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Adaptive Fixed Time Control for Generalized Synchronization of Mismatched Dynamical Systems With Parametric Estimations

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ABSTRACT This paper is concerned with the adaptive fixed time control problem for generalized synchronization of mismatched dynamical systems with parametric estimations. We first introduce a new lemma of the fixed-time stability and give a high accuracy estimation of the convergence time. Then, according to the new lemma, the adaptive control scheme for fixed-time synchronization between integer-order dynamical systems is mathematically derived by taking uncertain parameters into account. Meanwhile, the corresponding adaptation laws are designed to estimate the parameter uncertainties. Further, Adaptive fixed time sliding mode control strategy for generalized synchronization of fractional-order dynamical systems is proposed. A novel fractional-order integral sliding mode surface is presented and its fixed time stability to origin is analytically proved using the Lyapunov stability theory. In addition, by considering the parametric estimations in the controller, an appropriate adaptive law is constructed to obtain the expected results. Finally, compared with the existing finite-time stability method, some numerical simulations are conducted to demonstrate the validity and superiority of the proposed approach.

INDEX TERMS Adaptive tuning controller, fixed-time generalized synchronization, mismatched dynamical systems, parametric estimations, fractional-order sliding mode control.

I. INTRODUCTION

The synchronization problemis one of the emerging topics in nonlinear dynamic fields. It has attracted the widespread attention among scholars because of its application in various fields of science and engineering including secure communication, image encryption, signal transmission, neural network and other fields [1]–[4]. The key point of the synchronization is to construct an appropriate controller. It will control the state trajectories of the response system to follow the trajectories of the drive system asymptotically. A number of various control methods have been found in the previous works, such as the active control [5], optimal control [6], adaptive control [7], sliding mode control [8] and adaptive fuzzy control [9]. Very recently, the different types of synchronization including complete synchronization [10], lag synchronization [11], projective synchronization [12] and generalized synchronization [13] have been extensively studied in the existent literature. Among all kinds of synchronization, generalized synchronization between drive system and response system characterized by two optional functions could obtain desired types in practice application. Particularly, it can be used to extend the coexistence of different synchronization types, so generalized synchronization has attracted more and more attentions from a lot of scholars [14], [15]. Moreover, the generalized synchronization between the mismatched dynamical systems is investigated, Zhang *et al.* [13] presented a Lyapunov approach to obtain the expected results. Wang *et al.* [8] discussed the generalized synchronization among mismatched fractional-order chaotic and hyper-chaotic systems with different orders. Muthukumar *et al.* [15] investigated the generalized robust synchronization approach for mismatched fractional order dynamical systems with different dimensions via sliding

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mode control. Furthermore, by means of the proposed generalized synchronization criterion, in order to send or share voice messages secretly, a novel audio cryptosystem is proposed. Nevertheless, most of the above results on the basis of the Lyapunov asymptotical stability theory have not considered the effects of the uncertain parameters.

From an engineering application point of view, all of dynamical system parameters sometimes cannot be attained exactly in advance due to the various external factors such as linear approximation, measurement inaccuracy and modeling errors. In addition, it is also noted that all the previous methods cannot achieve finite-time synchronization and just accomplish asymptotical convergence, which implies that convergence time cannot be estimated in advance. Therefore, the finite-time synchronization of dynamical systems with unknown parameters has become a hot research topic. Some relevant results of the finite-time synchronization with and without parameter uncertainties have been found in [16]–[22]. According to the finite time stability method [17]. Aghababa and Aghababa [18] investigated the adaptive finite-time synchronization between chaotic systems with unknown parameters and nonlinear inputs. Robust adaptive finite-time controller was proposed in [19] to synchronize uncertain non-autonomous chaotic systems. Cai *et al.* [20] studied the generalized finite-time synchronization of chaotic systems with different orders for the first time. Reference [16] used the nonlinear feedback controller to realize the robust finite-time global synchronization. In [21], Zhao et al. further studied adaptive finite-time generalized synchronization between uncertain chaotic systems of different orders. Zhang *et al.* [22] discussed the global finite-time synchronization between different dimensional chaotic systems with and without the parameter uncertainties. However, it should be noteworthy that the convergence time of finite time stability method relies on the initial condition, so it will subject to a great inconvenience in the practical application.

In order to overcome the weakness of finite time control, Polyakov presented the fixed-time stability method in [23], which showed the following advantages, such as strong robustness, fast convergence speed and high precision performance. Due to these attractive properties, more and more attention has also been obtained [24]–[27]. In particular, Zuo proposed the fixed-time stability theory in [28]. Subsequently, the lemma was further applied to discuss the fixed-time stability of dynamical systems in [29]–[31]. The fixed-time stability of dynamical systems proposed by Hu *et al.* [29] and Xu *et al.* [30] is a faster convergence speed, higher precision and less conservative than the proposed method by Zuo [28]. A more general fixed-time stability theorem [31] was proposed by means of adding a constant term than the proposed method by Hu *et al.* [29]. Whereas, the research of fixed-time stability or synchronization is just at the primitive stage due to the lacking of the theory. And up to now, regarding the results of generalized synchronization are just proposed either were global finite-time synchronization or did not take the unknown parameters into account. Therefore, it should be meaningful to study the generalized synchronization in fixed-time between mismatched dynamical systems with parameter uncertainties.

Additionally, the control method of synchronization between fractional order dynamical systems with unknown parameters has also started to attract more attention. Jafari *et al.* [32] presented an adaptive fuzzy controller with compensation signal for synchronization and stabilization of a class of fractional order systems with uncertain nonlinearities. Behinfaraz *et al.* [33] provided an active adaptive control method for synchronization in fractional-order chaotic systems with parameter uncertainty. Then, in [34], they introduced the new synchronization method of fractional order chaotic systems by considering time-varying parameter and orders. The designed method has an expected performance. An adaptive back-stepping strategy is presented to control and synchronize a class of fractional order chaotic systems with unknown parameters in Ref. [35]. Meanwhile, an adaptive sliding mode control scheme of synchronization between fractional order dynamical systems with unknown parameters and external disturbances was studied in Ref [36]–[38]. Further, an adaptive synchronization law was applied in fractional order chaotic Arneodo system with unknown parameters in [39]. Nevertheless, all the above works mainly concerned the adaptive estimation of the uncertainties and disturbances based on Lyapunov asymptotic stability theory, neither of them have discussed the generalized synchronization with the upper bounds of fixed-time convergence in mismatched fractional order dynamical systems. To the best of our knowledge, there have also been no relevant results reporting the adaptive fixed-time control scheme with fractional derivatives. In fact, fractional order adaptive fixed time sliding mode control strategy provides a new way to deal with a class of synchronization problems.

Motivated by the above-mentioned works and ideas, this paper studies the adaptive fixed time control problem for generalized synchronization of mismatched dynamical systems in the presence of uncertain parameters. The main contributions of the proposed method here can be summarized as the following points:

(1) We first give the definition of generalized fixed-time synchronization and attempt to develop a new fixed-time stability lemma, which is an extension of the presented method by Hu *et al.* [29] and Xu *et al.* [30], and the stabilization time is shorter than his method.
(2) According to the new lemma, the adaptive control scheme for fixed-time synchronization between integer-order dynamical systems is mathematically derived by considering uncertain parameters. Meanwhile, the corresponding updated laws of parameter estimations are designed to guarantee the fixed-time stability of uncertain synchronization error system. Our approach has more advantages than those in [16], [21], [22].

TABLE 1. Annotations for the symbols.

Symbol	Nomenclature	Notation
α	fractional-order	$1 > \alpha > 0$
$\Gamma(\cdot)$	the gamma function	$\alpha\Gamma(\alpha) = \Gamma(\alpha+1)$
λ, η, μ	the control parameters	$\lambda, \eta \geq 0, 1 > \mu > 0$
p, q	positive odd integers	$p > 1+q, 0 \leq q < 1$
k	arbitrary constant	$k > 0$
m, n	The system dimensions	$n \neq m$
r	the rows of the Jacobin matrices	$r \leq min\{n, m\}$
$sig^{\epsilon}(\cdot)$	the sign function	$sig^{\epsilon}(\cdot) = \lvert\cdot\rvert^{\epsilon}\text{sign}(\cdot)$
ϕ, φ	the uncertain parameters of integer-order systems	$\phi \in R^{\rho}, \varphi \in R^{\gamma}$
χ, ψ	the uncertain parameters of fractional-order systems	$\chi \in R^{\rho}, \psi \in R^{\gamma}$
ρ, γ	Number of the uncertain parameters	$\rho \neq \gamma$
k_0, δ	the adjusted coefficient of siding surface	$k_0 > 0, \delta > 0$
k_1, k_2 σ, ϑ	the adjusted coefficient of reaching law	$0 < \sigma < 1, \vartheta > 1, k_1, k_2 > 0$

(3) Inspired by [8], [15], adaptive fixed time sliding mode control strategy for generalized synchronization of fractional-order dynamical systems is proposed. A novel fractional-order integral sliding mode surface is presented and its fixed time stability to origin is analytically proved using the Lyapunov stability theory. In addition, by considering the parametric estimations in the controller, an appropriate adaptive law is constructed to obtain the expected results.

(4) Compared with the existing finite-time stability method [15], [22], some simulations are conducted to demonstrate the validity and superiority of the proposed approach. Also, it can be further applied to the various fixed-time synchronization types between other dynamical systems.

The framework of this paper is formed as follows. Section 2 introduces some useful definitions and lemmas of fractional-order calculus, and develops a new global fixed-time stability lemma. In Section 3, the adaptive control scheme of fixed-time generalized synchronization between integer-order dynamical systems is mathematically derived. Then, adaptive fixed time sliding mode control strategy for generalized synchronization of fractional-order dynamical systems is proposed in Section 4. Some numerical examples are demonstrated with comparison of the finite time method in Section 5. Finally, Section 6 summarizes and discusses this paper.

II. PRELIMINARIES

The detailed annotations for the symbols are listed in Table 1.

For the global stability mathematical analysis, we introduce some necessary lemmas as follows.

A. PRELIMINARIES OF FRACTIONAL-ORDER CALCULUS

Definition 1 [40]: The αth-order Caputo fractional integral of a function $f(t)$ is described by

$$ {}_{t_0}^{C}I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} f(\tau) d\tau $$

Definition 2 [40]: The αth-order Caputo fractional derivative of a function $f(t)$ is defined as:

$$ {}_{t_0}^{C}D_t^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_{t_0}^{t} \dfrac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau, \\ \qquad m-1 < \alpha < m \\ \dfrac{d^m f(t)}{dt^m}, \quad \alpha = m \end{cases} $$

where, m is the smallest integer number.

