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Singular System Full-Order and Reduced-Order Fixed-Time Observer Design

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ABSTRACT This paper is concerned with the fixed-time observer (FTO) design for linear time-invariant singular systems. The advantage of the classical FTO technique is that it can reach accurate estimations of the states in an arbitrarily pre-defined time. However, the classical FTO method is only applied to non-singular systems. In order to overcome this restriction, new observer structures with respect to full-order and reduced-order FTOs respectively, are proposed. And, under the sufficient conditions we give, both the full-order and the reduced-order FTOs can indeed achieve accurate estimations in the fixed time we pre-set for the singular systems under consideration. Another advantage of the FTO is that both the estimation accuracy and the convergence time can be guaranteed regardless of whatever the initial values of the observer are. Finally, a numerical simulation validates the effectiveness of the proposed result.

INDEX TERMS Singular systems, fixed-time observer, full-order observer, reduced-order observer.

I. INTRODUCTION

As is known, the feedback control scheme plays important roles in both theory and applications in modern control community. However, in many situations, the state of the system under control is not available, which makes it impossible for the implementation of the feedback control. In order to deal with this contradiction, the idea of state estimation or observer design was proposed [1], and up till now, many observer design techniques have been developed such as unknown input observer technique [2]–[7], sliding mode observer technique [8]–[10], interval observer technique [11], [12], and so on.

The singular system, also called descriptor system, differential-algebraic system or generalized state-space system, is a kind of special dynamic system [13], [14]. Since the singular system consists of both dynamic equation and algebraic equation, it has superiority of describing systems with algebraic constraints such as circuit systems, mechanical systems and so on [15], [16]. In the past decades, a great amount of achievements regarding the singular system observer designs have been made in the literature [17]–[23]. To name a few, in Yip *et al.* [17] extended the classical concept of controllability, observability of general linear systems to singular systems. Hou *et al.* presented a new observer design method for singular systems, and meanwhile the sufficient and necessary conditions were discussed [19]. Based on the same conditions as in [19], reduced-order UIOs were studied in [20] and [21], respectively, and the method in [20] can also be applied to rectangular singular systems.

It should be pointed out that almost all the singular system observer designs mentioned above are referred to asymptotic convergence observer (ACO), i.e., the convergence of the state estimation error to zero is always asymptotic with time. For such type of observers, the convergence time and the convergence accuracy are usually not easy to be controlled, because they not only depend on the initial value errors between the original system and the observer system but also depend on the chosen of the observer gain matrices. General speaking, the initial condition of the original system is usually inaccessible. Therefore, we may not able to reduce the initial value errors. A possible approach is to place the observer poles at far left of the complex plane to make the observer error exponentially converge to zero fast. However, this approach requires larger observer gains. And hence, the saturation and increased bandwidth may lead observers to be more susceptible to sensor noises occurring at high frequencies [24]. Therefore, to find a feasible and easy implemented observer design method with fast estimation speed and exact estimation performance is desirable.

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In 2002, Engel *et al.* proposed a type of so-called fixed-time observer (FTO) technique [25]. The merits of the FTO are that it can reach an accurate estimation of original system states in an arbitrarily pre-defined time. By using this technique, Lee *et al.* considered the fault estimation problem [24], and Li *et al.* studied the fault fast reconstruction problem by geometric theory [26]. It should be noted that all the results regarding FTOs mentioned above are focused on non-singular system. However, to the best of the authors's knowledge, few works have been reported on the fixed-time observer design for singular systems. This observation motivates our research.

In this paper, we are dedicated to the full-order and reduced-order fixed-time observer designs for a class of singular systems. The contributions are summarized as follows:

(1) Existence conditions are given, and they are proven to be sufficient for both the full-order and the reduced-order FTO desigs.

(2) In order to overcome the restriction of the classical FTO technique on the singular systems, new observer structures with respect to full-order and reduced-order FTOs respectively, are proposed.

(3) For the singular system under consideration, both the full-order and the reduced-order FTOs can achieve accurate estimations in the pre-defined time.

The rest of the paper is organized as follows. In Section 2, background and some preliminaries are given. Section 3 presents the main results. In Section 4, a numerical simulation is given to verify the effectiveness of the proposed methods. Finally, we give the conclusion in Section 5.

II. PROBLEM FORMULATION AND PRELIMINARIES

A. PROBLEM FORMULATION

Consider the singular system in the form of as follows:

$$\begin{cases} E\dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are state, control input, measurement output vectors, respectively. Matrices $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$ are known constant matrices. Here, without loss of generality, we assume that matrices $\text{rank}(E) < n$ and $\text{rank}(C) = p$.

The main purpose of the present paper is to extend the classical fixed-time observer method introduced in [25] which is only adopted to non-singular systems to singular systems. Therefore, here we firstly present the classical FTO method in the following.

B. PRELIMINARIES

In literature [25], for the non-singular linear time-invariant system

$$\begin{cases} \dot{\breve{x}}(t) = \breve{A}\breve{x}(t) + \breve{B}\breve{u}(t) \\ \breve{y}(t) = \breve{C}\breve{x}(t) \end{cases} \tag{2}$$

with $\breve{x}(t)$, $\breve{u}(t)$, $\breve{y}(t)$ being the state, control input and measurement output, Engel *et al.* first studied the fixed-time observer design. The basic idea of the FTO is to express the state $\breve{x}(t)$ by using the combination of two sub-observers' states as well as a time delay. Specifically, under the assumption that $(\breve{A}, \breve{C})$ is observable, choose matrices $\breve{H}_i (i = 1, 2)$ such that $\breve{F}_i = \breve{A} - \breve{H}_i\breve{C}$ is Hurwitz, and any eigenvalue of $\breve{F}_1$ is different from that of $\breve{F}_2$. Denote

$$\breve{F} = \begin{bmatrix} \breve{F}_1 & 0 \\ 0 & \breve{F}_2 \end{bmatrix}, \quad \breve{G} = \begin{bmatrix} \breve{B} \\ \breve{B} \end{bmatrix}, \ \breve{H} = \begin{bmatrix} \breve{H}_1 \\ \breve{H}_2 \end{bmatrix}, \ \breve{T} = \begin{bmatrix} I_n \\ I_n \end{bmatrix}.$$

