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Iterative Learning Control for Discrete Distributed Parameter Systems With Randomly Varying Trial Lengths

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ABSTRACT In this paper, iterative learning control (ILC) is employed in discrete spatial-temporal parabolic distributed parameter systems (DPSs), where the trial lengths vary randomly. A distributed ILC strategy is proposed, in which containing spatial variable, utilizes all past tracking information to improve current performance. Through rigorous theoretical analysis, the convergence of the system output error is proved under mathematical expectation along the iteration axis. Finally, the proposed method is applied to numerical simulation to illustrate its effectiveness.

INDEX TERMS Iterative learning control, distributed parameter systems, partial difference equations, random, convergence.

I. INTRODUCTION

Any learning method needs an inherent requirement, that is repeatability. In a repetitive environment, the accumulated experience can be fully adopted to improve the current method for achieving the desired objective. Hence, it is reasonable to consider repeatability and accuracy as two major learning components in the control system [1], [2]. It is well known, as a powerful and simple control strategy, iterative learning control (ILC) received extensive attention since it was proposed in 1984 [3]. Classic ILC often requires the system can operate repeatedly so that the input signals can be continuously optimized along the iteration axis, and then achieve full tracking [4]–[10]. With the decades of development of ILC, it has been widely applied in distributed parameter systems (DPSs) [11]–[21]. Such as the author considers applying ILC to the high-order DPSs described by one-dimensional fourth order partial differential equations (PDEs) [11], and the boundedness of the system output errors is proved. In [12], for a class of parabolic DPSs, the paper proposes an ILC scheme based on the system eigenspectrum. In views of the ILC strategy, the desired trajectory can be repeatedly tracked by the system both in time and space. Furthermore, in the existing literature [13], an ILC problem for a class of MIMO distributed parameter systems consisting of second-order hyperbolic PDEs with uncertainties is considered. Unlike the lumped parameter systems, the distributed parameter systems described by the partial differential equations contain both time and space variables, which makes their research complicated. The above literature shows that the ILC is an effective methodology for the DPSs. However, the trial lengths all are fixed in this literature. In this paper, we apply ILC to DPSs governed by parabolic partial difference equations with non-uniform trial lengths. Traditional ILC requires that the control object must have fixed trial lengths under strictly repetitive environment; once it is not satisfied, the tracking error can only achieve bounded convergence [22].

In fact, the situation that trial lengths vary randomly exists in many practical applications of ILC, especially biomedical systems and anthropomorphic robots. The trial lengths are often non-uniform due to unknown dynamics and complex factors [23]. For example, in [24], for some patients with muscle atrophy and limb paralysis caused by the disuse of upper limb muscles, it can be relieved and cured by functional electrical stimulation (FES). Precise stimulation patterns are essential in FES, which requires us to know the detailed distribution of the atrophic muscles and apply accurate electrical

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stimulation therapy. For continuous learning of such stimulus distributions, it is theoretically feasible to use ILC algorithms that can achieve full tracking for the desired trajectory. Complex electrical stimulation therapy often requires multiple trial attempts. However, considering the patient's own physical condition and uncontrollable external factors in the trial, the actual output trajectory is likely to be affected and deviated. It should be noted that if the difference between the output trajectory and the desired trajectory is too high, the trial must be terminated immediately to ensure the safety of patients. Therefore, the time length is not fixed and changing randomly in each electrical stimulation. Nevertheless, if there is a way to address the information of these electrical stimulations with different time lengths, a lot of valuable data can be collected for learning. Similar examples are included in the gait-assisted FES process [24] and the anthropomorphic simulation of gait in [25]. These examples are almost due to various unavoidable reasons, resulting in different lengths of time in the learning process. In short, in these practical application examples, the requirement of classic ILC that trial lengths must be fixed in the iteration domain is no longer satisfied. It makes us need to consider the applicability of ILC in the context of non-uniform trial lengths.

Recently, some studies have been done on the stochastic ILC problems with trial lengths varying randomly in the learning process [23]–[31]. In the literature [24], the author first defines the maximum trial length as the full length and shorter than the maximum trial length as the incomplete length. Then, in order to satisfy the strict repeatability of classic ILC, the incomplete trial lengths are replenished as full length by filling errors of the shortage parts with 0. In [26], the author designs a new ILC algorithm with the iteration-average operator and proves the convergence of the system output errors under mathematical expectation. Also, [27] further considers the case of the system with continuous-time nonlinear based on the results of existing literature [26]. Moreover, in [28], the author reveals that the traditional P-type ILC scheme is robust for the factor of the trial lengths vary randomly, then the almost sure and mean-square convergence conditions of the output error are established without presupposing any probability distribution, etc.. However, the above examples all applied random ILC to lumped parameter systems which only define the system states by time variable but do not contain spatial variable. As far as I know, there are currently no works applying ILC to the distributed parameter system where the trial lengths vary randomly. In this paper, under the premise of considering both the spatial-temporal variables and the unfixed trial lengths in the system, a distributed ILC algorithm is introduced, and then the convergence of the system error is guaranteed under mathematical expectation.

The main contributions of this paper are given as follow:

(1) This paper first applies ILC to discrete parabolic distributed parameter systems where trial lengths are non-uniform. The research of discrete system provides a theoretical basis for digital computer process control. A distributed learning algorithm with the iteration-average operator is proposed.

(2) The detailed convergence analysis of system error under mathematical expectation in the sense of L^2 norm and effective numerical simulation are presented. It should be noted that the L^2 norm containing the space variables is used for the convergent analysis, which leads to the square terms will be involved in the proof process. Therefore, it makes the convergence analysis more complicated.

The structure of this paper is organized as follows: Firstly, in Section II, we formulate the ILC design problem and give the system description. Then, the learning algorithm design and convergence proof are presented in Section III. Further, in Section IV, an effective simulation example is given. We summarize this paper in Section V.

Notations: In this paper, N denotes the set of natural numbers, T_d, T_k, T_m are represented as the desired iteration length, the actual iteration length and the minimum iteration length respectively. In addition, $\| \cdot \|$ is denoted as L^2 norm that $\| g \| = (\sum_{\eta=1}^{I} g^2(\eta))^{\frac{1}{2}}$ where $g(\eta) \in \mathbb{R}$ for $1 \leqslant \eta \leqslant I$, and I is a given integer. $\| f_k \|^2_{(L^2,\lambda)} = \sup_{0 \leqslant \tau \leqslant T_d} \{\| f_k(\cdot, \tau) \|^2 \lambda^{\tau}\}$ as (L^2, λ) norm of a function $f_k(\eta, \tau) \in \mathbb{R}$ with $1 \leqslant \eta \leqslant I$, $0 \leqslant \tau \leqslant T_d$. Moreover, $\mathbb{E}\{\vartheta\}$ represents the expectation of stochastic variable ϑ, and $\mathbb{P}\{\zeta\}$ is defined as the probability of occurrence of event ζ.

II. PROBLEM FORMULATION

Consider the following discrete parabolic distributed parameter systems in a repeatable environment

$$\begin{cases} \Delta_2 x_k(\eta, \tau) = a\Delta_1^2 x_k(\eta-1, \tau) + g x_k(\eta, \tau) \\ \qquad\qquad + b u_k(\eta, \tau), \\ y_k(\eta, \tau) = c x_k(\eta, \tau), \end{cases} \tag{1}$$

where $k \in N$ stands for the iteration index, $1 \leqslant \eta \leqslant I$, $0 \leqslant \tau \leqslant T_d$ denote space and time variables, respectively, and I, T_d are given integers. $a > 0$ is a constant number. b, g and c are known constant numbers. Moreover, in the kth iteration, the state, input, and output of the system (1) are represented by $x_k(\eta, \tau)$, $u_k(\eta, \tau)$, $y_k(\eta, \tau) \in \mathbb{R}$, respectively.

Remark 1: As the main class of distributed parameter systems, parabolic distributed parameter systems have been extensively studied and have a broad industrial background [32]. Many practical continuous systems can be described by parabolic partial difference equations after discretization. For example, in [33], the author utilized the discrete parabolic partial difference equations to express a diffusion process with a domain control. Discrete-time models of parabolic DPSs are considered for application to the estimation of sulfur dioxide concentration in the atmosphere in [34]. Also, the system (1) is obtained by discretizing a continuous parabolic distributed parameter system consisting of partial differential equations.

