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2p-th Moment Global Exponential Stability of a Unique 2P-th Mean Almost Periodic Oscillation for Semi-Discrete Takagi-Sugeno Fuzzy Stochastic Cellular Neural Networks With Time Delays

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ABSTRACT By using semi-discrete and Takagi-Sugeno fuzzy methods, a new version of discrete analogue of stochastic fuzzy cellular neural networks is formulated, which gives a more accurate characterization for continuous-time stochastic model than that by Euler scheme. Firstly, the 2p-th moment global exponential stability for the obtained semi-discrete stochastic Takagi-Sugeno fuzzy model is studied with the help of Minkowski inequality and Hölder inequality. Secondly, the 2p-th mean almost periodic outputs of the model is investigated by using Krasnoselskii's fixed point theorem. Finally, illustrative examples and numerical simulations are given to demonstrate that our results are feasible.

INDEX TERMS Moment global exponential stability, semi-discrete method, stochastics, Takagi-Sugeno approach.

I. INTRODUCTION

In [1], cellular neural networks (CNNs), which have been widely applied in psychophysics, parallel computing, perception, robotics, adaptive pattern recognition, associative memory, image processing pattern recognition and combinatorial optimization. All of these applications heavily depend on the (almost) periodicity and global exponential stability. Specifically, many scholars had focused on the issues of the existence and global exponential stability of the equilibrium point, periodic and almost periodic solutions of CNNs with time delays in literatures [2]–[7]. For instance, Xu [7] considered the following CNNs with time delays:

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -a_i(t)x_i(t) + \sum_{j=1}^{n} b_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^{n} c_{ij}(t)g_j(x_j(t-\tau_{ij}(t))) + I_i(t), \qquad \text{(I.1)}$$

where n denotes the number of units in a neural network, $x_i(t)$ corresponds to the state of the ith unit at time t, $a_i > 0$ represents the passive decay rate at time t, f_j and g_j are the neuronal output signal functions, $b_{ij}(t)$ and $c_{ij}(t)$ denote the strength of the jth unit on the ith unit at time t, $I_i(t)$ denotes the external input at time t, the continuous function $\tau_{ij}(t)$ corresponds to the information transmission delay at time t, $i, j = 1, 2, \ldots, n$. In [7], the author studied the existence and exponential stability of anti-periodic solutions of system (I.1).

Uncertain models described by stochastic differential equations have received great attentions in recent years, since they have been widely applied in practice such as engineering, physics, chemistry and biology [8], [9]. In the actual situations, uncertainties have a consequence on the performance of neural networks. In neural networks, the connection weights of the neurons depend on certain resistance and capacitance values that include modeling errors or uncertainties during the parameter identification process. The uncertainties come mainly from the deviations and perturbations in parameters. In particular, when modeling neural networks, the parameter uncertainties should be taken

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into consideration. Therefore, we consider the following stochastic CNNs:

$$\begin{aligned} \mathrm{d}x_i(t) = \Big[& -a_i(t)x_i(t) + \sum_{j=1}^{n} b_{ij}(t)f_j(x_j(t)) \\ & + \sum_{j=1}^{n} c_{ij}(t)g_j(x_j(t-\tau_{ij}(t))) + I_i(t)\Big]\mathrm{d}t \\ & + \sum_{j=1}^{n} d_{ij}(t)\sigma_j(x_j(t-\eta_{ij}(t)))\mathrm{d}w_j(t), \end{aligned} \tag{I.2}$$

where d_{ij}, η_{ij} and σ_j are similarly specified as the corresponding parameters in system (I.1), w_j is the standard Brownian motion, $i, j = 1, 2, \ldots, n$.

The discrete-time neural networks become more important than the continuous-time counterparts when implementing the neural networks in a digital way. In order to investigate the dynamical characteristics with respect to digital signal transmission, it is essential to formulate the discrete analog of neural networks. A large number of literatures had been obtained for stability analysis of discrete-time determinant or stochastic neural networks formulated by Euler scheme [10]–[19]. The semi-discrete method [20] of determinant differential equations is very popular in recent years and it is widely utilized in the researches of continuous-time neural networks [20]–[25]. However, as far as the authors know, few scholars use this method to formulate the discrete analogue of stochastic differential equations. In order to get the discrete analogue of system (I.2) by the semi-discrete method [20], the following stochastic CNNs with piecewise constant arguments corresponding to system (I.2) has been taken into account:

$$\begin{aligned} \mathrm{d}x_i(t) = \Big[& -a_i([t])x_i(t) + \sum_{j=1}^{n} b_{ij}([t])f_j(x_j([t])) \\ & + \sum_{j=1}^{n} c_{ij}([t])g_j(x_j([t]-\tau_{ij}([t]))) \\ & + \sum_{j=1}^{n} d_{ij}([t])\sigma_j(x_j([t]-\eta_{ij}([t])))\Delta w_j([t]) \\ & + I_i([t])\Big]\mathrm{d}t, \end{aligned}$$

where $[t]$ denotes the integer part of t, $i = 1, 2, \ldots, n$. Here the discrete analogue of the stochastic part of system (I.2) is obtained by Euler scheme. For each $t \in \mathbb{R}$, there exists an integer $k \in \mathbb{Z}$ such that $k \le t < k+1$. Then the above equation becomes

$$\begin{aligned} \mathrm{d}x_i(t) = \Big[& -a_i(k)x_i(t) + \sum_{j=1}^{n} b_{ij}(k)f_j(x_j(k)) \\ & + \sum_{j=1}^{n} c_{ij}(k)g_j(x_j(k-\tau_{ij}(k))) \\ & + \sum_{j=1}^{n} d_{ij}(k)\sigma_j(x_j(k-\eta_{ij}(k)))\Delta w_j(k) + I_i(k)\Big]\mathrm{d}t, \end{aligned}$$

where $i = 1, 2, \ldots, n$. Integrating the above equation from k to t and letting $t \to k+1$, we achieve the discrete analogue of system (I.2) as follows:

$$\begin{aligned} & x_i(k+1) \\ & \quad = e^{-a_i(k)}x_i(k) \\ & \qquad + \frac{1-e^{-a_i(k)}}{a_i(k)}\Big[\sum_{j=1}^{n} b_{ij}(k)f_j(x_j(k)) \\ & \qquad + \sum_{j=1}^{n} c_{ij}(k)g_j(x_j(k-\tau_{ij}(k))) \\ & \qquad + \sum_{j=1}^{n} d_{ij}(k)\sigma_j(x_j(k-\eta_{ij}(k)))\Delta w_j(k) + I_i(k)\Big], \end{aligned} \tag{I.3}$$

where $k \in \mathbb{Z}$, $i = 1, 2, \ldots, n$.

Almost all dynamical models in real world cannot be represented by linear systems and have a nonlinear term. Simultaneously, linear control methods are applicable only to the linear models and sometimes the nonlinear models [26], [27] need to be linearized. In [28], Takagi and Sugeno presented a fuzzy method depicted by IF–THEN rules, which describe input-output relationships of nonlinear models. The major characteristic of Takagi-Sugeno fuzzy model is to show the local behaviour of each fuzzy rule by a linear model. As a matter of fact, Takagi-Sugeno fuzzy method can be applied to research universal approximators of almost all nonlinear models [29], [30]. For more researches on the dynamical behaviors of Takagi-Sugeno fuzzy neural networks, see [31]–[33].

Periodicity often appears in implicit ways in various natural phenomena. For instance, this is the case when one studies the effects of fluctuating environments. Though one can deliberately periodically fluctuate environmental parameters in controlled laboratory experiments, fluctuations in nature are hardly periodic. Almost periodicity is more likely to accurately describe natural fluctuations [34]–[44]. The concept of almost periodicity is important in probability especially for investigations on stochastic processes. The interest in such a notion lies in its significance and applications arising in engineering, statistics, etc., see [40]–[44].

Stimulated by the above discussions, the main purpose of this paper is to investigate the $2p$-th mean almost periodic outputs and moment global exponential stability of Takagi-Sugeno fuzzy model of system (I.3).

Research Highlights: The main contributions of this article are related as follows:

- A fuzzy model is obtained for semi-discrete stochastic CNNs by using Takagi-Sugeno fuzzy method.
- A discrete Volterra integral expression is obtained for semi-discrete stochastic fuzzy CNNs.
- A decision theorem for $2p$-th moment global exponential stability of semi-discrete stochastic fuzzy CNNs is derived.
- The existence of $2p$-th mean almost periodic oscillations for semi-discrete stochastic fuzzy CNNs is obtained.

- The problems solved in this paper can stimulate the studies of many other discrete stochastic fuzzy dynamic systems.

The paper is organized as follows. In Section 2, a fuzzy model of system (I.3) has been established by using Takagi-Sugeno fuzzy method, and some necessary lemmas are stated. In Section 3, the $2p$-th moment global exponential stability of the obtained semi-discrete stochastic Takagi-Sugeno fuzzy model is discussed. In Section 4, we employ Krasnoselskii's fixed point theorem to research the $2p$-th mean almost periodic outputs of the fuzzy model. In Section 5, two examples and computer simulations are also given to illustrate our main results. The conclusions and future developments of this paper are presented in Section 6.

II. TAKAGI-SUGENO FUZZY MODEL DESCRIPTION AND PRELIMINARIES

A. TAKAGI-SUGENO FUZZY MODEL DESCRIPTION

By the fuzzy method in [28], the rth rule of the Takagi-Sugeno fuzzy model of system (I.3) is of the following form:

Plant Rule l: IF θ_1 is P_{l1} and $\cdots$ and θ_q is P_{lq}, THEN

$$\begin{aligned} x_i(k+1) = {} & e^{-a_i(k)}x_i(k) \\ & + \frac{1-e^{-a_i(k)}}{a_i(k)}\Bigg[\sum_{j=1}^{n} b_{ij}(k)f_j^l(x_j(k)) \\ & + \sum_{j=1}^{n} c_{ij}(k)g_j^l(x_j(k-\tau_{ij}(k))) \\ & + \sum_{j=1}^{n} d_{ij}(k)\sigma_j^l(x_j(k-\eta_{ij}(k)))\Delta w_j(k) \\ & + I_i(k)\Bigg], \end{aligned} \tag{II.1}$$

where $\theta = (\theta_1, \ldots, \theta_p)^T$ is the known premise variable vector, $P_{ls}(l = 1, 2, \ldots, r, s = 1, 2, \ldots, q)$ are the fuzzy sets, r is the number of fuzzy IF-THEN rules; f_j^l, g_j^l and σ_j^l are the lth outputs of f_j, g_j and σ_j of system (I.3), respectively; $k \in \mathbb{Z}$, $i, j = 1, 2, \ldots, n$.