In this way, provided that

(I) $\breve{F}$ is Hurwitz

(II) $\det\left(\begin{bmatrix} \breve{T} & e^{\breve{F}\breve{\tau}}\breve{T} \end{bmatrix}\right) \neq 0$ ($\breve{\tau}$ is an arbitrarily pre-set)

the system

$$\begin{cases} \dot{\breve{z}}(t) = \breve{F}\breve{z}(t) + \breve{H}\breve{y}(t) + \breve{G}\breve{u}(t) \\ \hat{\breve{x}}(t) = \begin{bmatrix} I_n & 0 \end{bmatrix}\begin{bmatrix} \breve{T} & e^{\breve{F}\breve{\tau}}\breve{T} \end{bmatrix}^{-1}\left[\breve{z}(t) - e^{\breve{F}\tau}\breve{z}(t-\breve{\tau})\right] \end{cases} \tag{3}$$

is a fixed-time observer which provides a finite-time estimation $\hat{\breve{x}}(t)$, that is when $t \geq t_0 + \tau$, $\hat{\breve{x}}(t) \equiv \breve{x}(t)$.

In order to borrow the classical FTO method for the FTO design for the singular system (1), we give the following two assumptions firstly.

Assumption 1: For system (1), the following matrix rank condition holds:

$$\text{rank}\begin{bmatrix} E \\ C \end{bmatrix} = n.$$

Assumption 2: For system (1), for any complex number s, the following matrix rank condition holds:

$$\text{rank}\begin{bmatrix} sE - A \\ C \end{bmatrix} = n.$$

III. MAIN RESULTS

A. FULL-ORDER OBSERVER DESIGN

Since the classical FTO method is only adopted to non-singular systems, in order to borrow its idea for the FTO design for singular system (1), we have to transform system (1) into a non-singular system in advance. To this end, we give Lemma 1.

Lemma 1: Based on Assumption 1, there exist a non-singular matrix $G \in \mathbb{R}^{n\times n}$ and a matrix $H \in \mathbb{R}^{n\times p}$ such that

$$GE + HC = I_n. \tag{4}$$

Proof: For matrix C, there exists a non-singular matrix $T \in \mathbb{R}^{n\times n}$ such that $CT = \begin{bmatrix} I_p & 0 \end{bmatrix}$. Let

$$\begin{bmatrix} E_1 & E_2 \end{bmatrix} := ET,$$

where $E_1 \in \mathbb{R}^{n\times p}$ and $E_2 \in \mathbb{R}^{n\times(n-p)}$. Then, we have

$$n = \text{rank}\begin{bmatrix} E \\ C \end{bmatrix} = \text{rank}\left(\begin{bmatrix} E \\ C \end{bmatrix} T\right) = \text{rank}\begin{bmatrix} E_1 & E_2 \\ I_p & 0 \end{bmatrix}$$

which implies that $\text{rank}(E_2) = n - p$, i.e., E_2 has full column rank. Thus, there exists a non-singular matrix $\bar{G} \in \mathbb{R}^{n\times n}$ such that

$$\bar{G}E_2 = \begin{bmatrix} 0_{p\times(n-p)} \\ I_{n-p} \end{bmatrix}.$$

Let $\begin{bmatrix} E_{11} \\ E_{12} \end{bmatrix} := \bar{G}E_1$. Then, we have

$$\bar{G}\begin{bmatrix} E_1 & E_2 \end{bmatrix} = \begin{bmatrix} E_{11} & 0 \\ E_{12} & I_{n-p} \end{bmatrix}.$$

By this way, if choose $\bar{H} = \begin{bmatrix} (I_p - E_{11})^T & -E_{12}^T \end{bmatrix}^T$, we have

$$\bar{G}ET + \bar{H}CT = I_n \tag{5}$$

which, by pre-multiplying matrix T and meanwhile post-multiplying matrix T^{-1}, becomes

$$T\bar{G}E + T\bar{H}C = I_n. \tag{6}$$

Let $G = T\bar{G}$ and $H = T\bar{H}$, then we have $GE + HC = I_n$ with G being a non-singular matrix. The proof is completed. □

According to (4), system (1) can be equivalently transformed into

$$\begin{cases} \dot{x}(t) - H\dot{y}(t) = GAx(t) + GBu(t) \\ y(t) = Cx(t) \end{cases} \tag{7}$$

Remark 1: System (7) has already been a standard non-singular system in form. Thus, it can be seen from the preliminary part that for system (7), in order to use the conclusions of the classical FTO method to design a fixed-time observer, the matrix pair (GA, C) should be observable.

Lemma 2: Assumption 2 holds, if and only if (GA, C) is observable, i.e., for any complex number s, we have

$$\operatorname{rank}\begin{bmatrix} sI_n - GA \\ C \end{bmatrix} = n.$$

Proof: It follows from the Assumption 2 that for any complex number s

$$\begin{aligned} n &= \operatorname{rank}\begin{bmatrix} sE - A \\ C \end{bmatrix} \\ &= \operatorname{rank}\left(\begin{bmatrix} G & 0 \\ 0 & I_p \end{bmatrix}\begin{bmatrix} sE - A \\ C \end{bmatrix}\right) \\ &= \operatorname{rank}\begin{bmatrix} sGE - GA \\ C \end{bmatrix} \\ &= \operatorname{rank}\left(\begin{bmatrix} I_n & sH \\ 0 & I_p \end{bmatrix}\begin{bmatrix} sGE - GA \\ C \end{bmatrix}\right) \\ &= \operatorname{rank}\begin{bmatrix} sI_n - GA \\ C \end{bmatrix} \end{aligned}$$

which means that (GA, C) is observable. The proof is completed. □

Remark 2: Since (GA, C) is observable, we can choose gain matrices L_1 and L_2 such that any eigenvalue of $\mathcal{N}_1 = GA - L_1C$ is different from that of $\mathcal{N}_2 = GA - L_2C$.