The difference symbol Δ in system (1) are defined as follows

$$\begin{cases} \Delta_2 x_k(\eta,\tau) = x_k(\eta,\tau+1) - x_k(\eta,\tau), \\ \Delta_1^2 x_k(\eta-1,\tau) = x_k(\eta+1,\tau) - 2x_k(\eta,\tau) \\ \qquad\qquad + x_k(\eta-1,\tau). \end{cases} \tag{2}$$

The initial value and boundary value of the system (1) are set as

$$x_k(\eta,0) = x_d(\eta,0) = \phi(\eta), \quad 1 \leqslant \eta \leqslant I.\ k = 1,2\cdots, \tag{3}$$

$$x_k(0,\tau) = 0 = x_k(I+1,\tau), \quad 0 \leqslant \tau \leqslant T_d, \tag{4}$$

where $\phi(\eta)$ is a bounded function for $1 \leqslant \eta \leqslant I$.

For desired output $y_d(\eta,\tau)$, there exists a unique system input $u_d(\eta,\tau) \in \mathbb{R}$ satisfying that

$$\begin{cases} \Delta_2 x_d(\eta,\tau) = a\Delta_1^2 x_d(\eta-1,\tau) + g x_d(\eta,\tau) \\ \qquad\qquad + b u_d(\eta,\tau), \\ y_d(\eta,\tau) = c x_d(\eta,\tau), \end{cases} \tag{5}$$

where the $u_d(\eta,\tau)$ is uniformly bounded for all $1 \leqslant \eta \leqslant I$, $0 \leqslant \tau \leqslant T_d$.

This paper is mainly to investigate the problem that the lengths of time vary randomly in the iteration domain, which prompts us to consider the relationship between the desired iteration length T_d and the actual iteration length T_k. When $T_k < T_d$, the learning information is lost from $T_k + 1$ to T_d, which means that the system output errors are unmeasurable during this period, and we regard these output errors as 0. The other scenario is $T_k \geqslant T_d$, that is, the output errors are measurable from 0 to T_d, which can be used as valuable learning information. Further, it is worth noting, when $T_k > T_d$, the system errors from T_d+1 to T_k are useless for learning updates, so we also regard $T_k \geqslant T_d$ as $T_k = T_d$ (see [23], [26]). In addition, because of the existence of the minimum iteration length T_m, the actual iteration time lengths will not vary randomly from 0 to T_m, it also means that the system inputs $u_k(\eta,\tau)$ at $\tau \in \{0, 1 \cdots T_m\}$ are updated continuously, and the lengths of time vary on $\{T_m+1, \cdots, T_d\}$. Thus, in the subsequent analysis, we only need to consider the situation that the actual trial length T_k changes randomly from T_m+1 to T_d in the iteration process [25].

III. ALGORITHM DESIGN AND CONVERGENCE ANALYSIS

In the section, for describing the probability of the system error happens at each moment, this paper considers defining it by the probabilities of the random iteration lengths occur. Firstly, we set the value of the random variable $\tau \in \{T_m+1, T_m+2, \cdots, T_d\}$, which is the iteration length as an event A_τ. It is worth noting that when $\tau = T_k$, $T_k \in \{T_m+1, T_m+2, \cdots, T_d\}$, event A_{T_k} means the system errors on $\tau \in \{0, 1, \cdots, T_k\}$ is measurable, but the information of output errors is lost on $\{T_k+1, \cdots, T_d\}$. Further, because the range of random iteration length is $\{T_m+1, \cdots, T_d\}$, we have the equation $\sum_{\tau=T_m+1}^{T_d} \mathbb{P}(A_\tau) = 1$.

In addition, we define a random variable $\theta_k(\tau)$, $\tau \in \{0, 1, \cdots, T_d\}$ obeying the Bernoulli binomial distribution, when $\theta_k(\tau) = 1$ which stands for the system error is measurable at τ moment in the kth iteration. For example, if the system error is measurable at τ_0 moment which means $\theta_k(\tau_0) = 1$, $\tau_0 \in \{0, 1, \cdots, T_d\}$, that is, in the same trial, for $0 \leqslant \tau \leqslant \tau_0$, $1 \leqslant \eta \leqslant I$, the output errors $e_k(\eta,\tau)$ are measurable. For convenience, we rewrite the probability of $\theta_k(\tau) = 1$, $\mathbb{P}(\theta_k(\tau) = 1)$ as $p(\tau)$, $p(\tau) \in (0, 1]$. It is not hard to find $p(\tau) = 1$, $\tau \in \{0, 1, \cdots, T_m\}$ and $p(T_m+1) > p(T_m+2) > \cdots > p(T_d)$.

The other case is $\theta_k(\tau) = 0$, which means that the system error is unmeasurable at τ moment. It is also easy to realize that if the output error is unmeasurable at τ_0, in the same trial, for $\tau_0 \leqslant \tau \leqslant T_d$, $1 \leqslant \eta \leqslant I$, the information of output errors $e_k(\eta,\tau)$ is lost, and we use $1 - p(\tau)$ to represent the probability of $\theta_k(\tau) = 0$. According to the above explanation, in kth iteration, the expression that uses the probabilities of occurrences of the iteration lengths to describe the probability of the error which occurs at τ moment can be written as $p(\tau) = \sum_{T_k=\tau}^{T_d} \mathbb{P}(A_{T_k})$, $\tau \in \{0, 1, \cdots, T_d\}$.

Since the iteration lengths actually only vary on $\{T_m, T_m+1, \ldots, T_d\}$, we can divide the tracking error into two cases. One case is $T_k = T_d$, which means the actual iteration length is fixed and not varying random in iteration domain. In this case, the output errors are not affected by the random factor. The other case is $T_m < T_k < T_d$, which indicates that the actual iteration length is less than the desired iteration length. At this point, the information of system errors at $\{T_k, T_k+1, \cdots, T_d\}$ is lost and cannot be used for learning update, so the tracking errors during this period are set as 0.

From the above analysis, we can denote the actual output error $e_k^*(\eta,\tau) = \theta_k(\tau) e_k(\eta,\tau)$. When $T_k < T_d$ we obtain

$$e_k^*(\eta,\tau) = \begin{cases} e_k(\eta,\tau), & \tau \in \{0, 1, \cdots, T_k\}, \\ 0, & \tau \in \{T_k+1, T_k+2, \cdots, T_d\}, \end{cases} \tag{6}$$

where $\eta \in \{1, 2, \cdots, I\}$. When $T_k = T_d$, it follows that

$$e_k^*(\eta,\tau) = e_k(\eta,\tau), \quad \eta \in \{1, 2, \cdots, I\},\ \tau \in \{0, 1, \cdots, T_d\}, \tag{7}$$

where $e_k(\eta,\tau) = y_d(\eta,\tau) - y_k(\eta,\tau)$.

Since $\theta_k(\tau)$ is a random variable obeying the Bernoulli binomial distribution, we can get

$$\mathbb{E}\{\theta_k(\tau) = 1\} = (1 - p(\tau)) \cdot 0 + p(\tau) \cdot 1 = p(\tau). \tag{8}$$

For system (1), in order to design the ILC algorithm, we introduce the iteration-average operator [4]

$$\mathbb{A}\{f_k(\cdot)\} \triangleq \frac{1}{k+1} \sum_{s=0}^{k} f_s(\cdot), \tag{9}$$

The distributed ILC scheme is adopted as follow

$$u_{k+1}(\eta,\tau) = \mathbb{A}\{u_k(\eta,\tau)\} + \frac{k+2}{k+1}\gamma \sum_{s=0}^{k} e_s^*(\eta,\tau+1), \tag{10}$$

where γ is the learning gain, and $\eta \in [0, I]$, $\tau \in [0, T_d]$. We define $\bar{x}_k(\eta,\tau) \triangleq x_d(\eta,\tau) - x_k(\eta,\tau)$, $\bar{u}_k(\eta,\tau) \triangleq u_d(\eta,\tau) - u_k(\eta,\tau)$. $x_d(\eta,\tau)$, $u_d(\eta,\tau)$ respectively denote the desired state and desired input of the system (1). Combining these definitions and the system (1), we have

$$\begin{cases} \Delta_2 \bar{x}_k(\eta,\tau) = a\Delta_1^2 \bar{x}_k(\eta-1,\tau)+g\bar{x}_k(\eta,\tau) \\ \qquad +b\bar{u}_k(\eta,\tau), \\ e_k(\eta,\tau) = c\bar{x}_k(\eta,\tau). \end{cases} \tag{11}$$