Lemma 1 [40]: If the fractional-order derivative ${}_{t_0}^{C}D_t^{\alpha} x(t)$ is integrable, one can obtain:

$$ {}_{t_0}^{C}I_t^{\alpha} {}_{t_0}^{C}D_t^{\alpha} x(t) = x(t) - \sum\nolimits_{k=0}^{n-1} \frac{x^{(k)}(t_0)}{k!} (t-t_0)^k $$

Especially, for $0 < \alpha \leq 1$, then ${}_{t_0}^{C}I_t^{\alpha} {}_{t_0}^{C}D_t^{\alpha} x(t) = x(t) - x(t_0)$.

Lemma 2 [40]: Assume $p \in R$, then

$$ {}_{t_0}^{C}D_t^{\alpha} x^p(t) = \frac{\Gamma(1+p)}{\Gamma(1+p-\alpha)} x^{p-\alpha}(t) {}_{t_0}^{C}D_t^{\alpha} x(t) $$

Lemma 3 [41]: Suppose $x(t)$ a continuous and differentiable function, then it satisfies the following inequality

$$ {}_{t_0}^{C}D_t^{\alpha} |x(t)| \leq sign(x(t)) {}_{t_0}^{C}D_t^{\alpha} x(t) $$

Lemma 4 [31]: If $\varepsilon_i \in \mathbb{R}, i = 1, 2 \cdots n$ are arbitrary real numbers, the following inequalities satisfy:

$$ \begin{cases} \left(\displaystyle\sum_{i=1}^{n} |\varepsilon_i|\right)^{\xi} \leq \displaystyle\sum_{i=1}^{n} |\varepsilon_i|^{\xi} & 0 < \xi \leq 1 \\ n^{1-\xi} \left(\displaystyle\sum_{i=1}^{n} |\varepsilon_i|\right)^{\xi} \leq \displaystyle\sum_{i=1}^{n} |\varepsilon_i|^{\xi}, & 1 < \xi \end{cases} $$

B. GLOBAL FIXED TIME STABILITY

Lemma 5 [42]: Consider the continuous positive definite and radially unbounded function $V(e(t))$ satisfies the differential inequality

$$ \dot{V}(e(t)) \leq -\lambda V^{\mu}(e(t)) - \eta V(e(t)), \quad \forall t \geq t_0, \ V(e_0) \geq 0 $$

where, λ, $\eta > 0$, $1 > \mu > 0$, it meet $V(t) \equiv 0$, $\forall t > T_{max}^1$. The convergence time T_{max}^1 is given by $T_{max}^1 = t_0 + \frac{1}{(1-\mu)\eta} In\left(\frac{\lambda + \eta V^{1-\mu}(e_0)}{\lambda}\right)$.

Corollary 1: When $\eta = 0$, $\dot{V}(e(t)) \leq -\lambda V^{\mu}(e(t))$, $\forall t \geq t_0$, the global finite time should be rewritten as $T_{max}^2 = t_0 + \frac{V^{1-\mu}(e_0)}{(1-\mu)\lambda}$, and $T_{max}^1 \leq T_{max}^2$ for $V(e_0) \geq 0$.

Proof: When $\eta = 0$, it is easy to get $V^{1-\mu}(e(t)) \leq V^{1-\mu}(e(t_0)) - \lambda(1-\mu)(t-t_0)$, and $V(e(t)) \equiv 0$, for any $t \geq T^2_{max}$, the global finite time is given by $T^2_{max} = t_0 + \frac{V^{1-\mu}(e_0)}{(1-\mu)\lambda}$. Then

$$T^1_{max} - T^2_{max} = \frac{1}{(1-\mu)\eta} In\left(\frac{\lambda + \eta V^{1-\mu}(e_0)}{\lambda}\right) - \frac{V^{1-\mu}(e_0)}{(1-r)\lambda}$$
$$= \frac{1}{(1-\mu)}\left(\frac{1}{\eta} In\left(\frac{\lambda+\eta V^{1-\mu}(e_0)}{\lambda}\right) - \frac{V^{1-\mu}(e_0)}{\lambda}\right)$$

Let $\psi(\upsilon) = \frac{1}{\eta} In\left(\frac{\lambda+\eta\upsilon}{\lambda}\right) - \frac{\upsilon}{\lambda}$ and $\upsilon = V^{1-\mu}(e_0) \geq 0$, we have $\dot{\psi}(\upsilon) = \frac{1}{\lambda+\eta\upsilon} - \frac{1}{\lambda} < 0$, that is $T^1_{max} \leq T^2_{max}$ for $V(e_0) \geq 0$. The proof is completed. □

The purpose of this subsection is to develop some theoretical results of global fixed time stability and make some comparisons with the previous work provided by Hu *et al.* [29] and Xu *et al.* [30].

Lemma 6 [30]: Assume that there exists a continuous positive definite and radially unbounded function $V(e(t))$ and its right directional derivative satisfies the differential inequality:

$$\frac{d}{dt}V(e(t)) \leq -\lambda V^{\mu-p/q}(e(t)) - \eta V^{p/q}(e(t)), \quad e(0) = e_0$$

where, $\lambda, \eta > 0, \mu > (p+q)/q, p < q$, then the origin of the above system is globally fixed time stable and the upper bound of settling time $T(e_0)$ can be estimated by

$$\lim_{e_0\to\infty}[T(e_0)] \leq T^3_{max} = \frac{1}{\lambda(\mu - (p+q)/q)} + \frac{q}{\eta(q-p)}$$

Lemma 7 [29]: Assume that a continuous and positive definite function $V(e(t))$ satisfies differential inequality as follow:

$$\frac{d}{dt}V(e(t)) \leq -\left(\lambda V^p(e(t)) + \eta\right)^k, \quad e(0) = e_0$$

where, $\lambda, \eta, p, k > 0, pk > 1$, then the origin of the above system is globally fixed-time stable and the upper bound of settling time T (e_0) can be estimated by

$$\lim_{e_0\to\infty}[T(e_0)] \leq T^4_{max} = \frac{1}{\eta^k}\left(\frac{\eta}{\lambda}\right)^{\frac{1}{p}}\left(1 + \frac{1}{pk-1}\right)$$

Lemma 8: Consider the following differential inequality:

$$\frac{d}{dt}V(e(t)) \leq -\left(\lambda V^{p-q}(e(t)) + \eta V^q(e(t))\right)^k, \quad e(0) = e_0$$

where, $k > 0, p > 1+q, 0 < q < 1, qk < 1, pk > 1+qk$. The solution of the above inequality starting from arbitrary initial condition will converge to the equilibrium point in a fixed-time upper bounded by $\lim_{e_0\to\infty}[T(e_0)] \leq T^5_{max} = \left(\frac{\eta}{\lambda}\right)^{\frac{1-qk}{p-2q}}\left(\frac{1}{\eta^k((p-q)k-1)} + \frac{1}{\eta^k(1-qk)}\right)$.

Proof: Taking its time integration on both sides of the above inequality, one gets:

$$\lim_{e_0\to\infty}[T(e_0)] \leq T^5_{max} = \int_0^\infty \frac{dV(e(t))}{\left(\lambda V^{p-q}(e(t)) + \eta V^q(e(t))\right)^k}$$
$$= \int_0^l \frac{1}{\left(\lambda V^{p-q}(e(t))+\eta V^q(e(t))\right)^k} dV(e(t)) + \int_l^\infty \frac{1}{\left(\lambda V^{p-q}(e(t))+\eta V^q(e(t))\right)^k} dV(e(t))$$
$$\leq \int_0^l \frac{1}{\eta^k V^{qk}(e(t))} dV(e(t)) + \int_l^\infty \frac{1}{\lambda^k V^{(p-q)k}(e(t))} dV(e(t))$$
$$= \frac{l^{1-qk}}{\eta^k(1-qk)} + \frac{l^{1-(p-q)k}}{\lambda^k((p-q)k-1)}$$

where, $l > 0$ is an any constant, let $w(l) = \frac{l^{1-qk}}{\eta^k(1-qk)} + \frac{l^{1-(p-q)k}}{\lambda^k((p-q)k-1)}$, for $\dot{w}(l) = \frac{1}{\eta^k}l^{-qk} - \frac{1}{\lambda^k}l^{-(p-q)k} = 0$, i.e., $l = \left(\frac{\eta}{\lambda}\right)^{\frac{1}{p-2q}}$. We get its minimum value $w_{min}(l) = \left(\frac{\eta}{\lambda}\right)^{\frac{1-qk}{p-2q}}\left(\frac{1}{\eta^k((p-q)k-1)} + \frac{1}{\eta^k(1-qk)}\right)$. Hence, $T^5_{max} < w_{min}(l)$.

Remark 1: Evidently, when $q = 0$, Lemma 8 can derive the form of Lemma 7. In addition, as we all know, Lemma 7 is more effective and higher precision compared with the results given by Polyakov [23]. Hence, the proposed fixed time stability is less conservative in this paper.

Corollary 2: When $k = 1$, the upper bound of settling time $T(e_0)$ can satisfy

$$\begin{cases} T^5_{max} = T^3_{max}, & \lambda = \eta \\ T^5_{max} < T^3_{max}, & \lambda \neq \eta. \end{cases}$$

It is apparent that

$$T^5_{max} - T^3_{max} = \frac{1}{\eta\theta(1-q)}\left[(1+\theta)\left(\frac{\eta}{\lambda}\right)^{\frac{1}{1+\theta}} - \theta - \frac{\eta}{\lambda}\right]$$

where, $\theta = \frac{p-1}{1-q}$. Define $\varpi(\theta) = \left[(1+\theta)\left(\frac{\eta}{\lambda}\right)^{\frac{1}{1+\theta}} - \theta - \frac{\eta}{\lambda}\right]$, $\theta > 0$.