The final output of the fuzzy model (II.1) is inferred as follows:

$$\begin{aligned} x_i(k+1) = {} & \sum_{l=1}^{r} w_l(\theta)\Bigg\{e^{-a_i(k)}x_i(k) \\ & + \frac{1-e^{-a_i(k)}}{a_i(k)}\Bigg[\sum_{j=1}^{n} b_{ij}(k)f_j^l(x_j(k)) \\ & + \sum_{j=1}^{n} c_{ij}(k)g_j^l(x_j(k-\tau_{ij}(k))) \\ & + \sum_{j=1}^{n} d_{ij}(k)\sigma_j^l(x_j(k-\eta_{ij}(k)))\Delta w_j(k) \\ & + I_i(k)\Bigg]\Bigg\}, \end{aligned} \tag{II.2}$$

where $k \in \mathbb{Z}$, $i = 1, 2, \ldots, n$,

$$w_l(\theta) = \frac{e_l(\theta)}{\sum_{l=1}^{r} e_l(\theta)}, \quad e_l(\theta) = \prod_{s=1}^{q} P_{ls}(\theta_s),$$

$P_{ls}(\theta_s)$ denotes the grade of membership of θ_s in P_{ls}. Hence, we have

$$e_l(\theta) \geq 0, \quad \sum_{l=1}^{r} e_l(\theta) > 0, \quad w_l(\theta) \geq 0,$$
$$\sum_{l=1}^{r} w_l(\theta) = 1, \quad l = 1, 2, \ldots, r.$$

Then system (II.2) can be transformed to

$$\begin{aligned} & x_i(k+1) \\ & = e^{-a_i(k)}x_i(k) \\ & \quad + \frac{1-e^{-a_i(k)}}{a_i(k)}\sum_{l=1}^{r} w_l(\theta(k))\Bigg[\sum_{j=1}^{n} b_{ij}(k)f_j^l(x_j(k)) \\ & \quad + \sum_{j=1}^{n} c_{ij}(k)g_j^l(x_j(k-\tau_{ij}(k))) \\ & \quad + \sum_{j=1}^{n} d_{ij}(k)\sigma_j^l(x_j(k-\eta_{ij}(k)))\Delta w_j(k) \\ & \quad + I_i(k)\Bigg], \end{aligned} \tag{II.3}$$

where $k \in \mathbb{Z}$, $i = 1, 2, \ldots, n$.

B. PRELIMINARIES

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $BC(\mathbb{Z}; L^p(\Omega; \mathbb{R}^n))$ denote the vector space of all bounded continuous functions from $\mathbb{Z}$ to $L^p(\Omega; \mathbb{R}^n)$. Define $|X|_p = \max_{1\leq i\leq n}(E|x_i|^p)^{\frac{1}{p}}$, $\forall X = \{x_i\} := (x_1, x_2, \ldots, x_n)^T \in L^p(\Omega; \mathbb{R}^n)$. Then $L^p(\Omega; \mathbb{R}^n)$ is a Banach space equipped with $|\cdot|_p$. Define

$$\|X\|_p = \sup_{k\in\mathbb{Z}} |X|_p = \max_{1\leq i\leq n} \sup_{k\in\mathbb{Z}} (E|x_i(k)|^p)^{\frac{1}{p}},$$

$\forall X = \{x_i\} \in BC(\mathbb{Z}; L^p(\Omega; \mathbb{R}^n))$. Then $BC(\mathbb{Z}; L^p(\Omega; \mathbb{R}^n))$ is a Banach space equipped with $\|\cdot\|_p$ for $p \geq 1$.

Lemma 1 [[45] (Minkowski Inequality)]: Assume that $p \geq 1$, $E|\xi|^p < \infty$, $E|\eta|^p < \infty$, then

$$(E|\xi + \eta|^p)^{1/p} \leq (E|\xi|^p)^{1/p} + (E|\eta|^p)^{1/p}.$$

Lemma 2 [[45] (Hölder Inequality)]: Assume that $p > 1$, then

$$\sum_k |a_k b_k| \leq \Bigg[\sum_k |a_k|\Bigg]^{1-1/p}\Bigg[\sum_k |a_k||b_k|^p\Bigg]^{1/p}.$$

If $p = 1$, then $\sum_k |a_k b_k| \leq (\sum_k |a_k|)(\sup_k |b_k|)$.

Lemma 3: $X = \{x_i\}$ is the final output of Takagi-Sugeno fuzzy system (II.3) if and only if

$$\begin{aligned} x_i(k) &= \prod_{s=k_0}^{k-1} e^{-a_i(s)} x_i(k_0) + \sum_{v=k_0}^{k-1}\prod_{s=v+1}^{k-1} \frac{e^{-a_i(s)}[1-e^{-a_i(v)}]}{a_i(v)} \sum_{l=1}^{r} w_l(\theta(v)) \\ &\quad \Big[\sum_{j=1}^{n} b_{ij}(v) f_j^l(x_j(v)) + \sum_{j=1}^{n} c_{ij}(v) g_j^l(x_j(v-\tau_{ij}(v))) \\ &\quad + \sum_{j=1}^{n} d_{ij}(v)\sigma_j^l(x_j(v-\eta_{ij}(v)))\Delta w_j(v) + I_i(v)\Big], \qquad \text{(II.4)} \end{aligned}$$

where $k_0 \in \mathbb{Z}$, $k \in (k_0, +\infty)_{\mathbb{Z}}$, $i = 1, 2, \dots, n$.

Proof: Let

$$\begin{aligned} F_i^l(k,x) &:= \sum_{j=1}^{n} b_{ij}(k) f_j^l(x_j(k)) + \sum_{j=1}^{n} c_{ij}(k) g_j^l(x_j(k-\tau_{ij}(k))) \\ &\quad + \sum_{j=1}^{n} d_{ij}(k)\sigma_j^l(x_j(k-\eta_{ij}(k)))\Delta w_j(k) \\ &\quad + I_i(k), \quad k \in \mathbb{Z},\ i = 1,2,\dots,n,\ l = 1,2,\dots,r. \end{aligned}$$

By $\Delta[u(k)v(k)] = [\Delta u(k)]v(k) + u(k+1)[\Delta v(k)]$ and system (II.3), it gets

$$\Delta\Big[\prod_{s=0}^{k-1} e^{a_i(s)} x_i(k)\Big] = \prod_{s=0}^{k} \frac{e^{a_i(s)}[1-e^{-a_i(k)}]}{a_i(k)} \sum_{l=1}^{r} w_l(\theta(k)) F_i^l(k,x),$$

where $i = 1, 2, \dots, n$, $l = 1, 2, \dots, r$, $k \in \mathbb{Z}$. So

$$\sum_{v=k_0}^{k-1} \Delta\Big[\prod_{s=0}^{v-1} e^{a_i(s)} x_i(v)\Big] = \sum_{v=k_0}^{k-1}\prod_{s=0}^{v} \frac{e^{a_i(s)}[1-e^{-a_i(v)}]}{a_i(v)} \sum_{l=1}^{r} w_l(\theta(v)) F_i^l(v,x)$$

is equivalent to

$$\begin{aligned} \prod_{s=0}^{k-1} e^{a_i(s)} x_i(k) &= \prod_{s=0}^{k_0-1} e^{a_i(s)} x_i(k_0) \\ &\quad + \sum_{v=k_0}^{k-1}\prod_{s=0}^{v} \frac{e^{a_i(s)}[1-e^{-a_i(v)}]}{a_i(v)} \sum_{l=1}^{r} w_l(\theta(v)) F_i^l(v,x), \end{aligned}$$

where $i = 1, 2, \dots, n$, $l = 1, 2, \dots, r$, $k \in \mathbb{Z}$. By the above equations, we can easily derive (II.4). This completes the proof. □

Lemma 4 [9]: Suppose that $g \in L^2([a,b],\mathbb{R})$, then

$$E\Big[\sup_{t\in[a,b]}\Big|\int_a^t g(s)d\omega(s)\Big|^p\Big] \le C_p E\Big[\int_a^b |g(t)|^2 dt\Big]^{\frac{p}{2}},$$

where

$$C_p = \begin{cases} (32/p)^{p/2}, & 0 < p < 2, \\ 4, & p = 2, \\ \Big[\dfrac{p^{p+1}}{2(p-1)^{(p-1)}}\Big]^{\frac{p}{2}}, & p > 2. \end{cases}$$

Lemma 5: Assume that $\{x(k) : k \in \mathbb{Z}\}$ is real-valued stochastic process and $w(k)$ is the standard Brownian motion, then

$$E\big|x(k)\Delta w(k)\big|^p \le C_p E\big|x(k)\big|^p, \quad k \in \mathbb{Z},$$

where C_p is defined as that in Lemma 4, $p > 0$.

Proof: By Lemma 4, it follows that

$$\begin{aligned} E\big|x(k)\Delta w(k)\big|^p &= E\Big|\int_k^{k+1} x(k)\,\mathrm{d}w(s)\Big|^p \\ &\le C_p E\Big|\int_k^{k+1} x^2(k)\,\mathrm{d}s\Big|^{\frac{p}{2}} \le C_p E\big|x(k)\big|^p, \end{aligned}$$

This completes the proof. □

III. MOMENT GLOBAL EXPONENTIAL STABILITY OF TAKAGI-SUGENO MODEL

Set $\bar{f} = \sup_{k\in\mathbb{Z}} |f(k)|$ and $\underline{f} = \inf_{k\in\mathbb{Z}} |f(k)|$ for bounded function f defined on $\mathbb{Z}$. Define

$$\begin{aligned} \bar{a} &:= \max_{1\le i\le n} \bar{a}_i, \quad \underline{a} := \min_{1\le i\le n} \underline{a}_i, \\ r_{2p} &:= \frac{(1-e^{-\bar{a}})}{\underline{a}(1-e^{-\underline{a}})} \max_{1\le i\le n}\sup_{k\in\mathbb{Z}} \sum_{l=1}^{r} w_l(\theta(k)) \sum_{j=1}^{n}\Big[\bar{b}_{ij}L_j^l \\ &\quad + \bar{c}_{ij}K_j^l + \bar{d}_{ij}\Lambda_j^l C_{2p}^{\frac{1}{2p}}\Big], \quad \beta_{2p} := \frac{\alpha_{2p}}{1-r_{2p}}, \end{aligned}$$

where

$$\begin{aligned} \alpha_{2p} &:= \frac{(1-e^{-\bar{a}})}{\underline{a}(1-e^{-\underline{a}})} \max_{1\le i\le n}\sup_{k\in\mathbb{Z}} \sum_{l=1}^{r} w_l(\theta(k)) \Big[\sum_{j=1}^{n}\big(\bar{b}_{ij}|f_j^l(0)| \\ &\quad + \bar{c}_{ij}|g_j^l(0)|\big) + \bar{I}_i + \sum_{j=1}^{n} \bar{d}_{ij}\sigma_j^l(0) C_{2p}^{\frac{1}{2p}}\Big]. \end{aligned}$$