For proving the convergence of the output error under mathematical expectation, the operates $\mathbb{E}\{\}$, $\mathbb{A}\{\}$ are applied on both sides of (11), that is

$$\begin{cases} \mathbb{E}\{\mathbb{A}\{\Delta_2\bar{x}_k(\eta,\tau)\}\} = \mathbb{E}\{\mathbb{A}\{a\Delta_1^2\bar{x}_k(\eta-1,\tau)\}\} \\ \qquad +\mathbb{E}\{\mathbb{A}\{g\bar{x}_k(\eta,\tau)\}\} \\ \qquad +\mathbb{E}\{\mathbb{A}\{b\bar{u}_k(\eta,\tau)\}\}, & \text{(12-a)} \\ \mathbb{E}\{\mathbb{A}\{e_k(\eta,\tau)\}\} = \mathbb{E}\{\mathbb{A}\{c\bar{x}_k(\eta,\tau)\}\}. & \text{(12-b)} \end{cases}$$

By the definition of difference symbols (2), one yields

$$\begin{aligned} \mathbb{E}\{\mathbb{A}\{\Delta_2\bar{x}_k(\eta,\tau)\}\} &= \mathbb{E}\{\mathbb{A}\{\bar{x}(\eta,\tau+1)-\bar{x}(\eta,\tau)\}\} \\ &= \mathbb{E}\{\mathbb{A}\{\bar{x}(\eta,\tau+1)\}\}-\mathbb{E}\{\mathbb{A}\{\bar{x}(\eta,\tau)\}\}. \end{aligned} \tag{13}$$

Similarly, we can obtain

$$\begin{aligned} &\mathbb{E}\{\mathbb{A}\{\Delta_1^2\bar{x}_k(\eta-1,\tau)\}\} \\ &\quad = \mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta+1,\tau)-2\bar{x}_k(\eta,\tau)+\bar{x}_k(\eta-1,\tau)\}\} \\ &\quad = \mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta+1,\tau)\}\}-2\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau)\}\} \\ &\qquad +\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta-1,\tau)\}\}. \end{aligned} \tag{14}$$

Then, according to the initial value and boundary value assumptions of the system (1), there is

$$\begin{aligned} \mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,0)\}\} &= \mathbb{E}\{\mathbb{A}\{\phi(\eta)\}\}-\mathbb{E}\{\mathbb{A}\{\phi(\eta)\}\} \\ &= 0, \end{aligned} \tag{15}$$

$$\mathbb{E}\{\mathbb{A}\{\bar{x}_k(0,\tau)\}\}=0=\mathbb{E}\{\mathbb{A}\{\bar{x}_k(I+1,\tau)\}\}, \tag{16}$$

where $1 \leqslant \eta \leqslant I, 0 \le \tau \le T_d, k = 1, 2\cdots$.

Next, we introduce the four Lemmas which will be needed in the following proof process.

Lemma 1 [35]: Let $\{z(\eta)\}$, $\{R(\eta)\}$, $\{Q(\eta)\}$ be real sequences and $\eta \geqslant 0$, by the condition

$$z(\eta+1) \leqslant R(\eta)z(\eta)+Q(\eta), \quad R(\eta)\geqslant 0,\ \eta \geqslant 0, \tag{17}$$

we have

$$z(\tau)\leqslant \prod_{\eta=0}^{\tau-1}R(\eta)z(0)+\sum_{\eta=1}^{\tau-1}Q(\eta)\prod_{s=\eta+1}^{\tau-1}R(s),\ \forall\tau\leqslant 0. \tag{18}$$

Lemma 2 [36]: If the non-negative real number sequence μ_k satisfies that $\{\mu_{k+1} \leqslant \omega\mu_k + \nu_k\}$, where $0 \leqslant \omega < 1$ and $\lim_{k\to\infty} \nu_k = 0$, it follows $\lim_{k\to\infty} \mu_k = 0$.

Lemma 3: Consider equations (13), (14) and boundary value condition (16) of the system (12), we have the following equation

$$\sum_{\eta=1}^{I}\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau)\}\}\mathbb{E}\{\mathbb{A}\{\Delta_1^2\bar{x}_k(\eta-1,\tau)\}\} = -\sum_{\eta=1}^{I}(\mathbb{E}\{\mathbb{A}\{\Delta_1\bar{x}_k(\eta,\tau)\}\})^2. \tag{19}$$

Proof: Firstly, by the equation (14), we can express $\sum_{\eta=1}^{I}\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau)\}\}\mathbb{E}\{\mathbb{A}\{\Delta_1^2\bar{x}_k(\eta-1,\tau)\}\}$ as

$$\begin{aligned} &\sum_{\eta=1}^{I}\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau)\}\}\mathbb{E}\{\mathbb{A}\{\Delta_1^2\bar{x}_k(\eta-1,\tau)\}\} \\ &\quad = \sum_{\eta=1}^{I}\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau)\}\}[\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta+1,\tau)\}\} \\ &\qquad -\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau)\}\}]-\sum_{\eta=1}^{I}\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau)\}\} \\ &\qquad \times[\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau)\}\}-\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta-1,\tau)\}\}]. \end{aligned} \tag{20}$$

Then, in terms of equations (20) and (13), it follows that

$$\begin{aligned} &\sum_{\eta=1}^{I}\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau)\}\}\mathbb{E}\{\mathbb{A}\{\Delta_1^2\bar{x}_k(\eta-1,\tau)\}\} \\ &\quad = \mathbb{E}\{\mathbb{A}\{\bar{x}_k(I,\tau)\}\}\mathbb{E}\{\mathbb{A}\{\Delta_1\bar{x}_k(I,\tau)\}\} \\ &\qquad -\sum_{\eta=1}^{I-1}(\mathbb{E}\{\mathbb{A}\{\Delta_1\bar{x}_k(\eta,\tau)\}\})^2. \end{aligned} \tag{21}$$

In addition, considering $\mathbb{E}\{\mathbb{A}\{\bar{x}_k(I,\tau)\}\}\mathbb{E}\{\mathbb{A}\{\Delta_1\bar{x}_k(I,\tau)\}\}$ in (21), according to (16), we have

$$\begin{aligned} &\mathbb{E}\{\mathbb{A}\{\bar{x}_k(I,\tau)\}\}\mathbb{E}\{\mathbb{A}\{\Delta_1\bar{x}_k(I,\tau)\}\} \\ &\quad = \mathbb{E}\{\mathbb{A}\{\bar{x}_k(I,\tau)\}\}\mathbb{E}\{\mathbb{A}\{\Delta_1\bar{x}_k(I,\tau)\}\} \\ &\qquad -\mathbb{E}\{\mathbb{A}\{\bar{x}_k(I+1,\tau)\}\}\mathbb{E}\{\mathbb{A}\{\Delta_1\bar{x}_k(I,\tau)\}\} \\ &\quad = -(\mathbb{E}\{\mathbb{A}\{\Delta_1\bar{x}_k(I,\tau)\}\})^2. \end{aligned} \tag{22}$$

Further, Substituting (22) into (21), one yields

$$\sum_{\eta=1}^{I}\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau)\}\}\mathbb{E}\{\mathbb{A}\{\Delta_1^2\bar{x}_k(\eta-1,\tau)\}\} = -\sum_{\eta=1}^{I}(\mathbb{E}\{\mathbb{A}\{\Delta_1\bar{x}_k(\eta,\tau)\}\})^2. \tag{23}$$

The proof of Lemma 3 is finished.