(a) $\varpi(\theta) = 0$ for $\lambda = \eta, i.e. T^5_{max} = T^3_{max}$

(b) Then

$$\begin{cases} \dot{\varpi}(\theta) = \left(\frac{\eta}{\lambda}\right)^{\frac{1}{1+\theta}}\left[1 - \frac{1}{1+\theta} In\frac{\eta}{\lambda}\right] - 1 \\ \ddot{\varpi}(\theta) = \left(\frac{\eta}{\lambda}\right)^{\frac{1}{1+\theta}}\frac{1}{(1+\theta)^3}\left(In\frac{\eta}{\lambda}\right)^2 > 0 \end{cases}$$

We obtain $\lim_{\chi\to\infty}\dot{\varpi}(\theta) = \lim_{\theta\to\infty}\left[\left(\frac{\eta}{\lambda}\right)^{\frac{1}{1+\theta}}\left(1 - \frac{1}{1+\theta}In\frac{\eta}{\lambda}\right) - 1\right] = 0$, hence, $\dot{\varpi}(\theta) < 0$ for $\theta > 0$, i.e. $\varpi(\theta) < \varpi(0) = 0, T^5_{max} < T^3_{max}$ for $\lambda \neq \eta$. This completes the proof. □

Remark 2: From Lemma 6, the estimation bound of the convergence time obtained is more accurate and more widely in comparison with the classical results given in Zuo [28]. In Corollary 2, the condition is $k = 1$, so the parameter range is larger and the convergence time is faster than Lemma 6 in [30]. The superiority of the fixed-time stability in Lemma 8 will be further shown in comparison with Lemma 6.

Remark 3: In the study of finite time and fixed time stability or synchronization, a primary difference is that, whether their convergence time depends on the initial condition. Some scholars have referenced the fixed time stability theorem presented by Zuo [28]. Whereas, from Remark 1 and 2, the fixed time stability theorem of the dynamical system in this paper achieves a less conservative bound than the theorem proposed by Hu *et al.* [29] and Xu *et al.* [30].

III. DESCRIPTION OF INTEGER-ORDER DYNAMICAL SYSTEMS AND GENERALIZED SYNCHRONIZATION SCHEME

The following n-dimensional dynamical system is considered as the drive system

$$\dot{x} = F(x) + f(x)\phi \tag{1}$$

The corresponding m-dimensional response system is described by

$$\dot{y} = G(y) + g(y)\varphi + u(t) \tag{2}$$

where, $x = [x_1, x_2, \cdots, x_n]^T \in R^n$ and $y = [y_1, y_2, \cdots, y_m]^T \in R^m$ are the system state vector; $F(x) : R^n \to R^n$ and $G(y) : R^m \to R^m$ are the continuous nonlinear functions; $f(x) : R^n \to R^{n\times\rho}$ and $g(y) : R^m \to R^{m\times\gamma}$ are the function matrices of system parameters; $\phi \in R^\rho$ and $\varphi \in R^\gamma$ denote the unknown system parameters. $u(t) = [u_1(t), u_2(t), \cdots, u_m(t)]^T \in R^m$ is the control input.

Consider the system (1) and the system (2), suppose that there exist two arbitrary continuously differentiable functions $Q(x) : R^n \to R^r$ and $P(y) : R^m \to R^r$. Then the generalized synchronization error of the dynamical systems can be described as

$$e(t) = Q(x) - P(y) \tag{3}$$

where, if the origin is an equilibrium point of (3), there exists an open neighborhood $U_\varepsilon \in R^r$ of the origin, for every $e_0 = Q(x_0) - P(y_0) \in U_\varepsilon$.

Then the error dynamics can be given by

$$\begin{aligned}\dot{e}(t) &= J_Q(x)\dot{x} - J_P(y)\dot{y}\\ &= J_Q(x)(F(x) + f(x)\phi)\\ &\quad - J_P(y)(G(y) + g(y)\varphi + u(t))\end{aligned} \tag{4}$$

where $J_Q(x)$ and $J_P(y)$ denote the Jacobin matrices of the functions $Q(x)$ and $P(y)$, respectively, i.e.

$$J_Q(x) = \begin{bmatrix} \frac{\partial Q_1(x)}{\partial x_1} & \frac{\partial Q_1(x)}{\partial x_2} & \cdots & \frac{\partial Q_1(x)}{\partial x_n} \\ \frac{\partial Q_2(x)}{\partial x_1} & \frac{\partial Q_2(x)}{\partial x_2} & \cdots & \frac{\partial Q_2(x)}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial Q_r(x)}{\partial x_1} & \frac{\partial Q_r(x)}{\partial x_2} & \cdots & \frac{\partial Q_r(x)}{\partial x_n} \end{bmatrix},$$

$$J_P(y) = \begin{bmatrix} \frac{\partial P_1(y)}{\partial y_1} & \frac{\partial P_1(y)}{\partial y_2} & \cdots & \frac{\partial P_1(y)}{\partial y_m} \\ \frac{\partial P_2(y)}{\partial y_1} & \frac{\partial P_2(y)}{\partial y_2} & \cdots & \frac{\partial P_2(y)}{\partial y_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial P_r(y)}{\partial y_1} & \frac{\partial P_r(y)}{\partial y_2} & \cdots & \frac{\partial P_r(y)}{\partial} y_m \end{bmatrix} \tag{5}$$

Assumption 1: $r \le min\{n, m\}$, the matrix $J_P(y)$ is row full-rank, and $J_P^{-1}(y)$ denotes the generalized inverse matrix of $J_P(y)$ in this paper.

Definition 3: Consider the above systems (1) and (2), the origin of system (4) is said to be globally fixed time stable equilibrium. If it is globally stable in the convergence time function $T(e_0) : U_\varepsilon \to (0, +\infty)$, that is, there exists a bounded constant $\mathrm{T}_{\max}$, and

$$\begin{aligned}\lim_{t\to T(e_0)} \|e(t)\| &= \lim_{t\to T(e_0)} \|Q(x) - P(y)\| = 0\\ \|e(t)\| &\equiv 0, \quad \forall t \ge T(e_0), \; T(e_0) \le T_{max}\end{aligned} \tag{6}$$

where, $T_{max} = max_{e_0 \in U_\varepsilon} T(e_0) \in (0, +\infty)$ denotes the fixed convergence time. Then, the global synchronization of the systems (1) and (2) is realized with a fixed time and is called as generalized synchronization.

The generalized fixed time synchronization issue can be diverted to the fixed time stabilization issue of error system (4). Subsequently, the control objective is to construct an appropriate controller $u(t)$. Hence, in the strict sense, the generalized fixed time synchronization between dynamical systems with parameter uncertainties can be achieved within the upper bounded of the convergence time.

To further study generalized fixed time synchronization of two chaotic systems with parameter uncertainties. Therefore, the suitable update laws are provided to estimate the uncertain parameters ϕ and φ:

$$\begin{aligned}\dot{\hat{\phi}}(t) &= \left[J_Q(x)\cdot f(x)\right]^T e(t)\\ &\quad - \lambda sig^p\left(\tilde{\phi}\right) - \eta sig^q\left(\tilde{\phi}\right), \quad \hat{\phi}(0) = \hat{\phi}_0\\ \dot{\hat{\varphi}}(t) &= -[J_P(y)\cdot g(y)]^T e(t)\\ &\quad - \lambda sig^p(\tilde{\varphi}) - \eta sig^q(\tilde{\varphi}), \quad \hat{\varphi}(0) = \hat{\varphi}_0\end{aligned} \tag{7}$$

where $\tilde{\phi} = \hat{\phi} - \phi$, $\tilde{\varphi} = \hat{\varphi} - \varphi$, $p > 1$, $0 < q < 1$, $sig^\epsilon(\cdot) = |\cdot|^\epsilon sign(\cdot)$, and sign $(\cdot)$ denotes the sign function, $\hat{\phi}$ and $\hat{\varphi}$ are the estimations of ϕ and φ, respectively. And their initial values are $\hat{\phi}_0$ and $\hat{\varphi}_0$.

Then, the adaptive controller $u(t)$ can be designed as

$$u(t) = \begin{matrix} -G(y) - g(y)\hat{\varphi} \\ +J_P^{-1}(y)\left[\begin{matrix} J_Q(x)\left(F(x)+f(x)\hat{\phi}\right) \\ +\lambda sig^p\left(e(t)\right) + \eta sig^q\left(e(t)\right)\end{matrix}\right]\end{matrix} \quad (8)$$

The error dynamics (4) can be deduced as

$$\begin{aligned}\dot{e}(t) = -[J_Q(x)\cdot f(x)]\tilde{\phi} + [J_P(y)\cdot g(y)]\tilde{\varphi} \\ -\lambda sig^p\left(e(t)\right) - \eta sig^q\left(e(t)\right)\end{aligned} \quad (9)$$

Theorem 1: Consider the dynamical systems (1) and (2) satisfying Assumption 1, under the adaptive controller (8), the error dynamics (9) should be global fixed time stability at the origin. That is, the dynamical systems (1) and (2) are globally synchronized in a fixed time T, given by

$$T \le T_{max}^5 = \left(\frac{\eta}{\left(3^{1-\frac{p+1}{2}}\right)\lambda}\right)^{\frac{1-q}{p-q}}\left(\frac{1}{\eta\left(p-1\right)} + \frac{1}{\eta\left(1-q\right)}\right) \quad (10)$$

Proof: Choose a continuous positive definite function as follows

$$V_1(t)=\sum\nolimits_{i=1}^{r} e_i^2+\sum\nolimits_{j=1}^{\rho}\left(\hat{\phi}_j-\phi_j\right)^2+\sum\nolimits_{k=1}^{\gamma}\left(\hat{\varphi}_k-\varphi_k\right)^2 \quad (11)$$

Then, by taking time derivative of the function V (t), it can calculate

$$\dot{V}_1(t) = 2e^T(t)\dot{e}(t) + 2\left(\hat{\phi}-\phi\right)^T\dot{\hat{\phi}} + 2(\hat{\varphi}-\varphi)^T\dot{\hat{\varphi}} \quad (12)$$

Substituting ė(t) from (9) and inserting the adaptation laws from (7) into (12), we have

$$\begin{aligned}\dot{V}_1(t) = 2e^T(t)&\left\{\begin{matrix} -[J_Q(x)\cdot f(x)]\tilde{\phi} \\ +[J_P(y)\cdot g(y)]\tilde{\varphi} \\ -\lambda sig^p\left(e(t)\right) - \eta sig^q\left(e(t)\right)\end{matrix}\right\} \\ +2\tilde{\phi}^T&\left\{\begin{matrix}\left[J_Q(x)\cdot f(x)\right]^T e(t) \\ -\lambda sig^p\left(\tilde{\phi}\right) - \eta sig^q\left(\tilde{\phi}\right)\end{matrix}\right\} \\ +2\tilde{\varphi}^T&\left\{\begin{matrix} -\left[J_P(y)\cdot g(y)\right]^T e(t) \\ -\lambda sig^p\left(\tilde{\varphi}\right) - \eta sig^q\left(\tilde{\varphi}\right)\end{matrix}\right\}\end{aligned} \quad (13)$$

