} &= sym\left(L_1\bar{A}_iX + L_2W_{1i}\right) + \epsilon_iL_1L_1^T\\
&\quad + \frac{2}{\gamma^2}L_1\left(H_iH_i^T + \varepsilon_h^2\mathbf{I}_n\right)L_1^T + Q\\
\Omega_{12i} &= L_1\bar{A}_{\tau i}X + L_2W_{2i}\\
\Omega_{22i} &= -(1-\tau_d)Q
\end{aligned} \tag{18}$$

In addition, we can obtain the sliding surface matrix $\tilde{G} = L_2^TX^{-1}$.

Proof: The proof of this theorem is divided into two steps. Firstly, we prove that the system is regular and impulse-free.

From the condition (17), we can obtain

$$sym\left(L_1\bar{A}_iX + L_2W_{1i}\right) < 0 \tag{19}$$

Then, we multiply (17) by X^{-T} and X^{-1} on the left and right, respectively. We have

$$sym\left(X^{-T}L_1\bar{A}_i + X^{-T}L_2W_{1i}X^{-1}\right) < 0 \tag{20}$$

On the sliding surface, it is can be obtained that

$$s = \tilde{G}\bar{x} = L_2^T X^{-1}\bar{x} = 0 \tag{21}$$

then one has

$$\bar{x}^T(X^{-T}L_1\bar{A}_i + \bar{A}_i^T L_1^T X^{-1})\bar{x} < 0 \tag{22}$$

so, we have

$$X^{-T}L_1\bar{A}_i + \bar{A}_i^T L_1^T X^{-1} < 0 \tag{23}$$

Denote $X = P^{-1}$, the (16) and (23) are equivalent to the following forms

$$\bar{E}^T P = P^T\bar{E} \geq 0 \tag{24}$$

$$P^T L_1\bar{A}_i + \bar{A}_i^T L_1^T P < 0 \tag{25}$$

Due to $rank(\bar{E}) = rank(E) + m = q + m$, there always exist two nonsingular matrices $\mathbb{M} \in R^{(n+m)\times(n+m)}$, $\mathbb{N} \in R^{(n+m)\times(n+m)}$ such that

$$\mathbb{E} = \mathbb{M}\bar{E}\mathbb{N} = \begin{bmatrix} \mathbf{I}_{q+m} & 0 \\ 0 & 0 \end{bmatrix} \tag{26}$$

Denote

$$\mathbb{P} = \mathbb{M}^{-T}P\mathbb{N} = \begin{bmatrix} \mathbb{P}_1 & \mathbb{P}_2 \\ \mathbb{P}_3 & \mathbb{P}_4 \end{bmatrix}$$

$$\mathbb{M}\left(L_1\bar{A}_i\right)\mathbb{N} = \begin{bmatrix} J_1 & J_2 \\ J_3 & J_4 \end{bmatrix} \tag{27}$$

In the fact that $\mathbb{P}$ is also nonsingular, we can deduce from (24), (26)and (27) that $\mathbb{P}_1 > 0$, $\mathbb{P}_2 = 0$ and $\mathbb{P}_4$ is also nonsingular.

Then, pre-multiplying and post-multiplying (25) by $\mathbb{N}^T$ and $\mathbb{N}$, respectively, we have

$$\begin{pmatrix} * & * \\ * & \mathbb{P}_4^T J_4 + J_4^T\mathbb{P}_4 \end{pmatrix} < 0 \tag{28}$$

where $*$ represents the elements in matrix which are not related to next discussions.

Thus, we can imply that J_4 is nonsingular. Hence, according to Definition 1 and [24], we can deduce that the system (15) is regular and impulse-free.

Secondly, we will deduce that the system (15) is asymptotically stable. Select the Lyapunov function candidate as follows:

$$V(t) = \bar{x}^T\bar{E}^T P\bar{x} + \int_{t-\tau(t)}^{t} \bar{x}^T(s)\, Y\bar{x}(s)\, ds \tag{29}$$

Thus, the derivative of (29) is given by

$$\begin{aligned}
\dot{V}(t) &= 2\bar{x}^T P^T\bar{E}\dot{\bar{x}} + \bar{x}^T Y\bar{x} - (1-\dot{\tau}(t))\bar{x}_{\tau(t)}^T Y\bar{x}_{\tau(t)} \\
&\leq 2\sum_{i=1}^{r}\psi_i\bar{x}^T P^T\left\{L_1\left(\bar{A}_i + \Delta\bar{A}_i\right)\bar{x}\right. \\
&\quad \left. + L_1\left(\bar{A}_{\tau i} + \Delta\bar{A}_{\tau i}\right)\bar{x}_{\tau(t)} + L_1\bar{H}_i\omega\right\} + \bar{x}^T Y\bar{x} \\
&\quad - 2\sum_{i=1}^{r}\psi_i\bar{x}^T P^T L_2\left\{G\bar{A}_i\bar{x} + G\bar{A}_{\tau i}\bar{x}_{\tau(t)}\right\} \\
&\quad - (1-\tau_d)\bar{x}_{\tau(t)}^T Y\bar{x}_{\tau(t)} \\
&= 2\sum_{i=1}^{r}\psi_i\bar{x}^T P^T\left\{L_1\left(\bar{A}_i + \Delta\bar{A}_i\right)\bar{x}\right. \\
&\quad \left. + L_1\left(\bar{A}_{\tau i} + \Delta\bar{A}_{\tau i}\right)\bar{x}_{\tau(t)} + L_1\bar{H}_i\omega\right\} + \bar{x}^T Y\bar{x} \\
&\quad + 2\sum_{i=1}^{r}\psi_i\bar{x}^T P^T L_2\left\{K_{1i}\bar{x} + K_{2i}\bar{x}_{\tau(t)}\right\} \\
&\quad - (1-\tau_d)\bar{x}_{\tau(t)}^T Y\bar{x}_{\tau(t)}
\end{aligned} \tag{30}$$

where $K_{1i} = -G\bar{A}_i$, $K_{2i} = -G\bar{A}_{\tau i}$, $K_{1i}, K_{2i} \in R^{m\times(n+m)}$ are matrices to be determined. By using the Lemma 1, we can obtain