For system (7), another difficulty of the FTO design we have to deal with is that the unmeasurable signal $\dot{y}(t)$ is involved. Therefore, the classical FTO structure (3) cannot be adopted for system (7). In the following, we will overcome this problem by introducing the measurement signal $y(t)$ and $y(t-\tau)$ in the new FTO structure as follows:

$$\begin{cases} \dot{z}_1(t) = \mathcal{N}_1 z_1(t) + GBu(t) + L_1 y(t) + \mathcal{N}_1 Hy(t) \\ \dot{z}_2(t) = \mathcal{N}_2 z_1(t) + GBu(t) + L_2 y(t) + \mathcal{N}_2 Hy(t) \\ \hat{x}(t) = \mathcal{K}[z(t) - e^{\mathcal{N}\tau}z(t-\tau) + \mathcal{J}y(t) - e^{\mathcal{N}\tau}\mathcal{J}y(t-\tau)] \end{cases} \tag{8}$$

where $\mathcal{N} = \operatorname{diag}\{\mathcal{N}_1, \mathcal{N}_2\}$, $S = \begin{bmatrix} I_n^T & I_n^T \end{bmatrix}^T$ and $\mathcal{K} = \begin{bmatrix} I_n & 0_{n\times n} \end{bmatrix}\begin{bmatrix} S & e^{\mathcal{N}\tau}S \end{bmatrix}^{-1}$. Then, we give Theorem 1.

Theorem 1: Based on Assumptions 1-2 and lemmas 1-2, choose gain matrices L_1 and L_2 such that $\mathcal{N}_1$ and $\mathcal{N}_2$ are Hurwitz, and for any given $\rho_1 > \rho_2 > 0$

$$-\rho_2 < \lambda_i(\mathcal{N}_1) < \lambda_j(\mathcal{N}_2) < -\rho_1, \quad i, j = 1, 2, \ldots, n \tag{9}$$

holds. Then, system (8) or equivalently system

$$\begin{cases} \dot{z}(t) = \mathcal{N}z(t) + \mathcal{M}u(t) + \mathcal{L}y(t) + \mathcal{N}\mathcal{J}y(t) \\ \hat{x}(t) = \mathcal{K}[z(t) - e^{\mathcal{N}\tau}z(t-\tau) + \mathcal{J}y(t) - e^{\mathcal{N}\tau}\mathcal{J}y(t-\tau)] \end{cases} \tag{10}$$

is a fixed-time observer of system (1), i.e., $\hat{x}(t) \equiv x(t)$ for $t \geq t_0 + \tau$, where

$$\mathcal{M} = \begin{bmatrix} GB \\ GB \end{bmatrix}, \quad \mathcal{L} = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}, \quad \mathcal{J} = \begin{bmatrix} H \\ H \end{bmatrix},$$

$z = \begin{bmatrix} z_1^T & z_2^T \end{bmatrix}^T$ is the state of the observer, constant $\tau > 0$ is an arbitrarily pre-defined time delay, t_0 is the initial time, and $z(t), y(t) = 0$ when $-\tau \leq t < 0$.

Proof: According to Assumption 2, Lemma 2 and Remark 2 we can choose L_1 and L_2 such that (9) holds. Thus, according to the discussion in [25], for any $\tau > 0$ we have that $\begin{bmatrix} S & e^{\mathcal{N}\tau}S \end{bmatrix}$ is non-singular. Thus, matrix $\mathcal{K}$ in (8) or (10) exists.

Besides, according to (10) for any $t \geq t_0 + \tau$ we have

$$\begin{aligned} \left[\dot{z}(t) + \mathcal{J}\dot{y}(t) - S\dot{x}(t)\right] &= \mathcal{N}z(t) + \mathcal{M}u(t) + \mathcal{L}y(t) \\ &\quad + \mathcal{N}\mathcal{J}y(t) + \mathcal{J}\dot{y}(t) \\ &\quad - S\left[GAx(t) + GBu(t) + H\dot{y}(t)\right] \\ &= \mathcal{N}\left[z(t) + \mathcal{J}y(t) - Sx(t)\right] \end{aligned} \tag{11}$$

Equation (11) implies that for any $\tau > 0$, and $t \geq t_0 + \tau$

$$z(t) + \mathcal{J}y(t) - Sx(t) = e^{\mathcal{N}\tau}\left[z(t-\tau) + \mathcal{J}y(t-\tau) - Sx(t-\tau)\right]. \tag{12}$$

On the other hand, it is easy to verify that $\mathcal{K}S = I_n$ and $\mathcal{K}e^{\mathcal{N}\tau}S = 0$. This together with equation (12) yields

$$\begin{aligned} x(t) &= \mathcal{K}Sx(t) \\ &= \mathcal{K}\Big[z(t) + \mathcal{J}y(t) - e^{\mathcal{N}\tau}z(t-\tau) \\ &\quad - e^{\mathcal{N}\tau}\mathcal{J}y(t-\tau) + e^{\mathcal{N}\tau}Sx(t-\tau)\Big] \\ &= \mathcal{K}[z(t) - e^{\mathcal{N}\tau}z(t-\tau) + \mathcal{J}y(t) - e^{\mathcal{N}\tau}\mathcal{J}y(t-\tau)]. \end{aligned}$$

Therefore, when $t \geq t_0+\tau$, for the estimated state $\hat{x}$ given by FTO (10) we have $\hat{x}(t) \equiv x(t)$. The proof is completed. □

Remark 3: It can be seen from the proof of the Theorem 1 that once equations (11) and (12) hold, the state $x(t)$ can be expressed by the sub-observers' states $z_1(t)$ and $z_2(t)$. Therefore, in order to deal with the unmeasurable signal $\dot{y}(t)$ and then design FTO for system (7), what we should do is to design a new observer structure making that equation (11) holds. This just is the basic idea of the proposed FTO design method of the singular systems.