Since $e^T(t)[J_Q(x)\cdot f(x)]\tilde{\phi} = \tilde{\phi}^T\left[J_Q(x)\cdot f(x)\right]^T e(t)$ and $e^T(t)\left[J_P(y)\cdot g(y)\right]\tilde{\varphi} = \tilde{\varphi}^T[J_P(y)\cdot g(y)]^T e(t)$, simplifying (13), one has

$$\begin{aligned}\dot{V}_1(t) = 2e^T(t)&\left\{-\lambda sig^p\left(e(t)\right) - \eta sig^q\left(e(t)\right)\right\} \\ +2\tilde{\phi}^T&\left\{-\lambda sig^p\left(\tilde{\phi}\right) - \eta sig^q\left(\tilde{\phi}\right)\right\} \\ +2\tilde{\varphi}^T&\left\{-\lambda sig^p\left(\tilde{\varphi}\right) - \eta sig^q\left(\tilde{\varphi}\right)\right\}\end{aligned} \quad (14)$$

Since $e^T(t)\,\mathrm{sign}(e(t))\,|e(t)|^p = |e(t)|^{p+1}$, $\tilde{\phi}^T sign\left(\tilde{\phi}\right)\left|\tilde{\phi}\right|^p = \left|\tilde{\phi}\right|^{p+1}$, $\tilde{\varphi}^T sign\left(\tilde{\varphi}\right)\left|\tilde{\varphi}\right|^p = \left|\tilde{\varphi}\right|^{p+1}$, one can obtain

$$\begin{aligned}\dot{V}_1(t) = &-2\lambda\left(\left(e^2(t)\right)^{\frac{p+1}{2}} + \left(\tilde{\phi}^2\right)^{\frac{p+1}{2}} + \left(\tilde{\varphi}^2\right)^{\frac{p+1}{2}}\right) \\ &-2\eta\left(\left(e^2(t)\right)^{\frac{q+1}{2}} + \left(\tilde{\phi}^2\right)^{\frac{q+1}{2}} + \left(\tilde{\varphi}^2\right)^{\frac{q+1}{2}}\right)\end{aligned} \quad (15)$$

Obviously, the error dynamics (4) is asymptotically stable. Besides, in light of Lemma 4, Eq.(15) can be derived as

$$\begin{aligned}\dot{V}_1(t) \le &-3^{1-\frac{p+1}{2}}2\lambda\left(\begin{matrix}\sum\nolimits_{i=1}^{r} e_i^2(t) \\ +\sum\nolimits_{j=1}^{\rho}\left(\hat{\phi}_j-\phi_j\right)^2 \\ +\sum\nolimits_{k=1}^{\gamma}\left(\hat{\varphi}_k-\varphi_k\right)^2\end{matrix}\right)^{\frac{p+1}{2}} \\ &-2\eta\left(\begin{matrix}\sum\nolimits_{i=1}^{r} e_i^2(t)+\sum\nolimits_{j=1}^{\rho}\left(\hat{\phi}_j-\phi_j\right)^2 \\ +\sum\nolimits_{k=1}^{\gamma}\left(\hat{\varphi}_k-\varphi_k\right)^2\end{matrix}\right)^{\frac{q+1}{2}} \\ \le &-3^{1-\frac{p+1}{2}}2\lambda V_1^{\frac{p+1}{2}}(t) - 2\eta V_1^{\frac{q+1}{2}}(t)\end{aligned} \quad (16)$$

Based on Lemma 8, $k = 1, p-q = \frac{p+1}{2} > 1$, $q = \frac{q+1}{2} < 1$, the state trajectories of the error dynamics (4) will converge to zero in a given fixed time, determined by (10). Therefore, the proof is achieved completely here. □

Remark 4: In previous works, if $m = n, Q(x) = x(t), P(y) = y(t)$, the authors used Corollary 1 to study the finite-time synchronization in [18], which is a special case in this paper. Moreover, if $m \neq n$, Zhang et al. [22] applied Lemma 5 to investigate the global synchronization within finite time dependent of initial condition. The proposed adaptive control method combines the superiorities of fixed time stability theory and overcomes the weakness of convergence time dependent on initial value. Furthermore, the controller contains nonsingular term and has rapid and accurate convergence property. Meanwhile, we give the comparison in the following simulation result.

IV. ADAPTIVE FIXED TIME CONTROL SCHEME FOR GENERALIZED SYNCHRONIZATION OF FRACTIONAL-ORDER DYNAMICAL SYSTEMS

A class of fractional-order n-dimensional dynamical system can be expressed as follows:

$$D^\alpha x = f(x) + F(x)\chi \quad (17)$$

The corresponding fractional-order m-dimensional response system is given by

$$D^\alpha y = g(y) + G(y)\psi + U(t) \quad (18)$$

where, α is the fractional derivative order of the dynamical system, $x = [x_1, x_2, \cdots, x_n]^T \in R^n$ and $y = [y_1, y_2, \cdots, y_m]^T \in R^m$ are the system state vector; $f(x) : R^n \to R^n$ and $g(y) : R^m \to R^m$ are the continuous nonlinear functions; $F(x) : R^n \to R^{n\times\rho}$ and $G(y) : R^m \to R^{m\times\gamma}$ are the function matrices of system parameters; $\chi \in R^\rho$ and $\psi \in R^\gamma$ denote the unknown system parameters. $U(t) = [U_1(t), U_2(t), \cdots, U_m(t)]^T \in R^m$ is the control input.

According to Eq.(3), similarly, the generalized synchronization error of the fractional-order dynamical systems can be defined as

$$e(t) = P(y) - Q(x) \quad (19)$$

Then, the fractional order error dynamical system can be expressed as

$$\begin{aligned} D^{\alpha} e(t) &= D^{\alpha}\left[P(y) - Q(x)\right] \\ &= J_P(x) D^{\alpha} y - J_Q(x) D^{\alpha} x \\ &= J_P(y)\left(g(y) + G(y)\psi + U(t)\right) \\ &\quad - J_Q(x)(f(x) + F(x)\chi) \end{aligned} \tag{20}$$

where, the Jacobin matrices $J_Q(x)$ and $J_P(y)$ are written in the form of (19).

Based on Definition 3, an appropriate controller U (t) should be designed to achieve the fixed-time stabilization of the synchronization error. Here, a novel fractional integral siding surface is constructed as follows:

$$s(t) = D^{\alpha-1} e(t) + D^{-1}\left(k_0 sig^{\delta}\left(e(t)\right)\right) \tag{21}$$

where, $k_0 > 0$, $\delta > 0$ are the adjusted coefficient.

Remark 5: The fractional-order sliding surface has been widely applied in the integer-order and fractional-order dynamical systems [43], [44]. Especially in the work of Wang *et al.* [44], a fractional-order nonsingular terminal sliding mode (FONTSM) surface such as $s(t) = \dot{e}(t) + kD^{\lambda-1} sig^{a}\left(e(t)\right)$ was proposed, which was continuous and differentiable with respect to the error $e(t)$. But it is only suitable for integer-order dynamical systems and not for fractional-order dynamical systems. Nevertheless, we appreciate this idea and design a novel fractional integral siding surface (21). Furthermore, the error state, for any given initial condition, whether far from or close to the equilibrium point, can fast converge to $e(t) = 0$ in a given finite time, which will be mathematically deduced in the following part.

Then, the time derivative of the sliding surface (21) should satisfy the following equation

$$\begin{aligned} \dot{s}(t) &= D^{\alpha} e(t) + k_0 sig^{\delta}\left(e(t)\right) = 0 \\ &\rightarrow D^{\alpha} e(t) = -k_0 sig^{\delta}\left(e(t)\right) \end{aligned} \tag{22}$$

Remark 6: In order to make error dynamical system (20) arrive the sliding surface in the approaching motion process, the reaching law should be arranged in advance. As reported in [45], a fast-TSM-type reaching law was designed as $\dot{s}(t) = -k_1 s(t) - k_2 sig^{\sigma}\left(s(t)\right)$. This is exactly consistent with the form of **Lemma 5**, which is also a finite time comparison method. Meanwhile, according to **Corollary 1,** another common the sliding surface reaching law can be described as $\dot{s}(t) = -ksig^{\vartheta}\left(s(t)\right)$. They all can tend to zero in finite time. However, the stable time depends mainly on the initial condition of the system. Therefore, a new reaching law for fractional-order dynamical system is redesigned as $\dot{s}(t) = -k_1 sig^{\vartheta}\left(s(t)\right) - k_2 sig^{\sigma}\left(s(t)\right)$, that is, we can derive the following theorem.

Theorem 2: Consider the sliding mode dynamics (22). The error system will be global asymptotically stable and converge to the equilibrium e(t) = 0 within finite time upper bounded by:

$$T_1^{*} \leq T_{max}^{4} = t_0 + \left(\|e(t_0)\|_1^{\alpha-\delta} \frac{\Gamma(1-\delta)\Gamma(1+\alpha)}{k_0\Gamma(1+\alpha-\delta)}\right)^{\frac{1}{\alpha}} \tag{23}$$

Proof: Select the following Lyapunov function candidate:

$$V_2(t) = \|e(t)\|_1 = \sum\nolimits_{i=1}^{r} |e(t)| \tag{24}$$

By applying **Lemma 3**, one has

$$D^{\alpha} V_2(t) \leq \sum\nolimits_{i=1}^{r} sign(e(t)) D^{\alpha} e(t) \tag{25}$$

Substituting $D^{\alpha}e_i(t), i = 1, 2, \ldots, r$ from (22) into (25), and $sign(e_i) \times sign(e_i) = 1$, one obtains

$$\begin{aligned} D^{\alpha} V_2(t) &\leq -\sum\nolimits_{i=1}^{r} sign(e_i(t)) k_0 |e_i(t)|^{\delta} sign(e_i(t)) \\ &= -k_0 \sum\nolimits_{i=1}^{r} |e_i(t)|^{\delta} \end{aligned} \tag{26}$$

Using **Lemma 4** the following inequality $\sum_{i=1}^{r} |e_i|^{\delta} \geq \left(\sum_{i=1}^{r} |e_i|\right)^{\delta}$, one gets

$$D^{\alpha} V_2(t) \leq -\sum\nolimits_{i=1}^{r} k_0 |e_i(t)|^{\delta} \leq -k_0 V_1^{\delta}(t) < 0 \tag{27}$$