$$\begin{aligned}
&2\bar{x}^T P^T L_1\left(\Delta\bar{A}_i\bar{x} + \Delta\bar{A}_{\tau i}\bar{x}_{\tau(t)}\right) \\
&\quad \leq \bar{x}^T\left(\epsilon_i P^T L_1 L_1^T P + \epsilon_i^{-1}\varepsilon_{ai}^2\mathbf{I}_{n+m}\right)\bar{x} \\
&\qquad + \epsilon_i^{-1}\varepsilon_{bi}^2\bar{x}_{\tau(t)}^T\bar{x}_{\tau(t)}
\end{aligned} \tag{31}$$

and then, for any given $\epsilon_i > 0$, we have

$$\dot{V}(t) \leq \zeta^T\begin{bmatrix} \Gamma_i & P^T\left(L_1\bar{A}_{\tau i} + L_2K_{2i}\right) & P^T L_1\bar{H}_i \\ \star & \epsilon_i^{-1}\varepsilon_{bi}^2\mathbf{I}_{n+m} - (1-\tau_d)Y & 0 \\ \star & \star & 0 \end{bmatrix}\zeta \tag{32}$$

where

$$\begin{aligned}
\zeta &= \begin{bmatrix} \bar{x}^T & \bar{x}_{\tau(t)}^T & \omega^T \end{bmatrix}^T \\
\Gamma_i &= sym\left(P^T\left(L_1\bar{A}_i + L_2K_{1i}\right)\right) + \epsilon_i P^T L_1 L_1^T P \\
&\quad + \epsilon_i^{-1}\varepsilon_{ai}^2\mathbf{I}_{n+m} + Y
\end{aligned} \tag{33}$$

In the following, we will imply that system (15), with for all nonzero $\omega \in L_2[0,\infty)$ under the zero initial conditions $x(0) = 0$, is asymptotically stable.

$$\int_0^\infty \|z\|^2 dt < \gamma^2\int_0^\infty \|\omega\|^2 dt \tag{34}$$

Next, based on (32), we can infer the derivative of $V(t)$ in (29) holds

$$\begin{aligned}
&\dot{V}(t) + z^T z - \gamma^2\omega^T\omega \\
&\leq \sum_{i=i}^{r}\psi_i\zeta^T\left\{\begin{bmatrix} \Gamma_i & P^T\left(L_1\bar{A}_{\tau i} + L_2K_{2i}\right) & 0 \\ \star & \epsilon_i^{-1}\varepsilon_{bi}^2\mathbf{I}_{n+m} - (1-\tau_d)Y & 0 \\ \star & \star & 0 \end{bmatrix}\right. \\
&\quad \left. + \begin{bmatrix} \bar{C}_i^T\bar{C}_i & \bar{C}_i^T\bar{D}_i & P^T L_1\bar{H}_i \\ \star & \bar{D}_i^T\bar{D}_i & 0 \\ \star & \star & -\gamma^2\mathbf{I}_{n+m} \end{bmatrix}\right\}\zeta
\end{aligned} \tag{35}$$

And then, one has that $\dot{V}(t) + z^T z - \gamma^2 \omega^T \omega < 0$ if

$$\begin{bmatrix} \Gamma_i & P^T\left(L_1\bar{A}_{\tau i} + L_2 K_{2i}\right) & 0 \\ \star & \epsilon_i^{-1}\varepsilon_{bi}^2 \mathbf{I}_{n+m} - (1-\tau_d)Y & 0 \\ \star & \star & 0 \end{bmatrix} + \begin{bmatrix} \bar{C}_i^T \bar{C}_i & \bar{C}_i^T \bar{D}_i & P^T L_1 \bar{H}_i \\ \star & \bar{D}_i^T \bar{D}_i & 0 \\ \star & \star & -\gamma^2 \mathbf{I}_{n+m} \end{bmatrix} < 0 \quad (36)$$

Then, we multiply (36) by $diag\left(P^{-T}, P^{-T}, P^{-T}\right)$ and $diag\left(P^{-1}, P^{-1}, P^{-1}\right)$ on the left and right, respectively. We have

$$\begin{bmatrix} \Theta_i & \left(L_1\bar{A}_{\tau i} + L_2K_{2i}\right)P^{-1} & 0 \\ \star & \epsilon_i^{-1}\varepsilon_{bi}^2 P^{-T}P^{-1} - (1-\tau_d)P^{-T}YP^{-1} & 0 \\ \star & \star & 0 \end{bmatrix} + \begin{bmatrix} P^{-T}\bar{C}_i^T\bar{C}_iP^{-1} & P^{-T}\bar{C}_i^T\bar{D}_iP^{-1} & L_1\bar{H}_iP^{-1} \\ \star & P^{-T}\bar{D}_i^T\bar{D}_iP^{-1} & 0 \\ \star & \star & -\gamma^2P^{-T}P^{-1} \end{bmatrix} < 0 \quad (37)$$

where

$$\Theta_i = sym\left(\left(L_1\bar{A}_i + L_2K_{1i}\right)P^{-1}\right) + \epsilon_i L_1 L_1^T + \epsilon_i^{-1}\varepsilon_{ai}^2 P^{-T}P^{-1} + P^{-T}YP^{-1} \quad (38)$$

Denote $Y = P^T Q P$. By using Schur's complement, we can obtain

$$\begin{bmatrix} \Psi_i + \frac{2}{\gamma^2}L_1\left(H_iH_i^T + \varepsilon_h^2\mathbf{I}_n\right)L_1^T & \Psi_{12i} & \bar{X}^T\bar{C}_i^T \\ \star & \Psi_{22i} & X^T\bar{D}_i^T \\ \star & \star & -\mathbf{I}_k \end{bmatrix} < 0 \quad (39)$$

where

$$\begin{aligned} \Psi_i &= sym\left(L_1\bar{A}_iX + L_2W_{1i}\right) + \epsilon_iL_1L_1^T \\ &\quad + \epsilon_i\varepsilon_{ai}^2X^TX + Q \\ \Psi_{12i} &= \left(L_1\bar{A}_{\tau i} + L_2K_{2i}\right)X \\ \Psi_{22i} &= \epsilon_i^{-1}\varepsilon_{bi}^2X^TX - (1-\tau_d)Q \end{aligned} \quad (40)$$

It is worth mentioning that $\bar{H}_i\bar{H}_i^T \leq 2\left(H_iH_i^T + \varepsilon_h^2\mathbf{I}_n\right)$. In the fact that $W_{1i} = K_{1i}X$, $W_{2i} = K_{2i}X$, using Schur's complement, we can easily deduce that (39) is equivalent to (17).

