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Estimation and Forecasting of Sovereign Credit Rating Migration Based on Regime Switching Markov Chain

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ABSTRACT Our research aims to develop the regime switching Markov chain (RSMC), a discrete time Markov chain whose underlying regime is depending on a hidden Markov model, which express the dynamics of sovereign credit rating migration. Estimated based on a version of the Expectation-Maximization algorithm, the regime in RSMC indicates either economic expansion or contraction. Then, we apply RSMC to the monthly time series of the sovereign credit rating of 41 nations from January 1994 to December 2018. At first, we confirm that the estimation of RSMC is superior to a homogeneous Markov chain. It implies that the credit rating dynamics are subject to the underlying economic condition. Secondly, we observe that the second tier and non-investment credit ratings in economic contractions are likely to be downgraded. We also detect the continental clustering of economic contractions for the Asian currency and European sovereign debt crises. Lastly, we discover that the forecasting performance of RSMC is superior to that of the benchmark, especially for the second tier and non-investment credit ratings. In conclusion, we claim that RSMC can improve the management of sovereign credit risk exposures.

INDEX TERMS Credit migration, economic forecasting, hidden Markov models, Markov processes, regime switching, sovereign credit rating.

I. INTRODUCTION

For decades, the world economy has been suffering from various sovereign risks such as the Asian currency crisis in the late 1990s and the European sovereign debt crisis in the late 2000s. An inevitable consequence of such a risk is the degrading of the sovereign credit rating; both market participants and policy-makers have made efforts to respond promptly to risk exposure to credit migration, a transition probability matrix that represents the dynamics of credit ratings over time. For instance, the risk exposure of credit rating migration has been applied to develop a structural model for the pricing of bonds [1], credit derivatives [2], the term structure of credit spreads [3], and the investigation of global financial conditions [4]. Furthermore, the renovated regulatory requirements of the Basel Committee accentuate the efficiency of credit ratings as creditworthiness indicator of obligors and the accuracy and up-to-dateness of models for credit rating dynamics. Thus, sophisticated credit risk models are being developed or demanded by banks to assess the risk of their credit portfolio better by recognizing the different underlying sources of risk.

Consequently, the default probabilities for specific rating categories but also the probabilities for moving from one rating state to another are essential issues in credit risk management and pricing. Systematic changes in migration matrices have substantial influences on credit Value-at-Risk of a portfolio and the prices of credit derivatives like collateralized debt obligations. Therefore, rating transition matrices are of particular interest in determining the economic capital, expected loss, and VaR for credit portfolios within the Basel framework. They can also be helpful for the pricing of more complex credit products in the industry. Therefore, the development of a realistic model that accommodates the observed behavior of sovereign credit migration is essential in market practice and regulations.

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In the modeling perspective, a homogeneous Markov chain, which assumes homogeneity in time and population, has been the common instrument for credit rating migration. Fuertes and Kalotychou [5] applied markov model to sovereign credit migration rating. D'Amico *et al.* [6] predicted EU's credit rating inequality for 3 major companies credit rating using Markov chain model and dynamic entropy measure. Hu *et al.* [7] estimated transition matrices for sovereign credit ratings with various methods. Major rating agencies such as Fitch, Moody's, or Standard & Poor's periodically report the estimated transition matrix of the Markov chain. The estimated transition matrix serves as a key input element of some advanced risk management methodologies such as J.P. Morgan's Credit Metrics or McKinsey's Credit Portfolio View. However, much empirical literature has discovered non-Markovian behaviors in credit rating migration. For instance, Carty and Fons [8], Nickell *et al.* [9], and Bangia *et al.* [10] showed that business cycle affects the credit rate migrations; Fei *et al.* [11] suggested a Mixture of Markov Chains (MMC) model to explain credit rating migration risk in consideration of the stochastic business cycle effects; Frydman [12] and Frydman and Schuermann [13] claims population heterogeneity among bond issuers to better describe bond ratings migration; Siu *et al.* [14] and Ching *et al.* [15] developed multivariate Markov chain model describing inter-dependency and intra-dependency of credit ratings; D'Amico *et al.* [16]–[18] applied semi-Markov chain approach to credit rating migration with emphasis on the sojourn time modelling; Boreiko *et al.* [19] suggested a coupled Markov chain model to measure the effects of macroeconomic factors on credit-rating migration. Indeed, the credit rating migration model based on the HMM has been studied in previous researches such as korolkiewicz and Elliot [20], korolkiewicz [21], Elliott *et al.* [22], Petropoulos *et al.* [23]

A regime switching model has been widely applied for various financial time series. For instance, Hamilton [24], Pagan and Schwert [25], and Ang and Bekaert [26] considered economic states (regimes) that govern the dynamics of the observed time series; Goldfeld and Quandt [27] introduced Markov switching regressions in econometrics; Neftci [28], Hamilton and Susmel [29]; and Gray [30] worked for the extensions of autoregressive, ARCH, and GARCH models respectively. Also, the increasing importance of the model can be found in various financial domains including option pricing [31], [32], foreign exchange [33], and interest rates [34].

In this respect, the purpose of this paper is to develop a flexible and realistic credit migration model, called the Regime Switching Markov Chain (RSMC), that incorporates the non-Markovian behaviors using the hidden Markov Model (HMM). Similar to the concept of regime switching model, we consider a credit rating migration process as a non-homogeneous process in that the transition probabilities of credit ratings depend on the economic condition such as the periods of economic expansion or contraction. Also, we consider the credit rating as a discrete type stochastic migration process. Eventually, the model should possess a time-variant embedded (discrete time) Markov chain transition probabilities from one state to another for the different underlying economic condition. In RSMC, the credit rating migration is modelled by integrating a revised structure of HMM, which represents the unobserved economic regime switching process, into a discrete time Markov chain, which represents the observed credit migration process. Therefore, the credit migration probabilities in the economic expansion period are different from those in the contraction period. That is, the economic condition of expansion or contraction is regarded as an unobservable regime but affects the credit migration process, simultaneously. Then, we present a version of the Expectation-Maximization (EM) algorithm similar to the Baum-Welch algorithm [35] as a parameter estimation method. EM algorithm developed by Dempster *et al.* [36] is a recursive procedure to estimate maximum likelihood parameters in incomplete-data problems.

Our model is novel for its discrete time-variant nature based on the consolidation of the HMM into a Markov chain. Although we assume that the observed states are dependent on the underlying regime as in a standard HMM, our model considers that the observed process in a given regime has a Markov property. This assumption makes our model distinct from a single regime Markov chain and HMM. Furthermore, based on our model, we investigate the sovereign credit rating migration in terms of estimation and forecasting. In summary, the empirical results show that the performance of RSMC is superior to that of single regime Markov chain in both estimation and forecasting experiments, which suggests the possible implementation of RSMC in risk management of sovereign credit risk and related financial products.

This paper is organized as follows. Section 2 introduces the mathematical background of RSMC; Section 3 presents forward, and backward algorithms, and other preliminary results, all of which are used to derive a version of EM algorithm for parameter estimation; Section 4 evaluates the proposed model by applying the model to the S&P sovereign credit rating migration record; and Section 5 concludes. Note that we provide proofs for the equations used in Section 3 and present a model selection criterion to compare the fitting performance between the benchmark Markov chain and the proposed RSMC model in the Appendix.