Lemma 4: Consider the initial and boundary value conditions (15), (16) and system (12), we can get the inequality concerning $\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau)\}\}$ and $\mathbb{E}\{\mathbb{A}\{\bar{u}_k(\eta,\tau)\}\}$ as follow

$$\|(\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\cdot,\tau)\}\})\|^2 \leqslant \sum_{t=0}^{\tau-1}M_2\,\|(\mathbb{E}\{\mathbb{A}\{\bar{u}_k(\cdot,t)\}\})\|^2\,M_1^{(\tau-t-1)}, \tag{24}$$

where $M_1 = 1+2g+|b|+4(g-2a)^2+8a^2, M_2 = |b|+4a^2$.

Proof: By equation (12-a), we can obtain

$$\begin{aligned} &\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau+1)\}\} \\ &\quad = a\mathbb{E}\{\mathbb{A}\{\Delta_1^2\bar{x}_k(\eta-1,\tau)\}\} \\ &\qquad +(g+1)\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau)\}\}+b\mathbb{E}\{\mathbb{A}\{\bar{u}_k(\eta,\tau)\}\}. \end{aligned} \tag{25}$$

Multiplying on both sides of (25) by $\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau)\}\}$, there is

$$\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau)\}\}\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau+1)\}\}$$

$$= a\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau)\}\}\mathbb{E}\{\mathbb{A}\{\Delta_1^2\bar{x}_k(\eta-1,\tau)\}\} + (g+1)(\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau)\}\})^2 + b\mathbb{E}\{\mathbb{A}\{\bar{u}_k(\eta,\tau)\}\}\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau)\}\}. \quad (26)$$

According to the definition

$$(\mathbb{E}\{\mathbb{A}\{\Delta_2\bar{x}_k(\eta,\tau)\}\})^2 = (\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau+1)\}\})^2 - 2\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau)\}\}\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau+1)\}\} + (\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau)\}\})^2, \quad (27)$$

we can get

$$(\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau+1)\}\})^2 = (\mathbb{E}\{\mathbb{A}\{\Delta_2\bar{x}_k(\eta,\tau)\}\})^2 + 2\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau)\}\}\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau+1)\}\} - (\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau)\}\})^2. \quad (28)$$

Substituting (26) into (28), one obtains

$$(\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau+1)\}\})^2 = (\mathbb{E}\{\mathbb{A}\{\Delta_2\bar{x}_k(\eta,\tau)\}\})^2 + 2a\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau)\}\}\mathbb{E}\{\mathbb{A}\{\Delta_1^2\bar{x}_k(\eta-1,\tau)\}\} + (2g+1)(\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau)\}\})^2 + 2b(\mathbb{E}\{\mathbb{A}\{\bar{u}_k(\eta,\tau)\}\})(\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau)\}\}). \quad (29)$$

By summing up η from 1 to I on both sides of (29), $\sum_{\eta=1}^{I}(\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau+1)\}\})^2$ can be regarded as

$$\sum_{\eta=1}^{I}(\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau+1)\}\})^2 \triangleq \Re_1+\Re_2+\Re_3+\Re_4. \quad (30)$$

where

$$\Re_1 = \sum_{\eta=1}^{I}(\mathbb{E}\{\mathbb{A}\{\Delta_2\bar{x}_k(\eta,\tau)\}\})^2,$$
$$\Re_2 = \sum_{\eta=1}^{I} 2a\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau)\}\}\mathbb{E}\{\mathbb{A}\{\Delta_1^2\bar{x}_k(\eta-1,\tau)\}\},$$
$$\Re_3 = \sum_{\eta=1}^{I}(2g+1)(\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau)\}\})^2,$$
$$\Re_4 = \sum_{\eta=1}^{I} 2b(\mathbb{E}\{\mathbb{A}\{\bar{u}_k(\eta,\tau)\}\})(\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau)\}\}).$$

Then, equations (13), (16) and (12-a) are used to estimate $\Re_1$, that is

$$\Re_1 = \sum_{\eta=1}^{I}(\mathbb{E}\{\mathbb{A}\{\Delta_2\bar{x}_k(\eta,\tau)\}\})^2 \leqslant 4\sum_{\eta=1}^{I}\{[2a^2+(g-2a)^2](\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau)\}\})^2 + b^2(\mathbb{E}\{\mathbb{A}\{\bar{u}_k(\eta,\tau)\}\})^2\}. \quad (31)$$

Next, estimating $\Re_2$ by Lemma 3, we have

$$\Re_2 = \sum_{\eta=1}^{I} 2a\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau)\}\}\mathbb{E}\{\mathbb{A}\{\Delta_1^2\bar{x}_k(\eta-1,\tau)\}\} = -2a\sum_{\eta=1}^{I}(\mathbb{E}\{\mathbb{A}\{\Delta_1\bar{x}_k(\eta,\tau)\}\})^2 \leqslant 0. \quad (32)$$

Further, $\Re_4$ is estimated by Hölder inequality as follow

$$\Re_4 = 2b\sum_{\eta=1}^{I}\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau)\}\}\mathbb{E}\{\mathbb{A}\{\bar{u}_k(\eta,\tau)\}\} \leqslant |b|\sum_{\eta=1}^{I}[(\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau)\}\})^2 + (\mathbb{E}\{\mathbb{A}\{\bar{u}_k(\eta,\tau)\}\})^2]. \quad (33)$$

Substituting the estimated results of (31), (32) and (33) back into (30), one yields

$$\sum_{\eta=1}^{I}(\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau+1)\}\})^2 \leqslant [1+2g+|b|+4(g-2a)^2+8a^2]\sum_{\eta=1}^{I}(\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau)\}\})^2 + (|b|+4b^2)\sum_{\eta=1}^{I}(\mathbb{E}\{\mathbb{A}\{\bar{u}_k(\eta,\tau)\}\})^2. \quad (34)$$

Let $M_1 = 1+2g+|b|+4(g-2a)^2+8a^2$, $M_2 = |b|+4a^2$, and replacing them into (34), it follows

$$\sum_{\eta=1}^{I}(\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau+1)\}\})^2 \leqslant M_1\sum_{\eta=1}^{I}(\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau)\}\})^2 + M_2\sum_{\eta=1}^{I}(\mathbb{E}\{\mathbb{A}\{\bar{u}_k(\eta,\tau)\}\})^2. \quad (35)$$

According to Lemma 1, inequality (35) has

$$\sum_{\eta=1}^{I}(\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau)\}\})^2 \leqslant M_1^{\tau}\sum_{\eta=1}^{I}(\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,0)\}\})^2 + \sum_{t=0}^{\tau-1}M_2\sum_{\eta=1}^{I}(\mathbb{E}\{\mathbb{A}\{\bar{u}_k(\eta,\tau)\}\})^2 M_1^{(\tau-t-1)}. \quad (36)$$

Then, substituting (15) into (36) leads to

$$\|(\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\cdot,\tau)\}\})\|^2 \leqslant \sum_{t=0}^{\tau-1}M_2\,\|(\mathbb{E}\{\mathbb{A}\{\bar{u}_k(\cdot,t)\}\})\|^2\, M_1^{(\tau-t-1)}. \quad (37)$$

The proof of Lemma 4 is completed.

Theorem 1: Consider the systems (1) and the ILC scheme (10) under Assumptions (3)-(5), by the condition of the learning gain γ satisfies

$$\sup_{0\leqslant\tau\leqslant T_d}\{(1-p(\tau)\gamma cb)^2\}<\frac{1}{2}, \tag{38}$$

then, we have $\lim_{k\to\infty}\|\mathbb{E}\{e_k(\cdot,\tau)\}\|=0$. It is worth noting that the probability distribution of the trial length τ can be estimated through many past experiments in practice. Therefore, the probability $\mathbb{P}(A_\tau)$, $\tau\in\{0,1,\cdots,T_d\}$ can be regarded as known information to calculate $p(\tau)$, $\tau\in\{0,1,\cdots,T_d\}$ by equation $p(\tau)=\sum_{T_k=\tau}^{T_d}\mathbb{P}(A_{T_k})$.

Proof: The following proof about Theorem 1 can be roughly divided into the three steps. In Step 1, this paper considers using Lemma 4 to estimate $\|\mathbb{E}\{\mathbb{A}\{\bar{u}_{k+1}(\cdot,\tau)\}\}\|^2$. Contraction mapping principle and D'Alembert's principle are utilized to prove $\lim_{k\to\infty}\|\mathbb{E}\{\bar{u}_k(\cdot,\tau)\}\|^2=0$ in Step 2. In Step 3, by combining Lemma 2 and some conclusions obtained from the previous part of the certificate, we have finished the proof of Theorem 1.

