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Algebraic Expression and Construction of Control Sets of Graphs Using Semi-Tensor Product of Matrices

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ABSTRACT Using a new matrix analysis tool, called semi-tensor product of matrices (STP) developed in recent years, this paper investigates the problem of finding control sets (or dominating sets) of graphs mathematically. By defining characteristic vectors for vertex subsets of graphs, three sufficient and necessary conditions of control sets are proposed, based on which an algebraic algorithm that can find all the control sets of a graph is established. Further, the concepts of k-capacity and k-harmony control sets of graphs are proposed, which can model some real-world problems such as security monitoring of streets and wireless sensor networks. Several sufficient and necessary conditions are established to judge whether or not a vertex subset is a k-capacity control set; a necessity that a vertex subset is a k-harmony control set is proposed, which facilitates finding all the k-harmony control sets of graphs. The correctness and effectiveness of the results is finally examined in detail by examples. The approach of this paper may provide a new angle and means to understand and analyze the structure of graphs.

INDEX TERMS Algebraic approach, finite-valued systems, graphs, logical dynamical systems, logical systems, semi-tensor product of matrices.

I. INTRODUCTION

Graphs serve as mathematical models to analyze successfully many concrete real-world problems. Many problems in physics, chemistry, communications science, computer technology, genetics, and sociology can also be formulated as graphs [1]. Also many branches of mathematics, such as group theory, matrix theory, probability, and topology, have interactions with graph theory [2]. In fact, graph theory provides mathematical models for the systems comprised of binary relations. The rapid development of information and digital techniques has promoted the label theory of graphs, including labelling, controlling and coloring, to one of the fastest developing branches of graphs [3], [4].

A control set of a graph is a vertex subset that every vertex not in the subset is adjacent to (or controlled by) at least one vertex in the subset. Such a control set can model many actual problems such as computer communication network, making personnel assignment, constructing school bus route, locating radio station, and social network theory. A key issue in the control theory of graphs is finding the minimum control sets. This problem is NP-complete for arbitrary graphs. Existing results strongly suggest that the problems of determining the control number (i.e. the cardinality of a minimum control set) and the values of related parameters for most classes of graphs are inherently exponential in nature.

In recent years, many outstanding contributions have been made in the field of control sets of graphs from different perspectives and in different ways such as linear separation of connected dominating sets [5], description of graphs using dominating sets [6], construction of dominating sets [7], introduction of new types of dominating sets [8] [9], approaches to find minimum control sets of graphs [10]–[12] and searching algorithms [13]–[15].

We carried out the study of this paper mainly because of the following two reasons. In the aspect of theory of graphs, most of existing results lay emphasis on, as described as above, the values of some parameters such as control number.

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The works of designing algorithms only pay attention to the algorithm complexity not to how to formulate and find control sets of graphs mathematically. Another reason is the emergence of a new matrix analysis tool, semi-tensor product of matrices (STP), which was proposed by Cheng *et al.* [16]. STP generalizes the conventional product of matrices and retains all the major properties of the latter. Therefore, theoretically, STP can find its applications in great majority of science and engineering fields, especially in those can be modelled as discrete mathematical models. STP is well established and has been successfully applied in many areas such as neurosciences and neurology [17]–[19], automation control systems [20]–[22], systems science [23]–[25], computer science [26]–[28], and theory of graphs [29]–[34].

The STP was first applied to the field of graphs by Wang *et al.* [35]. He investigated the problems of the maximum weight stable set and vertex coloring, and presented several new results and algorithms, which can be used to study some problems of multiple agent systems such as the group consensus. The advantage of this method lies that some problems of graphs can be exactly described as an algebraic form of matrices. This method can express graph problems in a clear way and is helpful for further study of graph problems. Xu *et al.* [29], Xu and Wang [30], and Xu *et al.* [31] studied the coloring problem of graphs and its applications to timetabling and frequency assignment. In [29] the robust graph coloring problem with application to a kind of examination timetabling is investigated. Several new results and algorithms are presented by expressing the coloring problem into a kind of optimization problem; a practicable timetabling scheme is designed. In [30] the conflict-free coloring problem is considered and the results are applied to the problem of frequency assignment; several new methods are proposed for the problem. Fuzzy graph coloring is investigated in [31], a number of novel results are presented. Zhong *et al.* [32] investigated the externally stable sets, minimum stable sets, absolutely minimum externally stable set and cores of graphs. A number of new results and algorithms are presented including the algebraic descriptions and necessary and sufficient conditions of these subsets and algebraic algorithms of finding all of the structures for given graphs. The illustrative examples show the effective of the results. In [33] Meng et al. considered the transversal and covering problems of hypergraphs by STP. A necessary and sufficient criterion is established for hypergraph transversals, and an algorithm of finding the minimum transversal is designed. Further, the covering problem is reduced to be transversal problem and an algebraic algorithm is proposed for the covering problem. Yue *et al.* [34] considered the solvability conditions of k-track assignment problem in a mathematic manner via establishing a necessary and sufficient condition of k-internally stable sets of graphs by STP. Two sufficient and necessary conditions of the solvability of the k-track assignment problem are proposed. The results are quite different from existing methods and may provide a new angle and means to understand the inner mathematics of the k-track assignment problem.

Motivated by the above, one of the aims of this paper is to develop an algorithm to find all the control sets of graphs in a mathematical manner. To do this, we define a characteristic vector for a vertex subset, by which three sufficient and necessary conditions of control sets are obtained. Based on these conditions an algebraic algorithm is established to find all the control sets of graphs. Another aim is, inspired by some problems in real-world, to propose several new concepts, k-capacity control set and k-harmony control set, and further to establish algorithms to find these control sets of graphs mathematically.

The main contributions of our work are as follows. We established a new mathematical formulation for graph control sets and presented a set of new theoretical results and algorithms, which supports for the actual applications of graphs. One of the advantages of our method lies in the exact algebraic expressions of control sets, k-capacity control sets and k-harmony control sets. This is quite different from the existing approaches. Another advantage is the convenience of solving problems, to find all the control sets of a graph, one only needs to calculate a series of semi-tensor product of matrices, with which the control sets are obtained.

The remainder of the paper is organized as follows. Section 2 gives some preliminaries on STP and the problem statement. Section 3 presents the main results of this paper, including several necessary and sufficient conditions of control sets, k-capacity control sets, and k-harmony control sets, and an algorithm to find these control sets. Section 4 examines the correctness of the obtained results by a testing example; this is followed by a conclusion in Section 5.

II. PRELIMINARIES AND PROBLEM STATEMENT

In this section we first provide some necessary preliminaries on STP and graphs, and then the research problem of this paper is briefly narrated.

A. PRELIMINARIES ON STP AND GRAPHS

Definition 1: For $M \in M_{m\times n}$ and $N \in M_{p\times q}$, their STP, denoted by MN, is defined as [16]:

$$M \ltimes N := (M \otimes I_{s/n})(N \otimes I_{s/p}),$$

where s is the least common multiple of n and p, and $\otimes$ is the Kronecker product.

Remark 1: In many cases we use the following definition, which is a special case of Definition 1.

1. Let X be a row vector of dimension np, and Y be a column vector with dimension p. Split X into p equal-size blocks as $X_1, \cdots, X_p$, which are $1\times n$ row vectors. The semi-tensor product, denoted by, is defined as

$$\begin{cases} X \ltimes Y = \sum_{i=1}^{p} X_i y_i \in R^n, \\ Y^T \ltimes X^T = \sum_{i=1}^{p} y_i (X_i)^T \in R^n. \end{cases}$$

where $y_i \in R$ is the i-th component of Y.