II. REGIME SWITCHING MARKOV CHAIN MODEL

Our model aims to propose a discrete time and discrete value regime switching Markov chain, which incorporates time-varying property. We consider a discrete time Markov chain $\{X_k : k = 0, 1, 2, \ldots\}$ and a discrete time stochastic process $\{Y_k : k = 0, 1, 2, \ldots\}$. We assume that Y_k is observed credit rating at time k and X_k is a underlying economic condition which is unobservable. The state spaces of X_k and Y_k are $S_X = \{1, 2, \ldots, N\}$ and $S_Y = \{1, 2, \ldots, M\}$, respectively.

In Figure 1, the concepts of the Markov chain, HMM, and RSMC are sequentially illustrated in schematic diagrams. A standard Markov chain only possesses the observed process, Y_k, where the current credit rating of Y_k is only dependent on the previous rating Y_{k-1}. The standard HMM assumes that the hidden process X_k is a Markov chain and the dynamics of observation Y_k is governed by either X_k (or X_{k-1}). Hence, observations in HMM, Y_k, are assumed to be independent of each other. In contrast, our model consists of the hidden process, X_k, and the observed process, Y_k, where the current credit rating of Y_k is not only dependent on the previous rating Y_{k-1} but also affected by the previous economy condition of X_{k-1}.

As X_k is a Markov chain, $P(X_{k+1}|\mathcal{X}_k, \mathcal{Y}_k) = P(X_{k+1}|X_k)$ for $k = 0, \ldots, L-1$, where $\mathcal{X}_\ell = [X_0, X_1, \ldots, X_\ell]$ and $\mathcal{Y}_\ell = [Y_0, Y_1, \ldots, Y_\ell]$ for $\ell = 0, \ldots, L$, and we define the hidden economy state transition probability,

$$a_{ij} := P(X_{k+1} = j|X_k = i), \quad i, j \in S_X \tag{1}$$

Note that $\sum_{j\in S_X} a_{ij} = 1$ for all $i \in S_X$. For the probability of X_0, we define $a_0(i) := P(X_0 = i)$ and set $a_0(1) = 1$ and $a_0(i) = 0$ for $i \neq 1$ in our model.

The conditional probability of the observed process Y_k is $P(Y_{k+1}|\mathcal{X}_k, \mathcal{Y}_k) = P(Y_{k+1}|Y_k, X_k)$, and we define the credit rating transition probability given the economy state i as

$$e_i(r, s) := P(Y_{k+1} = s|Y_k = r, X_k = i), \quad i \in S_X, \ r, s \in S_Y \tag{2}$$

which satisfy $\sum_{s\in S_Y} e_i(r, s) = 1$ for all $i \in S_X$ and $r \in S_Y$. For the probability of Y_0, $e_0(r) := P(Y_0 = r)$, $r \in S_Y$ satisfying $\sum_{r\in S_Y} e_0(r) = 1$.

Our regime switching Markov chain model is built in the following parameter set,

$$\boldsymbol{\theta} := \{a_{ij}, e_i(r, s), a_0(i), e_0(r), i, j \in S_X, \ r, s \in S_Y\} \tag{3}$$

III. PARAMETER ESTIMATION FOR RSMC MODEL

A. FORWARD/BACKWARD EQUATIONS AND STATE FILTER/SMOOTHER

We describe how to estimate parameters in $\boldsymbol{\theta}$ of (3) using forward and backward equations (recursions), and also derive some useful and preliminary results which will be used in the parameter estimation procedure.

Let $f_j(\mathcal{Y}_k) := P(\mathcal{Y}_k, X_k = j)$, and its value is computed using the following forward equation, for $j \in S_X$

$$f_j(\mathcal{Y}_k) = \sum_{i=1}^{N} f_i(\mathcal{Y}_{k-1}) e_i(Y_{k-1}, Y_k) a_{ij}, \quad k = 1, \ldots, L \tag{4}$$

with the initial value $f_1(\mathcal{Y}_0) = 1$ and $f_j(\mathcal{Y}_0) = 0$ for $j = 2, \ldots, N$. The proof of (4) is provided in Appendix A.

Let $b_i(\mathcal{Y}^k) := P(\mathcal{Y}^{k+1}|Y_k, X_k = i)$ where $\mathcal{Y}^k = [Y_k, Y_{k+1}, \ldots, Y_L]$ for $k = 0, \ldots, L-1$, and its value is computed using the following backward equation, for $i \in S_X$

$$b_i(\mathcal{Y}^k) = e_i(Y_k, Y_{k+1}) \sum_{j\in S_X} a_{ij} b_j(\mathcal{Y}^{k+1}), \quad k = 0, \ldots, L-1 \tag{5}$$

For the start of recursive computation, we artificially set $b_1(\mathcal{Y}^L) = 1$ and $b_i(\mathcal{Y}^L) = 0$ for $i = 2, \ldots, N$. The proof of (5) is provided in Appendix A.

By applying $f_j(\mathcal{Y}_k)$ and $b_i(\mathcal{Y}^k)$, we can evaluate various quantities of interest. The probability $P(\mathcal{Y}_k)$ for $k = 0, \ldots, L$ is evaluated as

$$P(\mathcal{Y}_k) = \sum_{i\in S_X} P(\mathcal{Y}_k, X_k = i) = \sum_{i\in S_X} f_i(\mathcal{Y}_k) \tag{6}$$

The state filter for state i at time k, which is the conditional probability of hidden state i given $\mathcal{Y}_k$, is

$$\begin{aligned} P(X_k = i|\mathcal{Y}_k) &= P(X_k = i, \mathcal{Y}_k)/P(\mathcal{Y}_k) \\ &= f_i(\mathcal{Y}_k)/\sum_{i\in S_X} f_i(\mathcal{Y}_k) \end{aligned} \tag{7}$$

The state filter is the probability that the status of the economy at time k is i state (level) based on the history of credit ratings up to time k, and can be used for economic condition forecasting.

The state smoother for state i at time k, which is the conditional probability of hidden state i given $\mathcal{Y}_L$, is

$$P(X_k = i|\mathcal{Y}_L) = \frac{f_i(\mathcal{Y}_k) b_i(\mathcal{Y}^k)}{P(\mathcal{Y}_L)} \tag{8}$$

The state smoother in (8) is the probability that the status of economy at time k ($k \leq L$) is state (level) i based on the whole history of credit ratings throughout the observation period from time 0 to L, $\mathcal{Y}_L = [Y_0, \ldots, Y_L]$. For this reason, state smoother cannot provide the information on the current economic condition but can evaluate the past unobserved economic condition more accurately using the whole credit rating history. The proof of (8) is found in Appendix A.

Let $n_{ij} := \sum_{k=0}^{L-1} \mathbf{1}_{(i,j)}(X_k, X_{k+1})$ be the total count that X_k moves from state i to j, where $\mathbf{1}_z(Z)$ is an indicator function where it is unity if $Z = z$ and zero otherwise. Hence, n_{ij} is the number of times that the economic condition switched from state i to state j.