Then, the full-order FTO design method is summarized as Algorithm 1.

Algorithm 1.

Step 1. Check if Assumptions 1-2 hold, if so, go to next step; Otherwise, stop without results.

Step 2. According to the proof of Lemma 1, compute matrices T, G and H.

Step 3. According to (9), choose gain matrices L_1 and L_2.

Step 4. Compute matrices $\mathscr{N}$, $\mathscr{M}$, $\mathscr{L}$ and $\mathscr{J}$, and construct full-order FTO (10).

B. REDUCED-ORDER OBSERVER DESIGN

By using matrices T and $\bar{G}$ in subsection 3.1, let $\bar{x} = T^{-1}x$, then a new system regarding $\bar{x}$ is obtained as

$$\begin{cases} ET\dot{\bar{x}}(t) = AT\bar{x}(t) + Bu(t) \\ y(t) = CT\bar{x}(t) = \begin{bmatrix} I_p & 0 \end{bmatrix}\bar{x}(t) \end{cases} \tag{13}$$

Pre-multiplying matrix $\bar{G}$ in the two sides of the state equation of (13) leads to

$$\begin{cases} \bar{E}\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}u(t) \\ y(t) = \begin{bmatrix} I_p & 0 \end{bmatrix}\bar{x}(t) \end{cases} \tag{14}$$

where $\bar{E} = \bar{G}ET$, $\bar{A} = \bar{G}AT$ and $\bar{B} = \bar{G}B$. Then, decompose vector $\bar{x}$ and matrices $\bar{E}, \bar{A}, \bar{B}$ into block matrices as follows:

$$\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}, \quad \bar{E} = \begin{bmatrix} E_{11} & 0 \\ E_{12} & I_{n-p} \end{bmatrix},$$
$$\bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}, \ \bar{B} = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix}.$$

By this way, system (14) can be rewritten as

$$E_{11}\dot{\bar{x}}_1(t) = \bar{A}_{11}\bar{x}_1(t) + \bar{A}_{12}\bar{x}_2(t) + \bar{B}_1u(t) \tag{15}$$
$$E_{12}\dot{\bar{x}}_1(t) + \dot{\bar{x}}_2(t) = \bar{A}_{21}\bar{x}_1(t) + \bar{A}_{22}\bar{x}_2(t) + \bar{B}_2u(t) \tag{16}$$
$$y(t) = \bar{x}_1(t) \tag{17}$$

It can be seen from (17) that $\bar{x}_1$ is measurable. Thus, in order to estimate the entire x, we only need to estimate $\bar{x}_2$.

Choose gain matrices $\bar{L}_1$ and $\bar{L}_2$, and then the system (15) pre-multiplied by $-\bar{L}_1$ and $-\bar{L}_2$ adds to system (16) yields

$$\begin{aligned}(E_{12} - \bar{L}_1E_{11})\dot{\bar{x}}_1(t) + \dot{\bar{x}}_2(t) = (\bar{A}_{21} - \bar{L}_1\bar{A}_{11})\bar{x}_1(t) \\ + (\bar{A}_{22} - \bar{L}_1\bar{A}_{12})\bar{x}_2(t) \\ + (\bar{B}_2 - \bar{L}_1\bar{B}_1)u(t)\end{aligned} \tag{18}$$

and

$$\begin{aligned}(E_{12} - \bar{L}_2E_{11})\dot{\bar{x}}_1(t) + \dot{\bar{x}}_2(t) = (\bar{A}_{21} - \bar{L}_2\bar{A}_{11})\bar{x}_1(t) \\ + (\bar{A}_{22} - \bar{L}_2\bar{A}_{12})\bar{x}_2(t) \\ + (\bar{B}_2 - \bar{L}_2\bar{B}_1)u(t)\end{aligned} \tag{19}$$

respectively.

Remark 4: Systems (18) and (19) are two auxiliary systems which will be taken as the reference systems for the FTO design to estimate $\bar{x}_2$. Although with different dynamics, the state $\bar{x}_2$ in both the system(18) and (19) are as same as that in (15) and (16). It can be seen in the following part that after constructing such two auxiliary systems, we can easily design two sub-observers as parts of the FTO where each auxiliary system is taken as the reference system.

In order to construct two sub-observers or a FTO, the condition of $(\bar{A}_{22}, \bar{A}_{12})$ being observable is needed. Thus, we give Lemma 3.

Lemma 3: Assumption 2 holds, if and only if $(\bar{A}_{22}, \bar{A}_{12})$ is observable, i.e., for any complex number s, we have

$$\operatorname{rank}\begin{bmatrix} sI_{n-p} - \bar{A}_{22} \\ \bar{A}_{12} \end{bmatrix} = n - p. \tag{20}$$

Proof: Based on Assumption 2, for any complex number s we have

$$\begin{aligned} n &= \operatorname{rank}\begin{bmatrix} sE - A \\ C \end{bmatrix} \\ &= \operatorname{rank}\left(\begin{bmatrix} sE - A \\ C \end{bmatrix} T\right) \\ &= \operatorname{rank}\begin{bmatrix} sET - AT \\ \begin{bmatrix} I_p & 0_{p\times n} \end{bmatrix} \end{bmatrix} \\ &= \operatorname{rank}\left(\begin{bmatrix} \bar{G} & 0 \\ 0 & I_p \end{bmatrix}\begin{bmatrix} sET - AT \\ \begin{bmatrix} I_p & 0_{p\times n} \end{bmatrix} \end{bmatrix}\right) \\ &= \operatorname{rank}\begin{bmatrix} sE_{11} - \bar{A}_{11} & -\bar{A}_{12} \\ sE_{12} - \bar{A}_{21} & sI - \bar{A}_{22} \\ I_p & 0 \end{bmatrix} \\ &= p + \operatorname{rank}\begin{bmatrix} sI - \bar{A}_{22} \\ \bar{A}_{12} \end{bmatrix} \end{aligned}$$