2. Let $A \in R_{m\times n}$ and $B \in R_{p\times q}$. If either n is a factor of p or p is a factor of n, then the semi-tensor product of A and B,

denoted by $C = \{C_{ij}\} = A \ltimes B$, is defined as follows: C consists of $m \times q$ blocks and each block is defined as

$$C_{ij} = A^i \ltimes B_j, \quad i = 1, 2, \cdots, m; \quad j = 1, 2, \cdots, q,$$

where A^i is the i-th row of A and $B_j$is the j-th column of B.

The following is an illustration example.

1) let $X = [1, -2, 1, -3]$, $Y = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, then

$$X \ltimes Y = [1, -2] \cdot 2 + [1, -3] \cdot (-1) = [1, -1].$$

2) let $A = \begin{bmatrix} 2 & 2 & 4 & -1 \\ 4 & -1 & 0 & 2 \\ -1 & -2 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -3 & -2 \\ 2 & 1 \end{bmatrix}$, we have

$$A \ltimes B = \begin{bmatrix} [2, 2, 4, -1]\begin{bmatrix} -3 \\ 2 \end{bmatrix} & [2, 2, 4, -1]\begin{bmatrix} -2 \\ 1 \end{bmatrix} \\ [4, -1, 0, 2]\begin{bmatrix} -3 \\ 2 \end{bmatrix} & [4, -1, 0, 2]\begin{bmatrix} -2 \\ 1 \end{bmatrix} \\ [-1, 2, 1, 1]\begin{bmatrix} -3 \\ 2 \end{bmatrix} & [-1, 2, 1, 1]\begin{bmatrix} -2 \\ 1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & 0 & 1 \\ -12 & 2 & -8 & 4 \\ 5 & -4 & 3 & -3 \end{bmatrix}.$$

Remark 2: STP is a generalization of the conventional matrix product, when $n = p$, it reduces to the latter. Not only do almost all the major properties of the conventional matrix product remain true for STP, for instance, the associative law is that for $A \in \mathbf{M}_{m \times n}$, $B \in \mathbf{M}_{p \times q}$, and $C \in \mathbf{M}_{r \times s}$, we have $(A \ltimes B) \ltimes C = A \ltimes (B \ltimes C)$, but STP can overcome some defects of the conventional matrix product; the following is an example.

Let us look at an interesting phenomenon of the conventional matrix product, i.e., "legal operations" leads to "illegal terms". For example, let $X, Y, Z, W \in \mathbf{R}^n$ be column vectors. Since $Y^T Z$ is a real number, we have

$$(XY^T)(ZW^T) = X(Y^T Z)W^T = (Y^T Z)(XW^T) \in \mathbf{M}_{n \times n}.$$

Continuing application of the associativity will results in $(Y^T Z)(XW^T) = Y^T (ZX)W^T$. Now there is a question "what is the term ZX?". Obviously, it is an illegal term. This shows that the conventional matrix product is not complete "compatible". While in the context of the STP, such abnormal phenomenon turns to be natural and reasonable. For detailed descriptions, please refer to [16].

Definition 2: A swap matrix $W_{[m,n]}$ is an $mn \times mn$ matrix, which is defined as follows. Its rows and columns are labelled by double index (i, j), the rows are arranged by the ordered multi-index $\mathrm{Id}[j, i; n, m]$, and the columns are arranged by the ordered multi-index $\mathrm{Id}[i, j; m, n]$. Then the element at the position $[(I, J), (i, j)]$ is [16]:

$$W_{((I,J),(i,j))} = \delta_{i,j}^{I,J} = \begin{cases} 1, & I = i \text{ and } J = j, \\ 0, & \text{otherwise}. \end{cases}$$

From Definition 2, it is easy to see that [16]

$$\begin{aligned} W_{[m,n]} \ltimes X \ltimes Y &= Y \ltimes X, \\ W_{[n,m]} \ltimes Y \ltimes X &= X \ltimes Y, \end{aligned} \tag{1}$$

for any $X \in \mathbf{R}^m$ and $Y \in \mathbf{R}^n$.

Definition 3: The dummy operators, $E_{d\_latter}(m, n)$ and $E_{d\_former}(m, n)$, are defined as

$$E_{d\_latter}(m, n) = [\underbrace{I_n, \cdots, I_n}_{m}],$$
$$E_{d\_former}(m, n) = [\underbrace{I_m, \cdots, I_m}_{n}].$$

Assume that the vector form of a k-valued logical variable is $\delta_k^i, (1 \leq i \leq k)$. Then the dummy operators have the following property.

$$\begin{aligned} E_{d\_latter}(m, n) \ltimes u \ltimes v &= v, \\ E_{d\_former}(m, n) \ltimes W_{[n,m]} \ltimes u \ltimes v &= u. \end{aligned} \tag{2}$$

where $u \in \Delta_m$and $v \in \Delta_n$.

The following are some notations used throughout this paper.

$\mathbf{k}_n \in \mathbf{R}^n$ is a column vector with each element being k, $k \in \mathbf{R}$.

$\mathrm{col}_i(A)$ is the i-th column of matrix A.

δ_n^i is the i-th column of the identity matrix $\mathbf{I}_n \in \mathbf{M}_{n \times n}$, denote $\Delta_n := \{\delta_n^1, \cdots, \delta_n^n\}$.

A matrix $W = [\delta_n^{i_1}, \cdots, \delta_n^{i_k}]$ is denoted as $W = \delta_n[i_1, \cdots, i_k]$ for compactness.

For $X = [x_i] \in \mathbf{R}^n$ and $Y = [y_i] \in \mathbf{R}^n$, $X \geq Y$ means $x_i \geq y_i$, $i = 1, 2, \cdots, n$.

A graph G consists of a finite nonempty set V of objects called vertices and a set E of 2-element subsets of V called edges. The sets V and E are the vertex set and edge set of G, respectively. So a graph G is an ordered pair of two sets V and E, denoted by $G = (V, E)$. At times, it is useful to write $V(G)$ and $E(G)$ rather than V and E to emphasize that these are the vertex and edge sets of a particular graph G. For a graph $G = (V, E)$ with $V = \{v_1, v_2, \cdots, v_n\}$, if $e_{ij} = (v_i, v_j) \in E$ implies $e_{ji} = (v_j, v_i) \in E$, G is called an undirected graph; otherwise, G is called a directed graph. Vertex v_j is a neighbor of vertex v_i if $(v_j, v_i) \in E$. Vertex v_i is assumed to be itself neighbor. The neighbor set of v_i is denoted as $N(v_i) = \{v_j \in V | (v_j, v_i) \in E\}$.

Definition 4: A set S of vertices of G is a control set (or dominating set) of G if every vertex of G is controlled (or dominated) by some vertex in S. Equivalently, a set S of vertices of G is a control set of G if every vertex in $V(G) - S$ is adjacent to some vertex in S. A minimum control set in a graph G is a control set of minimum cardinality. The cardinality of a minimum control set is called the control number of G and is denoted by $\gamma(G)$ [1].

Remark 3: Definition 4 can be equivalently defined as follows. Let $G = (V, E)$ be a graph, a two-value function $f : V \to \{0, 1\}$ is called a control function of G if $\sum_{v \in N(u)} f(v) \geq 1$ for any $u \in V$. The control number of G is defined as $\gamma(G) = \min\{f(V) | f$ is a control function of $G\}$.

In this paper, vertex v_i can control vertex v_j, or say vertex v_j can be controlled by vertex v_i, means that there is at least one edge from v_i to v_j. For an undirected graph, vertex v_i can control vertex v_j implies vertex v_j can also control vertex v_i. We assume that vertex v_i can control itself. A subset D of V can control vertex v_j if there exists a vertex $v_i \in D$ such that v_i can control v_j.