Hence, if the LMIs (16) and (17) are satisfied, we have

$$\dot{V}(t) \leq -z^Tz + \gamma^2\omega^T\omega \quad (41)$$

For any nonzero $\omega \in L_2[0, \infty)$, $t > 0$, integrate both sides of (41) from 0 to $\mathcal{T} > 0$ results in

$$0 < V(\mathcal{T}) = \int_0^{\mathcal{T}} \dot{V}(t)\,dt \leq -\int_0^{\mathcal{T}} z^Tzdt + \gamma^2\int_0^{\mathcal{T}}\omega^T\omega dt \quad (42)$$

It is noted that $\int_0^{\mathcal{T}} z^Tzdt \leq \gamma^2\int_0^{\mathcal{T}}\omega^T\omega dt$. Thus, the system (15) is admissible in the sense of Definition1. So, the proof is end. □

Remark 3: Take the advantage that $L_2^TP\bar{x} = 0$ is on the sliding surface, the slack matrices $W_{1i}, W_{2i} (i = 1, 2, \cdots, r)$ are introduced in (17). Hence, the feasibility of the LMIs conditions can be improved.

Remark 4: In the fact, the equality constraints may have a little theoretical problem, but we find that it probably lead to a big trouble in simulations. Thus, the equality constraints are fragile and usually cannot be perfectly satisfied [31]. In order to solve this trouble, we can obtain the following theorem.

Theorem 3: Given a positive scalar γ, τ_d. The system (15) is robustly admissible, if there exist matrices S, $Q > 0$, $Z > 0$, and two sets of matrices $W_{1i}, W_{2i} (i = 1, 2, \cdots, r)$ with appropriate dimensions, a set of positive scalars $\epsilon_i (i = 1, 2, \cdots, r)$ such that the following LMIs are satisfied:

$$\begin{bmatrix} \digamma_i & \digamma_{12i} & \digamma_{13i} & \digamma_{14i} & 0 \\ \star & \digamma_{22i} & \digamma_{23i} & 0 & \digamma_{25i} \\ \star & \star & -\mathbf{I}_k & 0 & 0 \\ \star & \star & \star & -\epsilon_i\mathbf{I}_{n+m} & 0 \\ \star & \star & \star & \star & -\epsilon_i\mathbf{I}_{n+m} \end{bmatrix} < 0 \quad (43)$$

where, matrices V and U are full column rank and $\bar{E}V = 0$, $\bar{E}^TU = 0$,

$$\begin{aligned} \digamma_{11i} &= sym\left(L_1\bar{A}_i\left(Z\bar{E}^T + VSU^T\right) + L_2W_{1i}\right) \\ &\quad + \epsilon_iL_1L_1^T + \frac{2}{\gamma^2}L_1\left(H_iH_i^T + \varepsilon_h^2\mathbf{I}_n\right)L_1^T + Q \\ \digamma_{12i} &= L_1\bar{A}_{\tau i}\left(Z\bar{E}^T + VSU^T\right)^T + L_2W_{2i} \\ \digamma_{13i} &= \left(Z\bar{E}^T + VSU^T\right)^T\bar{C}_i^T \\ \digamma_{14i} &= \varepsilon_{ai}\left(Z\bar{E}^T + VSU^T\right)^T \\ \digamma_{22i} &= -(1-\tau_d)Q \\ \digamma_{23i} &= \left(Z\bar{E}^T + VSU^T\right)^T\bar{D}_i^T \\ \digamma_{25i} &= \varepsilon_{bi}\left(Z\bar{E}^T + VSU^T\right)^T \end{aligned} \quad (44)$$

In addition, the sliding surface matrix is $\tilde{G} = L_2^T(Z\bar{E}^T + VSU^T)^{-1}$.

Proof: Based on Lemma 2, by replacing $X = Z\bar{E}^T + VSU^T$ in the Theorem 2 that we can obtain inequality (43), it is easy to see that

$$\begin{aligned} \bar{E}X &= \bar{E}(Z\bar{E}^T + VSU^T) = \bar{E}Z\bar{E}^T \\ &= (\bar{E}Z + US^TV^T)\bar{E}^T = X^T\bar{E} \end{aligned} \quad (45)$$

Since so, we remove the equality constraint.

Remark 5: When $E = \mathbf{I}_n$, the T-S fuzzy singular systems turn into the T-S fuzzy systems, our results are also efficient.

Remark 6: One popular solution to eliminate chattering is to approximate discontinuous function $sgn(s) = \frac{s}{\|s\|}$ by some continuous and smooth functions. For example, it could be replaced by $\frac{s}{\|s\|+o}$, where o is a small positive scalar value. So, the chattering will be reduced [32]. In the simulations, we can observe that the proposed method has smaller chattering than the method in [28], [33]. In order to show the merit of our method, we have not used smooth functions to reduce chattering.

Remark 7: When we apply the method proposed in this paper to realtime applications, we can use the Euler method [34], [35], which has been widely used in sliding-mode control implementation. Moreover, the sampling time has to be very small in order to ensure that the discretized system approximates the continuous time system as closely as possible [36].

IV. SIMULATION EXAMPLES

In this section, we present two numerical examples and an inverted pendulum model to verify the merit and effectiveness of the approaches that are proposed in the previous sections, respectively.