This implies that (20) holds. The proof is completed. □

Remark 5: Lemma 3 demonstrates that we can choose $\bar{L}_1$ and $\bar{L}_2$ such that any eigenvalue of the $\bar{\mathscr{N}}_1 = \bar{A}_{22} - \bar{L}_1\bar{A}_{12}$ is different from that of $\bar{\mathscr{N}}_2 = \bar{A}_{22} - \bar{L}_2\bar{A}_{12}$.

Again, let

$$\begin{aligned} \bar{\mathscr{W}}_i &= \bar{A}_{21} - \bar{L}_i\bar{A}_{11} \\ \bar{\mathscr{M}}_i &= -\bar{L}_i\bar{B}_1 + \bar{B}_2 \\ \bar{\mathscr{J}}_i &= -E_{12} + \bar{L}_iE_{11} \end{aligned}$$

$i = 1, 2$ and

$$\bar{\mathscr{N}} = \begin{bmatrix} \bar{\mathscr{N}}_1 & 0 \\ 0 & \bar{\mathscr{N}}_2 \end{bmatrix}, \bar{\mathscr{W}} = \begin{bmatrix} \bar{\mathscr{W}}_1 \\ \bar{\mathscr{W}}_2 \end{bmatrix}, \bar{\mathscr{M}} = \begin{bmatrix} \bar{\mathscr{M}}_1 \\ \bar{\mathscr{M}}_2 \end{bmatrix}, \bar{\mathscr{J}} = \begin{bmatrix} \bar{\mathscr{J}}_1 \\ \bar{\mathscr{J}}_2 \end{bmatrix},$$
$$\bar{S} = \begin{bmatrix} I_{n-p} \\ I_{n-p} \end{bmatrix}.$$

Then, a reduced-order FTO of system (1) is constructed in Theorem 2.

Theorem 2: Based on Assumptions 1-2 and Lemma 1, Lemma 3, choose gain matrices $\bar{L}_1$ and $\bar{L}_2$ such that $\bar{\mathcal{N}}_1$ and $\bar{\mathcal{N}}_2$ are Hurwitz, and for any $\bar{\rho}_1 > \bar{\rho}_2 > 0$

$$-\bar{\rho}_2 < \lambda_i(\bar{\mathcal{N}}_1) < \lambda_j(\bar{\mathcal{N}}_2) < -\bar{\rho}_1 \tag{21}$$

$i, j = 1, 2, \ldots$ holds. Then, the following system

$$\begin{cases} \dot{\bar{z}}_1(t) = \bar{\mathcal{N}}_1\bar{z}_1(t) + \bar{\mathcal{W}}_1 y(t) + \bar{\mathcal{M}}_1 u(t) + \bar{\mathcal{N}}_1\bar{\mathcal{J}}_1 y(t) \\ \dot{\bar{z}}_2(t) = \bar{\mathcal{N}}_2\bar{z}_2(t) + \bar{\mathcal{W}}_2 y(t) + \bar{\mathcal{M}}_2 u(t) + \bar{\mathcal{N}}_2\bar{\mathcal{J}}_2 y(t) \\ \hat{\bar{x}}_2(t) = \bar{\mathcal{K}}[\bar{z}(t) - e^{\bar{\mathcal{N}}\tau}\bar{z}(t-\tau) + \bar{\mathcal{J}} y(t) - e^{\bar{\mathcal{N}}\tau}\bar{\mathcal{J}} y(t-\tau)] \end{cases} \tag{22}$$

or equivalently

$$\begin{cases} \dot{\bar{z}}(t) = \bar{\mathcal{N}}\bar{z}(t) + \bar{\mathcal{W}} y(t) + \bar{\mathcal{M}} u(t) + \bar{\mathcal{N}}\bar{\mathcal{J}} y(t) \\ \hat{\bar{x}}_2(t) = \bar{\mathcal{K}}[\bar{z}(t) - e^{\bar{\mathcal{N}}\tau}\bar{z}(t-\tau) + \bar{\mathcal{J}} y(t) - e^{\bar{\mathcal{N}}\tau}\bar{\mathcal{J}} y(t-\tau)] \end{cases} \tag{23}$$

is a fixed-time observer of state $\bar{x}_2$ with the order $n-p$, i.e., $\hat{\bar{x}}_2(t) \equiv \bar{x}_2(t)$ for $t \geq t_0 + \bar{\tau}$, where $\bar{z} = \begin{bmatrix}\bar{z}_1^T & \bar{z}_2^T\end{bmatrix}^T$ is the state of the observer,

$$\bar{\mathcal{K}} = \begin{bmatrix} I_{n-p} & 0_{(n-p)\times(n-p)} \end{bmatrix} \begin{bmatrix} \bar{S} & e^{\bar{\mathcal{N}}\bar{\tau}}\bar{S} \end{bmatrix}^{-1},$$

constant $\bar{\tau} > 0$ is an arbitrarily pre-defined time delay, t_0 is the initial time, and $\bar{z}(t), y(t) = 0$ when $-\tau \leq t < 0$. Furthermore, we have

$$\hat{x}(t) = T\begin{bmatrix} y(t) \\ \hat{\bar{x}}_2(t) \end{bmatrix} \tag{24}$$

with $\hat{x}(t) \equiv x(t)$ for $t \geq t_0 + \bar{\tau}$.