Remark 4: From definition 4 we can see that the concept of control between vertices is based on the concept of neighbor of vertices. More precisely, vertex v_i can control vertex v_j is equivalent to that v_i is a neighbor of v_j. Vertex v_i can control itself corresponding to that v_i is itself neighbor.

B. PROBLEM STATEMENT

Some problems in real-world expect that every object is controlled the same times by other objects. Take street security monitoring for example, the desired minimum number of monitoring cameras can be modelled by minimum control sets of graphs. But to save resources, we sometimes require that every intersection of streets is monitored by only one monitoring camera. There are also other applications that demand every object is controlled more than once. For example, in wireless sensor networks, it is better that every key sensor is connected twice by convergence sensors, which can maintain the network connected when a convergence node is of breakdown. Inspired by this, we propose the concept of k-capacity control set of graphs which can serve as the mathematical model of the above problems.

There are also other problems requiring that every object in an object set controls the same number of other objects, for instance, in wireless sensor networks, for a longer life it requests that convergence sensors should bear a balance burden of communication. We define, using graphical language, this concept as k-harmony control set of graphs.

One of the aims of this paper is to formulate the control sets, including control sets, k-capacity control sets, and k-harmony control sets, in an algebraic manner; the other one is to find out all these control sets of graphs mathematically. The main research tool is the semi-tensor product of matrices.

III. MAIN RESULTS

In this section we present the main findings of our work including algebraic algorithm of finding control sets, algebraic formulations and algorithms of k-capacity control sets and k-harmony control sets.

A. ALGEBRAIC ALGORITHM OF FINDING CONTROL SETS

In this subsection we use STP to investigate the problem of searching control sets mathematically and present the first part of the main results of this paper. We start with several concepts of graphs.

Consider a graph G with n vertices $V = \{v_1, v_2, \cdots, v_n\}$. The adjacency matrix of G, $B = [b_{ij}]$, is defined as

$$b_{ij} = \begin{cases} 1, & v_i \in N(v_j), \\ 0, & v_i \notin N(v_j). \end{cases} \tag{3}$$

The unitary adjacency matrix of G, $A = [a_{ij}]$, is defined as

$$a_{ij} = \begin{cases} 1, & i = j, \\ b_{ij}, & i \neq j. \end{cases} \tag{4}$$

From (3) and (4) it is easy to see that

$$a_{ij} = b_{ij} \vee \delta_{ij}, \tag{5}$$

where $\vee$ is the logic product operator, and δ_{ij} is the Kronecker symbol, i.e.,

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Thus $A = B \vee I_n$, i.e., $a_{ij} = b_{ij} \vee I_n(ij)$.

For a given subset $S \subseteq V$, we define its characteristic vector, $V_S = [x_1, x_2, \cdots, x_n]$, as

$$x_i = \begin{cases} 1, & x_i \in S, \\ 0, & x_i \notin S. \end{cases} \tag{6}$$

Let $y_i = [x_i, 1 - x_i]^T$, $i = 1, 2, \cdots, n$. Define

$$Y_S = \ltimes_{i=1}^{n} y_i. \tag{7}$$

Remark 5: Y_S and V_S are determined uniquely by each other because every y_i can be uniquely gained by calculating $y_i = S_i^n Y_S$, where S_i^n is defined in (17). Hence there exists a one-to-one mapping between Y_S and V_S. Y_S can also be therefore viewed as a characteristic variable of the vertex subset S. Thus once either Y_S or V_S is known, the subset S can be obtained consequently.

Theorem 1: Consider a graph $G = (V, E)$ with $A = [a_{ij}]$ as its unitary adjacency matrix. Let $V_S = [x_1, x_2, \cdots, x_n]$ be the characteristic vector of vertex subset $S \subseteq V$, then S is a control set of G if and only if

$$\sum_{j=1}^{n} a_{ij} x_j \geq 1, \quad (i = 1, 2, \cdots, n). \tag{8}$$

Proof: Define that $S \subseteq V$ controls vertex v_j just once means there exists exactly one vertex v_i in S that controls v_j. We first prove the following proposition.

Let $S \subseteq V$ be a subset of G and its characteristic vector be $V_S = [x_1, x_2, \cdots, x_n]$. And let the unitary adjacency matrix of G be $A = [a_{ij}]$. Then S controls $v_i \in V$ just once iff

$$\sum_{j=1}^{n} (a_{ij} x_j) = 1. \tag{9}$$

(Sufficiency). If $\sum_{j=1}^{n} (a_{ij} x_j) = 1$, then there is only one $j \in \{1, 2, \cdots, n\}$ such that $a_{ij} x_j = 1$. $a_{ij} x_j = 1$ implies that $a_{ij} = 1$ and $x_j = 1$, where $a_{ij} = 1$ means that v_j controls v_i, $x_j = 1$ implies that v_j belongs to S. This shows that S controls v_j just once. Thus the sufficiency is proved.

(Necessity). We first prove that $v_j \in S$ controls $v_i \in V$ is equivalent to $a_{ij} x_j = 1$. $v_j \in S$ implies $x_j = 1$, thus $a_{ij} x_j = 1$ is equivalent to $a_{ij} = 1$. $a_{ij} = 1$ means that v_j controls v_i.

If S controls $v_i \in V$ just once, it follows from $v_j \notin S$ implying $x_j = 0$ that there is only one $j \in \{1, 2, \cdots, n\}$ such that $a_{ij}x_j = 1$. Thus $\sum_{j=1}^{n}(a_{ij}x_j) = 1$. The necessity is obtained.

It is easy to see from the above that S controls $v_i \in V$ iff

$$\sum_{j=1}^{n}(a_{ij}x_j) \geq 1. \tag{10}$$

Now we return to the proof of Theorem 1. Recall that S is a control set of G means that S controls $v_i (i = 1, 2, \cdots, n)$. From (10) we get that S is a control set of G is equivalent to $\sum_{j=1}^{n} a_{ij}x_j \geq 1, (i = 1, 2, \cdots, n)$. The proof is then completed.

Condition (8) is a necessary and sufficient condition for a specific vertex subset of a graph. To find all the control sets of graphs, we need to describe whole graphs that contain control sets mathematically. Starting with this goal we begin the following condition.

Theorem 2: Consider a graph $G = (V, E)$ with $A = [a_{ij}]$ as its unitary adjacency matrix, $V = \{v_1, v_2, \cdots, v_n\}$. G contains a control set if and only if there is a $1 \leq j \leq 2^n$ such that

$$\mathrm{col}_j(M) \geq \mathbf{1}_n. \tag{11}$$

where

$$M = \begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_n \end{bmatrix}, \quad M_i = J \ltimes \sum_{i=1}^{n}(a_{ij}Q_j), i = 1, 2, \cdots, n,$$

$$J = [1, 0], \quad Q_j = (E_{d\_latter})^{n-j} \ltimes W_{[2,2^{n-j}]} \ltimes (E_{d\_latter})^{j-1},$$

Proof: Assume that G contains a control set S with $V_S = [x_1, x_2, \cdots, x_n]$ as its characteristic vector. From Theorem 1 we know

$$\sum_{j=1}^{n} a_{ij}x_j \geq 1, \quad (i = 1, 2, \cdots, n). \tag{12}$$

Define $y_j = [x_j, 1 - x_j]^T, j = 1, 2, \cdots, n$. By (2), we know that

$$\begin{aligned} y_j &= (E_{d\_latter})^{n-j} \ltimes y_{j+1} \ltimes y_{j+2} \ltimes \cdots \ltimes y_n y_j \\ &= (E_{d\_latter})^{n-j} \ltimes W_{[2,2^{n-j}]} \ltimes y_j \ltimes y_{j+1} \ltimes \cdots \ltimes y_n \\ &= (E_{d\_latter})^{n-j} W_{[2,2^{n-j}]} (E_{d\_latter})^{j-1} y_1 \cdots y_n \\ &:= Q_j \ltimes Y_S. \end{aligned}$$

where $Q_j = (E_{d\_latter})^{n-j} \ltimes W_{[2,2^{n-j}]}(E_{d\_latter})^{j-1}$.