Based on **Lemma 2**, the above expression can be rewritten as

$$D^{\alpha} V_2(t) = \frac{\Gamma(1-\delta)}{\Gamma(1+\alpha-\delta)} V_2^{\delta}(t) D^{\alpha} V_2^{\alpha-\delta}(t) \leq -k_0 V_2^{\delta}(t) \tag{28}$$

Simplifying (28), one has

$$D^{\alpha} V_2^{\alpha-\delta}(t) \leq -k_0 \frac{\Gamma(1+\alpha-\delta)}{\Gamma(1-\delta)} \tag{29}$$

It follows from **Lemma 1** that

$$V_2^{\alpha-\delta}(t) - V_2^{\alpha-\delta}(t_0) \leq {}_{t_0}^{C}I_t^{\alpha} \frac{-k_0\Gamma\left(1+\alpha-\delta\right)}{\Gamma\left(1-\delta\right)} \tag{30}$$

Considering **Definition 1**, it is easy to verify that

$$\begin{aligned} &{}_{t_0}^{C}I_t^{\alpha} \frac{-k_0\Gamma\left(1+\alpha-\delta\right)}{\Gamma\left(1-\delta\right)} \\ &= \frac{-k_0\Gamma\left(1+\alpha-\delta\right)}{\Gamma\left(1-\delta\right)} \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} d\tau \\ &= \frac{-k_0\Gamma\left(1+\alpha-\delta\right)}{\Gamma\left(1-\delta\right)\Gamma(\alpha)} \frac{(t-t_0)^{\alpha}}{\alpha} \\ &= \frac{-k_0\Gamma\left(1+\alpha-\delta\right)(t-t_0)^{\alpha}}{\Gamma\left(1-\delta\right)\Gamma(1+\alpha)} \end{aligned} \tag{31}$$

Combining (30) and (31), we can get

$$\begin{aligned} &V_2^{\alpha-\delta}(t) \leq V_2^{\alpha-\delta}(t_0) - \frac{k_0\Gamma\left(1+\alpha-\delta\right)}{\Gamma\left(1-\delta\right)} \frac{(t-t_0)^{\alpha}}{\Gamma(1+\alpha)}, \\ &\quad t_0 \leq t \leq T_1^{*} \end{aligned} \tag{32}$$

From (32), we obtain that $\lim_{t\to T_1^{*}} V_2(t) = 0$, such that $V_2(t) = 0$ for arbitrary $t \geq T_1^{*}$, that is $\lim_{t\to T_1^{*}} |e(t)| = 0$. T_1^{*} is the upper bound of convergence time, given by (23). This completes the proof. □

In the following, to guarantee the existence of the sliding motion in the presence of parameter uncertainties,

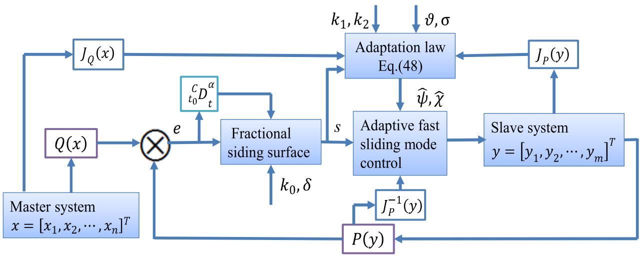

FIGURE 1. The control diagram of the generalized synchronization.

the control input $U(t)$ with the parametric estimations is designed as

$$U(t) = -g(y) - G(y)\hat{\psi} + J_P^{-1}(y)\begin{bmatrix} J_Q(x)\left(f(x) + F(x)\hat{\chi}\right) - k_0 sig^{\delta}(e(t)) \\ -k_1 sig^{\vartheta}(s(t)) - k_2 sig^{\sigma}(s(t)) \end{bmatrix} \tag{33}$$

where, $0 < \sigma < 1$, $\vartheta > 1$, $k_1, k_2 > 0$, $\hat{\psi}$ and $\hat{\chi}$ are estimations for ψ and χ respectively. i.e. $\tilde{\chi} = \hat{\chi} - \chi$, $\tilde{\psi} = \hat{\psi} - \psi$.

In order to eliminate all unknown parameters of fractional-order dynamical systems, the appropriate update laws are proposed as follows:

$$\begin{aligned} \dot{\hat{\chi}}(t) &= -\left[J_Q(x) \cdot F(x)\right]^T s(t) \\ &\quad - k_1 sig^{\vartheta}(\tilde{\chi}) - k_2 sig^{\sigma}(\tilde{\chi}), \quad \hat{\chi}(0) = \hat{\chi}_0 \\ \dot{\hat{\psi}}(t) &= \left[J_P(y) \cdot G(y)\right]^T s(t) \\ &\quad - k_1 sig^{\vartheta}\left(\tilde{\psi}\right) - k_2 sig^{\sigma}\left(\tilde{\psi}\right), \quad \hat{\psi}(0) = \hat{\psi}_0 \end{aligned} \tag{34}$$

The fractional order error dynamics (20) can be deduced as

$$\begin{aligned} D^{\alpha}e(t) = &-\left[J_P(y) \cdot G(y)\right]\tilde{\psi} + \left[J_Q(x) \cdot F(x)\right]\tilde{\chi} \\ &- k_0 sig^{\delta}(e(t)) - k_1 sig^{\vartheta}(s(t)) - k_2 sig^{\sigma}(s(t)) \end{aligned} \tag{35}$$

Consequently, from (22), it is easy to derive that $\dot{s}(t) = -\left[J_P(y) \cdot G(y)\right]\tilde{\psi} + \left[J_Q(x) \cdot F(x)\right]\tilde{\chi} - k_1 sig^{\vartheta}(s(t)) - k_2 sig^{\sigma}(s(t))$. The complete structure of the adaptive fast sliding mode control algorithm is demonstrated in Fig. 1

Theorem 3: Consider the fractional-order dynamical systems (17) and (18) satisfying Assumption 1, the error dynamics (35) should be global fixed-time stability at the origin by adding the adaptive controller (33). That is, the dynamical systems (17) and (18) are globally synchronized within fixed time upper bounded by:

$$T_2^* \le T_{max}^4 = \frac{1}{2^{\frac{\sigma-1}{2}}k_2}\left(\frac{2^{\frac{\sigma-\vartheta}{2}}k_2}{3^{\frac{1-\vartheta}{2}}k_1}\right)^{\frac{1-\sigma}{\vartheta-\sigma}}\left(\frac{1}{\vartheta-1} + \frac{1}{1-\sigma}\right) \tag{36}$$

Proof: The following continuous positive definite Lyapunov function is chosen as

$$V_3(t) = \sum_{i=1}^{r}\frac{1}{2}s_i^2 + \sum_{j=1}^{\rho}\frac{1}{2}\left(\hat{\chi}_j - \chi_j\right)^2 + \sum_{k=1}^{\gamma}\frac{1}{2}\left(\hat{\psi}_k - \psi_k\right)^2 \tag{37}$$

Combining (22) and (35), and introducing the update laws from (34) into (37), one gets

$$\begin{aligned} \dot{V}_3(t) = &\, s^T(t)\begin{Bmatrix} -\left[J_P(y) \cdot G(y)\right]\tilde{\psi} + \left[J_Q(x) \cdot F(x)\right]\tilde{\chi} \\ -k_1 sig^{\vartheta}(s(t)) - k_2 sig^{\sigma}(s(t)) \end{Bmatrix} \\ &+ \tilde{\chi}^T\begin{Bmatrix} -\left[J_Q(x) \cdot F(x)\right]^T s(t) \\ -k_1 sig^{\vartheta}(\tilde{\chi}) - k_2 sig^{\sigma}(\tilde{\chi}) \end{Bmatrix} \\ &+ \tilde{\psi}^T\begin{Bmatrix} \left[J_P(y) \cdot G(y)\right]^T s(t) \\ -k_1 sig^{\vartheta}\left(\tilde{\psi}\right) - k_2 sig^{\sigma}\left(\tilde{\psi}\right) \end{Bmatrix} \end{aligned} \tag{38}$$

Further, since $s^T(t)\left[J_Q(x) \cdot F(x)\right]\tilde{\chi} = \tilde{\chi}^T\left[J_Q(x) \cdot F(x)\right]^T s(t)$ and $s^T(t)\left[J_P(y) \cdot G(y)\right]\tilde{\psi} = \tilde{\psi}^T\left[J_P(y) \cdot G(y)\right]^T s(t)$, it follows

$$\begin{aligned} \dot{V}_3(t) = &\, s^T(t)\left\{-k_1 sig^{\vartheta}(s(t)) - k_2 sig^{\sigma}(s(t))\right\} \\ &+ \tilde{\chi}^T\left\{-k_1 sig^{\vartheta}(\tilde{\chi}) - k_2 sig^{\sigma}(\tilde{\chi})\right\} \\ &+ \tilde{\psi}^T\left\{-k_1 sig^{\vartheta}\left(\tilde{\psi}\right) - k_2 sig^{\sigma}\left(\tilde{\psi}\right)\right\} \end{aligned} \tag{39}$$

Rearranging (39) yields

$$\begin{aligned} \dot{V}_3(t) = &-2^{\frac{\vartheta+1}{2}}k_1\begin{pmatrix} \left(\frac{1}{2}s^2(t)\right)^{\frac{\vartheta+1}{2}} + \left(\frac{1}{2}\tilde{\chi}^2\right)^{\frac{\vartheta+1}{2}} \\ +\left(\frac{1}{2}\tilde{\psi}^2\right)^{\frac{\vartheta+1}{2}} \end{pmatrix} \\ &- 2^{\frac{\sigma+1}{2}}k_2\left(\left(\frac{1}{2}s^2(t)\right)^{\frac{\sigma+1}{2}} + \left(\frac{1}{2}\tilde{\chi}^2\right)^{\frac{\sigma+1}{2}} + \left(\frac{1}{2}\tilde{\psi}^2\right)^{\frac{\sigma+1}{2}}\right) \end{aligned} \tag{40}$$