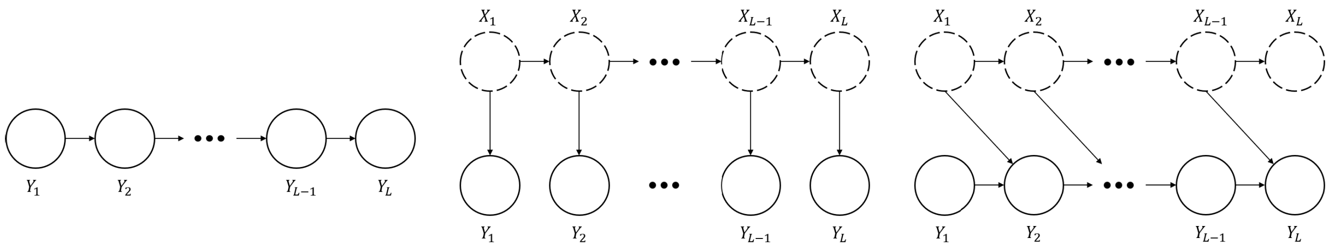
We can derive the conditional expectation of n_{ij} given $\mathcal{Y}_L$, $\hat{n}_{ij}$, as follows.

$$\begin{aligned} \hat{n}_{ij} &= E\left[n_{ij}|\mathcal{Y}_L\right] \\ &= \sum_{k=0}^{L-1} \frac{f_i(\mathcal{Y}_k) e_i(Y_k, Y_{k+1}) a_{ij} b_j(\mathcal{Y}^{k+1})}{P(\mathcal{Y}_L)} \end{aligned} \tag{9}$$

Also let $n_i(r, s) := \sum_{k=0}^{L-1} \mathbf{1}_{(r,s)}(Y_k, Y_{k+1}) \cdot \mathbf{1}_i(X_k)$ be the total count that Y_k moves from state r to state s under $X_k = i$. We can interpret $n_i(r, s)$ as the number of times the credit rating moves from state r into state s under the economy condition of level i. We can evaluate the conditional expectation of $n_i(r, s)$ given $\mathcal{Y}_L$, $\hat{n}_i(r, s)$, as follows.

$$\begin{aligned} \hat{n}_i(r, s) &= E\left[n_i(r, s)|\mathcal{Y}_L\right] \\ &= \frac{1}{P(\mathcal{Y}_L)} \sum_{k\in I_Y(r,s)} f_i(\mathcal{Y}_k) b_i(\mathcal{Y}^k). \end{aligned} \tag{10}$$

where $I_Y(r, s) := \{k : (Y_k, Y_{k+1}) = (r, s), k = 0, \ldots, L-1\}$. The proof of (9) and (10) are presented in Appendix A.



(a) Markov Chain (b) Hidden Markov Model (c) Regime Switching Markov Chain

FIGURE 1. Schematic diagrams using directed graph. Note that X_k in dashed circles and Y_k in solid circles represent state and observation processes, respectively, and arrows represent the dependency structure of the model.

B. EM ALGORITHM FOR RSMC MODEL

The EM algorithm, an iterative procedure to obtain maximum likelihood (ML) estimates of model parameters [36], works well in incomplete (missing) data problems where direct ML estimation is not easy. In Section 2, we assume that the economic state sequence $\{X_k\}$ is unobservable. Hence, we must estimate parameters based on observed data only. For this reason, EM algorithm is a valid method for our parameter estimation.

Let $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ be a set of parameters as in (3), and $\boldsymbol{\Theta}$ denotes the union of all possible parameter sets. Under the assumption that all data, $(\mathcal{X}_L, \mathcal{Y}_L)$, are known, the log-likelihood function, $\ln \mathcal{L}(\boldsymbol{\theta})$, can be written as

$$\begin{aligned} \ln \mathcal{L}(\boldsymbol{\theta}) &= \ln P_{\boldsymbol{\theta}}(\mathcal{X}_L, \mathcal{Y}_L) \\ &= \ln a_0(X_0) + \ln e_0(Y_0) \\ &\quad + \sum_{k=1}^{L} \left\{ \ln a_{X_{k-1}X_k} + \ln e_{X_{k-1}}(Y_{k-1}, Y_k) \right\} \\ &= \ln a_0(X_0) + \ln e_0(Y_0) + \sum_{i \in S_X} \sum_{j \in S_X} n_{ij} \ln a_{ij} \\ &\quad + \sum_{i \in S_X} \sum_{r \in S_Y} \sum_{s \in S_Y} n_i(r,s) \ln e_i(r,s) \end{aligned} \tag{11}$$

where n_{ij} and $n_i(r,s)$ are defined in Section 3.A. In our model, we refer to all sovereign credit ratings together and provide common model parameters applied to each sovereign rating.

Assuming that we have m nations having their own credit rating time series, we denote the credit rating of the hth country at time k as $Y_k^{(h)}$, and define nation h's credit rating time series, $\mathcal{Y}_L^{(h)} = [Y_0^{(h)}, \ldots, Y_L^{(h)}]$. The corresponding economy condition for the hth country at time k as $X_k^{(h)}$, and define nation h's economic condition time series, $\mathcal{X}_L^{(h)} = [X_0^{(h)}, \ldots, X_L^{(h)}]$. Then the log-likelihood function is extended to

$$\begin{aligned} \ln \mathcal{L}(\boldsymbol{\theta}) &= \sum_{h=1}^{m} \ln P_{\boldsymbol{\theta}}(\mathcal{X}_L^{(h)}, \mathcal{Y}_L^{(h)}) \\ &= \sum_{h=1}^{m} \Bigg(\ln a_0(X_0^{(h)}) + \ln e_0(Y_0^{(h)}) \\ &\quad + \sum_{k=1}^{L} \left\{ \ln a_{X_{k-1}^{(h)} X_k^{(h)}} + \ln e_{X_{k-1}^{(h)}}(Y_{k-1}^{(h)}, Y_k^{(h)}) \right\} \Bigg) \\ &= \sum_{h=1}^{m} \ln a_0(X_0^{(h)}) + \sum_{h=1}^{m} \ln e_0(Y_0^{(h)}) \\ &\quad + \sum_{h=1}^{m} \sum_{i \in S_X} \sum_{j \in S_X} n_{ij}^{(h)} \ln a_{ij} \\ &\quad + \sum_{h=1}^{m} \sum_{i \in S_X} \sum_{r \in S_Y} \sum_{s \in S_Y} n_i^{(h)}(r,s) \ln e_i(r,s) \end{aligned} \tag{12}$$

where $n_{ij}^{(h)} = \sum_{k=1}^{L} \mathbf{1}_{(i,j)}(X_{k-1}^{(h)}, X_k^{(h)})$ and $n_i^{(h)}(r,s) = \sum_{k=1}^{L} \mathbf{1}_{(r,s)}(Y_{k-1}^{(h)}, Y_k^{(h)}) \cdot \mathbf{1}_i(X_{k-1}^{(h)})$

However, in reality, $\{\mathcal{X}_L^{(h)} : h = 1, \ldots, m\}$ is unknown. To estimate $\boldsymbol{\theta}$ with incomplete data, we use the conditional expectation of $\ln \mathcal{L}(\boldsymbol{\theta})$ given $\{\mathcal{Y}_L^{(h)} : h = 1, \ldots, m\}$ under the set of parameters in (3) at the tth iteration of EM algorithm, $\boldsymbol{\theta}^{(t)} \in \boldsymbol{\Theta} = \{a_{ij}^{(t)}, e_i^{(t)}(r,s), a_0^{(t)}(i), e_0^{(t)}(r)\ i,j \in S_X,\ r,s \in S_Y\}$ for $t = 0, 1, 2, \ldots,$

$$\begin{aligned} Q(\boldsymbol{\theta}; \boldsymbol{\theta}^{(t)}) &:= E_{\boldsymbol{\theta}^{(t)}} \left[\ln \mathcal{L}(\boldsymbol{\theta}) |\, \mathcal{Y}_L^{(1)}, \ldots, \mathcal{Y}_L^{(m)} \right] \\ &= \sum_{h=1}^{m} \ln a_0(X_0^{(h)}) + \sum_{h=1}^{m} \ln e_0(Y_0^{(h)}) \\ &\quad + \sum_{h=1}^{m} \sum_{i \in S_X} \sum_{j \in S_X} \hat{n}_{ij}^{(h)} \ln a_{ij} \\ &\quad + \sum_{h=1}^{m} \sum_{i \in S_X} \sum_{r \in S_Y} \sum_{s \in S_Y} \hat{n}_i^{(h)}(r,s) \ln e_i(r,s) \end{aligned}$$

where $\hat{n}_{ij}^{(h)}$ and $\hat{n}_i^{(h)}(r,s)$ are $E_{\boldsymbol{\theta}^{(t)}}[n_{ij}^{(h)} | \mathcal{Y}_L^{(h)}]$ and $E_{\boldsymbol{\theta}^{(t)}}[n_i^{(h)}(r,s) | \mathcal{Y}_L^{(h)}]$ computed under $\boldsymbol{\theta}^{(t)}$ using (9) and (10) respectively.