Example 1: In this numerical example, a continuous T-S fuzzy singular systems is given by,

$$\begin{aligned} E\dot{x} &= \sum_{i=1}^{2} \psi_i(x_1)\{[A_i+\Delta A_i]x + [B_i+\Delta B_i]u \\ &\quad + [A_{\tau i}+\Delta A_{\tau i}]x_{\tau(t)} + [H_i+\Delta H_i]\omega\} \\ z &= \sum_{i=1}^{2} \psi_i(x_1)\left[C_i x + C_{id}u + D_i x_{\tau(t)}\right] \\ x &= \phi, \quad t\in[-\tau_M,0] \end{aligned} \tag{46}$$

The system matrices are given as follows:

$$E=\begin{bmatrix}1&0\\0&0\end{bmatrix},\quad A_1=\begin{bmatrix}0.1&0.2\\0.2&0.7\end{bmatrix},\quad A_2=\begin{bmatrix}0.1&0.1\\0&0.9\end{bmatrix}$$
$$A_{\tau 1}=\begin{bmatrix}-0.1&0\\0.1&0.1\end{bmatrix},\quad A_{\tau 2}=\begin{bmatrix}0&0\\0.1&0.1\end{bmatrix}$$
$$B_1=\begin{bmatrix}1.5&1.2\\1.6&1\end{bmatrix},\quad B_2=\begin{bmatrix}1.6&1\\1.2&0.9\end{bmatrix},\quad H_1=\begin{bmatrix}0.125\\0\end{bmatrix}$$
$$H_2=\begin{bmatrix}0.1227\\0\end{bmatrix},\quad C_1=C_2=\begin{bmatrix}0&0.1\\0.1&0\end{bmatrix}$$
$$C_{1d}=C_{2d}=\begin{bmatrix}0&0.1\\0.1&0\end{bmatrix},\quad D_1=D_2=\begin{bmatrix}0&0.1\\0.1&0\end{bmatrix}$$

Choose membership functions for rules: $\psi_1(x_1) = 1 - \frac{1}{1+e^{-2x_1}}$, $\psi_2(x_1) = \frac{1}{1+e^{-2x_1}}$.

When we assume that $\|[\Delta A_1, \Delta B_1]\| \le 0.0352$, $\|[\Delta A_2, \Delta B_2]\| \le 0.2447$, $\|\Delta A_{\tau 1}\| = \|\Delta A_{\tau 2}\| \le 0.0224$, $\Delta H_i \Delta H_i^T \le 0.05^2\mathbf{I_2} (i=1,2)$.

Let $\gamma = 0.9$, $\tau(t) = 0.4 + 0.3sin(t)$, and then $\tau_M = 0.7$, $\tau_d = 0.3$, and solve the LMIs in Theorem 3, the corresponding matrices are given by

$$\tilde{G}=\begin{bmatrix}0.2992&0&0.7755&-0.0109\\0.2846&0&-0.0109&0.7089\end{bmatrix}$$
$$G_u=\begin{bmatrix}0.7755&-0.0109\\-0.0109&0.7089\end{bmatrix}$$
$$G_x=\begin{bmatrix}0.2992&0\\0.2846&0\end{bmatrix},\quad G=\begin{bmatrix}0.3825&0\\0.4073&0\end{bmatrix}$$

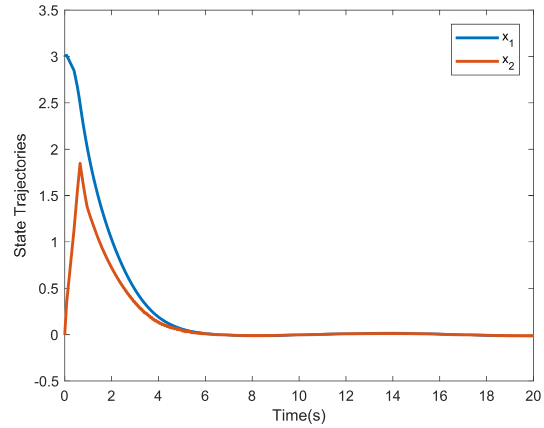

FIGURE 2. States trajectories of the system (46).

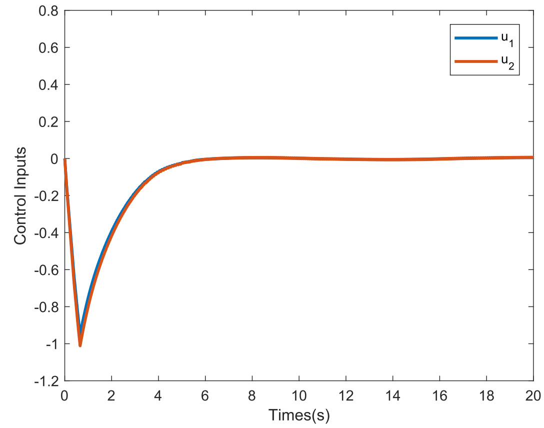

FIGURE 3. Sliding-mode controller of the system (46).

The sliding surface is given by

$$s=\begin{bmatrix}s_1\\s_2\end{bmatrix}=\begin{bmatrix}0.2992&0\\0.2846&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + \begin{bmatrix}0.7755&-0.0109\\-0.0109&0.7089\end{bmatrix}\begin{bmatrix}\eta_1\\\eta_2\end{bmatrix} \tag{47}$$

We can obtain the DSM controller as follows:

$$\begin{aligned} u &= \eta \\ \dot{\eta} &= -\sum_{i=1}^{2}\psi_i(x_1)\{G[A_i x(t) + A_{\tau i}x_{\tau(t)} \\ &\quad + B_i u] - G_u^{-1}(1+\xi(t))\,sgn(s)\} \\ \xi(t) &= \|G_x\|\left[\varepsilon_{ai}\|\bar{x}\| + \varepsilon_{bi}\left\|\bar{x}_{\tau(t)}\right\| \right.\\ &\quad \left. + 0.1\left(\|H_i\| + 0.05^2\right)\right] \end{aligned} \tag{48}$$

To demonstrate the effectiveness of design method, assume the initial conditions $x(0) = \begin{bmatrix}3 & -0.0016\end{bmatrix}^T$, the external disturbance $\omega = 0.1cos(0.5t)e^{-0.01t}$.