Suppose that $X = \{x_i\}$ with initial value $\varphi = \{\varphi_i\}$ and $X^* = \{x_i^*\}$ with initial value $\varphi^* = \{\varphi_i^*\}$ are arbitrary two solutions of system (II.3). For convenience, let

$$\begin{aligned} \gamma_{2p} &= \max_{1\le i\le n}\sup_{s\in[-\mu_0,0]_{\mathbb{Z}}} \{(E|\varphi_i(s) - \varphi_i^*(s)|^{2p})^{\frac{1}{2p}}\}, \\ \mu_0 &= \max_{(i,j)}\{\bar{\tau}_{ij}, \bar{\eta}_{ij}\}. \end{aligned}$$

Definition 6 [9]: System (II.3) *is said to be* 2p*-th moment global exponential stability if there are positive constants* M *and* λ *such that*

$$E|X(k) - X^*(k)|_{2p} < M\gamma_{2p}e^{-\lambda k}, \quad \forall k \in [-\mu_0, +\infty)_{\mathbb{Z}}.$$

The 2*-nd moment global exponential stability will be called square-mean moment global exponential stability.*

For the sake of deriving the moment global exponential stability of Takagi-Sugeno fuzzy model (*II.3*), some conditions are considered below.

(H_1) The lth outputs f_j^l, g_j^l and σ_j^l of f_j, g_j and σ_j satisfy the Lipschitz conditions, i.e., there exist several positive constants L_j^l, K_j^l and Λ_j^l such that

$$|f_j^l(u) - f_j^l(v)| \le L_j^l|u - v|,$$
$$|g_j^l(u) - g_j^l(v)| \le K_j^l|u - v|,$$
$$|\sigma_j^l(u) - \sigma_j^l(v)| \le \Lambda_j^l|u - v|,$$

for all $u, v \in \mathbb{R}$, where $j = 1, 2, \ldots, n$, $l = 1, 2, \ldots, r$.

(H_2) $r_{2p} < 1$.

Theorem 7: Assume that (H_1)*-*(H_2) *hold, then Takagi-Sugeno fuzzy model* (II.3) *is* 2p*-th moment globally exponentially stable.*

Proof: Suppose that $X = \{x_i\}$ with initial value $\varphi = \{\varphi_i\}$ and $X^* = \{x_i^*\}$ with initial value $\varphi^* = \{\varphi_i^*\}$ are arbitrary two solutions of model (II.3). Then it follows from Lemma 3 that

$$\begin{aligned}
&|x_i(k) - x_i^*(k)| \\
&\le \prod_{s=0}^{k-1} e^{-a_i(s)}|\varphi_i(0) - \varphi_i^*(0)| \\
&\quad + \frac{(1 - e^{-\bar{a}})}{\underline{a}} \sup_{t\in\mathbb{Z}} \sum_{l=1}^{r} w_l(\theta(t)) \sum_{v=0}^{k-1} \\
&\quad \prod_{s=v+1}^{k-1} e^{-a_i(s)} \sum_{j=1}^{n} \Big\{ \bar{b}_{ij}L_j^l|x_j(v) - x_j^*(v)| \\
&\quad + \bar{c}_{ij}K_j^l|x_j(v - \tau_{ij}(v)) - x_j^*(v - \tau_{ij}(v))| \\
&\quad + \bar{d}_{ij}\Lambda_j^l|x_j(v - \eta_{ij}(v)) - x_j^*(v - \eta_{ij}(v))||\Delta w_j(v)| \Big\} \\
&\le e^{-\underline{a}k}|\varphi_i(0) - \varphi_i^*(0)| + \frac{(1 - e^{-\bar{a}})}{\underline{a}} \sup_{t\in\mathbb{Z}} \sum_{l=1}^{r} w_l(\theta(t)) \\
&\quad \times \sum_{v=0}^{k-1} e^{-\underline{a}(k-v-1)} \sum_{j=1}^{n} \Big\{ \bar{b}_{ij}L_j^l|x_j(v) - x_j^*(v)| \\
&\quad + \bar{c}_{ij}K_j^l|x_j(v - \tau_{ij}(v)) - x_j^*(v - \tau_{ij}(v))| \\
&\quad + \bar{d}_{ij}\Lambda_j^l|x_j(v - \eta_{ij}(v)) \\
&\quad - x_j^*(v - \eta_{ij}(v))||\Delta w_j(v)| \Big\}, \qquad \text{(III.1)}
\end{aligned}$$

where $i = 1, 2, \ldots, n$, $k \in [-\mu_0, +\infty)_{\mathbb{Z}}$. For convenience, let

$$a_0 = \frac{1 - e^{-\bar{a}}}{\underline{a}},$$

and $Z = \{z_i\}$, $z_i(k) = x_i(k) - x_i^*(k)$, $i = 1, 2, \ldots, n$, $k \in \mathbb{Z}$. By Minkoswki inequality in Lemma 1, Hölder inequality in Lemma 2 and Lemma 5, it gets from (III.1) that

$$\begin{aligned}
&|Z(k)|_{2p} \\
&= |X(k) - X^*(k)|_{2p} \\
&\le e^{-\underline{a}k}\gamma_{2p} \\
&\quad + \sup_{t\in\mathbb{Z}} \sum_{l=1}^{r} w_l(\theta(t)) \max_{1\le i\le n} \sum_{j=1}^{n} a_0\bar{b}_{ij}L_j^l \\
&\quad \times \Bigg\{ \Big[\sum_{s=0}^{k-1} e^{-\underline{a}(k-s-1)} \Big]^{2p-1} \\
&\quad \times \sum_{s=0}^{k-1} e^{-\underline{a}(k-s-1)} E\big|x_j(s) - x_j^*(s)\big|^{2p} \Bigg\}^{\frac{1}{2p}} \\
&\quad + \sup_{t\in\mathbb{Z}} \sum_{l=1}^{r} w_l(\theta(t)) \max_{1\le i\le n} \sum_{j=1}^{n} \\
&\quad \times a_0\bar{c}_{ij}K_j^l \Bigg\{ \Big[\sum_{s=0}^{k-1} e^{-\underline{a}(k-s-1)} \Big]^{2p-1} \\
&\quad \times \sum_{s=0}^{k-1} e^{-\underline{a}(k-s-1)} E\big|x_j(s - \tau_{ij}(s)) \\
&\quad - x_j^*(s - \tau_{ij}(s))\big|^{2p} \Bigg\}^{\frac{1}{2p}} \\
&\quad + \sup_{t\in\mathbb{Z}} \sum_{l=1}^{r} w_l(\theta(t)) \max_{1\le i\le n} \sum_{j=1}^{n} a_0\bar{d}_{ij}\Lambda_j^l \\
&\quad \times \Bigg\{ \Big[\sum_{s=0}^{k-1} e^{-\underline{a}(k-s-1)} \Big]^{2p-1} \\
&\quad \times \sum_{s=0}^{k-1} e^{-\underline{a}(k-s-1)} E\big|[x_j(s - \eta_{ij}(s)) \\
&\quad - x_j^*(s - \eta_{ij}(s))]\Delta w_j(s)\big|^{2p} \Bigg\}^{\frac{1}{2p}} \\
&\le e^{-\underline{a}k}\gamma_{2p} + \sup_{t\in\mathbb{Z}} \sum_{l=1}^{r} w_l(\theta(t)) \Bigg\{ \max_{1\le i\le n} \sum_{j=1}^{n} a_0\bar{b}_{ij}L_j^l \\
&\quad \times \Bigg\{ \Big[\sum_{s=0}^{k-1} e^{-\underline{a}(k-s-1)} \Big]^{2p-1} \sum_{s=0}^{k-1} e^{-\underline{a}(k-s-1)} |Z(s)|_{2p}^{2p} \Bigg\}^{\frac{1}{2p}} \\
&\quad + \max_{1\le i\le n} \sum_{j=1}^{n} a_0\bar{c}_{ij}K_j^l \Bigg\{ \Big[\sum_{s=0}^{k-1} e^{-\underline{a}(k-s-1)} \Big]^{2p-1} \\
&\quad \times \sum_{s=0}^{k-1} e^{-\underline{a}(k-s-1)} |Z(s - \tau_{ij}(s))|_{2p}^{2p} \Bigg\}^{\frac{1}{2p}} \\
&\quad + \max_{1\le i\le n} \sum_{j=1}^{n} a_0 C_{2p}^{\frac{1}{2p}} \bar{d}_{ij}\Lambda_j^l \Bigg\{ \Big[\sum_{s=0}^{k-1} e^{-\underline{a}(k-s-1)} \Big]^{2p-1} \\
&\quad \times \sum_{s=0}^{k-1} e^{-\underline{a}(k-s-1)} |Z(s - \eta_{ij}(s))|_{2p}^{2p} \Bigg\}^{\frac{1}{2p}} \Bigg\}. \qquad \text{(III.2)}
\end{aligned}$$

By (H_2), there exists a constant $\lambda > 0$ small enough such that

$$\sup_{k\in\mathbb{Z}} \max_{1\le i\le n} \sum_{j=1}^{n} \frac{e^{\lambda} a_0}{1-e^{-(\underline{a}-2p\lambda)}} \sum_{l=1}^{r} w_l(\theta(k)) \Big[\bar{b}_{ij} L_j^l + e^{\mu_0\lambda} \bar{c}_{ij} K_j^l + e^{\mu_0\lambda} C_{2p}^{\frac{1}{2p}} \bar{d}_{ij} \Lambda_j^l \Big] \stackrel{\text{def}}{=} \rho \le 1.$$

Next, we claim that there exists a constant $M_0 > 1$ such that

$$|Z(k)|_{2p} \le M_0 \gamma_{2p} e^{-\lambda k}, \quad \forall k \in [-\mu_0, +\infty)_{\mathbb{Z}}. \tag{III.3}$$