It is obvious that the error dynamics (35) is asymptotically stable. Additionally, based on Lemma 4, Eq.(40) can obtain

$$\begin{aligned} &\dot{V}_3(t) \\ &\le -3^{1-\frac{\vartheta+1}{2}}2^{\frac{\vartheta+1}{2}}k_1\begin{pmatrix} \sum_{i=1}^{r}\frac{1}{2}s_i^2 \\ +\sum_{j=1}^{\rho}\frac{1}{2}\left(\hat{\chi}_j - \chi_j\right)^2 \\ +\sum_{k=1}^{\gamma}\frac{1}{2}\left(\hat{\psi}_k - \psi_k\right)^2 \end{pmatrix}^{\frac{\vartheta+1}{2}} \\ &\quad - 2^{\frac{\sigma+1}{2}}k_2\begin{pmatrix} \sum_{i=1}^{r}\frac{1}{2}e_i^2(t) + \sum_{j=1}^{\rho}\frac{1}{2}\left(\hat{\phi}_j - \phi_j\right)^2 \\ +\sum_{k=1}^{\gamma}\frac{1}{2}\left(\hat{\varphi}_k - \varphi_k\right)^2 \end{pmatrix}^{\frac{\sigma+1}{2}} \\ &\le -3^{1-\frac{\vartheta+1}{2}}2^{\frac{\vartheta+1}{2}}k_1 V_3^{\frac{\vartheta+1}{2}}(t) - 2^{\frac{\sigma+1}{2}}k_2 V_3^{\frac{\sigma+1}{2}}(t) \end{aligned} \tag{41}$$

Therefore, according to Lemma 8, $k = 1, p - q = \frac{\vartheta+1}{2} > 1$, $q = \frac{\sigma+1}{2} < 1$, $\lambda = 3^{1-\frac{\vartheta+1}{2}}2^{\frac{\vartheta+1}{2}}k_1$, $\eta = 2^{\frac{\sigma+1}{2}}k_2$. We have $\lim\limits_{t \to T_1^*} s_i(t) = 0, t \ge T_2^*, i = 1, 2 \ldots r$. It implies that the error dynamic trajectories (35) will converge to the predefined sliding surface $s_i(t) = 0$ within the fixed time upper bounded by (36). The proof is completed. □

Remark 7: On the basis of the Theorems 2 and 3, the slave system (18) will track the master system (17) within the fixed time $T = T_1^* + T_2^*$. From Eq. (23), the convergence times T_1^* not only depends of the initial value, but also will be determined by the fractional order α and parameters k_0, δ. And by (36), and the upper bound of T_2^* is only determined by preset parameters $k_1, k_2, \vartheta, \sigma$.

Remark 8: Most of adaptive studies for synchronization between commensurate fractional-order dynamical systems with known parameters, the synchronization errors can achieve asymptotically convergence, finite time convergence dependent on initial conditions and fixed time convergence by the design parameters. In this paper, based on sliding mode theory, we propose fractional order adaptive fixed-time control scheme for generalized synchronization in non-identical dynamical systems with parameter uncertainties. Singularity is still the main weakness of sliding mode control. Therefore, our control scheme combines the advantages of fixed-time stability theory to accomplish faster and more exact convergence with nonsingular control input.

Remark 9: From Remark 4, the generalized synchronization has an extension of synchronization types. For the various synchronization types between two identical or non-identical fractional order dynamical systems with all possibilities like different dimensions, fractional orders and with or without uncertainties and external disturbances [15], the proposed control approach is also entirely appropriate and can be further applied to the various fixed-time synchronization types among other dynamical systems such as neural networks.

V. NUMERICAL SIMULATIONS

In this section, the validity and superiority of our proposed control scheme is demonstrated by synchronizing two mismatched dynamical systems with parametric estimations. In addition, we give the comparison with finite time stability method.

A. ADAPTIVE FIXED TIME SYNCHRONIZATION OF INTEGER-ORDER DYNAMICAL SYSTEMS

It is assumed that the 3-D reverse butterfly-shaped chaotic system [15] is the master system and the 4-D hyper-chaotic Chen system [21] with the controller is the slave system. The dynamic equations of the master and slave systems are described in the form of systems (1) and (2) as follows.

Master system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} a_1(x_2 - x_1) \\ a_2 x_1 + a_3 x_1 x_3 \\ -x_1 x_2 - a_4 x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -x_1 x_2 \end{bmatrix} + \begin{bmatrix} x_2 - x_1 & 0 & 0 & 0 \\ 0 & x_1 & x_1 x_3 & 0 \\ 0 & 0 & 0 & -x_3 \end{bmatrix} \times \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \tag{42}$$

Slave system:

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{bmatrix} = \begin{bmatrix} b_1(y_2 - y_1) + y_4 + u_1 \\ b_2 y_1 + b_3 y_2 - y_1 y_3 + u_2 \\ -b_4 y_3 + y_1 y_2 + u_3 \\ b_5 y_4 + y_2 y_3 + u_4 \end{bmatrix} = \begin{bmatrix} y_4 \\ -y_1 y_3 \\ y_1 y_2 \\ y_2 y_3 \end{bmatrix} + \begin{bmatrix} y_2 - y_1 & 0 & 0 & 0 & 0 \\ 0 & y_1 & y_2 & 0 & 0 \\ 0 & 0 & 0 & -y_3 & 0 \\ 0 & 0 & 0 & 0 & y_4 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \tag{43}$$

In order to exhibit the chaotic behavior of the systems (42) and (43), the parameters are chosen as $a_1 = 10$, $a_2 = 40$, $a_3 = 16$, $a_4 = 2.5$, $b_1 = 35$, $b_2 = 7$, $b_3 = 12$, $b_4 = 3$, $b_5 = 0.5$. Assume the continuous differentiable functions of the systems (42) and (43) are

$$Q(x) = \begin{pmatrix} x_1 x_2 \\ x_1 - x_3 \end{pmatrix}, \quad P(y) = \begin{pmatrix} 0.5y_1 - 0.5y_2 \\ 0.5y_3 - 0.5y_4 \end{pmatrix} \tag{44}$$

Then, we obtain $\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} x_1 x_2 - 0.5y_1 + 0.5y_2 \\ x_1 - x_3 - 0.5y_3 + 0.5y_4 \end{bmatrix}$, moreover, one has

$$J_Q(x) = \begin{bmatrix} x_2 & x_1 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \quad J_P(y) = \begin{bmatrix} 0.5 & -0.5 & 0 & 0 \\ 0 & 0 & 0.5 & -0.5 \end{bmatrix}, \quad J_P^{-1}(y) = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \tag{45}$$

According to (8), the adaptive controller $u_i(t)(i = 1, 2, 3, 4)$ can be written as the following

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = -\begin{bmatrix} \hat{b}_1(y_2 - y_1) + y_4 \\ \hat{b}_2 y_1 + \hat{b}_3 y_2 - y_1 y_3 \\ -\hat{b}_4 y_3 + y_1 y_2 \\ \hat{b}_5 y_4 + y_2 y_3 \end{bmatrix} + \begin{bmatrix} \hat{a}_1 x_2(x_2 - x_1) + x_1(\hat{a}_2 x_1 + \hat{a}_3 x_1 x_3) \\ -\hat{a}_1 x_2(x_2 - x_1) - x_1(\hat{a}_2 x_1 + \hat{a}_3 x_1 x_3) \\ \hat{a}_1(x_2 - x_1) + x_1 x_2 + \hat{a}_4 x_3 \\ -\hat{a}_1(x_2 - x_1) - x_1 x_2 - \hat{a}_4 x_3 \end{bmatrix} + \begin{bmatrix} \lambda sig^p(e_1) + \eta sig^q(e_1) \\ -\lambda sig^p(e_1) - \eta sig^q(e_1) \\ \lambda sig^p(e_2) + \eta sig^q(e_2) \\ -\lambda sig^p(e_2) - \eta sig^q(e_2) \end{bmatrix} \tag{46}$$

Considering (9), the error dynamics can be derived as

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \end{bmatrix} = -\begin{bmatrix} x_2(x_2-x_1)\tilde{a}_1 + x_1^2\tilde{a}_2 + x_1^2x_3\tilde{a}_3 \\ (x_2-x_1)\tilde{a}_1 + x_3\tilde{a}_4 \end{bmatrix} + \begin{bmatrix} 0.5(y_2-y_1)\tilde{b}_1 - 0.5y_1\tilde{b}_2 - 0.5y_2\tilde{b}_3 \\ -0.5y_3\tilde{b}_4 - 0.5y_4\tilde{b}_5 \end{bmatrix} - \begin{bmatrix} \lambda sig^p(e_1) + \eta sig^q(e_1) \\ \lambda sig^p(e_1) + \eta sig^q(e_1) \end{bmatrix} \tag{47}$$

Then, regarding (7), the appropriate updating laws can be determined as

$$\begin{bmatrix} \dot{\hat{a}}_1 \\ \dot{\hat{a}}_2 \\ \dot{\hat{a}}_3 \\ \dot{\hat{a}}_4 \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} (x_2-x_1)(x_2e_1+e_2) \\ -\lambda sig^p(\tilde{a}_1) - \eta sig^q(\tilde{a}_1) \end{pmatrix} \\ x_1^2e_1 - \lambda sig^p(\tilde{a}_2) - \eta sig^q(\tilde{a}_2) \\ x_1^2x_3e_1 - \lambda sig^p(\tilde{a}_3) - \eta sig^q(\tilde{a}_3) \\ x_3e_2 - \lambda sig^p(\tilde{a}_4) - \eta sig^q(\tilde{a}_4) \end{bmatrix} \tag{48}$$

and

$$\begin{bmatrix} \dot{\hat{b}}_1 \\ \dot{\hat{b}}_2 \\ \dot{\hat{b}}_3 \\ \dot{\hat{b}}_4 \\ \dot{\hat{b}}_5 \end{bmatrix} = \begin{bmatrix} 0.5(y_1-y_2)e_1 - \lambda sig^p\left(\tilde{b}_1\right) - \eta sig^q\left(\tilde{b}_1\right) \\ 0.5y_1e_1 - \lambda sig^p\left(\tilde{b}_2\right) - \eta sig^q\left(\tilde{b}_2\right) \\ 0.5y_2e_1 - \lambda sig^p\left(\tilde{b}_3\right) - \eta sig^q\left(\tilde{b}_3\right) \\ 0.5y_3e_2 - \lambda sig^p\left(\tilde{b}_4\right) - \eta sig^q\left(\tilde{b}_4\right) \\ 0.5y_4e_2 - \lambda sig^p\left(\tilde{b}_5\right) - \eta sig^q\left(\tilde{b}_5\right) \end{bmatrix} \tag{49}$$