Fig.2 shows the state response curve of the closed loop system, Fig.3 displays the dynamics of the dynamic sliding mode controller, and Fig.4 depicts the dynamics of sliding surface function. It can be observed that by using the dynamic

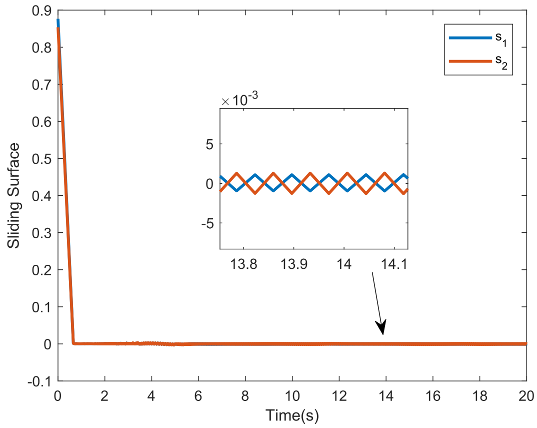

FIGURE 4. Sliding surface of the system (46).

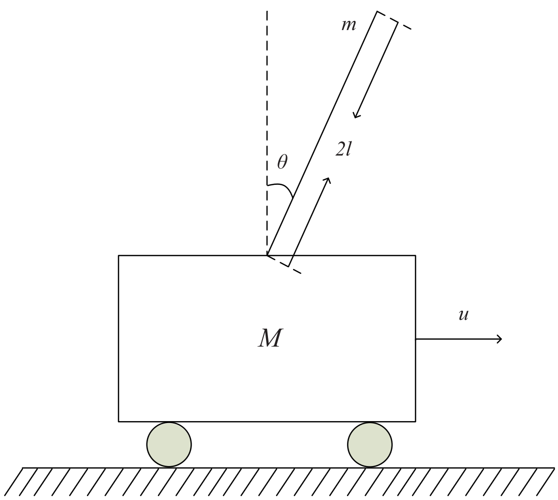

FIGURE 5. The model of Inverted pendulum on a cart.

sliding mode controller, the closed loop system is regular, impulse free and stable.

In this example, we can observe that the T-S fuzzy singular systems do not need every subsystem's input matrix is identical. So, the SMC design methods proposed in [24], [26], [27] cannot be applied.

Example 2: Take into account the inverted pendulum system in Fig.5. The continuous time model of the inverted pendulum plant is given by [28]

$$\begin{aligned}
\dot{x}_1 &= x_2\\
\dot{x}_2 &= \frac{(\mu mlx_4cos(x_1) + (M+m)mgx_5)}{(M+m)(J+ml^2) - m^2l^2cos^2(x_1)}\\
&\quad - \frac{(mlcos(x_1)(u + mx_2^2x_5))}{(M+m)(J+ml^2) - m^2l^2cos^2(x_1)}\\
\dot{x}_3 &= x_4\\
\dot{x}_4 &= \frac{(-\mu(J+ml^2)x_4 - m^2lgx_5cos(x_1))}{(M+m)(J+ml^2) - m^2l^2cos^2(x_1)}\\
&\quad + \frac{((J+ml^2)(u + mx_2^2x_5))}{(M+m)(J+ml^2) - m^2l^2cos^2(x_1)}\\
0 &= lsin(x_1) - x_5
\end{aligned} \tag{49}$$

where

- x_1 is the angle of the pendulum,
- x_3 is the displacement of the pivot,
- x_5 is the horizontal position of the pendulum center relative to the pivot,
- $M = 8\ kg$ is the mass of the cart,
- $m = 2\ kg$ is the mass of the pendulum,
- $2l = 1\ m$ is the length of the pendulum,
- $g = 9.8\ m/s^2$ is the gravity constant

Define $x = col(x_1, x_2, x_3, x_4, x_5))$ and a compact set $\Delta = \{x : |x_i| \le \phi_i, i = 1, 2, \cdots, 5\}$, where $\phi_1 = \frac{5\pi}{18}$ and ϕ_2, ϕ_3, ϕ_4, ϕ_5 are appropriate positive constants. Consider the external disturbances ω, we can obtain the following T-S fuzzy singular model:

$$\begin{aligned}
E\dot{x} &= \sum_{i=1}^{8} \psi_i(x_1)\left\{A_ix + B_i(u + 2x_2^2x_5) + H_i\omega\right\}\\
z &= \sum_{i=1}^{8} \psi_i(x_1)\{C_ix + C_{id}u\}
\end{aligned} \tag{50}$$

where, the premise variables are $\vartheta_1(t) = cos(x_1)$, $\vartheta_2(t) = 1/(2 - 0.3cos^2(x_1))$, and $\vartheta_3(t) = sin(x_1)$. The membership functions are $\psi_i(x_1) = \theta_k(x_1)\eta_l(x_1)\nu_j(x_1)$, $i = j + 2(l-1) + 4(k-1)$, $j, l, k = 1, 2$ with $\theta_1(x_1 = (\vartheta_1(t) - a_2)/(a_1 - a_2)$, $\theta_2(x_1) = 1 - \theta_1(x_1)$, $\eta_1(x_1) = (\vartheta_2(t) - b_2)/(b_1 - b_2)$, $\eta_2(x_1) = 1 - \eta_1(x_1)$, $\nu_1(x_1) = (\vartheta_3(t) - c_2arcsin(\vartheta_3(t)))/((c_1 - c_2)arcsin(\vartheta_3(t)))$, $\nu_2(x_1(t)) = 1 - \nu_1(x_1)$, $a_1 = 1$, $a_2 = cos(\phi_1)$, $b_1 = \frac{1}{1.7}$, $b_2 = 1/(2 - 0.3cos^2(\phi_1))$, $c_1 = 1$, and $c_2 = (sin(\phi_1)/\phi_1)$. The system matrices are given as follows:

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_i = \begin{bmatrix} 0\\ -0.3a_kb_l\\ 0\\ 0.2b_l\\ 0 \end{bmatrix}$$

$$A_i = \begin{bmatrix} 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0.15a_kb_l & 58.8b_l\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -0.1b_l & -5.88a_kb_l\\ 0.5c_j & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$C_i = \begin{bmatrix} 2 & 0.1 & 0.1 & 0.1 & 1 \end{bmatrix}, \quad C_{id} = 0.2, \ H_i = B_i$$

where $i = j + 2(l-1) + 4(k-1)$ and $j, l, k = 1, 2$.