Proof: According to the state transformation, we have

$$x(t) = T\begin{bmatrix} y(t) \\ \bar{x}_2(t) \end{bmatrix}. \tag{25}$$

From (24) and (25) we know that in order to show $\hat{x}(t) \equiv x(t)$ for $t \geq t_0 + \bar{\tau}$, we only need to show $\hat{\bar{x}}_2(t) \equiv \bar{x}_2(t)$ for $t \geq t_0 + \bar{\tau}$. While, this process is very similar with that in Subsection 3.1, and here we omit it. □

The reduced-order FTO design method is concluded as Algorithm 2.

Algorithm 2.

Step 1. Based on the chosen of matrix T, compute matrices $\bar{E}, \bar{A}, \bar{B}$ and their block matrices.

Step 2. According to (21), compute gain matrices $\bar{L}_1$ and $\bar{L}_2$.

Step 3. Compute matrices $\bar{\mathcal{N}}$, $\bar{\mathcal{M}}$, $\bar{\mathcal{W}}$ and $\bar{\mathcal{J}}$, and then construct reduced-order FTO (23).

IV. NUMERICAL SIMULATION

In this section, one numerical example is given to verify the effectiveness of the proposed methods.

Consider system (1) with the matrices as follows:

$$E = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & -1 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

and the control input is $u(t) = \begin{bmatrix} 2.8sin(5.2t) & 4.9cos(6.7t) \end{bmatrix}^T$. In the following, state estimations will be given by using full-order and reduced-order FTOs, respectively.

A. STATE ESTIMATION VIA FULL-ORDER FTO

According to Algorithm 1, computing G and H obtains

$$G = \begin{bmatrix} 1.0000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0000 \\ 0 & 0.5000 & -0.5000 & 0 \\ 0 & 0.5000 & 1.5000 & 0 \end{bmatrix}$$

$$H = \begin{bmatrix} 1.0000 & 0 & -1.0000 \\ 0 & 0 & 1.0000 \\ 0 & 0.5000 & 0.5000 \\ 0 & -0.5000 & -1.5000 \end{bmatrix}$$

Then, choose gain matrices

$$L_1 = \begin{bmatrix} 0.2335 & 0.0273 & -1.0283 \\ 0.8043 & 0.7993 & 2.7013 \\ -0.3645 & 0.6302 & -0.4006 \\ -0.7483 & 1.4350 & 0.8370 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} 3.6053 & 0.0140 & -1.0145 \\ 0.5992 & 1.9509 & 8.3500 \\ 0.4203 & -1.7280 & -8.0744 \\ -1.7785 & 8.3726 & 10.7310 \end{bmatrix}$$

such that the eigenvalues of the matrices $N_1 = GA - L_1C$ and $N_2 = GA - L_2C$ are $\{-1.1, -1.2, -1.3, -1.4\}$ and $\{-4.5, -4.6, -4.7, -4.8\}$, respectively. Following this, without loss of generality, choose the observer convergence time $\tau = 0.5s$, we can compute matrices $\mathcal{N}$, $\mathcal{M}$, $\mathcal{L}$ and $\mathcal{J}$ (due to the length limited, we omit them here), and then the full-order FTO (10) can be constructed.

For the purpose of simulation, set the initial state of system (1) as $x(0) = \begin{bmatrix} -1.3 & 0.2 & -1 & -0.3 \end{bmatrix}^T$ and FTO's initial value $z(0) = \begin{bmatrix} 1 & 0.2 & 0.2 & -0.1 & -0.1 & 0.2 & -0.2 & 0.3 \end{bmatrix}^T$. We plot the estimations in Fig. 1 and the estimation errors in Fig. 2, respectively, from which we can see that via the full-order FTO, the accurate estimation can be reached within the pre-defined time $\tau = 0.5s$.

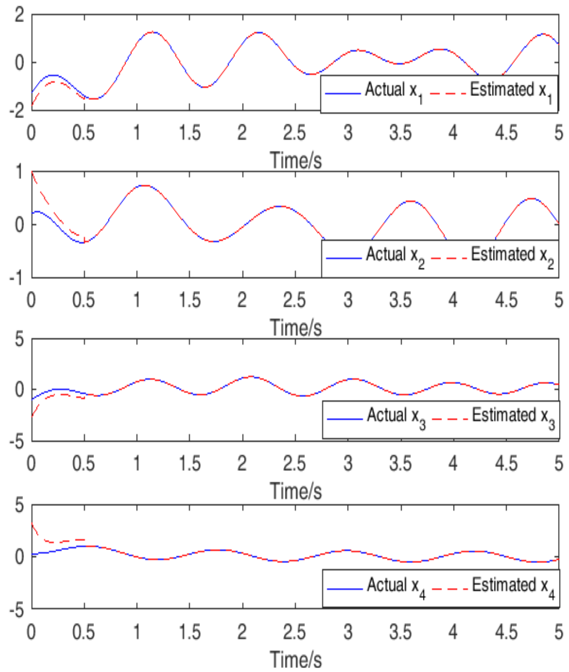

FIGURE 1. State estimations of $x_1 - x_4$ via full-order FTO.

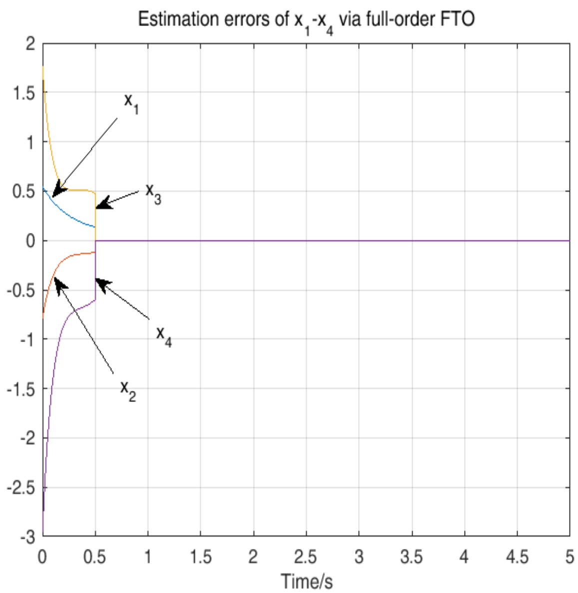

FIGURE 2. State estimation errors of $x_1 - x_4$ via full-order FTO.