In numerical simulation, the initial conditions of the systems (42) and (43) are chosen as $x_1(0) = 2$, $x_2(0) = -1$, $x_3(0) = 1$ and $y_1(0) = 2$, $y_2(0) = -2$, $y_3(0) = 2$, $y_4(0) = 1$, respectively. Vectors [5, 5, 5, 5] and [2, 2, 2, 2, 2] are set as the initial values of the parameters estimation $\hat{a}_i(0)$ $(i = 1, 2, 3, 4)$ and $\hat{b}_i(0)$ $(i = 1, 2, 3, 4, 5)$, respectively. The fixed-time synchronization control parameters are given as $\lambda = \eta = 10$, $p = 9/5$, $q = 5/9$. Based on Theorem 1, the estimation upper bound of convergence time can be calculated by $T \le T^5_{max} = 0.3717$. The state trajectories of the generalized fixed-time synchronization between master and slave systems are plotted in Fig. 2. Subsequently, in the finite-time stability Lemma 5, the parameters are selected as $\lambda = 5$, $\eta = 2$, $\mu = 0.5$, where the convergence time can be estimated as $T^1_{max} \le 3.0652$. It is observed from Figs.2 that the proposed controller has faster global convergence speed and better tracking performance than the finite-time controllers. The comparison results are plotted in Figs. 3 and 4. As shown in Fig. 3, it is obvious that the generalized synchronization error states of the system (42) and (43) converge to the zero within 1.00 s, which implies that it is in good agreement with the calculation results. The time response curves of parameter estimations $\hat{a}_i(0)$ $(i = 1, 2, 3, 4)$ and $\hat{b}_i(0)$ $(i = 1, 2, 3, 4, 5)$ are illustrated in Fig. 4, respectively. From Fig. 4, Note that the proposed control method has less steady state errors and smaller overshoot than the finite-time method. The simulation results of two methods all show that the expected value of parameter estimate has been achieved

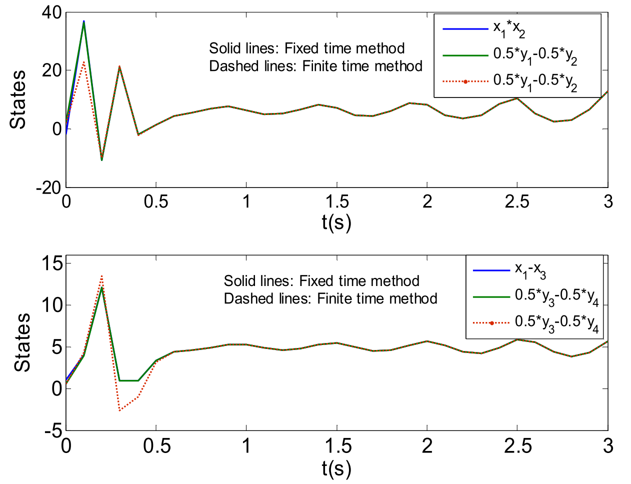

FIGURE 2. Synchronized states of master system (42) and slave system (43).

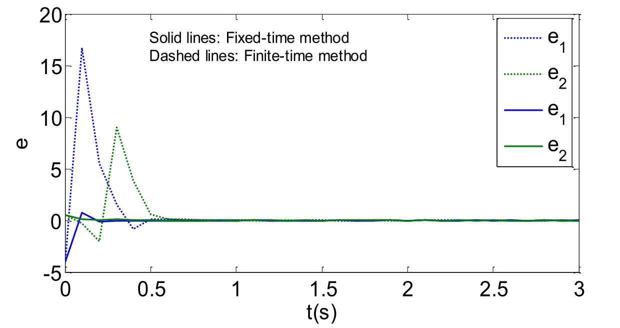

FIGURE 3. Time responses of synchronization errors (47).

after 2s by using a straight line. In addition, from the comparisons of these results above, it is easy to find that the chattering phenomenon is well suppressed and the convergence time is more accurate in fixed-time control method, which means its higher superiority than finite time control method.

B. ADAPTIVE SLIDING MODE SYNCHRONIZATION OF FRACTIONAL-ORDER DYNAMICAL SYSTEMS

The proposed control strategy could be implemented in fractional order chaotic systems. In this case, we choose chaotic fractional order Rössler system (FORS) as a master system, and hyper-chaotic fractional order Lorenz system (FOLS) as a slave system. The mathematical expressions of the two systems are shown as the following forms:

FORS:

$$\begin{bmatrix} D^\alpha x_1 \\ D^\alpha x_2 \\ D^\alpha x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 \\ x_1 + c_1x_2 \\ x_1x_3 - c_2x_3 + c_3 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 \\ x_1 \\ x_1x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ x_2 & 0 & 0 \\ 0 & -x_3 & 1 \end{bmatrix} \begin{bmatrix} c \\ c_2 \\ c_3 \end{bmatrix} \tag{50}$$

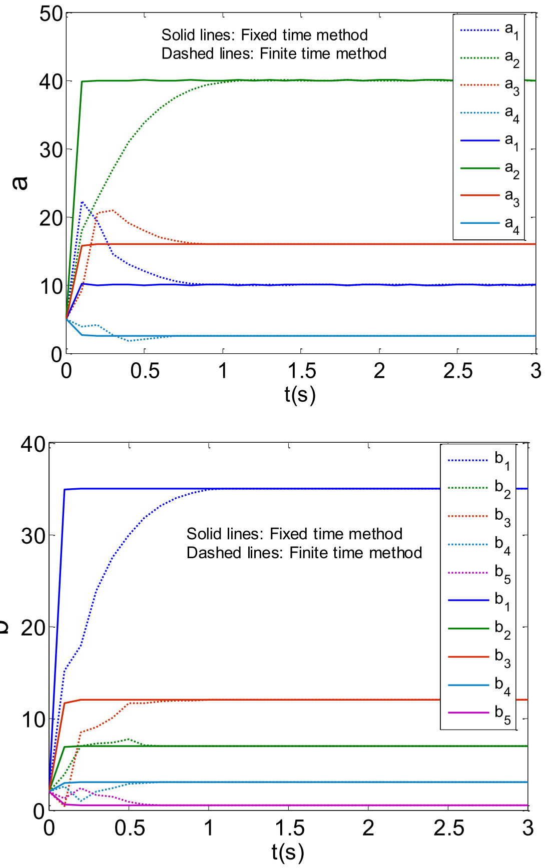

FIGURE 4. Time evolution of the adaptive parameters $\hat{a}_i$ $(i = 1, 2, 3, 4)$ and $\hat{b}_i$ $(i = 1, 2, 3, 4, 5)$.

FOLS:

$$\begin{bmatrix} D^\alpha y_1\\ D^\alpha y_2\\ D^\alpha y_3\\ D^\alpha y_4\end{bmatrix} = \begin{bmatrix} d_1(y_2-y_1)+y_4+u_1\\ d_2y_1 - y_2 - y_1y_3+u_2\\ -d_3y_3+y_1y_2+u_3\\ -y_4-y_2y_3+u_4\end{bmatrix} = \begin{bmatrix} y_4\\ -y_2-y_1y_3\\ y_1y_2\\ -y_4-y_2y_3\end{bmatrix} + \begin{bmatrix} y_2-y_1&0&0\\0&y_1&0\\0&0&-y_3\\0&0&0\end{bmatrix}\begin{bmatrix}d_1\\d_2\\d_3\end{bmatrix}+\begin{bmatrix}u_1\\u_2\\u_3\\u_4\end{bmatrix} \tag{51}$$

In order to exhibit chaotic behavior, parameters of the systems are set as $c_1 = 0.4, c_2 = 10, c_3 = 0.2$ and $d_1 = 10, d_2 = 28, d_3 = {}^8/_3$. Further, the continuous functions$Q(x)$ and $P(y)$ are given in the form of $\begin{pmatrix} x_1\\ x_2\\ x_3\end{pmatrix}$ and $\begin{pmatrix} 0.5y_1\\ y_2-y_3\\ 0.5y_4\end{pmatrix}$. Then, we have $J_Q(x) = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$, $J_P(y) = \begin{bmatrix}0.5&0&0&0\\0&1&-1&0\\0&0&0&0.5\end{bmatrix}$, $J_P^{-1}(y) = \begin{bmatrix}2&0&0\\0&0.5&0\\0&-0.5&0\\0&0&2\end{bmatrix}$.

TABLE 2. Various parameters for the FORS and FOLS.

Indices	Parameters	Values
FORS	c_1, c_2, c_3	0.4,10,2
FOLS	d_1,d_2, d_3	10,28,8/3
Proposed controller	k_0,k_1,k_2	5,8,5
	δ,ϑ,σ	0.5,1.8,0.8
	$\hat{c}_1(0)$,$\hat{c}_2(0)$, $\hat{c}_3(0)$	1,1,1
	$\hat{d}_1(0)$,$\hat{d}_2(0)$,$\hat{d}_3(0)$	5,5,5

From (20), the fractional order error dynamics can be written as

$$\begin{bmatrix} D^\alpha e_1\\ D^\alpha e_2\\ D^\alpha e_3\end{bmatrix} = -\begin{bmatrix} 0.5(y_2-y_1)\tilde{d}_1\\ y_1\tilde{d}_2+y_3\tilde{d}_3\\ 0\end{bmatrix} + \begin{bmatrix}0\\ x_2\tilde{c}_1\\ -x_3\tilde{c}_2+\tilde{c}_3\end{bmatrix} - \begin{bmatrix} k_0 sig^\delta(e_1)\\ k_0 sig^\delta(e_2)\\ k_0 sig^\delta(e_3)\end{bmatrix} - \begin{bmatrix} k_1 sig^\vartheta(s_1) - k_2 sig^\sigma(s_1)\\ k_1 sig^\vartheta(s_2) - k_2 sig^\sigma(s_2)\\ k_1 sig^\vartheta(s_3) - k_2 sig^\sigma(s_3)\end{bmatrix} \tag{52}$$