It is clear that the following equations hold.

$$\begin{aligned} x_j &= J \ltimes y_j, \\ a_{ij}x_j &= a_{ij}J \ltimes y_j = a_{ij}J \ltimes Q_j \ltimes Y_S. \end{aligned}$$

Then we have

$$\sum_{j=1}^{n}(a_{ij}x_j) = J \ltimes \sum_{j=1}^{n}(a_{ij}Q_j) \ltimes Y_S := M_i \ltimes Y_S.$$

where $M_i = J \ltimes \sum_{j=1}^{n}(a_{ij}Q_j)$.

Therefore, for $i \in \{1, 2, \cdots, n\}$ the following statements are equivalent to each other.

1) $\sum_{j=1}^{n} a_{ij}x_j \geq 1$.
2) $M_i \ltimes Y_S \geq \mathbf{1}_n$.
3) There is a column of M_i is $\mathbf{1}_n$.
4) There exists a $1 \leq j \leq 2^n$ such that $\mathrm{col}_j(M_i) \geq 1$.

Note that (12) is equivalent to

$$\begin{cases} \sum_{j=1}^{n} a_{1j}x_j \geq 1, \\ \sum_{j=1}^{n} a_{2j}x_j \geq 1, \\ \vdots \\ \sum_{j=1}^{n} a_{nj}x_j \geq 1. \end{cases} \tag{13}$$

It is evident that (13) has a solution $[x_1, x_2, \cdots, x_n]$ if and only if there exists a $1 \leq j \leq 2^n$ such that

$$\mathrm{col}_j(M) \geq \mathbf{1}_n.$$

where

$$M = \begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_n \end{bmatrix}.$$

From the equivalence of (12) and (13), we obtain Theorem 2. The proof is completed.

Remark 6: Theorem 2 is a necessary and sufficient condition describing graphs that has control sets. If we further want to find all the control sets of a given graph, it will be needed to mathematically characterize a vertex subset to be a control set, which is presented as the following Theorem 3.

Theorem 3: Consider a graph $G = (V, E)$ with $A = [a_{ij}]$ as its unitary adjacency matrix. For a given vertex subset $S \subseteq V$ with $V_S = [x_1, x_2, \cdots, x_n]$ as its characteristic vector and $Y_S =_{i=1}^{n} y_i = \delta_{2^n}^{k}, y_i = [x_i, 1 - x_i]^T$. Then S is a control set of G if and only if

$$\mathrm{col}_k(M) \geq \mathbf{1}_n. \tag{14}$$

where M is given in Theorem 2.

Proof: The proof is very similar to that of Theorem 2 in the proving idea and means. Comparing the result of Theorem 3, i.e., equation (14), with the result of Theorem 2, i.e., equation (11), it follows from the conditions of Theorems 2 and 3 that it is only needed to prove that the kth-column of M is greater than or equal to $\mathbf{1}_n$, i.e., the kth-element of $M_i, i = 1, 2, \cdots, n$ is greater than or equal to 1.

Based on the proof of Theorem 2, we know that $M_i \ltimes Y_S \geq 1$ is equivalent to "there exists a $1 \leq j \leq 2^n$ such that $\mathrm{col}_j(M_i) \geq 1$." It is sufficient to show that k satisfies the condition. In fact, M_i and Y_S are of 1×2^n and $2^n \times 1$ dimensions, respectively; recall that in this case the semi-tensor product of matrices reduces to the conventional product of matrices. Therefore $M_i \ltimes Y_S$ is just the kth-column of M_i. Naturally, $M_i \ltimes Y_S \geq 1$ is equivalent to $\mathrm{col}_k(M_i) \geq 1$. This completes the proof.

From the proof of Theorem 3, Corollary 1 is obvious.

Corollary 1: For a given graph $G = (V, E)$ with $A = [a_{ij}]$ as its unitary adjacency matrix, assign each vertex $v_i \in V$ a characteristic variable $x_i \in \{0, 1\}$ and set $y_i = [x_i, 1 - x_i]^T$. Then G contains a control set if and only if the following inequality has solution(s).

$$M \ltimes_{i=1}^{n} y_i \geq \mathbf{1}_n. \tag{15}$$

Furthermore the number of solutions is just the number of control sets of G. By Remark 5, each solution determines a control set. In other words, the number of control sets equals to the number of columns of M that are greater than or equal to $\mathbf{1}_n$. The number of 0_n in M is the number of vertex subsets that cannot be treated as control sets.

Remark 7: The matrices M_i and M in Theorems 2 and 3 can be calculated by a MATLAB toolbox easily. This toolbox is developed by Professors Daizhan Cheng and Hongsheng Qi to handle the related calculations on the STP, which can be obtained at http://lsc.amss.ac.cn/~dcheng/stp.

By the proof of Theorem 3 and Corollary 1, we can establish an algebraic algorithm to find all the control sets of graphs in a mathematical manner.

Algorithm 1: For a given graph $G = (V, E)$ with $A = [a_{ij}]$ as its unitary adjacency matrix, assign each vertex $v_i \in V$ a characteristic variable $x_i \in \{0, 1\}$ and define $y_i = [x_i, 1-x_i]^T$. Take the following steps, we can get all the control sets and minimum control sets of G.

Step 1: Compute the matrix M in Theorem 3.

Step 2: Check whether there exists a column in M that is greater than or equal to $\mathbf{1}_n$. If not, G has no control set and the algorithm comes to end. Otherwise, set

$$K = \{i | \mathrm{col}_i(M) \geq \mathbf{1}_n\}. \tag{16}$$

Step 3: For each l in K, consider the equation $\ltimes_{i=1}^{n} y_i = \delta_{2^n}^{l}$. According to reference [16], let

$$\begin{cases} S_1^n = \delta_2[\underbrace{1 \cdots 1}_{2^{n-1}} \underbrace{2 \cdots 2}_{2^{n-1}}], \\ S_2^n = \delta_2[\underbrace{1 \cdots 1}_{2^{n-2}} \underbrace{2 \cdots 2}_{2^{n-2}} \underbrace{1 \cdots 1}_{2^{n-2}} \underbrace{2 \cdots 2}_{2^{n-2}}], \\ \vdots \\ S_n^n = \delta_2[\underbrace{\underbrace{12}_{2} \cdots \underbrace{12}_{2}}_{2^{n-1}}]. \end{cases} \tag{17}$$

Then y_i can be obtained by calculating $y_i = S_i^n \delta_{2^n}^l$, $i = 1, 2, \cdots, n$. Choose $y_i = \delta_2^1$ and set

$$S_l = \{v_i | y_i = \delta_2^1\}.$$

S_l is a control set of G. All the control sets are

$$\{S_l | l \in K\}.$$

Step 4: The control number of G is

$$\gamma(G) = \min_{l \in K}\{|S_l|\},$$

where $|S_l|$ is the cardinality of S_l. All the minimum control sets of G are

$$\zeta = \{S_l | \, |S_l| = \gamma(G)\}.$$

Remark 8: The above algorithm can find all the control sets of a graph, we can therefore obtain the control sets of some special features, such as, every vertex is controlled by a control set only once, exactly twice, and so on. About this we have a lot to say in the following subsection. In a word, finding all the control sets provides us a complete picture for the structure of a graph in terms of control sets.