B. STATE ESTIMATION VIA REDUCED-ORDER FTO

According to Algorithm 2, compute matrices

$$\bar{E} = \begin{bmatrix} E_{11} & 0_{3\times 1} \\ E_{12} & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 & 0 \\ -1 & -1 & -1 & 2 \\ 1 & -1 & 0 & 2 \\ 0 & -1 & -1 & 1 \end{bmatrix}$$

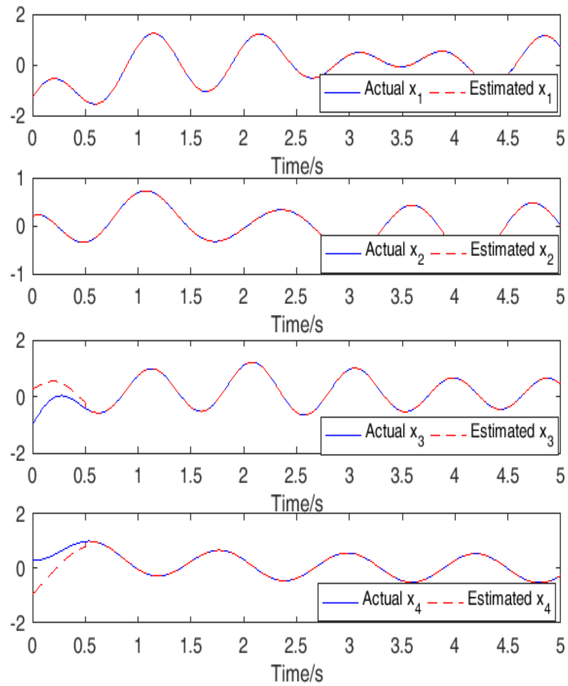

FIGURE 3. State estimations of $x_1 - x_4$ via reduced-order FTO.

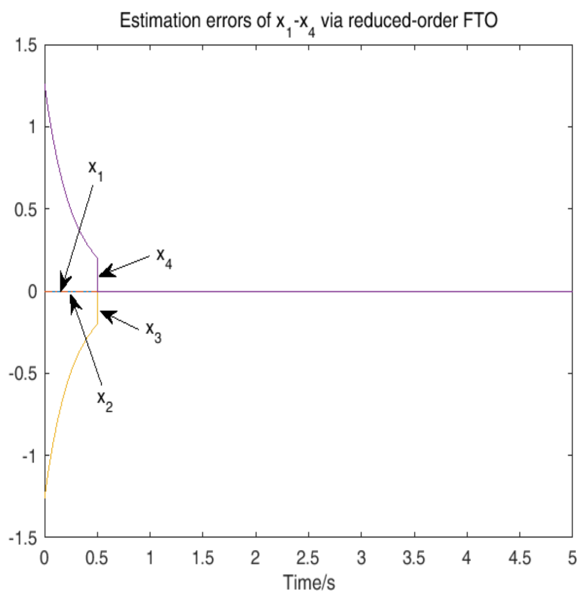

FIGURE 4. State estimation errors of $x_1 - x_4$ via reduced-order FTO.

$$\bar{B} = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Choose gain matrices $\bar{L}_1 = \begin{bmatrix} 0 & 0.5 & 0.5 \end{bmatrix}$ and $\bar{L}_2 = \begin{bmatrix} 0 & 1.25 & 1.25 \end{bmatrix}$ such that the eigenvalues of matrices $\bar{N}_1 = \bar{A}_{22} - \bar{L}_1\bar{A}_{12}$ and $\bar{N}_2 = \bar{A}_{22} - \bar{L}_2\bar{A}_{12}$ are -1 and -4, respectively. And based on these results, the gain matrices regarding the reduced-order FTO (23) are

$$\bar{\mathscr{N}} = \begin{bmatrix} -1.0000 & 0 \\ 0 & -4.0000 \end{bmatrix}, \quad \bar{\mathscr{M}} = \begin{bmatrix} 0 & -0.5000 \\ 0 & -1.2500 \end{bmatrix}$$

$$\bar{\mathscr{W}} = \begin{bmatrix} 0 & -0.0000 & -0.5000 \\ 0 & 1.5000 & 0.2500 \end{bmatrix}$$

$$\bar{\mathscr{J}} = \begin{bmatrix} 0 & 0.5000 & -0.5000 \\ 0 & 1.2500 & 0.2500 \end{bmatrix}$$

Following the above computations, we do the simulation. Set the initial condition of system (1) as same as that of the previous subsection, and the reduced-order FTO's initial value is $\bar{z}(0) = \begin{bmatrix}1 & 0.2\end{bmatrix}^T$. The estimation performances are illustrated in Figs. 3-4, which indicates that the reduced-order FTO can also give accurate estimations within the pre-defined time $\tau = 0.5s$.

V. CONCLUSION

In this paper, the full-order and reduced-order fixed-time observer designs for linear time-invariant singular systems are studied. Two new observer structures are developed, and it has been proven that under the sufficient conditions we give, both the full-order and the reduced-order FTOs can achieve accurate estimations in the fixed time we pre-set. In view of the superiority of the FTO, how to extend the present methods to T-S fuzzy systems will be our future consideration.

REFERENCES

[1] D. G. Luenberger, "Observing the state of a linear system," *IEEE Trans. Mil. Electron.*, vol. MIL-8, no. 2, pp. 74–80, Apr. 1964.