The related EM algorithm is described as follows.

- E-step: Compute the conditional log-likelihood $Q(\boldsymbol{\theta}; \boldsymbol{\theta}^{(t)})$.
- M-step: Choose the new $\boldsymbol{\theta}' \in \boldsymbol{\Theta}$ to maximize $Q(\boldsymbol{\theta}; \boldsymbol{\theta}^{(t)})$.
- Update $t \leftarrow t + 1$ and $\boldsymbol{\theta}^{(t+1)} \leftarrow \boldsymbol{\theta}'$.

First, initialize the counter index $t = 0$ and $\boldsymbol{\theta}^{(0)} \in \boldsymbol{\Theta}$. Then repeat the following expectation-step (E-step) and maximization-step (M-step) until the given stopping criterion is satisfied. Our way to set $\boldsymbol{\theta}^{(0)}$ for EM algorithm is described in Appendix B.

Some typical stopping criteria are as follows. The algorithm stops when the change in total log-likelihood is sufficiently small or when the difference between the updated parameter value and the previous one is smaller than a specified value. Baum-Welch algorithm [37] is a special version of the EM algorithm for the standard HMM depicted in Figure 1-(b).

Under the Karush-Kuhn-Tucker condition, we obtain $\boldsymbol{\theta}'$ maximizing $Q(\boldsymbol{\theta}; \boldsymbol{\theta}^{(m)})$ as follows.

$$\hat{a}_{ij} = \frac{\sum_{h=1}^{m} \hat{n}_{ij}^{(h)}}{\sum_{h=1}^{m} \sum_{j \in S_X} \hat{n}_{ij}^{(h)}} \tag{13}$$

$$\hat{e}_i(r, s) = \frac{\sum_{h=1}^{m} \hat{n}_i^{(h)}(r, s)}{\sum_{h=1}^{m} \sum_{s \in S_Y} \hat{n}_i^{(h)}(r, s)} \tag{14}$$

From (13) and (14), our EM algorithm is described in Algorithm 1.

Algorithm 1 EM Algorithm for the Estimation of a_{ij} and $e_i(r, s)$ in $\boldsymbol{\theta}$

Input : Initial value of a_{ij}, $e_i(r, s)$ for $i, j \in S_X$ and $r, s \in S_Y$
Output : a_{ij} and $e_i(r, s)$ for $i, j \in S_X$ and $r, s \in S_Y$ maximizing the log-likelihood

Initialize a_{ij}, $e_i(r, s)$ for $i, j \in S_X$ and $r, s \in S_Y$
Set $f_1(\mathcal{Y}_0) = 1$ and $f_j(\mathcal{Y}_0) = 0$ for $j = 2, \ldots, N$
Set $b_1(\mathcal{Y}^L) = 1$ and $b_i(\mathcal{Y}^L) = 0$ for $i = 2, \ldots, N$
while *stopping criteria are unsatisfied* **do**
  Expectation step
    Compute $f_j(\mathcal{Y}_k)$ using (4)
      for $j \in S_X$, $k = 1, \ldots, L$ (in ascending order)
    Compute $b_i(\mathcal{Y}^k)$ using (5)
      for $i \in S_X$, $k = L-1, \ldots, 0$ (in descending order)
  Maximization step
    Compute $\hat{a}_{ij}$ and $\hat{e}_i(r, s)$ using (9),(10),(13) and (14)
    Update $\hat{a}_{ij} \rightarrow a_{ij}$ and $\hat{e}_i(r, s) \rightarrow e_i(r, s)$
end
return a_{ij} and $e_i(r, s)$ for $i, j \in S_X$ and $r, s \in S_Y$

IV. APPLICATION TO SOVEREIGN CREDIT RATING MIGRATION

A. DATA DESCRIPTION

We apply our model to sovereign credit rating migration. The series of sovereign credit rating of 41 nations evaluated by Standard & Poor's during 300 months from January 1994 through December 2018 are used. The S&P's original rating scheme consists of major rating categories and additional modifiers. The major rating categories range from AAA, AA, A, BBB, BB, B, CCC, CC, C, to D in decreasing order of creditworthiness of an obligor, and ratings from AA to CCC are classified in more detail using plus (+) or a minus (−) to show relative standing. In our research, we classified 22 original rating states into 14 coarser rating states: AAA, AA+, AA, AA−, A+, A, A−, BBB+, BBB, BBB−, BB, B (including B+, B, and B−), C (including CCC, CC, and C grades), and Others (below C−).

TABLE 1. Parameter estimation of the hidden economy state transition probability of RSMC model.

	State 1	State 2
State 1	0.9969	0.0031
State 2	0.0400	0.9600

B. ESTIMATION RESULTS FROM 1994 TO 2017

Based on the estimates obtained from the algorithm in Section 3.B, we evaluate the fitting performance using a likelihood-ratio (LR) test in (15) (see Appendix C). The null hypothesis of LR test states that the fitting performance of the homogeneous Markov chain is better than that of RSMC. The chi-squared test statistic $-2 \ln \Lambda$ in (15) with the degrees of freedom of 38 is 125.19, and the corresponding p-value of the LR test is below 10^{-10}. The result suggests a strong rejection of the null hypothesis, which eventually claims that the process of sovereign credit rating migration should be modeled in RSMC rather than a single regime Markov chain.

In our model, we assume two hidden states, $S_X = \{1, 2\}$, where state 1 and 2 stand for the economic expansion and contraction regimes, respectively. Table 1 shows the estimated results for the monthly hidden economy state transition probability matrix of RSMC. The result shows that each state is likely to remain in each state after a month. The probability of switching after a month from state 1 to state 2 is 0.31%, and the probability from state 2 to state 1 is 4.00%. It means that the probability from the economic contraction regime to the economic expansion regime after a month is higher than in the opposite case.

Table 2 summarizes the estimated results for the benchmark Markov chain, which is constructed based on a single regime. The estimated results for our regime switching Markov chain (RSMC) are shown in Table 3, where the first and second transition probability matrices correspond to the hidden states 1 and 2, respectively. Each state is classified into three groups: the investment credit rating group (AAA, AA+, AA, AA−, A+, A, A−, BBB+, BBB, BBB−), non-investment credit rating group (BB, B, C), and default group (Others). In the investment credit rating group, there are three layers: the first-tier has (AAA, AA+, AA, AA−), the second tier includes (A+, A, A−), and the bottom-tier consists of (BBB+, BBB, BBB−).

On the whole, the diagonal elements, which are the probabilities to maintain the current credit ratings, of the matrix for hidden state 1 corresponding to the economic expansion regime are larger than those of the matrix for hidden state 2 corresponding to the economic contraction regime. There is still some chance for rating upgrade in the economic expansion regime, while the chance is close to zero in the economic

TABLE 2. Parameter estimation of the benchmark (homogenous) Markov chain.