If (III.3) is invalid, then there exist $k_0 \in (0, +\infty)_{\mathbb{Z}}$ such that

$$|Z(k_0)|_{2p} > M_0 \gamma_{2p} e^{-\lambda k_0} \tag{III.4}$$

and

$$|Z(k)|_{2p} \le M_0 \gamma_{2p} e^{-\lambda k}, \quad \forall k \in [-\mu_0, k_0)_{\mathbb{Z}}. \tag{III.5}$$

In view of (III.2), it follows from (III.5) that

$$\begin{aligned}
&|Z(k_0)|_{2p} \\
&\le e^{-\underline{a}k_0}\gamma_{2p} \\
&\quad + \sup_{t\in\mathbb{Z}} \sum_{l=1}^{r} w_l(\theta(t)) \Bigg\{ \max_{1\le i\le n} \sum_{j=1}^{n} a_0 \bar{b}_{ij} L_j^l M_0 \gamma_{2p} \\
&\quad \times \Bigg\{ \Bigg[\sum_{s=0}^{k-1} e^{-\underline{a}(k_0-s-1)} \Bigg]^{2p-1} \sum_{s=0}^{k-1} e^{-\underline{a}(k_0-s-1)} e^{-2p\lambda s} \Bigg\}^{\frac{1}{2p}} \\
&\quad + \max_{1\le i\le n} \sum_{j=1}^{n} a_0 M_0 \gamma_{2p} \Big[\bar{c}_{ij} K_j^l + C_{2p}^{\frac{1}{2p}} \bar{d}_{ij} \Lambda_j^l \Big] \\
&\quad \times \Bigg\{ \Bigg[\sum_{s=0}^{k-1} e^{-\underline{a}(k_0-s-1)} \Bigg]^{2p-1} \\
&\quad \times \sum_{s=0}^{k-1} e^{-\underline{a}(k_0-s-1)} e^{-2p\lambda(s-\mu_0)} \Bigg\}^{\frac{1}{2p}} \Bigg\} \\
&\le e^{-\underline{a}k_0}\gamma_{2p} \\
&\quad + \sup_{t\in\mathbb{Z}} \sum_{l=1}^{r} w_l(\theta(t)) \Bigg\{ \max_{1\le i\le n} \sum_{j=1}^{n} a_0 \bar{b}_{ij} L_j^l M_0 \gamma_{2p} \\
&\quad \times e^{-\lambda k_0} e^{\lambda} \Bigg[\frac{1-e^{-\underline{a}k_0}}{1-e^{-\underline{a}}} \Bigg]^{1-\frac{1}{2p}} \Bigg[\sum_{s=0}^{k_0-1} e^{-(\underline{a}-p\lambda)(k_0-s-1)} \Bigg]^{\frac{1}{2p}} \\
&\quad + \max_{1\le i\le n} \sum_{j=1}^{n} a_0 M_0 \gamma_{2p} \Big[\bar{c}_{ij} K_j^l + C_{2p}^{\frac{1}{2p}} \bar{d}_{ij} \Lambda_j^l \Big] \\
&\quad \times e^{-\lambda k_0} e^{(\mu_0+1)\lambda} \Bigg[\frac{1-e^{-\underline{a}k_0}}{1-e^{-\underline{a}}} \Bigg]^{1-\frac{1}{2p}} \\
&\quad \times \Bigg[\sum_{s=0}^{k_0-1} e^{-(\underline{a}-p\lambda)(k_0-s-1)} \Bigg]^{\frac{1}{2p}} \Bigg\} \\
&\le e^{-\underline{a}k_0}\gamma_{2p} + \sup_{k\in\mathbb{Z}} \sum_{l=1}^{r} w_l(\theta(k)) \max_{1\le i\le n} \sum_{j=1}^{n} a_0 M_0 \gamma_{2p} \\
&\quad \times e^{-\lambda k_0} \Big[\bar{b}_{ij} L_j^l + e^{\mu_0\lambda} \bar{c}_{ij} K_j^l + e^{\mu_0\lambda} C_{2p}^{\frac{1}{2p}} \bar{d}_{ij} \Lambda_j^l \Big] \\
&\quad \times e^{\lambda} \Bigg[\frac{1-e^{-\underline{a}k_0}}{1-e^{-\underline{a}}} \Bigg]^{1-\frac{1}{2p}} \Bigg[\frac{1-e^{-(\underline{a}-2p\lambda)k_0}}{1-e^{-(\underline{a}-2p\lambda)}} \Bigg]^{\frac{1}{2p}} \\
&\le M_0 \gamma_{2p} e^{-\lambda k_0} \Bigg\{ \frac{1}{M_0} e^{-(\underline{a}-\lambda)k_0} \\
&\quad + \sup_{k\in\mathbb{Z}} \sum_{l=1}^{r} w_l(\theta(k)) \max_{1\le i\le n} \sum_{j=1}^{n} a_0 \Big[\bar{b}_{ij} L_j^l + e^{\mu_0\lambda} \bar{c}_{ij} K_j^l \\
&\quad + e^{\mu_0\lambda} C_{2p}^{\frac{1}{2p}} \bar{d}_{ij} \Lambda_j^l \Big] \frac{e^{\lambda}[1-e^{-(\underline{a}-\lambda)k_0}]}{1-e^{-(\underline{a}-2p\lambda)}} \Bigg\} \\
&\le M_0 \gamma_{2p} e^{-\lambda k_0} \Big\{ e^{-(\underline{a}-\lambda)k_0} + \rho[1-e^{-(\underline{a}-\lambda)k_0}] \Big\} \\
&\le M_0 \gamma_{2p} e^{-\lambda k_0}.
\end{aligned} \tag{III.6}$$

In the fourth inequality from the bottom of (III.6), we use the facts $[1-e^{-\underline{a}k_0}]^{1-\frac{1}{2p}}[1-e^{-(\underline{a}-p\lambda)k_0}]^{\frac{1}{2p}} \le 1-e^{-(\underline{a}-\lambda)k_0}$ and $[1-e^{-\underline{a}}]^{\frac{1}{2p}} \ge [1-e^{-(\underline{a}-p\lambda)}]^{\frac{1}{2p}}$. (III.6) contradicts (III.4). Hence, (III.3) is satisfied. Therefore, Takagi-Sugeno fuzzy model (II.3) is $2p$-th moment globally exponentially stable. This completes the proof. □

IV. MEAN ALMOST PERIODIC OUTPUTS OF TAKAGI-SUGENO MODEL

Lemma 8 [46]: Assume that Λ is a closed convex nonempty subset of a Banach space $\mathbb{X}$. Suppose further that $\mathcal{B}$ and $\mathcal{C}$ map Λ into $\mathbb{X}$ such that

(1) *$x, y \in \Lambda$ implies that $\mathcal{B}x + \mathcal{C}y \in \Lambda$;*
(2) *$\mathcal{B}$ is continuous and $\mathcal{B}\Lambda$ is contained in a compact set;*
(3) *$\mathcal{C}$ is a contraction mapping.*

Then there exists a $z \in \Lambda$ with $z = \mathcal{B}z + \mathcal{C}z$.

Definition 9 [8]: A stochastic process $X \in BC(\mathbb{Z}; L^p(\Omega; \mathbb{R}^n))$ is said to be p-th mean almost periodic sequence if for each $\epsilon > 0$, there exists an integer $l(\epsilon) > 0$ such that each interval of length $l(\epsilon)$ contains an integer ω for which

$$\sup_{k\in\mathbb{Z}} E|X(k+\omega) - X(k)|_p < \epsilon.$$

A stochastic process X, which is 2-nd mean almost periodic sequence will be called square-mean almost periodic sequence. Like for classical almost periodic functions, the number ω will be called an ϵ-translation of X.

Theorem 10: Assume that (H_1)-(H_2) and the following conditions are satisfied:

(H_3) *All of the coefficients in system (I.3) are almost periodic sequences.*
(H_4) *The known premise variable $\theta(k) = (\theta_1(k), \ldots, \theta_p(k))^T$ is almost periodic and membership function $P_{ls}(\theta)$ is uniformly continuous in θ, $k \in \mathbb{Z}$, $l = 1, 2, \ldots, r$, $s = 1, 2, \ldots, q$.*

Then Takagi-Sugeno fuzzy model (II.3) of system (I.3) outputs a $2p$-th mean almost periodic oscillation X with $\|X\|_{2p} \le \beta_{2p}$, $p \ge 1$.

Proof: Let $\Lambda \subseteq BC(\mathbb{Z}; L^{2p}(\Omega; \mathbb{R}^n))$ be the collection of all 2p-th mean almost periodic sequences $X = \{x_i\}$ satisfying the inequality $\|X\|_{2p} \le \beta_{2p}$.

Firstly, $X = \{x_i\}$ is described by

$$\begin{aligned} &x_i(k) \\ &= \sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_i(s)}[1-e^{-a_i(v)}]}{a_i(v)} \sum_{l=1}^{r} w_l(\theta(v)) \\ &\Big[\sum_{j=1}^{n} b_{ij}(v) f_j^l(x_j(v)) + \sum_{j=1}^{n} c_{ij}(v) g_j^l(x_j(v-\tau_{ij}(v))) \\ &+ \sum_{j=1}^{n} d_{ij}(v)\sigma_j^l(x_j(v-\eta_{ij}(v)))\Delta w_j(v) + I_i(v)\Big], \quad \text{(IV.1)} \end{aligned}$$

where $i = 1, 2, \ldots, n$, $k \in \mathbb{Z}$. By Lemma 3, (IV.1) is well defined and satisfies (II.4). So we define $\Phi X(k) = \mathcal{B}X(k) + \mathcal{C}X(k)$, where

$$\begin{aligned} \Phi X(k) &= ((\Phi X)_1(k), (\Phi X)_2(k), \ldots, (\Phi X)_n(k))^T, \\ (\Phi X)_i(k) &= (\mathcal{B}X)_i(k) + (\mathcal{C}X)_i(k), \quad \text{(IV.2)} \\ (\mathcal{B}X)_i(k) &= \sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_i(s)}[1-e^{-a_i(v)}]}{a_i(v)} \\ &\quad \times \sum_{l=1}^{r} w_l(\theta(v)) \Big[\sum_{j=1}^{n} b_{ij}(v) f_j^l(x_j(v)) \\ &\quad + \sum_{j=1}^{n} c_{ij}(v) g_j^l(x_j(v-\tau_{ij}(v))) + I_i(v)\Big], \quad \text{(IV.3)} \\ (\mathcal{C}X)_i(k) &= \sum_{v=\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_i(s)}[1-e^{-a_i(v)}]}{a_i(v)} \sum_{l=1}^{r} w_l(\theta(v)) \\ &\quad \times \sum_{j=1}^{n} d_{ij}(v)\sigma_j^l(x_j(v-\eta_{ij}(v)))\Delta w_j(v), \quad \text{(IV.4)} \end{aligned}$$

where $i = 1, 2, \ldots, n$, $k \in \mathbb{Z}$.