B. ALGEBRAIC FORMULATION AND ALGORITHM OF K-CAPACITY CONTROL SETS

In this subsection we first extract the concept of k-capacity control sets of graphs from real-world problems such as security monitoring of streets, and then present the second part of the main results of this paper.

Definition 5: A vertex subset of G, S, is a k-capacity control set of G if every vertex of G is controlled by k vertices of S. Equivalently, a set S of vertices of G is a k-capacity control set of G if every vertex in $V(G) - S$ is adjacent to k vertices in S. A minimum k-capacity control set in a graph G is a k-capacity control set with minimum cardinality. The cardinality of a k-capacity minimum control set is called the k-capacity control number of G.

We have the following results about the k-capacity control sets.

Theorem 4: Consider a graph $G = (V, E)$ with $A = [a_{ij}]$ as its unitary adjacency matrix.

1) Let $V_S = [x_1, x_2, \cdots, x_n]$ be the characteristic vector of vertex subset $S \subseteq V$, then S is a k-capacity control set of G if and only if

$$\sum_{j=1}^{n} a_{ij} x_j = k, \quad (i = 1, 2, \cdots, n). \tag{18}$$

2) G contains a k-capacity control set if and only if there is a $1 \leq j \leq 2^n$ such that

$$\mathrm{col}_j(M) = \mathbf{k}_n. \tag{19}$$

3) For a given vertex subset $S \subseteq V$ with $V_S = [x_1, x_2, \cdots, x_n]$ as its characteristic vector and $Y_S = \ltimes_{i=1}^{n} y_i = \delta_{2^n}^{r}$, $y_i = [x_i, 1 - x_i]^T$. Then S is a k-capacity control set of G if and only if

$$\mathrm{col}_r(M) = \mathbf{k}_n. \tag{20}$$

where M is the same as the M in (11).

Proof: Based on Theorems 1-3, comparing the definition of control set with that of k-capacity control set, to prove Theorem 4, we only have to prove the following proposition.

Let $S \subseteq V$ be a vertex subset of G and its characteristic vector be $V_S = [x_1, x_2, \cdots, x_n]$. And let the unitary adjacency matrix of G be $A = [a_{ij}]$. Then S controls vertex $v_i \in V$ k times if and only if

$$\sum_{j=1}^{n}(a_{ij}x_j) = k. \tag{21}$$

(Sufficiency). Consider $\sum_{j=1}^{n}(a_{ij}x_j) = k$ first. $\sum_{j=1}^{n}(a_{ij}x_j) = k$ implies that the number of $j \in \{1, 2, \cdots, n\}$ such that $a_{ij}x_j = 1$ is k, because $a_{ij}x_j$ takes values in $\{0, 1\}$. Since both a_{ij} and x_j are 1 or 0, we know that $a_{ij}x_j = 1$ indicates $a_{ij} = 1$ and $x_j = 1$, where $a_{ij} = 1$ denotes that vertex v_j controls vertex v_i, and $x_j = 1$ means that vertex v_j belongs to S. This together with the number of j satisfying $a_{ij}x_j = 1$ is k implies that S controls the vertex $v_j k$ times. Thus the sufficiency is proved.

(Necessity). Consider the vertices of S. Since $v_j \in S$ suggests that $x_j = 1$, $a_{ij}x_j = 1$ if and only if $a_{ij} = 1$, where $a_{ij} = 1$ means that vertex v_j controls vertex v_i. We thus get that vertex $v_j \in S$ controls $v_i \in V$ is equivalent to $a_{ij}x_j = 1$.

If S controls vertex $v_i \in V$ k times, that is, the number of $v_j \in S$ that controls vertex v_i is k, we then know that there are k indexes, $j \in \{1, 2, \cdots, n\}$, such that $a_{ij}x_j = 1$ because the vertices $v_j \notin S$ imply $x_j = 0$.

Thus we have

$$\sum_{j=1}^{n}(a_{ij}x_j) = k.$$

The necessity is obtained. Then the proof of Theorem 4 is completed.

Similarly to Corollary 1, the following result follows Theorem 4 immediately.

Corollary 2: Consider a graph $G = (V, E)$ with $A = [a_{ij}]$ as its unitary adjacency matrix. Assign each vertex $v_i \in V$ a characteristic variable $x_i \in \{0, 1\}$ and set $y_i = [x_i, 1 - x_i]^T$. Then G contains a k-capacity control set is equivalent to the following equation is solvable.

$$M \ltimes_{i=1}^{n} y_i = \mathbf{k}_n. \tag{22}$$

Furthermore the number of solutions is just the number of k-capacity control sets of G. Remark 5 tells us that each solution determines a k-capacity control set. That is, the number of k-capacity control sets equals to the number of $\mathbf{k}_n$ in the set of columns of M.

Remark 9: Comparing the proofs of Theorems 4, 2 and 3, it is easy to know that to find all the k-capacity control sets of a graph, we only need to replace Step 2 of Algorithm 1 as follows.

Check whether there exists a column in M that is $\mathbf{k}_n$. If not, G has no k-capacity control set and the algorithm stops. Otherwise, set $K = \{i | \mathrm{col}_i(M) = \mathbf{k}_n\}$.

C. ALGEBRAIC FORMULATION AND ALGORITHM OF K-HARMONY CONTROL SETS

In this subsection we define the concept of k-harmony control set of graphs which is abstracted from wireless sensor networks (WSN) where some key nodes are required to bear balanced communication and energy consumption, and then present the third part of the main results of this paper.

First we denote the set of vertices controlled by vertex v_i as $C(v_i)$, i.e., $C(v_i) = \{v_j \in V | (v_i, v_j) \in E\}$.

Definition 6: In a graph G, a vertex subset S is called a k-harmony control set of G if S satisfies the following three conditions.

1) S is a control set.
2) Each vertex of S controls k vertices of $G - S$.
3) $C(v_i) \cap C(v_j) = \emptyset$ for any $i \neq j$.

In other words, a subset S of vertices is a k-harmony control set if each vertex of S serves as neighbors of k distinct vertices of $V(G) - S$. A minimum k-harmony control set in a graph G is a k-harmony control set of minimum cardinality.

Remark 10: Comparing the above three vertex subsets, control set, k-capacity control set, and k-harmony control set with each other. We know that both k-capacity control set and k-harmony control set are special kinds of control set. The k-capacity control set emphasizes that all the vertices of G can be controlled by the same times, while the k-harmony control set lays emphasis on control sets and requires every vertex of a control set controls the same number of vertices.

For the k-harmony control set, we have the following conclusion.

Theorem 5: Consider a graph $G = (V, E)$ with $B = [b_{ij}]$ as its adjacency matrix. Let $V_S = [x_1, x_2, \cdots, x_n]$ be the characteristic vector of k-harmony control set S, then

$$\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}x_i\bar{x}_j = k|S|, \tag{23}$$

where $\bar{x}_j = 1 - x_j$.

Proof: Assume that S contains m vertices, denoted as $S = \{v_{i_1}, v_{i_2}, \cdots, v_{i_m}\}$. Then its characteristic vector is

$$V_S = [0, \cdots, 0, \underbrace{1}_{i_1}, 0, \cdots, 0, \underbrace{1}_{i_2}, 0, \cdots, 0, \underbrace{1}_{i_m}, 0, \cdots, 0].$$

We first consider $\sum_{j=1}^{n} b_{ij}x_i\bar{x}_j$.