	AAA	AA+	AA	AA-	A+	A	A-	BBB+	BBB	BBB-	BB	B	C	Others
AAA	0.9971	0.0026	0.0003											
AA+	0.0078	0.9868	0.0054											
AA		0.0078	0.9830	0.0091										
AA-			0.0090	0.9795	0.0090	0.0026								
A+				0.0101	0.9782	0.0084	0.0034							
A				0.0019	0.0185	0.9648	0.0111	0.0037						
A-						0.0128	0.9757	0.0057	0.0029	0.0014		0.0014		
BBB+						0.0020	0.0160	0.9681	0.0080	0.0040	0.0020			
BBB							0.0022	0.0242	0.9626	0.0110				
BBB-									0.0179	0.9746	0.0060	0.0015		
BB										0.0072	0.9888	0.0040		
B											0.0077	0.9750	0.0154	0.0019
C												0.0328	0.9180	0.0492
Others												0.0488	0.0244	0.9268

TABLE 3. Parameter estimation of RSMC.

(a)

	AAA	AA+	AA	AA-	A+	A	A-	BBB+	BBB	BBB-	BB	B	C	Others
AAA	0.9993	0.0004	0.0003											
AA+	0.0081	0.9905	0.0014											
AA		0.0079	0.9874	0.0047										
AA-			0.0085	0.9832	0.0083									
A+				0.0097	0.9880	0.0023	0.0000							
A				0.0021	0.0187	0.9769	0.0023	0.0000						
A-						0.0135	0.9865	0.0000		0.0000				
BBB+						0.0023	0.0188	0.9742	0.0047	0.0000				
BBB							0.0024	0.0259	0.9717	0.0000				
BBB-									0.0227	0.9747	0.0008	0.0018		
BB										0.0081	0.9893	0.0026		
B											0.0033	0.9860	0.0107	0.0000
C												0.0470	0.8826	0.0704
Others												0.0360	0.0250	0.9390

(b)

	AAA	AA+	AA	AA-	A+	A	A-	BBB+	BBB	BBB-	BB	B	C	Others
AAA	0.9265	0.0735	0.0000											
AA+	0.0000	0.9512	0.0488											
AA		0.0000	0.9504	0.0496										
AA-			0.0000	0.9179	0.0271	0.0551								
A+				0.0000	0.0000	0.6065	0.3935							
A				0.0000	0.0000	0.7420	0.1837	0.0743						
A-						0.0000	0.7602	0.1203	0.0597	0.0302		0.0296		
BBB+						0.0000	0.0000	0.9282	0.0286	0.0287	0.0145			
BBB							0.0000	0.0000	0.8378	0.1622				
BBB-									0.0000	0.9732	0.0268			
BB										0.0000	0.9845	0.0155		
B											0.0821	0.7824	0.0998	0.0357
C												0.0000	1.0000	0.0000
Others												0.4961	0.0000	0.5039

contraction regime. The tendency to maintain the current credit rating is the highest in the first tier of investment credit rating group (AAA, AA+, AA, AA−) in state 1, and that is the lowest in the second tier of investment credit rating group (A+, A, A−) in state 2. It implies that the credit ratings in the top tier of investment credit rating group in the economic expansion regime are most unlikely to change, and those in the second tier of investment credit rating group in the economic contraction regime are most likely to be downgraded.

Notably, the downgrade of A+ rating in the economic contraction regime is almost sure. Except for the second tier of investment credit rating group, the probabilities to keep incumbent credit ratings tend to be lower as ratings become poorer in the economic contraction regime while this trend is not observed in the economic expansion regime. About 3% of A− credit rating cases in the economic contraction regime fall to non-investment credit rating grade, B, and 1.45% of BBB+ credit rating cases descend to non-investment credit rating grade, BB. The downgrade and default probabilities in non-investment credit rating group (BB, B, C) tend to increase in state 2, the economic contraction regime, compared with state 1, the economic expansion regime.

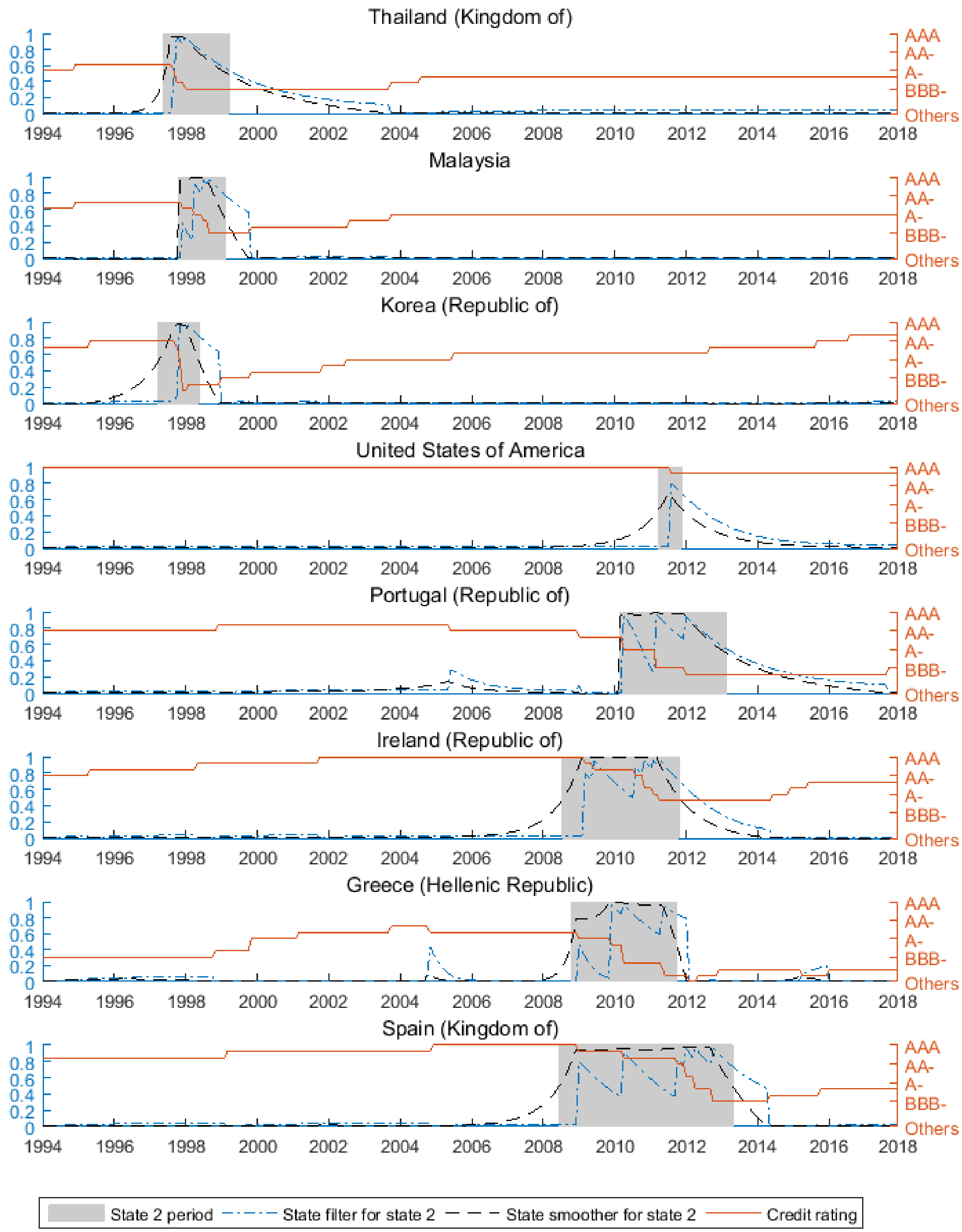


FIGURE 2. Examples of state filter and smoother based on the probability of state 2.