A. DSMC Technique: Choose $\gamma = 1$, and solve the LMIs in Theorem 3, the corresponding matrices are given by:

$$\begin{aligned}
\tilde{G} &= \begin{bmatrix} -4.2778 & -1.0429 & -0.0156 - 0.0517 & 0 & 0.0012 \end{bmatrix}\\
G_x &= \begin{bmatrix} -4.2778 & -1.0429 & -0.0156 & -0.0517 & 0 \end{bmatrix}\\
G_u &= 0.0012, \quad G_u^{-1} = 818.9213, \quad G = G_u^{-1}G_x
\end{aligned}$$

The sliding surface can be written as

$$s = -4.2778x_1 - 1.0429x_2 - 0.0156x_3 - 0.0517x_4 + 0.0012v \tag{51}$$

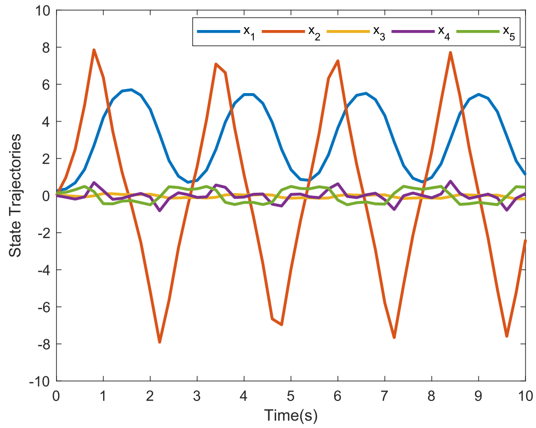

FIGURE 6. Time response of the open-loop system (50).

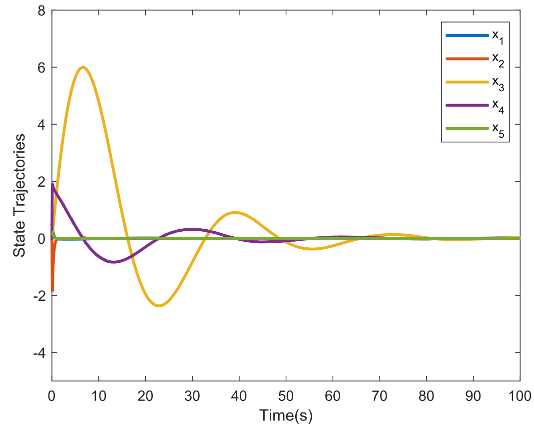

FIGURE 7. Time response of the closed-loop system (50) by this paper.

We can obtain the DSM controller as follows:

$$\begin{aligned} u &= -2x_2^2x_5 + v \\ v &= \eta \\ \dot{\eta} &= -\sum_{i=1}^{8} \psi_i(x_1)\{G[A_ix + B_iv]\} \\ &\quad - 818.9213 \times \left(\alpha + \sum_{i=1}^{8} \psi_i(x_1)\|H_i\|\right) sgn(s) \end{aligned} \tag{52}$$

B. Classical Integral SMC Technique [28]: The corresponding matrices are given by

$$\begin{aligned} S &= \begin{bmatrix} 0 & -4.3638 & 0 & 3.5418 & 0 \end{bmatrix} \\ K1 &= \begin{bmatrix} 399.6547 & 139.9611 & 2.3698 & 42.3405 & 346.7144 \end{bmatrix} \end{aligned}$$

The sliding surface can be obtained as follows:

$$\begin{aligned} s &= -4.3638x_2 + 3.5418x_4 \\ &\quad - \int_0^t \sum_{i=1}^{8} \psi_i(x_1(\tau))S(A_i + B_iK_1)x(\tau)d\tau \end{aligned} \tag{53}$$

We can obtain the classical integral sliding-mode controller as follows:

$$\begin{aligned} u &= -2x_2^2x_5 + K_1x - 2\left(\sum_{i=1}^{8} \psi_i(x_1)SB_i\right)^{-1} s \\ &\quad - \frac{\left(\sum_{i=1}^{8} \psi_i(x_1)SB_i\right)^T s}{\left\|\left(\sum_{i=1}^{8} \psi_i(x_1)SB_i\right)^T s\right\|} \end{aligned} \tag{54}$$

To illustrate the merits of our results, some simulations have been presented. Under the initial condition $x(0) = \begin{bmatrix} \pi/6 & 0 & 0 & 0 & 0.25 \end{bmatrix}$ and external disturbances $\omega = cos(0.5t)e^{-0.01t}$, Fig.6 shows the time of responses of the open loop system (50), Fig.7 and Fig.8 show the time responses of the closed-loop system (50) by the DSM controller (52) in this paper and the classical integral sliding-mode controller (54) [28], respectively, from Fig.8 and Fig.9, both controllers in this paper and [28] are efficient. Fig.9 shows the time responses of sliding-mode controller, we can observe that the DSM controller (52) has smaller amplitude of chattering than the classical integral sliding-mode controller (54), so, the chattering of SMC can

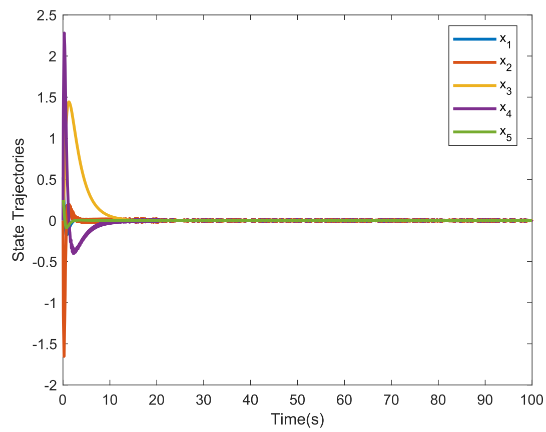

FIGURE 8. Time response of the closed-loop system (50) by [28].