[2] P. Kudva, N. Viswanadham, and A. Ramakrishna, "Observers for linear systems with unknown inputs," *IEEE Trans. Autom. Control*, vol. 25, no. 1, pp. 113–115, Feb. 1980.

[3] F. Zhu, "State estimation and unknown input reconstruction via both reduced-order and high-order sliding mode observers," *J. Process Control*, vol. 22, no. 1, pp. 296–302, Jan. 2012.

[4] J. Zhang and F. Zhu, "On the observer matching condition and unknown input observer design based on the system left-invertibility concept," *Trans. Inst. Meas. Control*, vol. 40, no. 9, pp. 2887–2900, 2018.

[5] J. Zhang, X. Zhao, F. Zhu, and H. Karimi, "Reduced-order observer design for switched descriptor systems with unknown inputs," *IEEE Trans. Autom. Control*, to be published. doi: 10.1109/TAC.2019.2913050.

[6] X. Li, C. K. Ahn, D. Lu, and S. Guo, "Robust simultaneous fault estimation and nonfragile output feedback fault-tolerant control for Markovian jump systems," *IEEE Trans. Syst., Man, Cybern., Syst.*, to be published. doi: 10.1109/TSMC.2018.2828123.

[7] M. Pazera and M. Witczak, "Towards robust simultaneous actuator and sensor fault estimation for a class of nonlinear systems: Design and comparison," *IEEE Access*, vol. 7, pp. 97143–97158, 2019.

[8] K. Kalsi, J. Lian, S. Hui, and S. H. Żak, "Sliding-mode observers for systems with unknown inputs: A high-gain approach," *Automatica*, vol. 46, no. 2, pp. 347–353, Feb. 2010.

[9] X. Wang, C. P. Tan, and D. Zhou, "A novel sliding mode observer for state and fault estimation in systems not satisfying matching and minimum phase conditions," *Automatica*, vol. 79, pp. 290–295, May 2017.

[10] S. Yin, H. Yang, and O. Kaynak, "Sliding mode observer-based FTC for Markovian jump systems with actuator and sensor faults," *IEEE Trans. Autom. Control*, vol. 62, no. 7, pp. 3551–3558, Jul. 2017.

[11] Y. Wang, D. M. Bevly, and R. Rajamani, "Interval observer design for LPV systems with parametric uncertainty," *Automatica*, vol. 60, pp. 79–85, Oct. 2015.

[12] H. Oubabas, S. Djennoune, and M. Bettayeb, "Interval sliding mode observer design for linear and nonlinear systems," *J. Process Control*, vol. 61, pp. 12–22, Jan. 2018.

[13] J. Zhang, F. Zhu, J. Li, and X. Li, "Discrete-time linear descriptor system unknown input observer design: An auxiliary output-based approach," *Int. J. Control, Autom. Syst.*, vol. 15, no. 6, pp. 2599–2607, 2017.

[14] J. Zhang, F. Zhu, H. R. Karimi, and F. Wang, "Observer-based sliding mode control for T-S fuzzy descriptor systems with time-delay," *IEEE Trans. Fuzzy Syst.*, to be published. doi: 10.1109/TFUZZ.2019.2893220.

[15] Z. Feng and P. Shi, "Admissibilization of singular interval-valued fuzzy systems," *IEEE Trans. Fuzzy Syst.*, vol. 25, no. 6, pp. 1765–1776, Dec. 2017.

[16] Z. Feng and P. Shi, "Sliding mode control of singular stochastic Markov jump systems," *IEEE Trans. Autom. Control*, vol. 62, no. 8, pp. 4266–4273, Aug. 2017.

[17] E. Yip and R. Sincovec, "Solvability, controllability, and observability of continuous descriptor systems," *IEEE Trans. Autom. Control*, vol. 26, no. 3, pp. 702–707, Jun. 1981.

[18] Z. Wang, M. Rodrigues, D. Theilliol, and Y. Shen, "Fault estimation filter design for discrete-time descriptor systems," *IET Control Theory Appl.*, vol. 9, no. 10, pp. 1587–1594, 2015.

[19] M. Hou and P. C. Muller, "Observer design for descriptor systems," *IEEE Trans. Autom. Control*, vol. 44, no. 1, pp. 164–169, Jan. 1999.

[20] M. K. Gupta, N. K. Tomar, and S. Bhaumik, "Full- and reduced-order observer design for rectangular descriptor systems with unknown inputs," *J. Franklin Inst.*, vol. 352, no. 3, pp. 1250–1264, Mar. 2015.

[21] M. Darouach, M. Zasadzinski, and M. Hayar, "Reduced-order observer design for descriptor systems with unknown inputs," *IEEE Trans. Autom. Control*, vol. 41, no. 7, pp. 1068–1072, Jul. 1996.

[22] J. Lin and Z. Gao, "Observers design for switched discrete-time singular time-delay systems with unknown inputs," *Nonlinear Anal., Hybrid Syst.*, vol. 18, pp. 85–99, Nov. 2015.

[23] J. Lin, Y. Shi, J. Ding, and Z. Gao, "Functional observer for switched discrete-time singular systems with time delays and unknown inputs," *IET Control Theory Appl.*, vol. 9, no. 14, pp. 2146–2156, 2015.

[24] K.-S. Lee and T.-G. Park, "New results on fault reconstruction using a finite-time converging unknown input observer," *IET Control Theory Appl.*, vol. 6, no. 9, pp. 1258–1265, Jun. 2012.

[25] R. Engel and G. Kreisselmeier, "A continuous-time observer which converges in finite time," *IEEE Trans. Autom. Control*, vol. 47, no. 7, pp. 1202–1204, Jul. 2002.

[26] F. Li, G. Zhao, and J. Huang, "Fast fault estimation for linear systems with unmatched faults," *Proc. Inst. Mech. Eng. G, J. Aerosp. Eng.*, vol. 230, no. 3, pp. 554–565, 2016.

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