Then, the estimated transition probability matrices for RSMC is used to compute the state filter and smoother for the entire countries. At first, Figure 2 displays the state filter and smoother based on the evolution of the probability of state 2 for eight countries: Thailand, Malaysia, Korea (Asia), United States of America (Key currency), Portugal, Ireland, Greece and Spain (Europe, PIGS). For this case, we classify the current state as state 2 (economic contraction) if the state smoother of state 2 is greater or equal to that of state 1, and state 1 (economic

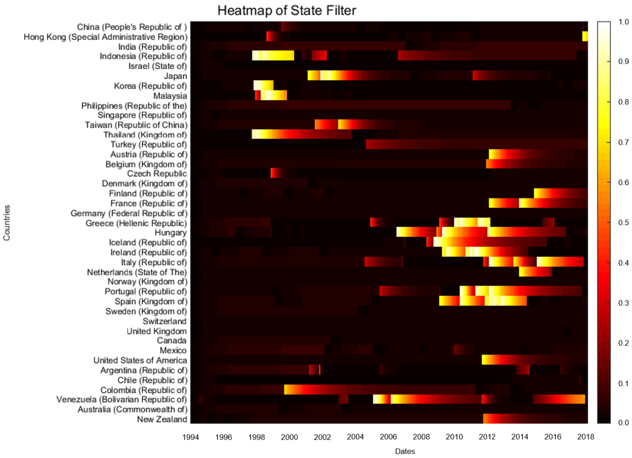


FIGURE 3. Heatmap of state filter based on the probability of state 2.

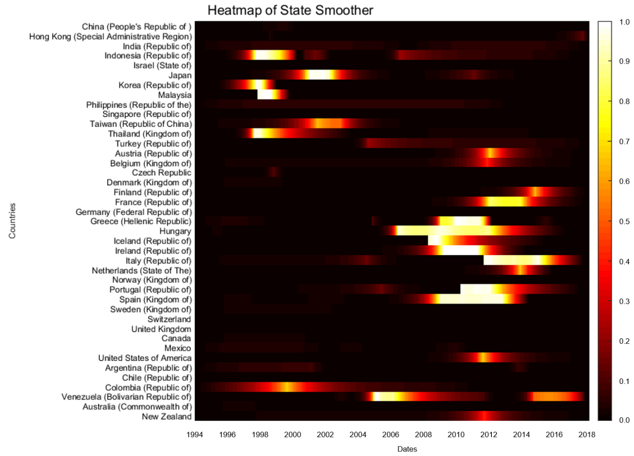


FIGURE 4. Heatmap of state smoother based on the probability of state 2.

expansion) otherwise. Note that the state 2 is shaded in Figure 2.

The results of Asia show the economic contraction from 1997 to 1999, which indicates the outbreak of the Asian currency crisis. Note that Thailand, Malaysia, and Korea are directly involved and damaged during the crisis. The result of the United States shows that the state smoother indicates the economic contraction regime in 2011. As the state smoother is dependent on the S&P credit ratings which changed the US rating only once from AAA into AA+ in August 2011, state smoother succeeds to capture the drop earlier. The result of Portugal exhibits the economic contraction period for three years, 2010 to 2012, whereas those of Ireland, Greece, and Spain exhibit the economic contraction period from 2008 or 2009 and remain in the regime until 2011, 2011, and 2013, respectively. Note that this indicates the European sovereign debt crisis. Overall, the state filter and smoother well detect the economic contraction based on the downgrades of credit ratings, which yields a nation-wide clustering for corresponding sovereign risks. Also, it seems that RSMC can detect the contraction period promptly for the countries with volatile credit migration than a continuously steady one. For more details, we plot heatmaps of state filter and smoother based on the evolution of the probability of state 2 for the entire 41 countries in Figure 3 and 4, respectively.

The order of the countries in Figure 3 and 4 is arranged alphabetically from top to bottom in Asia, Europe, North America, South America, and Oceania. Asia has 13 countries from China to Turkey; Europe has 19 countries from Austria to the United Kingdom; North America has 4 countries including Canada, Mexico, and United States; South America has 4 countries from Argentina to Venezuela; and Oceania has 2 countries including Australia and New Zealand. The heatmaps show monthly values of state filter and smoother for state 2 from 1994 to 2017 where the color scheme becomes brighter as its value approaches from 0 to 1. That is, the brighter the color, the more likely the country is in the economic contraction regime. As a result, we discover similar results based on the color patterns in both heatmaps. The Asian currency crisis caused a higher probability of state 2 in various Asian countries from 1997 to 2000, whereas the European sovereign debt crisis caused the same phenomenon in European countries from 2009 to 2012. The color suddenly changes for the filter, whereas it gradually becomes bright for smoother. It refers that the filter for state 2 defines the economic contraction at the change of credit rating with a value of 1 then it gradually returns to 0. It seems that the gradual decrease of the probability of the contraction period can be used to define the duration of sovereign risks.

In case of smoother for state 2, the future information of credit migration is reflected, so it displays the highest probability at the moment of the degrading of credit rating by gradually increasing the probability in advance. However, since the filter only uses the present data, the value suddenly changes when the degrading emerges. Specifically, the state smoother is constructed based on the observations during the whole sample period while state filter is determined using the observations up to the current time point. Note that the state filter and smoother for hidden state 1 are $P(X_k = 1|\mathcal{Y}_k)$ in (7) and $P(X_k = 1|\mathcal{Y}_{288})$ in (8) for $k = 1, \ldots, 288$, ($L = 288$), respectively. Therefore, the state smoother, which precedes state filter for the whole sample period, shows better performance to distinguish economic contraction regime from economic expansion regime than state filter. However, we conclude that the state smoother cannot play a forecasting role like the state filter due to its ex-post nature. Lastly, the proportion of time spent in each state is listed in Table 4.

The results of both state filter and smoother indicate the dominance of state 1 (96% in filter and 97% in smoother) against state 2 (4% in filter and 3% in smoother). Such dominance retains for different continents where the proportions of state 1 in filter (smoother) for Asia, North America, South America, Europe, and Oceania are 97% (97%), 99% (99%), 96% (94%), 95% (93%), and 99% (100%), respectively.

TABLE 4. Proportion of state 1 and 2 during the estimation.