Let $X^0 = \{x_i^0\}$ be defined as

$$\begin{aligned} x_i^0(k) &= \sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_i(s)}[1-e^{-a_i(v)}]}{a_i(v)} \\ &\quad \times \sum_{l=1}^{r} w_l(\theta(v)) \Big[\sum_{j=1}^{n} b_{ij}(v) f_j^l(0) + \sum_{j=1}^{n} c_{ij}(v) g_j^l(0) \\ &\quad + \sum_{j=1}^{n} d_{ij}(v)\sigma_j^l(0)\Delta w_j(v) + I_i(v)\Big], \end{aligned}$$

where $i = 1, 2, \ldots, n$, $k \in \mathbb{Z}$. By Minkoswki inequality in Lemma 1 and Hölder inequality in Lemma 2, we have

$$\begin{aligned} &\|X^0\|_{2p} \\ &\le \sup_{t\in\mathbb{Z}} \sum_{l=1}^{r} w_l(\theta(t)) \max_{1\le i\le n} \sup_{k\in\mathbb{Z}} \Bigg\{\Bigg[E\Bigg|\sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \\ &\times \frac{e^{-a_i(s)}[1-e^{-a_i(v)}]}{a_i(v)} \sum_{j=1}^{n} \big(\bar{b}_{ij} f_j^l(0) + \bar{c}_{ij} g_j^l(0)\big)\Bigg|^{2p}\Bigg]^{\frac{1}{2p}} \\ &+ \Bigg[E\Bigg|\sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_i(s)}[1-e^{-a_i(v)}]}{a_i(v)} \sum_{j=1}^{n} \bar{d}_{ij}\sigma_j^l(0) \\ &\times \Delta w_j(v)\Bigg|^{2p}\Bigg]^{\frac{1}{2p}} + \Bigg[E\Bigg|\sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \\ &\times \frac{e^{-a_i(s)}[1-e^{-a_i(v)}]}{a_i(v)} \bar{I}_i\Bigg|^{2p}\Bigg]^{\frac{1}{2p}}\Bigg\}. \quad \text{(IV.5)} \end{aligned}$$

It gets from (IV.5) that

$$\begin{aligned} &\|X^0\|_{2p} \\ &\le \sup_{t\in\mathbb{Z}} \sum_{l=1}^{r} w_l(\theta(t)) \max_{1\le i\le n} \sup_{k\in\mathbb{Z}} \Bigg\{\frac{(1-e^{-\bar{a}})}{\underline{a}(1-e^{-\underline{a}})} \\ &\quad \times \Bigg[\sum_{j=1}^{n} \big(\bar{b}_{ij}|f_j^l(0)| + \bar{c}_{ij}|g_j^l(0)|\big) + \bar{I}_i\Bigg] \\ &\quad + \sum_{j=1}^{n} \bar{d}_{ij}\sigma_j^l(0) \Bigg[\sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_i(s)}[1-e^{-a_i(v)}]}{a_i(v)}\Bigg]^{1-\frac{1}{2p}} \\ &\quad \times \Bigg[\sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_i(s)}[1-e^{-a_i(v)}]}{a_i(v)} E|\Delta w_j(v)|^{2p}\Bigg]^{\frac{1}{2p}}\Bigg\} \\ &\le \frac{(1-e^{-\bar{a}})}{\underline{a}(1-e^{-\underline{a}})} \max_{1\le i\le n} \sup_{k\in\mathbb{Z}} \sum_{l=1}^{r} w_l(\theta(k)) \\ &\quad \times \Bigg[\sum_{j=1}^{n} \big(\bar{b}_{ij}|f_j^l(0)| + \bar{c}_{ij}|g_j^l(0)|\big) + \bar{I}_i + \sum_{j=1}^{n} \bar{d}_{ij}\sigma_j^l(0) C_{2p}^{\frac{1}{2p}}\Bigg] \\ &:= \alpha_{2p}. \quad \text{(IV.6)} \end{aligned}$$

It follows (IV.2), (IV.3) and (IV.4) that

$$\begin{aligned} &\|\Phi X - X^0\|_{2p} \\ &\le \sup_{t\in\mathbb{Z}} \sum_{l=1}^{r} w_l(\theta(t)) \max_{1\le i\le n} \sup_{k\in\mathbb{Z}} \sum_{j=1}^{n} \Bigg\{\bar{b}_{ij} L_j^l \Bigg\{E\Bigg[\sum_{v=-\infty}^{k-1} \\ &\quad \times \prod_{s=v+1}^{k-1} \frac{e^{-a_i(s)}[1-e^{-a_i(v)}]}{a_i(v)} |x_j(v)|\Bigg]^{2p}\Bigg\}^{\frac{1}{2p}} \\ &\quad + \bar{c}_{ij} K_j^l \Bigg\{E\Bigg[\sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_i(s)}[1-e^{-a_i(v)}]}{a_i(v)} \\ &\quad \times |x_j(v-\tau_{ij}(v))|\Bigg]^{2p}\Bigg\}^{\frac{1}{2p}} \\ &\quad + \bar{d}_{ij} \Lambda_j^l \Bigg\{E\Bigg[\sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_i(s)}[1-e^{-a_i(v)}]}{a_i(v)} \\ &\quad \times |x_j(v-\eta_{ij}(v))\Delta w_j(v)|\Bigg]^{2p}\Bigg\}^{\frac{1}{2p}}\Bigg\}, \end{aligned}$$

which yields from Lemmas 1 and 2 that

$$\begin{aligned}
&\|\Phi X - X^0\|_{2p} \\
&\le \sup_{t\in\mathbb{Z}} \sum_{l=1}^{r} w_l(\theta(t)) \max_{1\le i\le n} \sup_{k\in\mathbb{Z}} \sum_{j=1}^{n} \Bigg\{ \bar{b}_{ij} L_j^l \Bigg\{ \Bigg[\sum_{v=-\infty}^{k-1} \\
&\quad \times \prod_{s=v+1}^{k-1} \frac{e^{-a_i(s)}[1-e^{-a_i(v)}]}{a_i(v)} \Bigg]^{2p-1} \\
&\quad \times \sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_i(s)}[1-e^{-a_i(v)}]}{a_i(v)} E\big|x_j(v)\big|^{2p} \Bigg\}^{\frac{1}{2p}} \\
&\quad + \bar{c}_{ij} K_j^l \Bigg\{ \Bigg[\sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_i(s)}[1-e^{-a_i(v)}]}{a_i(v)} \Bigg]^{2p-1} \\
&\quad \times \sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_i(s)}[1-e^{-a_i(v)}]}{a_i(v)} \\
&\quad \times E\big|x_j(v-\tau_{ij}(v))\big|^{2p} \Bigg\}^{\frac{1}{2p}} \\
&\quad + \bar{d}_{ij} \Lambda_j^l \Bigg\{ \Bigg[\sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_i(s)}[1-e^{-a_i(v)}]}{a_i(v)} \Bigg]^{2p-1} \\
&\quad \times \sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_i(s)}[1-e^{-a_i(v)}]}{a_i(v)} \\
&\quad \times E\big|x_j(v-\eta_{ij}(v))\Delta w_j(v)\big|^{2p} \Bigg\}^{\frac{1}{2p}} \Bigg\}.
\end{aligned}$$

Applying Lemma 5 to the above inequality, it derives

$$\begin{aligned}
\|\Phi X - X^0\|_{2p} &\le \frac{(1-e^{-\bar{a}})}{\underline{a}(1-e^{-\underline{a}})} \max_{1\le i\le n} \sup_{k\in\mathbb{Z}} \sum_{l=1}^{r} w_l(\theta(k)) \\
&\quad \times \sum_{j=1}^{n} \Big[\bar{b}_{ij} L_j^l + \bar{c}_{ij} K_j^l + \bar{d}_{ij} \Lambda_j^l C_{2p}^{\frac{1}{2p}} \Big] \|X\|_{2p} \\
&= r_{2p} \|X\|_{2p} \le \frac{r_{2p}\alpha_{2p}}{1-r_{2p}}. \qquad \text{(IV.7)}
\end{aligned}$$

Hence, $\forall X = \{x_i\} \in \Lambda$, it leads from (IV.6) and (IV.7) to

$$\begin{aligned}
\|\Phi X\|_{2p} &\le \|X^0\|_{2p} + \|\Phi X - X^0\|_{2p} \\
&\le \alpha_{2p} + \frac{r_{2p}\alpha_{2p}}{1-r_{2p}} = \frac{\alpha_{2p}}{1-r_{2p}} := \beta_{2p}. \quad \text{(IV.8)}
\end{aligned}$$

From (IV.8), $\mathcal{B}\Lambda$ is uniformly bounded. Together with the continuity of $\mathcal{B}$, for any bounded sequence $\{\varphi_n\}$ in Λ, we know that there exists a subsequence $\{\varphi_{n_k}\}$ in Λ such that $\{\mathcal{B}(\varphi_{n_k})\}$ is convergent in $\mathcal{B}(\Lambda)$. Therefore, $\mathcal{B}$ is compact on Λ. Then condition (2) of Lemma 8 is satisfied.