Step 1: From the definition of the iteration-average operator (9), $\mathbb{A}\{\bar{u}_{k+1}(\eta,\tau)\}$ can be rewritten as

$$\mathbb{A}\{\bar{u}_{k+1}(\eta,\tau)\}=\frac{1}{k+2}[\bar{u}_{k+1}(\eta,\tau)+(k+1)\mathbb{A}\{\bar{u}_k(\eta,\tau)\}]. \tag{39}$$

In addition, it should be noted that $u_d(\eta,\tau)=\mathbb{A}\{u_d(\eta,\tau)\}$. Both sides of the learning law (9) are subtracted by $u_d(\eta,\tau)$, we can get

$$\begin{aligned}&\bar{u}_{k+1}(\eta,\tau)\\&\quad=\mathbb{A}\{u_d(\eta,\tau)-u_k(\eta,\tau)\}-\frac{k+2}{k+1}\gamma\sum_{s=0}^{k}e_s^*(\eta,\tau+1)\\&\quad=\mathbb{A}\{\bar{u}_k(\eta,\tau)\}-\frac{k+2}{k+1}\gamma\sum_{s=0}^{k}e_s^*(\eta,\tau+1).\end{aligned} \tag{40}$$

Further, substituting (40) into (39), we obtain

$$\begin{aligned}&\mathbb{A}\{\bar{u}_{k+1}(\eta,\tau)\}\\&\quad=\frac{1}{k+2}[\mathbb{A}\{\bar{u}_k(\eta,\tau)\}-\frac{k+2}{k+1}\gamma\sum_{s=0}^{k}e_s^*(\eta,\tau+1)\\&\qquad+(k+1)\mathbb{A}\{\bar{u}_k(\eta,\tau)\}]\\&\quad=\mathbb{A}\{\bar{u}_k(\eta,\tau)\}-\gamma\mathbb{A}\{\bar{e}_k^*(\eta,\tau+1)\}.\end{aligned} \tag{41}$$

Because $\mathbb{E}\{\cdot\}$ and $\mathbb{A}\{\cdot\}$ are both linear operators, the order of operations can be exchanged. Taking expectations on both sides of (41) and using $e_k^*(\eta,\tau)=\theta_k(\tau)e_k(\eta,\tau)$, it follows

$$\begin{aligned}&\mathbb{E}\{\mathbb{A}\{\bar{u}_{k+1}(\eta,\tau)\}\}\\&\quad=\mathbb{E}\{\mathbb{A}\{\bar{u}_k(\eta,\tau)\}\}-\gamma\mathbb{E}\{\mathbb{A}\{\bar{e}_k^*(\eta,\tau+1)\}\}\\&\quad=\mathbb{E}\{\mathbb{A}\{\bar{u}_k(\eta,\tau)\}\}-p(\tau+1)\gamma\mathbb{E}\{\mathbb{A}\{e_k(\eta,\tau+1)\}\}.\end{aligned} \tag{42}$$

Then, combing $e_k(\eta,\tau+1)=c\bar{x}_k(\eta,\tau+1)$ and (42), we have

$$\begin{aligned}\mathbb{E}\{\mathbb{A}\{\bar{u}_{k+1}(\eta,\tau)\}\}&=\mathbb{E}\{\mathbb{A}\{\bar{u}_k(\eta,\tau)\}\}\\&\quad-p(\tau+1)\gamma c\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau+1)\}\}.\end{aligned} \tag{43}$$

Next, substituting (25) into (43), one yields

$$\begin{aligned}&\mathbb{E}\{\mathbb{A}\{\bar{u}_{k+1}(\eta,\tau)\}\}\\&\quad=(1-p(\tau+1)\gamma cb)\mathbb{E}\{\mathbb{A}\{\bar{u}_k(\eta,\tau)\}\}\\&\qquad-p(\tau+1)\gamma ca\mathbb{E}\{\mathbb{A}\{\Delta_1^2\bar{x}_k(\eta-1,\tau)\}\}\\&\qquad-p(\tau+1)\gamma c(g+1)\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau)\}\}.\end{aligned} \tag{44}$$

Simultaneously squaring both sides of (44) and according to $(a+b)^2\leqslant 2a^2+2b^2$, we obtain

$$\begin{aligned}&(\mathbb{E}\{\mathbb{A}\{\bar{u}_{k+1}(\eta,\tau)\}\})^2\\&\quad\leqslant 2(1-p(\tau+1)\gamma cb)^2(\mathbb{E}\{\mathbb{A}\{\bar{u}_k(\eta,\tau)\}\})^2\\&\qquad+4(p(\tau+1)\gamma ca)^2(\mathbb{E}\{\mathbb{A}\{\Delta_1^2\bar{x}_k(\eta-1,\tau)\}\})^2\\&\qquad+4(p(\tau+1)\gamma c(g+1))^2(\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau)\}\})^2.\end{aligned} \tag{45}$$

Further, summing up η from 1 to I on both sides of the (45), we can get

$$\begin{aligned}&\sum_{\eta=1}^{I}(\mathbb{E}\{\mathbb{A}\{\bar{u}_{k+1}(\eta,\tau)\}\})^2\\&\quad\leqslant 2(1-p(\tau+1)\gamma cb)^2\sum_{\tau=1}^{I}(\mathbb{E}\{\mathbb{A}\{\bar{u}_k(\eta,\tau)\}\})^2\\&\qquad+4(p(\tau+1)\gamma ca)^2\sum_{\tau=1}^{I}(\mathbb{E}\{\mathbb{A}\{\Delta_1^2\bar{x}_k(\eta-1,\tau)\}\})^2\\&\qquad+4(p(\tau+1)\gamma c(g+1))^2\sum_{\tau=1}^{I}(\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau)\}\})^2.\end{aligned} \tag{46}$$

Let $\sup_{0\leqslant\tau\leqslant T_d}\{1-p(\tau+1)\gamma cb\}=\rho_1$, $\sup_{0\leqslant\tau\leqslant T_d}\{p(\tau+1)\gamma ca\}=\rho_2$ and $\sup_{0\leqslant\tau\leqslant T_d}\{p(\tau+1)\gamma c(g+1)\}=\rho_3$, there is

$$\begin{aligned}&\sum_{i=1}^{I}(\mathbb{E}\{\mathbb{A}\{\bar{u}_{k+1}(\eta,\tau)\}\})^2\\&\quad\leqslant 2\rho_1^2\sum_{\eta=1}^{I}(\mathbb{E}\{\mathbb{A}\{\bar{u}_k(\eta,\tau)\}\})^2\\&\qquad+4\rho_2^2\sum_{\eta=1}^{I}(\mathbb{E}\{\mathbb{A}\{\Delta_1^2\bar{x}_k(\eta-1,\tau)\}\})^2\\&\qquad+4\rho_3^2\sum_{\eta=1}^{I}(\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau)\}\})^2.\end{aligned} \tag{47}$$

According to the equations (14) and (16), we can rewrite $\sum_{\eta=1}^{I}(\mathbb{E}\{\mathbb{A}\{\Delta_1^2\bar{x}_k(\eta-1,\tau)\}\})^2$ as follows

$$\begin{aligned}&\sum_{\eta=1}^{I}(\mathbb{E}\{\mathbb{A}\{\Delta_1^2\bar{x}_k(\eta-1,\tau)\}\})^2\\&\quad\leqslant 3\sum_{\eta=1}^{I}[(\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta+1,\tau)\}\})^2+4(\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau)\}\})^2\\&\qquad+(\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta-1,\tau)\}\})^2]\\&\quad\leqslant 18\sum_{\eta=1}^{I}(\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\eta,\tau)\}\})^2.\end{aligned} \tag{48}$$

Combining (47) and (48), we obtain

$$\begin{aligned}&\|(\mathbb{E}\{\mathbb{A}\{\bar{u}_{k+1}(\cdot,\tau)\}\})\|^2\\&\quad\leqslant 2\rho_1^2\ \|(\mathbb{E}\{\mathbb{A}\{\bar{u}_k(\cdot,\tau)\}\})\|^2\\&\qquad+(72\rho_2^2+4\rho_3^2)\ \|(\mathbb{E}\{\mathbb{A}\{\bar{x}_k(\cdot,\tau)\}\})\|^2 .\end{aligned}\tag{49}$$