Continent	Country	State Filter		State Smoother	
		State 1	State 2	State 1	State 2
Asia	China (People's Republic of)	100%	0%	100%	0%
	Hong Kong (Special Administrative Region)	99%	1%	100%	0%
	India (Republic of)	100%	0%	100%	0%
	Indonesia (Republic of)	90%	10%	91%	9%
	Israel (State of)	100%	0%	100%	0%
	Japan	91%	9%	91%	9%
	Korea (Republic of)	95%	5%	95%	5%
	Malaysia	93%	7%	94%	6%
	Philippines (Republic of the)	100%	0%	100%	0%
	Singapore (Republic of)	100%	0%	100%	0%
	Taiwan (Republic of China)	97%	3%	93%	7%
	Thailand (Kingdom of)	93%	7%	92%	8%
	Turkey (Republic of)	100%	0%	100%	0%
	Average	97%	3%	97%	3%
North America	Canada	100%	0%	100%	0%
	Mexico	100%	0%	100%	0%
	United States of America	97%	3%	97%	3%
	Average	99%	1%	99%	1%
South America	Argentina (Republic of)	100%	0%	100%	0%
	Chile (Republic of)	100%	0%	100%	0%
	Colombia (Republic of)	97%	3%	95%	5%
	Venezuela (Bolivarian Republic of)	89%	11%	82%	18%
	Average	96%	4%	94%	6%
Europe	Austria (Republic of)	97%	3%	97%	3%
	Belgium (Kingdom of)	99%	1%	100%	0%
	Czech Republic	100%	0%	100%	0%
	Denmark (Kingdom of)	100%	0%	100%	0%
	Finland (Republic of)	97%	3%	97%	3%
	France (Republic of)	93%	7%	88%	12%
	Germany (Federal Republic of)	100%	0%	100%	0%
	Greece (Hellenic Republic)	91%	9%	88%	12%
	Hungary	84%	16%	72%	28%
	Iceland (Republic of)	93%	7%	92%	8%
	Ireland (Republic of)	87%	13%	86%	14%
	Italy (Republic of)	88%	12%	82%	18%
	Netherlands (State of The)	97%	3%	98%	2%
	Norway (Kingdom of)	100%	0%	100%	0%
	Portugal (Republic of)	89%	11%	88%	13%
	Spain (Kingdom of)	83%	17%	80%	20%
	Sweden (Kingdom of)	100%	0%	100%	0%
	Switzerland	100%	0%	100%	0%
	United Kingdom	100%	0%	100%	0%
	Average	95%	5%	93%	7%
Oceania	Australia (Commonwealth of)	100%	0%	100%	0%
	New Zealand	99%	1%	100%	0%
	Average	99%	1%	100%	0%
Total		96%	4%	95%	5%

C. FORECASTING RESULTS FOR 2018

Based on the state filter, we perform forecasting in sovereign credit rating migration as described in Frydman and Schuermann [13]. If the history of credit ratings of a country is known for time k, its credit rating at $k+1$ can be forecasted based on the conditional transition matrix of the regime at k. In this regard, we measure the forecast error based on the current transition matrix for next month's credit rating. Note that the performance of the out-of-sample test based on the standard Markov chain is used as a benchmark. Specifically, the Markov chain computes the distribution of credit rating at $k+1$ using a single transition probability matrix at k. However, RSMC computes the distribution of credit rating at $k+1$ using a transition probability matrix for each hidden state determined based on the value of state filter. Note that we use the same criterion for state filter

TABLE 5. Nation-wide forecasting performances in 2018.

Continent	Country	Markov Chain Forecast Error	RSMC Forecast Error	Reduced Error rate
Asia	China (People's Republic of)	0.0214	0.0123	42.4%
	Hong Kong (Special Administrative Region)	0.0130	0.0316	-143.6%
	India (Republic of)	0.0244	0.0250	-2.2%
	Indonesia (Republic of)	0.0244	0.0250	-2.2%
	Israel (State of)	0.1016	0.0950	6.5%
	Japan	0.0214	0.0123	42.4%
	Korea (Republic of)	0.0164	0.0123	25.2%
	Malaysia	0.0240	0.0133	44.4%
	Philippines (Republic of the)	0.0387	0.0307	20.7%
	Singapore (Republic of)	0.0028	0.0006	77.4%
	Taiwan (Republic of China)	0.0202	0.0167	17.6%
	Thailand (Kingdom of)	0.0326	0.0267	18.1%
	Turkey (Republic of)	0.0975	0.0939	3.7%
	Average	0.0337	0.0304	9.8%
North America	Canada	0.0028	0.0006	77.4%
	Mexico	0.0326	0.0267	18.1%
	United States of America	0.0130	0.0096	26.4%
	Average	0.0161	0.0123	23.6%
South America	Argentina (Republic of)	0.0245	0.0136	44.3%
	Chile (Republic of)	0.0214	0.0123	42.4%
	Colombia (Republic of)	0.0244	0.0254	-3.9%
	Venezuela (Bolivarian Republic of)	0.0687	0.0546	20.5%
	Average	0.0348	0.0265	23.8%
Europe	Austria (Republic of)	0.0130	0.0096	26.4%
	Belgium (Kingdom of)	0.0164	0.0123	25.2%
	Czech Republic	0.0202	0.0167	17.6%
	Denmark (Kingdom of)	0.0028	0.0006	77.4%
	Finland (Republic of)	0.0130	0.0096	26.4%
	France (Republic of)	0.0164	0.0123	25.2%
	Germany (Federal Republic of)	0.0028	0.0006	77.4%
	Greece (Hellenic Republic)	0.0245	0.0136	44.3%
	Hungary	0.0244	0.0250	-2.2%
	Iceland (Republic of)	0.0348	0.0230	34.1%
	Ireland (Republic of)	0.0214	0.0123	42.4%
	Italy (Republic of)	0.0387	0.0307	20.7%
	Netherlands (State of The)	0.0028	0.0006	77.4%
	Norway (Kingdom of)	0.0028	0.0006	77.4%
	Portugal (Republic of)	0.0244	0.0250	-2.2%
	Spain (Kingdom of)	0.1052	0.0960	8.7%
	Sweden (Kingdom of)	0.0028	0.0006	77.4%
	Switzerland	0.0028	0.0006	77.4%
	United Kingdom	0.0164	0.0123	25.2%
	Average	0.0203	0.0159	21.7%
Oceania	Australia (Commonwealth of)	0.0028	0.0006	77.4%
	New Zealand	0.0164	0.0123	25.2%
	Average	0.0096	0.0064	32.9%
Total		0.0251	0.0208	17.2%

as state smoother in the previous section. Since both models are discrete, we consider the probability of the particular credit rating is realized at $k+1$ as the correct answer, whereas the sum of the other probabilities as the prediction error. For instance, let's suppose the credit rating of a country at k is AA+ given that the current transition matrix yields 1%, 97%, 2% for being AAA, AA+, AA, respectively. The forecast error is $3\%(= 0.03)$ if the credit rating stays in $AA+$ at $k+1$, whereas the forecast error is 99% if it migrates to AAA. The experiment is conducted in all countries for 12 months from January 2018 to December 2018. The average forecast error for each country is summarized in Table 5.

Based on the reduced error rate in the third column, which calculates the percentage of reduced error from the Markov Chain to RSMC, 35 out of 41 countries show the lower forecast errors in RSMC given that the average reduced error rate for all countries is 17.2%. It is difficult to directly convert the percent of error reduction in equivalent amount of economic impact. However, the advantage of error reduction can be described in the example of portfolio management for the sovereign credit default swap products. When investing in CDS products, it is important to forecast the country's credit rating to manage and hedge the corresponding risk. It is essential to forecast the credit rating of a country, and

TABLE 6. Monthly forecasting performances in 2018. * indicates 0.1% statistical significance.**

Forecasting Month	Markov Chain Forecast Error	RSMC Forecast Error	Reduced Error rate	P-value
201801	0.0197	0.0159	19.6%	9.6E-04***
201802	0.0196	0.0158	19.7%	8.7E-04***
201803	0.0427	0.0390	8.8%	9.5E-04***
201804	0.0193	0.0154	20.3%	6.6E-04***
201805	0.0192	0.0153	20.5%	5.8E-04***
201806	0.0191	0.0152	20.6%	5.0E-04***
201807	0.0190	0.0151	20.8%	4.3E-04***
201808	0.0666	0.0614	7.8%	6.7E-06***
201809	0.0193	0.0143	25.5%	1.0E-10***
201810	0.0192	0.0142	25.6%	1.2E-10***
201811	0.0191	0.0142	25.7%	1.3E-10***
201812	0.0190	0.0141	25.8%	1.4E-10***
Average	0.0251	0.0208	17.2%	

if it is mistaken, it will cause huge damage in the portfolio performance. Therefore, it is worthwhile to raise the accuracy by even 0.1% to appropriately distribute the asset and to hedge the risk. Therefore, we believe that the result of 17.2% error reduction in average by the forecasting of RSMC is encouraging result.