The next step is proving condition (1) of Lemma 8. Now, we consist in proving the $2p$-th mean almost periodicity of $\mathcal{B}X(\cdot)$ and $\mathcal{C}X(\cdot)$. Since $X(\cdot)$ is a $2p$-th mean almost periodic sequence and all the coefficients in system (I.3) are almost periodic sequences, for any $\epsilon > 0$, there exist $l_\epsilon > 0$ and ω in every interval of length l_ϵ such that

$$\begin{aligned}
&\big[E|x_i(k+\omega) - x_i(k)|^{2p}\big]^{\frac{1}{2p}} < \epsilon, \quad |a_i(k+\omega) - a_i(k)| < \epsilon, \\
&\Big|\sum_{l=1}^{r} w_l(\theta(k+\omega)) - \sum_{l=1}^{r} w_l(\theta(k))\Big| < \epsilon, \\
&|b_{ij}(k+\omega) - b_{ij}(k)| < \epsilon, \quad |c_{ij}(k+\omega) - c_{ij}(k)| < \epsilon, \\
&|d_{ij}(k+\omega) - d_{ij}(k)| < \epsilon, \quad |\tau_{ij}(k+\omega) - \tau_{ij}(k)| < \epsilon, \\
&|\eta_{ij}(k+\omega) - \eta_{ij}(k)| < \epsilon, \quad |I_i(k+\omega) - I_i(k)| < \epsilon,
\end{aligned}$$

where $i, j = 1, 2, \ldots, n$, $k \in \mathbb{Z}$. By (IV.3) and (IV.4), we could easily find a positive constant M such that

$$\begin{aligned}
&\big[E|(\mathcal{B}X)_i(k+\omega) - (\mathcal{B}X)_i(k)|^{2p}\big]^{\frac{1}{2p}} \\
&\quad \le M \max_{1\le i\le n} \sup_{k\in\mathbb{Z}} \big[E|x_i(k+\omega) - x_i(k)|^{2p}\big]^{\frac{1}{2p}} \\
&\quad < M\epsilon, \qquad \text{(IV.9)} \\
&\big[E|(\mathcal{C}X)_i(k+\omega) - (\mathcal{C}X)_i(k)|^{2p}\big]^{\frac{1}{2p}} \\
&\quad \le M \max_{1\le i\le n} \sup_{k\in\mathbb{Z}} \big[E|x_i(k+\omega) - x_i(k)|^{2p}\big]^{\frac{1}{2p}} \\
&\quad < M\epsilon, \qquad \text{(IV.10)}
\end{aligned}$$

where $i = 1, 2, \ldots, n$, $k \in \mathbb{Z}$. From (IV.9) and (IV.10), $\mathcal{B}X(\cdot)$ and $\mathcal{C}X(\cdot)$ are $2p$-th mean almost periodic processes. Further, by (IV.8), it is easy to obtain that $\mathcal{B}X + \mathcal{C}Y \in \Lambda$, $\forall X, Y \in \Lambda$. Then condition (1) of Lemma 8 holds.

Finally, $\forall X = \{x_i\}$, $Y = \{y_i\} \in \Lambda$, from (IV.4), it yields

$$\begin{aligned}
&\|\mathcal{C}X - \mathcal{C}Y\|_{2p} \\
&\le \sup_{t\in\mathbb{Z}} \sum_{l=1}^{r} w_l(\theta(t)) \frac{[1-e^{-\bar{a}}]}{\underline{a}} \max_{1\le i\le n} \sup_{k\in\mathbb{Z}} \\
&\quad \times \Bigg\{ E\Bigg[\sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} e^{-\underline{a}} \sum_{j=1}^{n} d_{ij}(v) \big(\sigma_j^l(x_j(v-\eta_{ij}(v))) \\
&\quad - \sigma_j^l(y_j(v-\eta_{ij}(v))) \big) \Delta w_j(v) \Bigg]^{2p} \Bigg\}^{\frac{1}{2p}} \\
&\le \sup_{t\in\mathbb{Z}} \sum_{l=1}^{r} w_l(\theta(t)) \frac{[1-e^{-\bar{a}}]}{\underline{a}} \max_{1\le i\le n} \sup_{k\in\mathbb{Z}} \sum_{j=1}^{n} \bar{d}_{ij} \Lambda_j^l \\
&\quad \times \Bigg\{ \Bigg[\sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} e^{-\underline{a}} \Bigg]^{2p-1} \\
&\quad \times \sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} e^{-\underline{a}} E\big|[x_j(v-\eta_{ij}(v)) \\
&\quad - y_j(v-\eta_{ij}(v))]\Delta w_j(v)\big|^{2p} \Bigg\}^{\frac{1}{2p}} \\
&\le \max_{1\le i\le n} \sup_{k\in\mathbb{Z}} \sum_{l=1}^{r} w_l(\theta(k)) \\
&\quad \times \sum_{j=1}^{n} \bar{d}_{ij} \Lambda_j^l \frac{C_{2p}^{\frac{1}{2p}} (1-e^{-\bar{a}})}{\underline{a}(1-e^{-\underline{a}})} \|X - Y\|_{2p} \\
&\le r_{2p} \|X - Y\|_{2p}.
\end{aligned}$$

In view of (H_2), $\mathcal{C}$ is a contraction mapping. Hence condition (3) of Lemma 8 is satisfied. Therefore, all the conditions in Lemma 8 hold. By Lemma 8, Takagi-Sugeno fuzzy model (II.3) outputs a $2p$-th mean almost periodic oscillation. This completes the proof. □

Together with Theorems 7 and 10, we have

Theorem 11: Assume that all conditions in Theorem 10 *hold, then Takagi-Sugeno fuzzy model* (II.3) *outputs a* $2p$*-th mean almost periodic oscillation, which is* $2p$*-th moment globally exponentially stable.*

Proof: By Theorem 7, the solution of model (II.3) is $2p$-th moment globally exponentially stable. By Theorem 10, model (II.3) has a $2p$-th mean almost periodic solution. Together with them, so we get Theorem IV.2. This completes the proof. □

Remark 12: In view of Theorems 7 *and* 10*, the bounded information transmission delays in* (II.3) *have no effect on the existence of* $2p$*-th mean almost periodic oscillations and moment global exponential stability of Takagi-Sugeno fuzzy model* (II.3)*. The results of this paper can also apply to the systems with other types of bounded delays.*

Remark 13: It is worth mentioning that Minkoswki inequality in Lemma 1*, Hölder inequality in Lemma* 2 *and Lemma* 5 *are crucial to the computing processes of Theorems* 7 *and* 10*. It can be viewed from the computations of (III.2) and (IV.5) in Theorems* 7 *and* 10*, respectively.*

V. ILLUSTRATIVE EXAMPLES WITH NUMERICAL SIMULATIONS

Example 14: In the last decades, the following cellular neural networks were employed for the studies of image detection [48], [49], image encryption [50], skull stripping in brain [51], and template decomposition [52] etc.

$$\frac{\mathrm{d}x_{ij}(t)}{\mathrm{d}t} = -x_{ij}(t) + \sum_{k,l\in S_{ij}(r)} a_{kl}f_{kl}(x_{kl}(t)) + I_{ij}, \tag{V.1}$$

where x_{ij}, f_{kl} *and* I_{ij} *denote the state, output and input, respectively;*

$$S_{ij}(r) = \{x_{kl} : \max\{|k-i|, |l-j|\} \le r,\ 1 \le k \le M, 1 \le l \le N\},$$

$i, k = 1, 2, \ldots, M, j, l = 1, 2, \ldots, N.$

The output $y_{kl} = f_{kl}(x)$ (also known as activation function) usually has different types as follows:

For example, Italcura *et al.* [53] considered the problem of image reconstruction via CNNs with linear function. Cuevas *et al.* [48] studied a issue of image detection by CNNs with triangular function. Mosa *et al.* [49] discussed a problem of truck detection by CNNs with sigmoid, tangent sigmoid and radial basis functions. These applications mainly depend on the property of stablity of CNNs. Therefore, the stability of CNNs received widespread attentions in the last decades, see [2]–[4], [20]–[22], [54]. In [54], Mo et al. studied the stability of CNNs with linear, sigmoid and tangent sigmoid functions.

TABLE 1. Different types of activation function.

Name of activation function	Function equation
Linear function	$y_{kl} = x$
Sigmoid function	$y_{kl} = \frac{1}{1+e^{-x}}$
Triangular function	$y_{kl} = \frac{1}{2}(\lvert x+1\rvert - \lvert x-1\rvert)$
Radial basis function	$y_{kl} = e^{-x^2}$
Tangent sigmoid function	$y_{kl} = \tanh(x)$

To make it to easily understand for readers, we consider the following simple CNNs with radial basis function:

$$\begin{cases} \dfrac{\mathrm{d}x_{11}(t)}{\mathrm{d}t} = -x_{11}(t) + 0.1e^{-x_{12}^2(t)} + 0.2, \\ \dfrac{\mathrm{d}x_{12}(t)}{\mathrm{d}t} = -x_{12}(t) + 0.2e^{-x_{11}^2(t)} + 0.1, \end{cases} \tag{V.2}$$

where $t \in \mathbb{R}$.

(1) Semi-discrete model: By the semi-discrete method in Section I, it obtains from (V.2) that

$$\begin{cases} x_{11}(k+1) = e^{-1}x_{11}(k) \\ \qquad\qquad + (1-e^{-1})\Big[0.1e^{-x_{12}^2(k)} + 0.2\Big], \\ x_{12}(k+1) = e^{-1}x_{12}(k) \\ \qquad\qquad + (1-e^{-1})\Big[0.2e^{-x_{11}^2(k)} + 0.1\Big], \end{cases} \tag{V.3}$$

where $k \in \mathbb{Z}$.

(2) Takagi-Sugeno fuzzy model: Corresponding to system (V.1), $f_{11} = e^{-x_{11}^2}$ and $f_{12} = e^{-x_{12}^2}$ in model (V.3) are nonlinear. By numerical calculation of Matlab, $x_{11} \in [-0.3, 0.3]$ and $x_{12} \in [-0.3, 0.3]$. Let $\theta_1(k) = x_{11}(k)$, $\theta_2(k) = x_{12}(k)$, $\forall k \in \mathbb{Z}$. Then $\theta_1, \theta_2 \in [-0.3, 0.3]$. θ_1 and θ_2 can be represented by membership functions M_1, M_2, N_1 and N_2 as follows:

$$\theta_1(k) = M_1(\theta_1(k)) \cdot 0.3 + M_2(\theta_1(k)) \cdot (-0.3),$$
$$\theta_2(k) = N_1(\theta_2(k)) \cdot 0.3 + N_2(\theta_2(k)) \cdot (-0.3),$$

where $M_1(\theta_1) + M_2(\theta_1) = 1$, $N_1(\theta_2) + N_2(\theta_2) = 1$. Hence, the membership functions can be obtained as follows:

$$M_1(\theta_1(k)) = \frac{\theta_1(k)+0.3}{0.6}, \quad M_2(\theta_1(k)) = \frac{0.3-\theta_1(k)}{0.6},$$
$$N_1(\theta_2(k)) = \frac{\theta_2(k)+0.3}{0.6}, \quad N_2(\theta_2(k)) = \frac{0.3-\theta_2(k)}{0.6}.$$

Let $P_{11} = P_{21} = M_1$, $P_{31} = P_{41} = M_2$, $P_{12} = P_{32} = N_1$, $P_{22} = P_{42} = N_2$. Therefore, the nonlinear functions f_{11} and f_{12} are modeled by the following IF-THEN rules:

Model Rule 1: IF θ_1 is P_{11} and θ_2 is P_{12}, THEN

$$\begin{bmatrix} f_{11}^1(x_{11}(k)) \\ f_{12}^1(x_{12}(k)) \end{bmatrix} = \begin{bmatrix} \dfrac{e^{-0.3^2}}{0.3+1.3}(x_{11}(k)+1.3) \\ \dfrac{e^{-0.3^2}}{0.3+1.3}(x_{12}(k)+1.3) \end{bmatrix}.$$

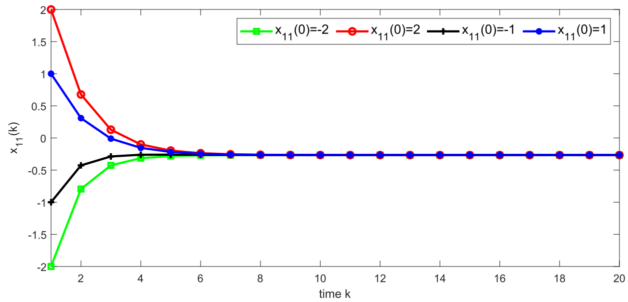

FIGURE 1. Square-mean global exponential stability of state variable $x_{11}(k)$ of model (V.4).