An application of Lemma 4 to (49), it follows

$$\begin{aligned}&\|\mathbb{E}\{\mathbb{A}\{\bar{u}_{k+1}(\cdot,\tau)\}\}\|^2\\&\quad\leqslant 2\rho_1^2\ \|\mathbb{E}\{\mathbb{A}\{\bar{u}_k(\cdot,\tau)\}\}\|^2+(72\rho_2^2\\&\qquad+4\rho_3^2)\sum_{t=0}^{\tau-1}M_2\ \|\mathbb{E}\{\mathbb{A}\{\bar{u}_k(\cdot,t)\}\}\|^2\ M_1^{(\tau-t-1)}.\end{aligned}\tag{50}$$

Then, multiplying on both sides of (50) by $\lambda^\tau(0<\lambda<1)$, according to the definition of (L^2,λ) norm, there is

$$\begin{aligned}&\|\mathbb{E}\{\mathbb{A}\{\bar{u}_{k+1}(\cdot,\tau)\}\}\|^2\ \lambda^\tau\\&\quad\leqslant 2\rho_1^2\ \|\mathbb{E}\{\mathbb{A}\{\bar{u}_k\}\}\|^2_{(L^2,\lambda)}+(72\rho_2^2\\&\qquad+4\rho_3^2)\sum_{t=0}^{\tau-1}M_2\ \|\mathbb{E}\{\mathbb{A}\{\bar{u}_k\}\}\|^2_{(L^2,\lambda)}\ (\lambda M_1)^{(\tau-t-1)}\lambda\\&\quad\leqslant(2\rho_1^2+(72\rho_2^2\\&\qquad+4\rho_3^2)M_2\frac{\lambda}{1-\lambda M_1})\ \|\mathbb{E}\{\mathbb{A}\{\bar{u}_k\}\}\|^2_{(L^2,\lambda)} .\end{aligned}\tag{51}$$

Let $\rho_0=2\rho_1^2+(72\rho_2^2+4\rho_3^2)M_2\frac{\lambda}{1-\lambda M_1}$ which can be replaced into (51), we have

$$\|\mathbb{E}\{\mathbb{A}\{\bar{u}_{k+1}\}\}\|^2_{(L^2,\lambda)}\leqslant\rho_0\ \|\mathbb{E}\{\mathbb{A}\{\bar{u}_k\}\}\|^2_{(L^2,\lambda)} .\tag{52}$$

Step 2: When λ is sufficiently small, from (52) we have $0<\rho_0<1$. Then, in views of contraction mapping principle, we obtain

$$\lim_{k\to\infty}\|\mathbb{E}\{\mathbb{A}\{\bar{u}_k\}\}\|^2_{(L^2,\lambda)}=0.\tag{53}$$

Multiplying on both sides of (52) by $(k+2)^2$, yields

$$\|\mathbb{E}\{\sum_{s=0}^{k+1}\bar{u}_s\}\|^2_{(L^2,\lambda)}\leqslant\rho_0(\frac{k+2}{k+1})^2\ \|\mathbb{E}\{\sum_{s=0}^{k}\bar{u}_s\}\|^2_{(L^2,\lambda)} .\tag{54}$$

In addition, let both sides of inequality (54) be divided by $\|\mathbb{E}\{\sum_{s=0}^k\bar{u}_s\}\|^2_{(L^2,\lambda)}$, we can get

$$\frac{\xi_{k+1}}{\xi_k}\leqslant\rho_0(\frac{k+2}{k+1})^2,\tag{55}$$

where $0<\rho_0<1$, $\xi_{k+1}=\|\mathbb{E}\{\sum_{s=0}^{k+1}\bar{u}_s\}\|^2_{(L^2,\lambda)}$ and $\xi_k=\|\mathbb{E}\{\sum_{s=0}^{k}\bar{u}_s\}\|^2_{(L^2,\lambda)}$. According to D'Alembert's principle and inequality (55), we can obtain $\lim_{k\to\infty}\xi_k=0$, that is

$$\lim_{k\to\infty}\|\mathbb{E}\{\sum_{s=0}^{k}\bar{u}_s\}\|^2_{(L^2,\lambda)}=0.\tag{56}$$

Moreover

$$\begin{aligned}&\|\mathbb{E}\{\sum_{s=0}^{k}\bar{u}_s(\cdot,\tau)\}\|^2=\lambda^{-\tau}\lambda^\tau\ \|\mathbb{E}\{\sum_{s=0}^{k}\bar{u}_s(\cdot,\tau)\}\|^2\\&\quad\leqslant\sup_{0\leqslant\tau\leqslant T_d}\{\|\mathbb{E}\{\sum_{s=0}^{k}\bar{u}_s(\cdot,\tau)\}\|^2\ \lambda^\tau\}\lambda^{-T_d}\\&\quad=\lambda^{-T_d}\ \|\mathbb{E}\{\sum_{s=0}^{k}\bar{u}_s\}\|^2_{(L^2,\lambda)} .\end{aligned}\tag{57}$$

Combing (56) and (57), we have

$$\lim_{k\to\infty}\|\mathbb{E}\{\sum_{s=0}^{k}\bar{u}_s(\cdot,\tau)\}\|^2=0.\tag{58}$$

Consider the definition of L^2 norm, we can rewrite (58) as follow

$$\lim_{k\to\infty}\sum_{\eta=1}^{I}(\mathbb{E}\{\sum_{s=0}^{k}\bar{u}_s(\eta,\tau)\})^2=0.\tag{59}$$

Based on (59), we express $\lim_{k\to\infty}\sum_{\eta=1}^{I}(\mathbb{E}\{\bar{u}_k(\eta,\tau)\})^2$ as

$$\begin{aligned}&\lim_{k\to\infty}\sum_{\eta=1}^{I}(\mathbb{E}\{\bar{u}_k(\eta,\tau)\})^2\\&\quad=\lim_{k\to\infty}\sum_{\eta=1}^{I}(\mathbb{E}\{\sum_{s=0}^{k}\bar{u}_s(\eta,\tau)\}-\mathbb{E}\{\sum_{s=0}^{k-1}\bar{u}_s(\eta,\tau)\})^2\\&\quad\leqslant\lim_{k\to\infty}\sum_{\eta=1}^{I}[2(\mathbb{E}\{\sum_{s=0}^{k}\bar{u}_s(\eta,\tau)\})^2+2(\mathbb{E}\{\sum_{s=0}^{k-1}\bar{u}_s(\eta,\tau)\})^2]\\&\quad=0.\end{aligned}\tag{60}$$

From (60), we can get

$$\lim_{k\to\infty}\|\mathbb{E}\{\bar{u}_k(\cdot,\tau)\}\|^2=0.\tag{61}$$

Step 3: Next, we consider proving $\lim_{k\to\infty}\|\mathbb{E}\{\bar{e}_k(\cdot,\tau)\}\|^2=0$ by using (61). Firstly, multiplying on both sides of (37) by λ^τ, it follows that

$$\frac{1-\lambda M_1}{M_2}\ \|\mathbb{E}\{\mathbb{A}\{\bar{x}_k\}\}\|^2_{(L^2,\lambda)}\leqslant\lambda\ \|\mathbb{E}\{\mathbb{A}\{\bar{u}_k\}\}\|^2_{(L^2,\lambda)} .\tag{62}$$

Then, from (62) we have

$$\frac{1-\lambda M_1}{M_2}\ \|\mathbb{E}\{\mathbb{A}\{\bar{x}_{k+1}\}\}\|^2_{(L^2,\lambda)}\leqslant\lambda\ \|\mathbb{E}\{\mathbb{A}\{\bar{u}_{k+1}\}\}\|^2_{(L^2,\lambda)} .\tag{63}$$