From $V_S = [0, \cdots, 0, \underbrace{1}_{i_1}, 0, \cdots, 0, \underbrace{1}_{i_2}, 0, \cdots, 0, \underbrace{1}_{i_m}, 0, \cdots, 0]$, it is easy to see that $b_{ij}x_i\bar{x}_j = 0$ for these is which are out of the set $\{i_1, i_2, \cdots, i_m\}$. Therefore, we have

$$\begin{aligned}\sum_{j=1}^{n} b_{ij}x_i\bar{x}_j &= b_{i1}x_i\bar{x}_1 + b_{i2}x_i\bar{x}_2 + \cdots + b_{in}x_i\bar{x}_n \\ &= b_{i1}\bar{x}_1 + b_{i2}\bar{x}_2 + \cdots + b_{in}\bar{x}_n. \end{aligned} \tag{24}$$

We then consider $b_{i_rj}\bar{x}_j$, $r = 1, 2, \cdots, m$. $b_{i_rj} = 1$ implies vertex v_{i_r} is a neighbor of vertex v_j, which together with

$C(v_{i_r}) \cap C(v_j) = \emptyset$ suggests that vertex v_j is not in S. Thus $x_j = 0, \bar{x}_j = 1$. We therefore get that if $b_{i_r j} = 1$, then $\bar{x}_j = 1$. Since each vertex of S controls k vertices of $G - S$, that is, the number of i_r such that $b_{i_r j} = 1$ is k. With this and (24) we know

$$\sum_{j=1}^{n} b_{ij} x_i \bar{x}_j = k. \tag{25}$$

Last we consider $\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij} x_i \bar{x}_j$. Because $b_{ij} x_i \bar{x}_j = 0$ for the is out of the set $I = \{i_1, i_2, \cdots, i_m\}$, further $\sum_{j=1}^{n} b_{ij} x_i \bar{x}_j = 0$ for these is. Based on (25), we then have

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij} x_i \bar{x}_j &= \sum_{i\in I}^{n}\sum_{j=1}^{n} b_{ij} x_i \bar{x}_j \\ &= \sum_{i\in I}^{n} k = km = k|S|.\end{aligned}$$

The proof is completed.

Similarly to the relation between Theorems 1 and 2, Theorem 5 can also be presented as follows.

Theorem 6: Consider the graph G in Theorem 5. If G contains a k-harmony control set then there is a $1 \le j \le 2^n$ such that

$$\mathrm{col}_j(N) = k, \tag{26}$$

where $N = H \ltimes \left(\sum_{i=1}^{n}(\sum_{i<j} b_{ij} T_{ij} + \sum_{i>j} b_{ij} U_{ij})\right)$, in which

$$\begin{aligned} H &= [0\ 1\ 0\ 0], \\ T_{ij} &= (E_{d\_latter})^{n-2} \ltimes W_{[2^j, 2^{n-j}]} \ltimes W_{[2^i, 2^{j-i-1}]}, \\ U_{ij} &= (E_{d\_latter})^{n-2} \ltimes W_{[2^i, 2^{n-i}]} \ltimes W_{[2^j, 2^{i-j-1}]}. \end{aligned}$$

Proof: Assume that G contains a k-harmony control set S with $V_S = [x_1, x_2, \cdots, x_n]$ as its characteristic vector. Define $y_i = [x_i, \bar{x}_i]^T$ and $y_j = [x_j, \bar{x}_j]^T$, then it follows from straightforward calculations that $x_i \bar{x}_j = H y_i y_j$.

When $i < j$, using (2), we can get (for writing ease, symbol $\ltimes$ is omitted)

$$\begin{aligned} y_i y_j &= (E_{d\_latter})^{n-2} y_{j+1} \cdots y_n y_{i+1} \cdots y_{j-1} y_1 \cdots y_{i-1} y_i y_j \\ &= (E_{d\_latter})^{n-2} W_{[2^j, 2^{n-j}]} y_{i+1} \cdots y_{j-1} y_1 \cdots y_{i-1} y_i y_j \\ &\quad \times y_{j+1} \cdots y_n \\ &= (E_{d\_latter})^{n-2} W_{[2^j, 2^{n-j}]} W_{[2^i, 2^{j-i-1}]} y_1 \cdots y_{i-1} y_i \\ &\quad \times y_{i+1} \cdots y_{j-1} y_j y_{j+1} \cdots y_n \\ &= (E_{d\_latter})^{n-2} W_{[2^j, 2^{n-j}]} W_{[2^i, 2^{j-i-1}]} Y_S \\ &:= T_{ij} Y_S. \end{aligned}$$

where

$$T_{ij} = (E_{d\_latter})^{n-2} \ltimes W_{[2^j, 2^{n-j}]} \ltimes W_{[2^i, 2^{j-i-1}]}.$$

Similarly, when $i > j$ we can get

$$\begin{aligned} y_i y_j &= U_{ij} \ltimes Y_S, \\ U_{ij} &= (E_{d\_latter})^{n-2} \ltimes W_{[2^i, 2^{n-i}]} \ltimes W_{[2^j, 2^{i-j-1}]}. \end{aligned}$$

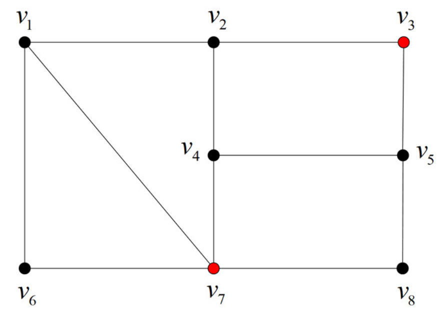

FIGURE 1. The graph of Example 1.

Since when $i = j$, $b_{ij} = 0$, we know $b_{ij} x_j \bar{x}_j = 0$ for $i = j$. Thus we have

$$\begin{aligned} &\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij} x_i \bar{x}_j \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} H \ltimes b_{ij} y_i \ltimes y_j \\ &= \sum_{i=1}^{n}\sum_{i<j} H \ltimes b_{ij} y_i \ltimes y_j + \sum_{i=1}^{n}\sum_{i>j} H \ltimes b_{ij} y_i \ltimes y_j \\ &= H \ltimes \left(\sum_{i=1}(\sum_{i<j}^{n} b_{ij} T_{ij} + \sum_{i>j}^{n} b_{ij} U_{ij})\right) \ltimes Y_S \\ &= N Y_S. \end{aligned}$$

Because $Y_S \in \Delta_{2^n}$, from (23), we know that if S is a k-harmony control set, then there must exists a $1 \le j \le 2^n$ such that

$$\mathrm{col}_j(N) = k.$$

We then obtain the conclusion.

Remark 11: Theorem 5 (or 6) is a necessary condition of judging whether a vertex subset is a k-harmony control set or not. Algorithm 1 provides us a way to find all the control sets of graphs, therefore, based on Algorithm 1, Theorem 5 (or 6) can help us to find the k-harmony control sets of a graph. In fact, we only need to check whether a control set obtained by Algorithm 1 satisfies Theorem 5 (or 6), if yes, the control set is a k-harmony one, otherwise, it is not.

IV. EFFECTIVENESS OF THE RESULTS

This section uses an example in [36] {Xu, 2008 #253} to illustrate in detail the effectiveness and correctness of the presented results (Theorems 1-6) and to show how to use our proposed algorithm (Algorithm 1 and Remark 9) to search all the control sets and k-capacity control sets of a graph.

Example 1: Consider the graph $G = (V, E)$ shown in Fig.1. We first use Algorithm 1 to find out all the control sets of G.