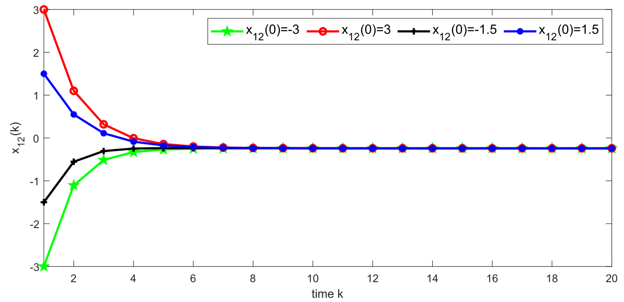

FIGURE 2. Square-mean global exponential stability of state variable $x_{12}(k)$ of model (V.4).

Model Rule 2: IF θ_1 is P_{21} and θ_2 is P_{22}, THEN

$$\begin{bmatrix} f_{11}^2(x_{11}(k)) \\ f_{12}^2(x_{12}(k)) \end{bmatrix} = \begin{bmatrix} \dfrac{e^{-0.3^2}}{0.3+1.3}(x_{11}(k)+1.3) \\ \dfrac{e^{-(-0.3)^2}}{-0.3+1.3}(x_{12}(k)+1.3) \end{bmatrix}.$$

Model Rule 3: IF θ_1 is P_{31} and θ_2 is P_{32}, THEN

$$\begin{bmatrix} f_{11}^3(x_{11}(k)) \\ f_{12}^3(x_{12}(k)) \end{bmatrix} = \begin{bmatrix} \dfrac{e^{-(-0.3)^2}}{-0.3+1.3}(x_{11}(k)+1.3) \\ \dfrac{e^{-0.3^2}}{0.3+1.3}(x_{12}(k)+1.3) \end{bmatrix}.$$

Model Rule 4: IF θ_1 is P_{41} and θ_2 is P_{42}, THEN

$$\begin{bmatrix} f_{11}^4(x_{11}(k)) \\ f_{12}^4(x_{12}(k)) \end{bmatrix} = \begin{bmatrix} \dfrac{e^{-(-0.3)^2}}{-0.3+1.3}(x_{11}(k)+1.3) \\ \dfrac{e^{-(-0.3)^2}}{-0.3+1.3}(x_{12}(k)+1.3) \end{bmatrix}.$$

Then the following linear model can be derived out of defuzzification process:

$$\begin{cases} x_{11}(k+1) = e^{-1}x_{11}(k) + (1-e^{-1}) \\ \qquad \displaystyle\sum_{l=1}^{4} w_l(\theta(k))\Big[0.1f_{12}^l(x_{12}(k)) + 0.2\Big], \\ x_{12}(k+1) = e^{-1}x_{12}(k) + (1-e^{-1}) \\ \qquad \displaystyle\sum_{l=1}^{4} w_l(\theta(k))\Big[0.2f_{11}^l(x_{11}(k)) + 0.1\Big], \end{cases} \tag{V.4}$$

where $w_1 = M_1N_1$, $w_2 = M_1N_2$, $w_3 = M_2N_1$, $w_4 = M_2N_2$. Taking the known premise variables $\theta_1 = 0.1$ and $\theta_2 = 0.2$, then $M_1(\theta_1) = 0.67$, $M_2(\theta_1) = 0.33$, $N_1(\theta_2) = 0.83$, $N_2(\theta_2) = 0.17$. And $w_1(\theta) = 0.56$, $w_2(\theta) = 0.11$, $w_3(\theta) = 0.27$, $w_4(\theta) = 0.06$. Let $p = 1$, by simple calculation, $r_2 < 1$. It is easy to verify that all conditions in Theorem 10 are valid. According to Theorem 10, Takagi-Sugeno fuzzy model (V.4) is square-mean globally exponentially stable, see Figures 1-2.

Remark 15: Takagi-Sugeno fuzzy model (V.4) is the approximate model for non-fuzzy model (V.3) via 4 fuzzy rules. Figures 3-4 depict the time responses of solution $(x_{11}, x_{12})^T$ of (V.3) and (V.4), respectively. By computation of Matlab, the absolute error of x_{11} between (V.3) and (V.4) is 0.0258, and the absolute error of x_{12} between (V.3) and (V.4) is 0.0426. If more fuzzy rules are used, one could obtain a more precise fuzzy model for model (V.3).

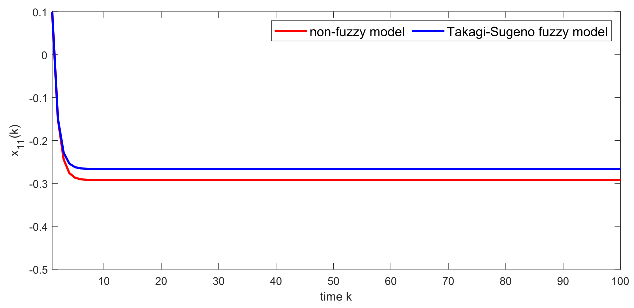

FIGURE 3. Time responses of $x_{11}(k)$ of non-fuzzy model (V.3) and Takagi-Sugeno fuzzy model (V.4).

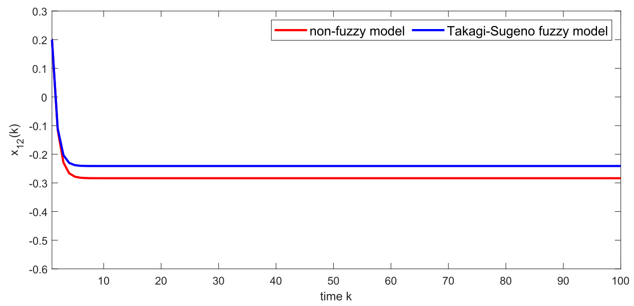

FIGURE 4. Time responses of $x_{12}(k)$ of non-fuzzy model (V.3) and Takagi-Sugeno fuzzy model (V.4).

Example 16: Considering the following two-neuron stochastic cellular neural networks:

$$\begin{cases} dx_1(t) = \Big[-x_1(t) + 0.1\sin(\sqrt{5}t)\sin(x_1(t))x_1(t) \\ \qquad + 0.2\sin(\sqrt{7}t)x_2(t-1) \\ \qquad + 0.01\cos^2(\sqrt{17}t)\Big]dt + 0.1dw_1(t), \\ dx_2(t) = \Big[-x_2(t) + 0.2\cos(\sqrt{5}t)\cos(x_2(t))x_2(t) \\ \qquad + 0.1\cos(\sqrt{2}t)x_1(t-1) \\ \qquad - 0.02|\sin(\sqrt{33}t)|\Big]dt + 0.2dw_2(t), \end{cases} \tag{V.5}$$

where $t \in \mathbb{R}$.

(1) Semi-discrete model:

$$\begin{cases} x_1(k+1) \\ = e^{-1}x_1(k) \\ \quad +(1-e^{-1})\Big[0.1\sin(\sqrt{5}k)\sin(x_1(k))x_1(k) \\ \quad +0.2\sin(\sqrt{7}k)x_2(k-1)+0.1\Delta w_1(k) \\ \quad +0.01\cos^2(\sqrt{17}k)\Big], \\ x_2(k+1) \\ = e^{-1}x_2(k) \\ \quad +(1-e^{-1})\Big[0.2\cos(\sqrt{5}k)\cos(x_2(k))x_2(k) \\ \quad +0.1\cos(\sqrt{2}k)x_1(k-1)+0.2\Delta w_2(k) \\ \quad -0.02|\sin(\sqrt{33}k)|\Big], \end{cases} \tag{V.6}$$

where $k \in \mathbb{Z}$. Corresponding to system (*I.3*), we have

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} 0.1\sin(\sqrt{5}k) & 0 \\ 0 & 0.2\cos(\sqrt{5}k) \end{bmatrix},$$

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 0.01\cos^2(\sqrt{17}k) \\ -0.02|\sin(\sqrt{33}k)| \end{bmatrix},$$

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0.2\sin(\sqrt{7}k) \\ 0.1\cos(\sqrt{2}k) & 0 \end{bmatrix},$$

$$\begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad \begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$\begin{bmatrix} f_1(x_1(k)) \\ f_2(x_2(k)) \end{bmatrix} = \begin{bmatrix} \sin(x_1(k))x_1(k) \\ \cos(x_2(k))x_2(k) \end{bmatrix},$$

$$\begin{bmatrix} g_1(x_1(k)) \\ g_2(x_2(k)) \end{bmatrix} = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, \quad k \in \mathbb{Z}.$$

(2) Takagi-Sugeno fuzzy model: Clearly, f_1 and f_2 in model (V.6) are nonlinear. They need to be linearized by using Takagi-Sugeno fuzzy method. With the help of Matlab, $x_1 \in [-0.3057, 0.3144]$ and $x_2 \in [-0.4656, 0.5733]$. Let $\theta_1(k) = x_1(k)$, $\theta_2(k) = x_2(k)$, $\forall k \in \mathbb{Z}$. Then $\theta_1 \in [-0.3057, 0.3144]$ and $\theta_2 \in [-0.4656, 0.5733]$. θ_1 and θ_2 can be represented by membership functions M_1, M_2, N_1 and N_2 as follows:

$$\theta_1(k) = M_1(\theta_1(k)) \cdot 0.3144 + M_2(\theta_1(k)) \cdot (-0.3057),$$
$$\theta_2(k) = N_1(\theta_2(k)) \cdot 0.5733 + N_2(\theta_2(k)) \cdot (-0.4656),$$

where $M_1(\theta_1) + M_2(\theta_1) = 1$, $N_1(\theta_2) + N_2(\theta_2) = 1$. Hence, the membership functions can be obtained as follows:

$$M_1(\theta_1(k)) = \frac{\theta_1(k) + 0.3057}{0.6201},$$
$$M_2(\theta_1(k)) = \frac{0.3144 - \theta_1(k)}{0.6201},$$
$$N_1(\theta_2(k)) = \frac{\theta_2(k) + 0.4656}{1.0389},$$
$$N_2(\theta_2(k)) = \frac{0.5733 - \theta_2(k)}{1.0389}.$$

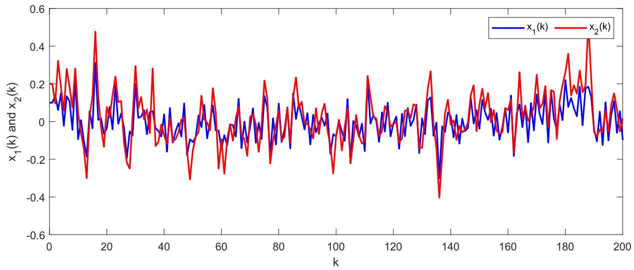

FIGURE 5. Square-mean almost periodicity of $(x_1(k), x_2(k))^T$ of model (V.7).