By multiplying on both sides of (49) by λ^τ, we have

$$\begin{aligned}&\|\mathbb{E}\{\mathbb{A}\{\bar{u}_{k+1}\}\}\|^2_{(L^2,\lambda)}\\&\quad\leqslant 2\rho_1^2\ \|\mathbb{E}\{\mathbb{A}\{\bar{u}_k\}\}\|^2_{(L^2,\lambda)}\\&\qquad+(72\rho_2^2+4\rho_3^2)\ \|\mathbb{E}\{\mathbb{A}\{\bar{x}_k\}\}\|^2_{(L^2,\lambda)} .\end{aligned}\tag{64}$$

In addition, combing (63) and (64) leads to

$$\begin{aligned}&\|\mathbb{E}\{\mathbb{A}\{\bar{x}_{k+1}\}\}\|^2_{(L^2,\lambda)}\\&\quad\leqslant\frac{(72\rho_2^2+4\rho_3^2)\lambda M_2}{1-\lambda M_1}\ \|\mathbb{E}\{\mathbb{A}\{\bar{x}_k\}\}\|^2_{(L^2,\lambda)}\\&\qquad+\frac{2\rho_1^2\lambda M_2}{1-\lambda M_1}\ \|\mathbb{E}\{\mathbb{A}\{\bar{u}_k\}\}\|^2_{(L^2,\lambda)} .\end{aligned}\tag{65}$$

By denoting $\frac{\lambda M_2(72\rho_2^2+4\rho_3^2)}{1-\lambda M_1} = \rho_4$, $\frac{2\rho_1^2\lambda M_2}{1-\lambda M_1} = \rho_5$ respectively, we can express (65) as

$$\| \mathbb{E}\{\mathbb{A}\{\bar{x}_{k+1}\}\|^2_{(L^2,\lambda)} \leqslant \rho_4 \| \mathbb{E}\{\mathbb{A}\{\bar{x}_k\}\} \|^2_{(L^2,\lambda)} + \rho_5 \| \mathbb{E}\{\mathbb{A}\{\bar{u}_k\}\} \|^2_{(L^2,\lambda)} . \quad (66)$$

It is not hard to find $0 < \rho_4 < 1$, $0 < \rho_5 < 1$ when λ is small enough. Since $\lim_{k\to\infty} \| \mathbb{E}\{\mathbb{A}\{\bar{u}_k\}\} \|^2_{(L^2,\lambda)} = 0$, combing (66) with Lemma 2, we can obtain

$$\lim_{k\to\infty} \| \mathbb{E}\{\mathbb{A}\{\bar{x}_k\}\} \|^2_{(L^2,\lambda)} = 0. \quad (67)$$

According to $e_k(\eta,\tau) = c\bar{x}_k(\eta,\tau)$, multiplying on both sides of (66) by c^2, there is

$$\| \mathbb{E}\{\mathbb{A}\{\bar{e}_{k+1}\}\} \|^2_{(L^2,\lambda)} \leqslant \rho_4 \| \mathbb{E}\{\mathbb{A}\{e_k\}\} \|^2_{(L^2,\lambda)} + \rho_6 \| \mathbb{E}\{\mathbb{A}\{\bar{u}_k\}\} \|^2_{(L^2,\lambda)}, \quad (68)$$

where $\rho_6 = c^2\rho_5$.

Then, combining (53) and (68), we have

$$\lim_{k\to\infty} \| \mathbb{E}\{\mathbb{A}\{e_k\}\} \|^2_{(L^2,\lambda)} = 0. \quad (69)$$

Multiplying on both sides of (68) by $(k+2)^2$, there is

$$\begin{aligned} &\| \mathbb{E}\{\sum_{s=0}^{k+1} e_s\} \|^2_{(L^2,\lambda)} \\ &\leqslant 2\rho_4 \| \mathbb{E}\{\sum_{s=0}^{k} e_s\} \|^2_{(L^2,\lambda)} + 2\rho_4 \| \mathbb{E}\{A\{e_k\}\} \|^2_{(L^2,\lambda)} \\ &\quad + 2\rho_6 \| \mathbb{E}\{\sum_{s=0}^{k} \bar{u}_s\} \|^2_{(L^2,\lambda)} + 2\rho_6 \| \mathbb{E}\{A\{\bar{u}_k\}\} \|^2_{(L^2,\lambda)} . \end{aligned} \quad (70)$$

Based on some conclusions obtained from the previous part of the certificate such as (53), (56) and (69), when λ is chosen to be small enough and k is close to infinity, we can directly derive

$$\lim_{k\to\infty} \| \mathbb{E}\{\sum_{s=0}^{k} e_s(\cdot,\tau)\} \|^2 = 0. \quad (71)$$

Consider the definition of L^2 norm, (71) can be rewritten as

$$\lim_{k\to\infty} \sum_{\eta=1}^{I} (\mathbb{E}\{\sum_{s=0}^{k} e_s(\eta,\tau)\})^2 = 0. \quad (72)$$

Next, by (72), we have

$$\begin{aligned} &\lim_{k\to\infty} \sum_{\eta=1}^{I} (\mathbb{E}\{e_k(\eta,\tau)\})^2 \\ &= \lim_{k\to\infty} \sum_{\eta=1}^{I} (\mathbb{E}\{\sum_{s=0}^{k} e_s(\eta,\tau)\} - \mathbb{E}\{\sum_{s=0}^{k-1} e_s(\eta,\tau)\})^2 \\ &\leqslant \lim_{k\to\infty} \sum_{\eta=1}^{I} [2(\mathbb{E}\{\sum_{s=0}^{k} e_s(\eta,\tau)\})^2 \\ &\quad + 2(\mathbb{E}\{\sum_{s=0}^{k-1} e_s(\eta,\tau)\})^2] = 0. \end{aligned} \quad (73)$$

From (73) we can get

$$\lim_{k\to\infty} \| \mathbb{E}\{\bar{e}_k(\cdot,\tau)\} \|^2 = 0. \quad (74)$$

The proof of Theorem 1 is finished.

IV. NUMERICAL SIMULATIONS

Consider the following discrete parabolic distributed parameter systems

$$\begin{cases} \Delta_2 x_k(\eta,\tau) = 0.31\Delta_1^2 x_k(\eta-1,\tau) - 0.34 x_k(\eta,\tau) \\ \qquad + 0.9 u_k(\eta,\tau), \\ y_k(\eta,\tau) = 0.91 x_k(\eta,\tau), \end{cases} \quad (75)$$

where $(\eta,\tau) \in [1, 10] \times [0, 80]$.

The conditions of initial value and boundary value in the system (73) are set as follows

$$x_k(\eta,0) = x_d(\eta,0) = \phi(\eta), \quad 1 \leqslant \eta \leqslant I.\ k = 1, 2\cdots, \quad (76)$$

$$x_k(0,\tau) = 0 = x_k(I+1,\tau), \quad 0 \le \tau \le T_d, \quad (77)$$

where $\varphi(\eta)$ is bounded function for $\eta \in [1, I]$.

The desired tracking trajectory is given as

$$y_d(\eta,\tau) = 0.8\sin(\tau)\sin(\frac{\eta-1}{50}\pi)\sin((2(I+1-\eta)). \quad (78)$$

It should be remarked that in the simulation, the minimum iteration length is chosen as $T_m = 75$, the full iteration length is assumed to be $T_d = 80$, and the actual iteration time T_k follows a uniform probability distribution at 76 to 85. This means the probability of the actual iteration length $\mathbb{P}(A_\tau) = 1/10$, $\tau \in [76, 85]$, then, according to the definition of the probability $p(\tau)$ occurring at a specific moment in this paper, we can express it as

$$p(\tau) = \begin{cases} 1, & \tau \in \{0, 1, \cdots, 75\}, \\ \frac{86-\tau}{10}, & \tau \in \{76, 77, \cdots, 80\}. \end{cases} \quad (79)$$

The distributed learning algorithm is designed as follow

$$u_{k+1}(\eta,\tau) = \mathbb{A}\{u_k(\eta,\tau)\} + \frac{k+2}{k+1}\gamma \sum_{s=0}^{k} e_s^*(\eta,\tau+1), \quad (80)$$

where the learning gain $\gamma = 0.8$, which renders to $\sup_{0\leqslant\tau\leqslant T_d} \{(1-p(\tau)\gamma cb)^2\} < \frac{1}{2}$, the condition of Theorem 1 is satisfied.