The unitary adjacency matrix of G is

$$A=\begin{bmatrix}1&1&0&0&0&1&1&0\\1&1&1&1&0&0&0&0\\0&1&1&0&1&0&0&0\\0&1&0&1&1&0&1&0\\0&0&1&1&1&0&0&1\\1&0&0&0&0&1&1&0\\1&0&0&1&0&1&1&1\\0&0&0&0&1&0&1&1\end{bmatrix}$$

Step 1: Compute the matrix M in Algorithm 1.

By Theorem 3, we can get

$$\begin{aligned}
Q_1&=\delta_2[\underbrace{1\cdots1}_{128}\underbrace{2\cdots2}_{128}],\\
Q_2&=\delta_2[\underbrace{1\cdots1}_{64}\underbrace{2\cdots2}_{64}\cdots\underbrace{1\cdots1}_{64}\underbrace{2\cdots2}_{64}],\\
Q_3&=\delta_2[\underbrace{1\cdots1}_{32}\underbrace{2\cdots2}_{32}\cdots\underbrace{1\cdots1}_{32}\underbrace{2\cdots2}_{32}],\\
Q_4&=\delta_2[\underbrace{1\cdots1}_{16}\underbrace{2\cdots2}_{16}\cdots\underbrace{1\cdots1}_{16}\underbrace{2\cdots2}_{16}],\\
Q_5&=\delta_2[\underbrace{1\cdots1}_{8}\underbrace{2\cdots2}_{8}\cdots\underbrace{1\cdots1}_{8}\underbrace{2\cdots2}_{8}],\\
Q_6&=\delta_2[\underbrace{1\cdots1}_{4}\underbrace{2\cdots2}_{4}\cdots\underbrace{1\cdots1}_{4}\underbrace{2\cdots2}_{4}],\\
Q_7&=\delta_2[\underbrace{11}_{2}\underbrace{22}_{2}\cdots\underbrace{11}_{2}\underbrace{22}_{2}],\\
Q_8&=\delta_2[\underbrace{12\cdots12}_{256}],
\end{aligned}$$

Further, we have M_i, $i=1,2,\cdots,8$ as follows.

$$\begin{aligned}
M_1=[&e_{11},e_{11},e_{11},e_{11},e_{12},e_{12},e_{12},e_{12},e_{12},e_{12},e_{12},\\
&e_{12},e_{13},e_{13},e_{13},e_{13}],
\end{aligned}$$

where

$$\begin{aligned}
e_{11}&=[4,4,3,3,3,3,2,2,4,4,3,3,3,3,2,2],\\
e_{12}&=[3,3,2,2,2,2,1,1,3,3,2,2,2,2,1,1],\\
e_{13}&=[2,2,1,1,1,1,0,0,2,2,1,1,1,1,0,0].\\
M_2&=[e_{21},e_{22},e_{22},e_{23},e_{22},e_{23},e_{23},e_{24},e_{22},e_{23},e_{23},\\
&\quad e_{24},e_{23},e_{24},e_{24},e_{25}],
\end{aligned}$$

where

$$e_{21}=[\underbrace{4,\cdots,4}_{16}],\quad e_{22}=[\underbrace{3,\cdots,3}_{16}],\quad e_{23}=[\underbrace{2,\cdots,2}_{16}],$$
$$e_{24}=[\underbrace{1,\cdots,1}_{16}],\quad e_{25}=[\underbrace{0,\cdots,0}_{16}].$$

$$\begin{aligned}
M_3=[&e_{31},e_{31},e_{32},e_{32},e_{32},e_{32},e_{33},e_{33},e_{31},e_{31},e_{32},\\
&e_{32},e_{32},e_{32},e_{33},e_{33}],
\end{aligned}$$

where

$$\begin{aligned}
e_{31}&=[\underbrace{3,\cdots,3}_{8},\underbrace{2,\cdots,2}_{8}],\\
e_{32}&=[\underbrace{2,\cdots,2}_{8},\underbrace{1,\cdots,1}_{8}],\\
e_{33}&=[\underbrace{1,\cdots,1}_{8},\underbrace{0,\cdots,0}_{8}].\\
M_4&=[e_{41},e_{42},e_{41},e_{42},e_{42},e_{43},e_{42},e_{43},e_{41},e_{42},e_{41},\\
&\quad e_{42},e_{42},e_{43},e_{42},e_{43}],
\end{aligned}$$

where

$$\begin{aligned}
e_{41}&=[4,4,3,3,4,4,3,3,3,3,2,2,3,3,2,2],\\
e_{42}&=[3,3,2,2,3,3,2,2,2,2,1,1,2,2,1,1],\\
e_{43}&=[2,2,1,1,2,2,1,1,1,1,0,0,1,1,0,0].\\
M_5&=[e_{51},e_{52},e_{52},e_{53},e_{51},e_{52},e_{52},e_{53},e_{51},e_{52},e_{52},\\
&\quad e_{53},e_{51},e_{52},e_{52},e_{53}],
\end{aligned}$$

where

$$\begin{aligned}
e_{51}&=[4,3,4,3,4,3,4,3,3,2,3,2,3,2,3,2],\\
e_{52}&=[3,2,3,2,3,2,3,2,2,1,2,1,2,1,2,1],\\
e_{53}&=[2,1,2,1,2,1,2,1,1,0,1,0,1,0,1,0].\\
M_6&=[e_{61},e_{61},e_{61},e_{61},e_{61},e_{61},e_{61},e_{61},e_{62},e_{62},e_{62},e_{62},\\
&\quad e_{62},e_{62},e_{62},e_{62}],
\end{aligned}$$

where

$$\begin{aligned}
e_{61}&=[3,3,2,2,2,2,1,1,3,3,2,2,2,2,1,1],\\
e_{62}&=[2,2,1,1,1,1,0,0,2,2,1,1,1,1,0,0].\\
M_7&=[e_{71},e_{72},e_{71},e_{72},e_{71},e_{72},e_{71},e_{72},e_{71},e_{72},e_{72},\\
&\quad e_{73},e_{72},e_{73},e_{72},e_{73}],
\end{aligned}$$

where

$$\begin{aligned}
e_{71}&=[5,4,4,3,4,3,3,2,5,4,4,3,4,3,3,2],\\
e_{72}&=[4,3,3,2,3,2,2,1,4,3,3,2,3,2,2,1],\\
e_{73}&=[3,2,2,1,2,1,1,0,3,2,2,1,2,1,1,0].\\
M_8&=[e_{81},e_{81},e_{81},e_{81},e_{81},e_{81},e_{81},e_{81},e_{81},e_{81},e_{81},\\
&\quad e_{81},e_{81},e_{81},e_{81},e_{81}],
\end{aligned}$$

where

$$e_{81}=[3,2,2,1,3,2,2,1,2,1,1,0,2,1,1,0].$$

We then get

$$M=\begin{bmatrix}M_1\\M_2\\\vdots\\M_n\end{bmatrix}.$$

Step 2: From M we know that there are 165 columns greater than or equal to 1_n in M and can get the following set K.

$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 29, 30, 31, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 45, 46, 47, 49, 50, 51, 52, 53, 54, 55, 56, 57, 59, 61, 63, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 77, 78, 79, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 93, 94, 97, 98, 99, 100, 101, 102, 103, 104, 113, 114, 115, 116, 117, 118, 119, 120, 129, 130, 131, 132, 133, 134, 137, 138, 139, 141, 142, 145, 146, 147, 148, 149, 150, 153, 154, 155, 157, 158, 161, 162, 163, 164, 165, 166, 169, 170, 171, 173, 174, 177, 178, 179, 180, 181, 182, 185, 187, 189, 193, 194, 195, 196, 197, 198, 201, 202, 203, 205, 206, 209, 210, 211, 212, 213, 214, 217, 218, 221, 222, 225, 226, 227, 228, 229, 230\}$.