By (33), the adaptive controller can be obtained as

$$\begin{bmatrix} U_1\\ U_2\\ U_3\\ U_4\end{bmatrix} = -\begin{bmatrix} \hat{d}_1(y_2-y_1)+y_4\\ \hat{d}_2y_1-y_2-y_1y_3\\ -\hat{d}_3y_3+y_1y_2\\ -y_4-y_2y_3\end{bmatrix} + \begin{bmatrix} 2(-x_2-x_3) - 2k_0 sig^\delta(e_1)\\ 0.5\left(x_1+\hat{c}_1x_2\right) - 0.5k_0 sig^\delta(e_2)\\ -0.5\left(x_1+\hat{c}_1x_2\right)+0.5k_0 sig^\delta(e_2)\\ 2\left(x_1x_3-\hat{c}_2x_3+\hat{c}_3\right) - 2k_0 sig^\delta(e_3)\end{bmatrix} + \begin{bmatrix} -2k_1 sig^\vartheta(s_1) - 2k_2 sig^\sigma(s_1)\\ -0.5k_1 sig^\vartheta(s_2) - 0.5k_2 sig^\sigma(s_2)\\ 0.5k_1 sig^\vartheta(s_2)+0.5k_2 sig^\sigma(s_2)\\ -2k_1 sig^\vartheta(s_3) - 2k_2 sig^\sigma(s_3)\end{bmatrix} \tag{53}$$

Considering (34), the updating laws are described as the following

$$\begin{bmatrix} \dot{\hat{c}}_1\\ \dot{\hat{c}}_2\\ \dot{\hat{c}}_3\end{bmatrix} = \begin{bmatrix} -x_2s_2 - k_1 sig^\vartheta(\tilde{c}_1) - k_2 sig^\sigma(\tilde{c}_1)\\ x_3s_3 - k_1 sig^\vartheta(\tilde{c}_2) - k_2 sig^\sigma(\tilde{c}_2)\\ -s_3 - k_1 sig^\vartheta(\tilde{c}_3) - k_2 sig^\sigma(\tilde{c}_3)\end{bmatrix}$$
$$\begin{bmatrix} \dot{\hat{d}}_1\\ \dot{\hat{d}}_2\\ \dot{\hat{d}}_3\end{bmatrix} = \begin{bmatrix} \begin{pmatrix} 0.5(y_2-y_1)s_1 - k_1 sig^\vartheta\left(\tilde{d}_1\right)\\ -k_2 sig^\sigma\left(\tilde{d}_1\right)\end{pmatrix}\\ y_1s_2 - k_1 sig^\vartheta\left(\tilde{d}_2\right) - k_2 sig^\sigma\left(\tilde{d}_2\right)\\ y_3s_2 - k_1 sig^\vartheta\left(\tilde{d}_3\right) - k_2 sig^\sigma\left(\tilde{d}_3\right)\end{bmatrix} \tag{54}$$

Meanwhile, the fractional order α is equal 0.95, the initial conditions of the systems (50) and (51) are selected as $x_2(0) = 1.5, x_3(0 = 0.1, x_1(0) = 0.5$, $y_1(0) = 2$, $y_2(0) = -2$, $y_3(0) = 4$, $y_4(0) = 1$. The various values of design parameters can be provided from Table 2. In the light of Theorem 3, the estimation value of convergence time

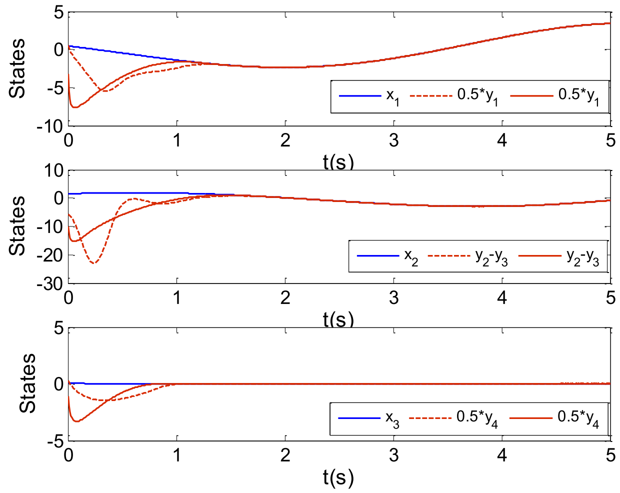

FIGURE 5. Synchronized states of the systems (50) and (51). (Solid lines: Fixed time method; Dashed lines: Finite time method).

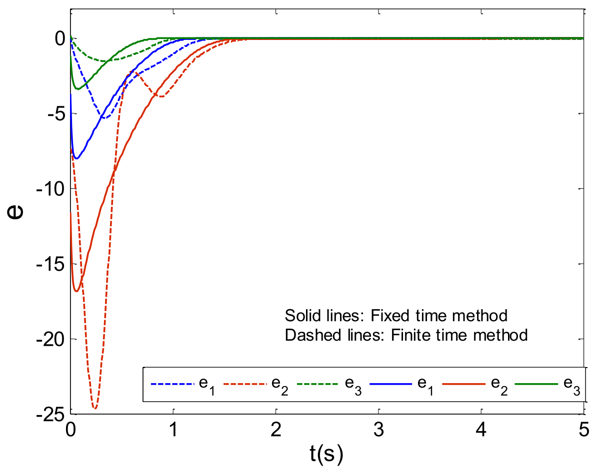

FIGURE 6. Time responses of synchronization errors (52).

can be obtained by $T_2^* \leq 1.4994$. Furthermore, by calculation from Theorem 2, the synchronization errors will converge to zero in finite time upper bounded by $T_1^* \leq 0.9728$. The state trajectories of the generalized synchronization in fixed time between the systems (50) and (51) are depicted in Fig. 5. We can see that the state trajectories of the slave system will track the trajectories of the master system within 1.5s. Subsequently, Time responses of the corresponding synchronization errors and the estimated parameters are displayed in Figs. 6 and 7. From Fig. 6, one can see that the synchronization errors converge to zero after a short transient. It is observed from Figs. 7 that parameter estimations have a short transient response and almost no chattering phenomenon, which is quite similar to the result in Fig. 4. Fig. 8 exhibits the curves of the sliding surfaces and further reveals the fast convergence of the system. As seen from these figures, the proposed control approach by using fixed-time stability theory has been fully validated.

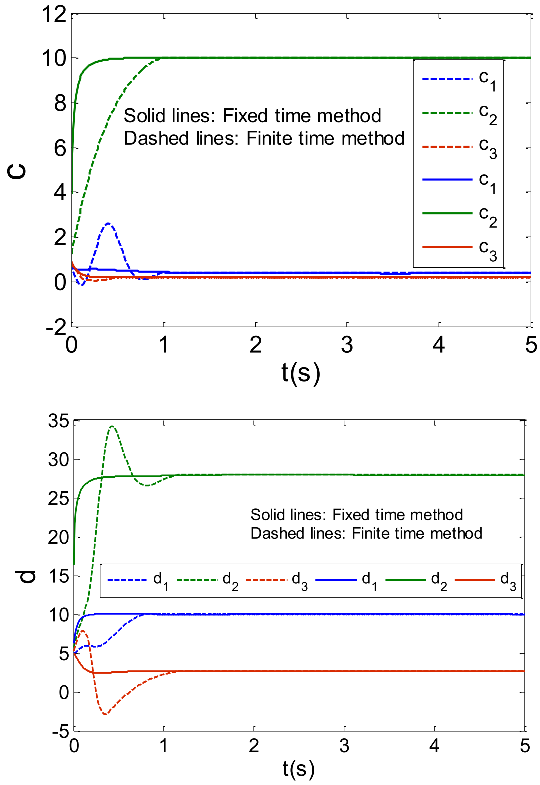

FIGURE 7. Time evolution of the adaptive parameters $\hat{c}_i$ ($i = 1, 2, 3$) and $\hat{d}_i$ ($i = 1, 2, 3$).

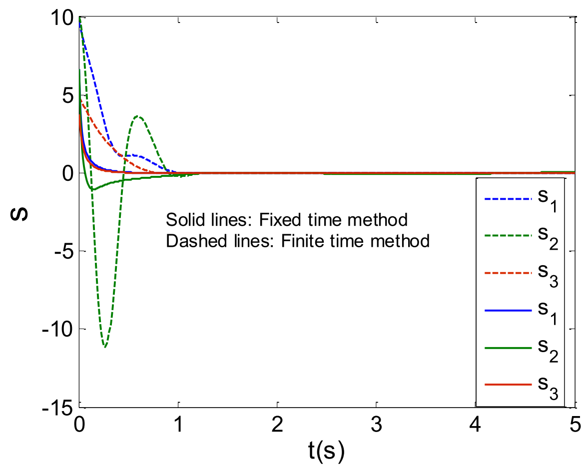

FIGURE 8. The curves of the sliding surfaces.

In addition, to better demonstrate the fine performance of the fixed time method, and finite-time stability method is taken into account in the simulations for the comparison purpose. The same initial conditions are chosen in the comparative results. Consequently, according to the finite-time stability Lemma 5, the control parameters are selected as $\lambda = 3, \eta = 2, \mu = 0.2$. The setting time can be calculated by $T^1_{max} \leq 2.7895$. As shown in Figures 5 to 8, the overall simulation results are plotted graphically, respectively. It is very clearly seen that the amplitude of chattering is well weakened and there is a faster steady rate and high-precision estimation in fixed time control method. It verifies that the proposed control method has superior performance versus finite time control, too.

VI. CONCLUSION

This paper has focused on the adaptive fixed time control problem for the generalized synchronization in mismatched integer-order and fractional-order dynamical systems with parameter uncertainties. First, we give the definition of generalized synchronization, introduce the fixed time stability lemma and obtain a high accuracy estimation of the convergence time. Second, based on the new lemma, the adaptive control strategy of fixed-time synchronization between integer-order dynamical systems is mathematically deduced by considering uncertain parameters. Meanwhile, the suitable updating laws are proposed and the corresponding adaptive controller is constructed to guarantee the fixed time stability of uncertain synchronization error system. Furthermore, consider the generalized synchronization in fractional-order dynamical systems, a novel fractional-order integral sliding mode surface and adaptive fixed time sliding mode control strategy are proposed. And due to the parametric estimations in the controller, an appropriate adaptive law is designed to obtain the expected results and its fixed time stability to origin is analytically proved by the aid of the Lyapunov stability theory. Finally, compared with the existing finite-time stability method, the validity and superior performance of our proposed fixed time control scheme is demonstrated in some numerical simulations. It is noted that the proposed scheme here can be further implemented to the various fixed time synchronization types between other dynamical systems. And the fixed time stability theory can be also extended to investigate other nonlinear systems, such as neural networks, complex networks, delayed systems and stochastic systems.

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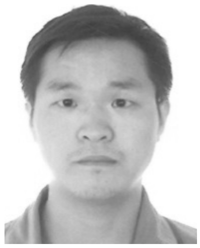

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