From Fig.1∼Fig.7, it is showed that the system errors under mathematical expectation decay with the iteration axis. Because the trial lengths vary randomly in the iteration process, unlike the desired tracking surface Fig.1 with full trial length, there is a lack of tracking information in part-time length in the 80th output surface Fig.7. When $k = 80$ in Fig.5, the maximum absolute value of output error under mathematical expectation decreases to 4.32×10^{-3}. Since the iteration length varies randomly from 76 to 85, which means that the system errors at these moments are not all measurable during each iteration. However, at period form 0 to 75, which is not affected by the random factor, useful learning information

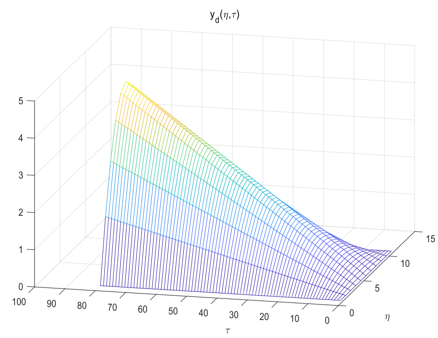

FIGURE 1. Desired output surface $y_d(\eta, \tau)$.

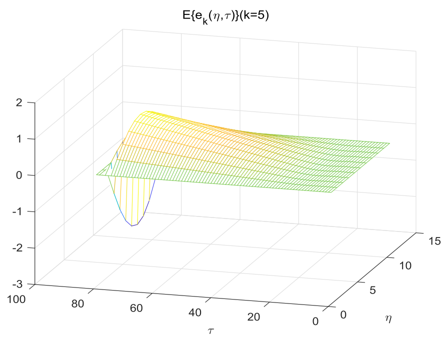

FIGURE 2. The expectation of error surface $E\{e_k(\eta, \tau)\}(k = 5)$.

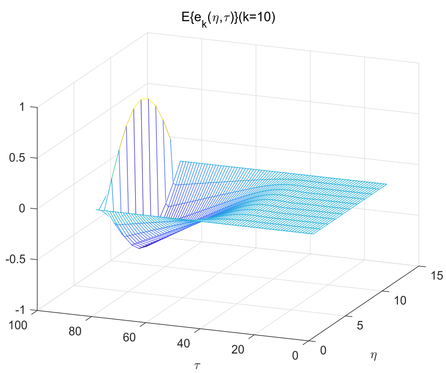

FIGURE 3. The expectation of error surface $E\{e_k(\eta, \tau)\}(k = 10)$.

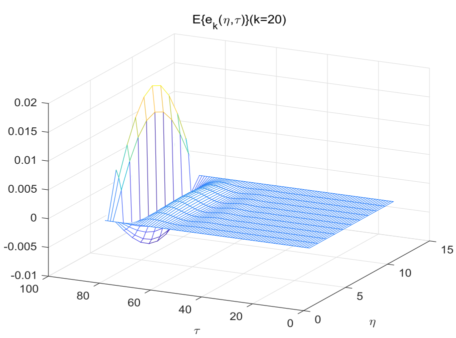

FIGURE 4. The expectation of error surface $E\{e_k(\eta, \tau)\}(k = 20)$.

can be obtained and continuously used in updating the system inputs during each iteration. Therefore, the convergence effects of system errors under mathematical expectation are quite different in these two periods (see Fig.4∼Fig.6).

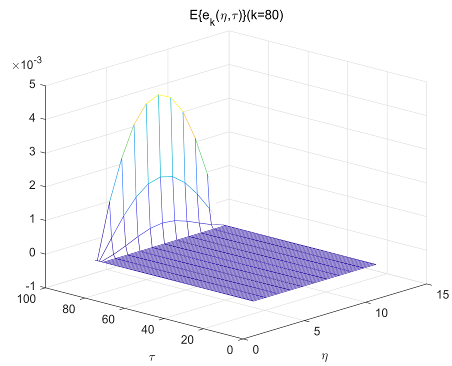

FIGURE 5. The expectation of error surface $E\{e_k(\eta, \tau)\}(k = 80)$.

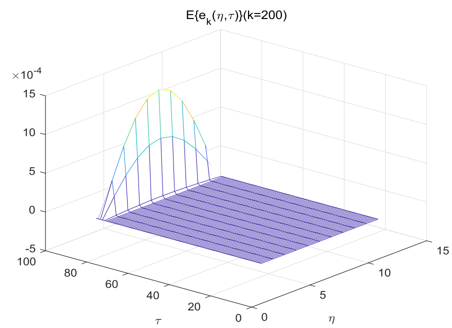

FIGURE 6. The expectation of error surface $E\{e_k(\eta, \tau)\}(k = 200)$.

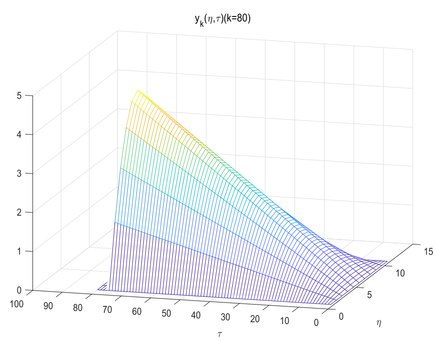

FIGURE 7. Output surface $y_k(\eta, \tau)(k = 80)$.

Consider the situation that trial lengths are fixed, based on the system (75) and learning algorithm (80), we give the maximum tracking error curve Fig.8. It should be noted that in this case, the probability of trial lengths $p(\tau) = 1, \tau \in \{0, 1, \cdots, T_d\}$ and the random variable $\theta_k(\tau) = 1, \tau \in \{0, 1, \cdots, T_d\}$ are always established. In this paper, we can only get the surfaces of system error under mathematical expectation Fig.2∼Fig.6 in the situation that the trial lengths change randomly. However, in the situation of fixed time lengths, we can obtain the simulation result of maximum tracking error curve Fig.8. In addition, we can see that under the condition of the fixed trial lengths, the convergence rate of system error is faster than that when the batch lengths vary randomly. This is because that the information of system errors from 0 to T_d can be utilized to improve the system inputs during each iteration when the trial lengths are fixed.

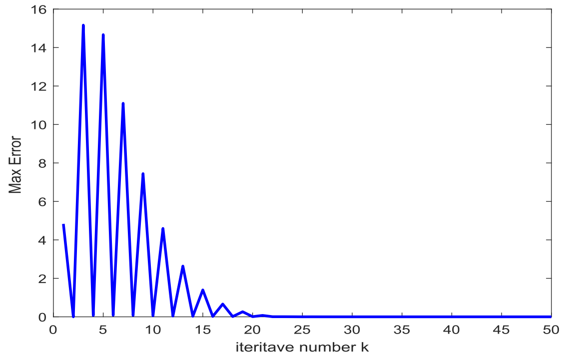

FIGURE 8. Maximum tracking error curve under the fixed trial lengths.

Remark 2: In [26], in the case of non-uniform trial lengths, only the convergence of the tracking error under mathematical expectation is proved. Similarly, in this paper, we also get only the convergence results of system error under mathematical expectation in the situation that trial lengths vary randomly. Therefore, we consider that it is reasonable to give the simulation surfaces of system error under mathematical expectation to illustrate the effectiveness of the distributed learning algorithm. In addition, for the simulation results of the tracking error under the mathematical expectation in this paper, since the systems we use are DPSs that contain time and space variables, the simulation results are different from the simulation curves obtained in [26], but presented in the form of surfaces (see Fig.2∼Fig.6).

V. CONCLUSION

In this paper, we consider applying ILC to discrete spatial-temporal parabolic distributed parameter systems consisting of partial difference equations, where the trial lengths are non-uniform. The iteration-average operator is used to design the distributed ILC algorithm. By rigorous theoretical analysis, the convergence of the output error under mathematical expectation is proved and the effectiveness of the algorithm is illustrated by numerical simulation.

VI. REFERENCE EXAMPLES

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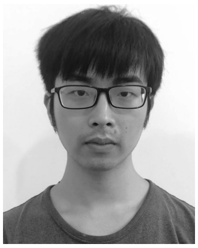

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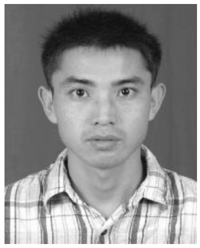

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