Step 3: For each element $l \in K$, we can get y_i, $i = 1, 2, \cdots, 8$, by solving $y_i = S_i^8 \delta_{256}^l$, where S_i^8 are defined as in (17). Then a control set $S_l = \{v_i | y_i = \delta_2^1\}$, corresponding to l, can be determined.

Take $l = 222$ for example, we can get

$$\begin{aligned} y_1 &= S_1^8 \ltimes \delta_{256}^{222} = \delta_2^2, \quad y_2 = S_2^8 \ltimes \delta_{256}^{222} = \delta_2^2,\\ y_3 &= S_3^8 \ltimes \delta_{256}^{222} = \delta_2^1, \quad y_4 = S_4^8 \ltimes \delta_{256}^{222} = \delta_2^2,\\ y_5 &= S_5^8 \ltimes \delta_{256}^{222} = \delta_2^2, \quad y_6 = S_6^8 \ltimes \delta_{256}^{222} = \delta_2^2,\\ y_7 &= S_7^8 \ltimes \delta_{256}^{222} = \delta_2^1, \quad y_8 = S_8^8 \ltimes \delta_{256}^{222} = \delta_2^2. \end{aligned}$$

This shows that $l = 222$ determines a control set $S = \{v_3, v_7\}$, which is shown in red in Fig.1.

Similarly, all the other control sets can be obtained by other ls, some of them are listed in the following.

$$\begin{aligned} S &= \{v_1, v_4, v_5\} \text{ corresponding to } l = 104.\\ S &= \{v_2, v_6, v_8\} \text{ corresponding to } l = 187.\\ S &= \{v_3, v_4, v_7\} \text{ corresponding to } l = 206.\\ S &= \{v_4, v_5, v_7\} \text{ corresponding to } l = 230. \end{aligned}$$

The above results illustrate that Algorithm 1 is correct. Because Algorithm 1 is based on Theorems 1-3, the correctness of them can also be guaranteed by the above analysis.

Next, we examine the correctness of Theorems 4-6. According to Theorem 4, each column of M that equals to $\mathbf{k}_n$ determines a k-capacity control set. From M we know that there are two columns equal to $\mathbf{1}_n$ and one column equals to 2_n in M. By Remark 9, we can get two 1-capacity control sets and one 2-capacity control set of G.

In the case of $k = 1$, the index set of Step 2 is $K = \{120, 222\}$.

For $l = 120$, we get

$$\begin{aligned} y_1 &= S_1^8 \ltimes \delta_{256}^{120} = \delta_2^1, \quad y_2 = S_2^8 \ltimes \delta_{256}^{120} = \delta_2^2,\\ y_3 &= S_3^8 \ltimes \delta_{256}^{120} = \delta_2^2, \quad y_4 = S_4^8 \ltimes \delta_{256}^{120} = \delta_2^2,\\ y_5 &= S_5^8 \ltimes \delta_{256}^{120} = \delta_2^1, \quad y_6 = S_6^8 \ltimes \delta_{256}^{120} = \delta_2^2,\\ y_7 &= S_7^8 \ltimes \delta_{256}^{120} = \delta_2^2, \quad y_8 = S_8^8 \ltimes \delta_{256}^{120} = \delta_2^2. \end{aligned}$$

This means that $l = 120$ determines a 1-capacity control set $S = \{v_1, v_5\}$ shown in red in Fig.2.

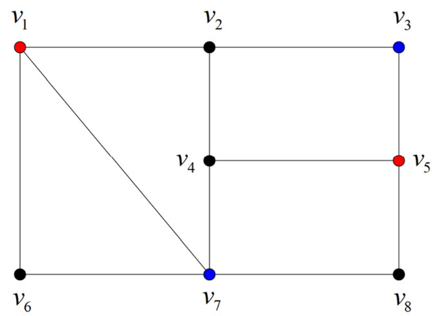

FIGURE 2. 1-capacity control sets of Example 1.

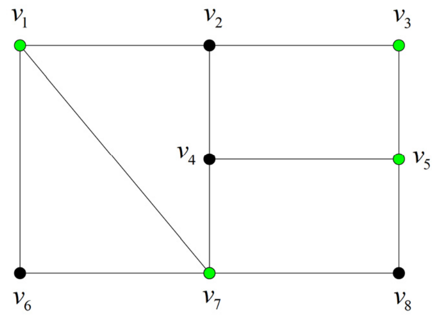

FIGURE 3. 2-capacity control set of Example 1.

For $l = 222$, we get

$$\begin{aligned} y_1 &= S_1^8 \ltimes \delta_{256}^{222} = \delta_2^2, \quad y_2 = S_2^8 \ltimes \delta_{256}^{222} = \delta_2^2,\\ y_3 &= S_3^8 \ltimes \delta_{256}^{222} = \delta_2^1, \quad y_4 = S_4^8 \ltimes \delta_{256}^{222} = \delta_2^2,\\ y_5 &= S_5^8 \ltimes \delta_{256}^{222} = \delta_2^2, \quad y_6 = S_6^8 \ltimes \delta_{256}^{222} = \delta_2^2,\\ y_7 &= S_7^8 \ltimes \delta_{256}^{222} = \delta_2^1, \quad y_8 = S_8^8 \ltimes \delta_{256}^{222} = \delta_2^2. \end{aligned}$$

This shows that $l = 222$ determines a 1-capacity control set $S = \{v_3, v_7\}$ shown in blue in Fig.2.

For $k = 2$, the index set $K = \{86\}$. We get

$$\begin{aligned} y_1 &= S_1^8 \ltimes \delta_{256}^{86} = \delta_2^1, \quad y_2 = S_2^8 \ltimes \delta_{256}^{86} = \delta_2^2,\\ y_3 &= S_3^8 \ltimes \delta_{256}^{86} = \delta_2^1, \quad y_4 = S_4^8 \ltimes \delta_{256}^{86} = \delta_2^2,\\ y_5 &= S_5^8 \ltimes \delta_{256}^{86} = \delta_2^1, \quad y_6 = S_6^8 \ltimes \delta_{256}^{86} = \delta_2^2,\\ y_7 &= S_7^8 \ltimes \delta_{256}^{86} = \delta_2^1, \quad y_8 = S_8^8 \ltimes \delta_{256}^{86} = \delta_2^2. \end{aligned}$$

This means there is a 2-capacity control set $S = \{v_1, v_3, v_5, v_7\}$, which is shown in green in Fig.3. Since there is no column equals to $\mathbf{k}_n$ in M ($k \geq 3$), the graph G contains no k-capacity control set for $k \geq 3$.

Observing Figure 1, we know that the vertex subset $S = \{v_1, v_5\}$ is a 1-harmony control set. By substituting the related parameters into (23) and (26), the correctness of Theorems 5 and 6 can be testified easily in a way similar to the above.

V. CONCLUSION

STP is a powerful mathematical tool to analyze the discrete mathematical structures such as graphs. Graphs provide discrete mathematical models for systems comprised of binary relations. This paper introduces STP to the field of graph theory and uses it to investigate the problems of searching control sets, k-capacity control sets and k-harmony control sets. A set of new results are obtained with the help of STP, including three sufficient and necessary conditions of control sets, an algorithm to find all the control sets of a graph. Moreover, two new kinds of control sets, k-capacity control sets and k-harmony control sets, are proposed, which come from some real problems. Using STP, several new conclusions that can help us to find the k-harmony control sets of graphs are found. These new results of this paper may provide a new angle and means to understand and analyze the structures of graphs.

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