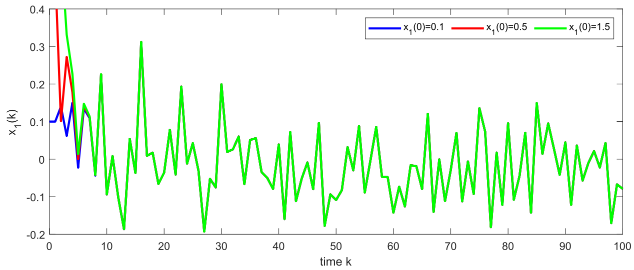

FIGURE 6. Square-mean global exponential stability of state variable $x_1(k)$ of model (V.7).

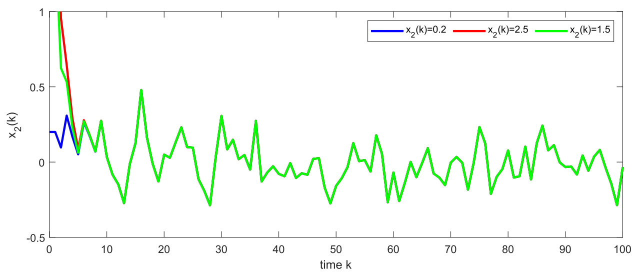

FIGURE 7. Square-mean global exponential stability of state variable $x_2(k)$ of model (V.7).

Let $P_{11} = P_{21} = M_1$, $P_{31} = P_{41} = M_2$, $P_{12} = P_{32} = N_1$, $P_{22} = P_{42} = N_2$. Therefore, the nonlinear functions f_1 and f_2 are modeled by the following IF-THEN rules:

Model Rule 1: IF θ_1 is P_{11} and θ_2 is P_{12}, THEN

$$\begin{bmatrix} f_1^1(x_1(k)) \\ f_2^1(x_2(k)) \end{bmatrix} = \begin{bmatrix} \sin(0.3144)x_1(k) \\ \cos(0.5733)x_2(k) \end{bmatrix}.$$

Model Rule 2: IF θ_1 is P_{21} and θ_2 is P_{22}, THEN

$$\begin{bmatrix} f_1^2(x_1(k)) \\ f_2^2(x_2(k)) \end{bmatrix} = \begin{bmatrix} \sin(0.3144)x_1(k) \\ \cos(-0.4656)x_2(k) \end{bmatrix}.$$

Model Rule 3: IF θ_1 is P_{31} and θ_2 is P_{32}, THEN

$$\begin{bmatrix} f_1^3(x_1(k)) \\ f_2^3(x_2(k)) \end{bmatrix} = \begin{bmatrix} \sin(-0.3057)x_1(k) \\ \cos(0.5733)x_2(k) \end{bmatrix}.$$

Model Rule 4: IF θ_1 is P_{41} and θ_2 is P_{42}, THEN

$$\begin{bmatrix} f_1^4(x_1(k)) \\ f_2^4(x_2(k)) \end{bmatrix} = \begin{bmatrix} \sin(-0.3057)x_1(k) \\ \cos(-0.4656)x_2(k) \end{bmatrix}.$$

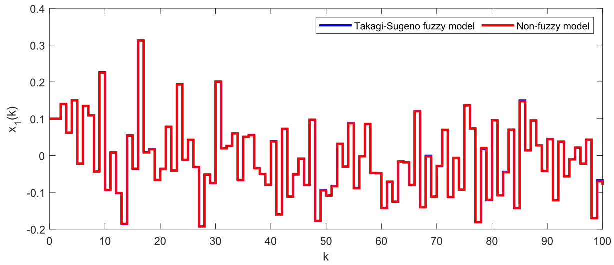

FIGURE 8. Time responses of $x_1(k)$ of model (V.6) and fuzzy model (V.7).

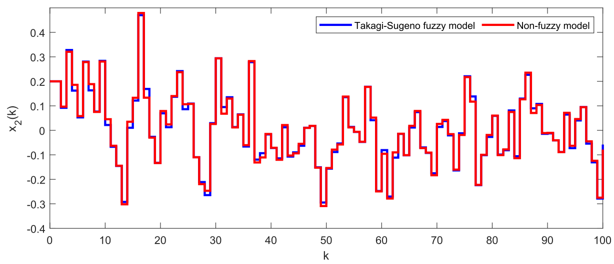

FIGURE 9. Time responses of $x_2(k)$ of model (V.6) and fuzzy model (V.7).

Then the following linear model can be derived out of defuzzification process:

$$\begin{cases} x_1(k+1) = e^{-1}x_1(k) + (1-e^{-1})\sum_{l=1}^{4} w_l(\theta(k)) \\ \qquad \Big[0.1\sin(\sqrt{5}k)f_1^l(x_1(k)) \\ \qquad + 0.2\sin(\sqrt{7}k)x_2(k-1) + 0.1\Delta w_1(k) \\ \qquad + 0.01\cos^2(\sqrt{17}k)\Big], \\ x_2(k+1) = e^{-1}x_2(k) + (1-e^{-1})\sum_{l=1}^{4} w_l(\theta(k)) \\ \qquad \Big[0.2\cos(\sqrt{5}k)f_2^l(x_2(k)) \\ \qquad + 0.1\cos(\sqrt{2}k)x_1(k-1) + 0.2\Delta w_2(k) \\ \qquad - 0.02|\sin(\sqrt{33}k)|\Big], \end{cases} \tag{V.7}$$

where w_i is defined as that in Example 14, $i = 1, 2, 3, 4$.

Taking the known premise variables $\theta_1 = 0.1$ and $\theta_2 = 0.2$, then $M_1(\theta_1) = 0.6542$, $M_2(\theta_1) = 0.3458$, $N_1(\theta_2) = 0.6407$, $N_2(\theta_2) = 0.3593$. And $w_1(\theta) = 0.4192$, $w_2(\theta) = 0.2351$, $w_3(\theta) = 0.2215$, $w_4(\theta) = 0.1242$. Let $p = 1$, by simple calculation, $r_2 < 1$. It is easy to verify that all conditions in Theorems 7 and 10 hold. By Theorems 7 and 10, Takagi-Sugeno fuzzy model (V.7) outputs a square-mean almost periodic sequence solution, which is square-mean globally exponentially stable, see Figures 5-7. Figures 8-9 compare the time responses of the original model (V.6) with its fuzzy approximation model (V.7).

VI. CONCLUSION AND FUTURE DEVELOPMENTS

In recent years, the semi-discrete method [20] of differential equations has been applied into the investigations of determinant neural networks [20], [21]. But few people employ this method to study stochastic neural networks. In this paper, we formulate a kind of discrete analogue of stochastic CNNs by using semi-discrete method, which gives a more accurate characterization for continuous-time stochastic CNNs than that by Euler scheme [16], [17]. Based on the above semi-discrete model, a class of discrete-time stochastic fuzzy CNNs is obtained with the help of Takagi-Sugeno fuzzy method, which gives an approximate version of the above semi-discrete stochastic CNNs. Next, we investigate the $2p$-th mean almost periodic outputs and moment global exponential stability of a semi-discrete stochastic Takagi-Sugeno fuzzy CNNs with the help of Minkowski inequality, Hölder inequality, Krasnoselskii's fixed point theorem and the proof of contradiction.

Looks over the entire paper, the major achievements of this paper are detailedly summarized below.

(1) A kind of discrete analogue of stochastic CNNs is derived by using semi-discrete method, which gives a more accurate characterization of continuous-time model than that by Euler scheme, see model (I.3).

(2) A class of semi-discrete stochastic fuzzy CNNs is obtained with the help of Takagi-Sugeno fuzzy method, which gives an approximate version of the above semi-discrete stochastic CNNs, see model (II.3). By applying Takagi-Sugeno fuzzy method, a nonlinear model can be approximated by a corresponding Takagi-Sugeno fuzzy model.

(3) Theorems 7 and 10 provide possible technique to study $2p$-th mean almost periodic oscillations and moment global exponential stability of semi-discrete stochastic Takagi-Sugeno fuzzy CNNs. They can be applied to research the other realistic models described by the discrete stochastic Takagi-Sugeno fuzzy systems, see Sections 3-4.

(4) In Section 5, a two-neuron stochastic CNNs is considered. By means of semi-discrete and Takagi-Sugeno fuzzy methods, we obtained the corresponding semi-discrete stochastic Takagi-Sugeno fuzzy model (V.7). With the help of numerical simulations of Matlab, Takagi-Sugeno fuzzy model (V.7) gives a better approximate version of the semi-discrete model (V.6), see Figures 8-9.

Of course, there are some developments in this article to explore further. For instance,

(1) In semi-discrete model (I.3), the discrete analogue of stochastic part is obtained by Euler scheme, but not by semi-discrete method.

(2) In fuzzy model (II.3), w_l is defined by product of membership functions. One could consider w_l by minimum of membership functions.

(3) Other dynamic behaviours of fuzzy model (II.3) need further discussion.

(4) Other realistic models described by the semi-discrete stochastic Takagi-Sugeno fuzzy models need further study.

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