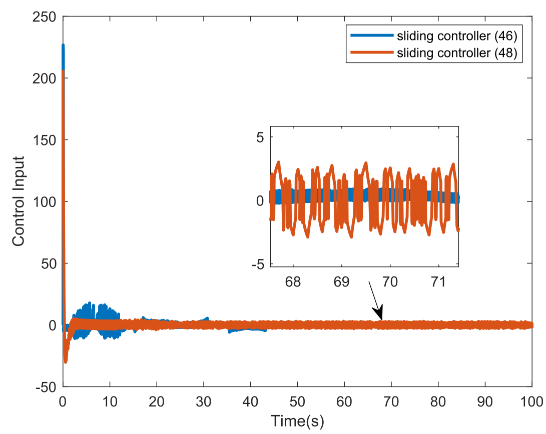

FIGURE 9. Time response of the control inputs.

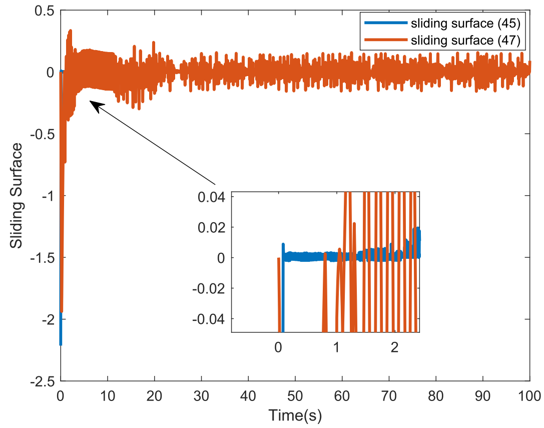

FIGURE 10. Time response of the sliding surface.

be alleviated better compared with the method from [28]. Fig.10 shows the time responses of sliding surface. First, it shows that the sliding surface (51) can be reached more quickly than the sliding surface (53). In other words, this means that the DSMC technique proposed in this paper can make the system to turn into sliding motion earlier than the classical integral SMC technique in [28]. Second, the sliding surface (51) also has smaller amplitude of chattering than the sliding surface (53).

Example 3: Consider a T-S fuzzy uncertain systems with time-delay in the following form:

$$\begin{aligned}\dot{x} &= \sum_{i=1}^{2} \psi_i(x_2)\{[A_i + \Delta A_i]x + [B_i + \Delta B_i]u \\ &\quad + [A_{\tau i} + \Delta A_{\tau i}]x_{\tau(t)} + H_i\omega\} \\ z &= \sum_{i=1}^{2} \psi_i(x_2) C_i x \\ x &= \phi, \quad t \in [-\tau_M, 0] \end{aligned} \tag{55}$$

The model parameters are given as $A_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & -2 \end{bmatrix}$, $A_2 = \begin{bmatrix} -0.3 & 0 \\ 1 & -3 \end{bmatrix}$, $A_{\tau 1} = \begin{bmatrix} 0.1 & 0.1 \\ 0 & 0.1 \end{bmatrix}$, $A_{\tau 2} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix}$, $B_1 = B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $H_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $H_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $C_1 = C_2 = \begin{bmatrix} 2 & 0 \\ 0 & 1.5 \end{bmatrix}$. The uncertainties are set to be $\Delta A_1 = \begin{bmatrix} 0 & 0.08sint \\ 0 & 0.06sint \end{bmatrix}$,
$\Delta A_2 = \begin{bmatrix} 0.06sint & 0 \\ 0.02sint & 0.06sint \end{bmatrix}$, $\Delta A_{\tau 1} = \begin{bmatrix} 0 & 0.06sint \\ 0 & 0.06sint \end{bmatrix}$,
$\Delta A_{\tau 2} = \begin{bmatrix} 0.01sint & 0 \\ 0 & 0.06sint \end{bmatrix}$, $\Delta B_1 = \Delta B_2 = \begin{bmatrix} 0.1sint \\ 0.1sint \end{bmatrix}$.
So, we can obtain $\varepsilon_{a1} = 0.1728$, $\varepsilon_{a2} = 0.1528$, $\varepsilon_{b1} = 0.0849$, $\varepsilon_{b1} = 0.06$.

Choose membership functions for rules: $\psi_1(x_2) = sin^2(x_2)$, $\psi_2(x_2) = cos^2(x_2)$, and the time-varying delay $\tau(t) = 0.2 + 0.2sint$.

TABLE 1. Performance indexes of IAE and ITAE.

sliding-mode controller	IAE	ITAE
sliding-mode controller (57)	8.4152	60.7037
sliding-mode controller (59)	14.8518	130.9583

A. DSMC Technique: The corresponding matrices can be obtain

$$\begin{aligned}\bar{G} &= \begin{bmatrix} 0.1319 & -0.0107 & 0.0061 \end{bmatrix} \\ G_x &= \begin{bmatrix} 0.1319 & -0.0107 \end{bmatrix} \\ G_u &= 0.0061, \quad G = \begin{bmatrix} 21.7189 & -1.7571 \end{bmatrix}\end{aligned}$$

The sliding surface is given by

$$s = 0.1319x_1 - 0.0107x_2 + 0.0061\eta \tag{56}$$

We can obtain the DSM controller as follows

$$\begin{aligned} u &= \eta \\ \dot{\eta} &= -\sum_{i=1}^{2} \psi_i(x_2)\{G[A_i x + A_{\tau i}x_{\tau(t)} \\ &\quad + B_i u] - 164.6590(1 + \xi(t))sgn(s)\} \\ \xi(t) &= \|G_x\| [\varepsilon_{ai}\|\bar{x}\| + \varepsilon_{bi}\|\bar{x}_{\tau(t)}\| + \|H_i\|] \end{aligned} \tag{57}$$

B. Sliding-mode Technique in [33]: The corresponding matrices can be obtain

$$G = \begin{bmatrix} 6.3117 & 10.4480 \end{bmatrix}$$

The sliding surface is given by

$$s = 6.3117x_1 + 10.4480x_2 \tag{58}$$

We can obtain the DSM controller as follows

$$\begin{aligned} u(t) &= -1.7x_1 - 5.7x_2 + u_r(t) \\ u_r(t) &= -\sum_{i=1}^{2} \psi_i(x_2) G[A_i x + A_{\tau i}x_{\tau(t)}] \\ &\quad - \sum_{i=1}^{2} \psi_i(x_2)\rho_i(x,t)sgn(s) \\ \rho_1(x,t) &\geq 2.0202(30.4837\|x\| + 6.7609\|x_{\tau(t)}\| \\ &\quad + 12.6235\|s\|\|\omega\| + 1) \\ \rho_2(x,t) &\geq 2.0202(43.1401\|x\| + 4.4110\|x_{\tau(t)}\| \\ &\quad + 0.3766\|s\|\|\omega\| + 1) \end{aligned} \tag{59}$$

Utilize the DSMC technique (56) and (57) in this paper and sliding-mode control scheme (58) and (59) in [33], under the initial condition $x(0) = \begin{bmatrix} 3 & -3 \end{bmatrix}^T$, the time responses of the closed loop system, sliding-mode controller and sliding surface are shown in Figs.15-18. From Fig.15 and Fig.16, both controllers in this paper and [33] are efficient. First, we can obviously observe that the DSMC technique has smaller amplitude of chattering than the sliding-mode control scheme in [33]. Second, compared with the controller (59), the sliding mode controller (57) presents better control performance. It has less oscillation and the control force is smaller than [33].

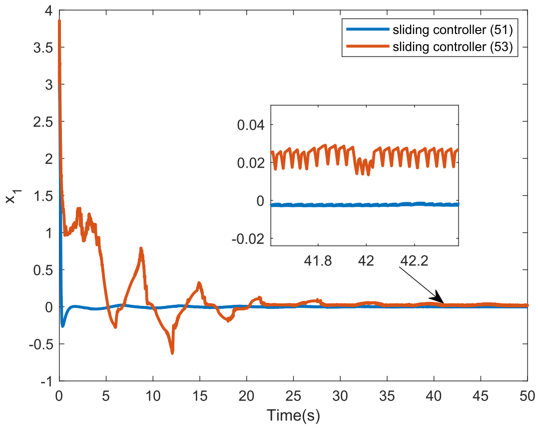

FIGURE 11. Time response of the closed-loop system (55).

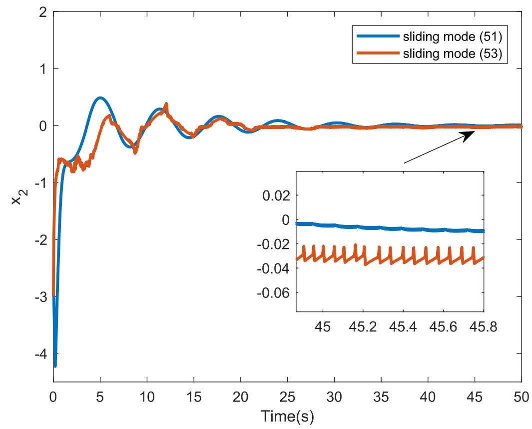

FIGURE 12. Time response of the closed-loop system (55).

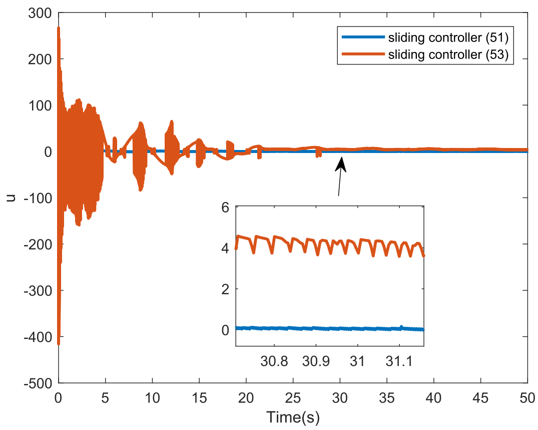

FIGURE 13. Time response of the control inputs.

Furthermore, the performance indexes of integral of the absolute value of error (IAE) $\sum_{i=1}^{2}\int_0^t |0 - x_i(s)|\, ds$ and integral of time multiplied by the absolute value of error (ITAE)$\sum_{i=1}^{2}\int_0^t t\,|0 - x_i(s)|\, ds$ with different SMC technique are shown in Table 1. It shows that the IAE and ITAE values using the sliding-mode controller (57) is less than (59). It is

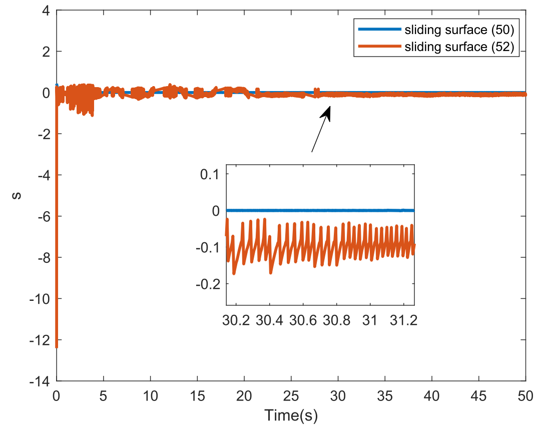

FIGURE 14. Time response of the sliding surface.

obvious that the DSMC technique proposed in this paper is more efficient and feasible than existing method [33].

V. CONCLUSION

A DSMC technique has been proposed for a class of nonlinear singular time-delay systems in the form of T-S fuzzy model with robust H_∞ control in this paper. The DSMC strategy remove the restrictive assumption that every subsystem's input matrix is identical. A novel sliding surface function and a new DSM controller are developed. A set of LMIs are feasible to guarantee that the system state trajectories reach onto the predefined sliding surfaces in finite time, and the sliding motion is admissible with H_∞ performance. In the end, three examples are used to verify the merit and effectiveness of the proposed DSMC technique. In the future, we will extend the results to hybrid electric vehicle [37], [38]

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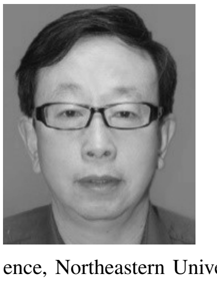

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