Hong Kong, India, Indonesia, Columbia, Hungary and Portugal are 6 countries with the higher forecast errors in RSMC. Although the difference of forecasting performance between the Markov Chain and RSMC for Hong Kong is large, that of other 5 countries are relatively small. Also, RSMC shows lower forecast errors in all continents given that the average reduced error rates of Asia, North America, South America, Europe, and Oceania are 9.5%, 23.6%, 23.8%, 21.7%, and 32.9%, respectively. Again, such low reduction in Asia is caused by Hong Kong. The credit rating of Hong Kong is downgraded from AAA to AA+ on September 2017. Specifically, Hong Kong has maintained the best quality grade (AAA) for almost 7 years before the downgrade, which is rarely observed phenomenon. In the state filter of Hong Kong, RSMC classifies the downgrade as State 2 (economic contraction), which then holds until August 2018. Although the downgrade from AAA to AA+ is a negative event, AA+ is still a second-best quality. However, our model exhibits the limitation that a sudden downgrade can incur the switching to State 2 regardless of the relative quality of the credit rating, which causes the high error rate of Hong Kong. In summary, the performance of RSMC can be worse than HMM if the high-quality credit rating is suddenly downgraded while maintaining the same rating for such a long period. However, except for such special case, RSMC shows much lower error rate than that of Markov chain model. For more details, we analyze the results for different months and credit ratings in Table 6 and Figure 5, respectively.

Table 6 shows the average forecast error for different months. The results of paired t-test indicate the rejection of the null hypothesis, which states that the forecast error of the Markov chain is not larger than that of RSMC, at 0.1% statistical significance. Therefore, we claim that the forecasting performance of RSMC is significantly superior to that of the Markov chain.

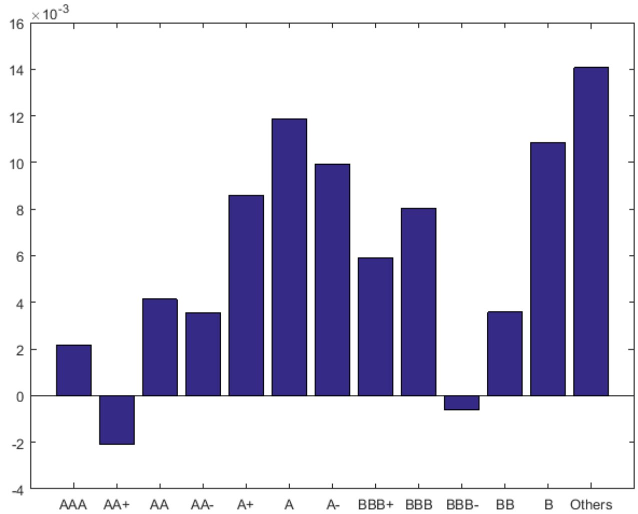


FIGURE 5. Reduced forecast errors for different credit ratings in 2018.

Figure 5 illustrates the average reduced forecast errors for different credit ratings. Note that we determine the credit rating of a country for 2018 based on its median. Given that the positive value indicates the improvement by RSMC, 11 out of 13 credit ratings show the lower forecast errors in RSMC. Exceptions are AA+(Hong Kong) and BBB−(India, Indonesia, Columbia, Hungary, and Portugal), caused by the poor performances of six countries as in Table 5. Note that AA+ and BBB− include 4 and 6 countries, respectively, in 2018. Besides two exceptions, the lower the credit rating, the more improvement in the forecasting by RSMC. Therefore, we expect that RSMC is applicable in managing the credit products regarding countries with higher risk of credit rating migration.

V. CONCLUSION

Throughout this paper, we propose the RSMC, a discrete time regime switching Markov model where the underlying regime changes in the way of HMM, for modeling the sovereign credit rating migration. We carefully define and evaluate the mathematical models of RSMC, a version of the EM algorithm for parameter estimation, and the fitting performance using an LR test. Then, we perform the estimation of credit rating migration in S&P sovereign credit ratings from January 1994 to December 2017 and its utilization for forecasting in 2018. Based on the comparison between the performances of a single regime (homogeneous) Markov chain and RSMC, we discover the following results.

For estimation, we detect that the sovereign credit rating migration can be classified into two regimes: economic expansion (state 1) and contraction (state 2). Note that the relatively high chance in state 1 and almost zero chance in state 2 for the upgrade of credit rating ensure the credibility of two regimes. Also, the result of the LR test provides evidence of improved modeling for the dynamics of sovereign credit rating migration via RSMC. Given that the probability of regime switching from economic expansion to contraction is 0.31%, whereas that from economic contraction to expansion is 4.00%, it seems that the economic condition tends to stay in the same state where the contraction due to the sovereign credit risk is a rare event in the economic cycle. Secondly, we observe that the top tier credit ratings (AAA, AA+, AA, AA−) in the economic expansion are most unlikely to be changed, whereas the second tier (A+, A, A−) in the economic contraction are most likely to be downgraded. In the meantime, the downgrade transition probability in non-investment credit rating group (BB, B, C) is highly susceptible to the change in the economic condition where the probabilities of downgrade and default ratings tend to increase in the economic contraction. These findings agree with the business cycle effect reported in empirical literature [9], [38], [39] on credit migration. Lastly, we discover the association between the events of sovereign credit risks and the periods of economic contraction detected by RSMC. Based on the heatmaps of state filter and smoother, we detect the continental clustering for the Asian currency crisis and European sovereign debt crisis.

For forecasting, we confirm that the overall forecast error of RSMC is lower than that of a single regime Markov chain. Specifically, RSMC reduces the forecast error by 17.2% where 35 out of 41 countries exhibit the improved forecast errors. Note that the improvement is also detected for all continents and forecasting months. Interestingly, the forecasting for different credit ratings exhibits more improvements in lower credit ratings where the improvements in forecast errors in the second tier and non-investment credit ratings are relatively higher than those of others.

In best of our knowledge, this study is the first attempt to apply a discrete time Markov chain that consolidates the HMM whose observed process in a given regime inherits a Markov property for the estimation and forecasting of sovereign credit rating migration. The economic contribution of our model lies in its estimation and forecasting ability to the second tier and non-investment credit ratings whose investment should be reconsidered and rebalanced when the sovereign credit risk arises. Note that those credit ratings are sensitive to the changes in economic condition. Once the forecasting is executed, the investment decision for credit portfolio becomes possible. Hence, we believe that our model is suitable to be implemented to manage the risk exposures of sovereign credit risk and related financial products in both practical and governing purposes.

Despite its discoveries, some limitations should be addressed in future studies. In our research, we only consider two states, economic contraction, and expansion regimes, as hidden states. When our model is applied to the rating migration analysis for various bonds, more than two of multiple hidden states may be needed for the sophisticated modeling to reflect the unobserved condition. Also, we assume that the latent variable for time is not related to potential exogenous variables. Thus, the study on the adequate structure of hidden states and utilization of macro-economic variables are left as the areas of further extension of our research.

APPENDIX A
PROOFS

Proof of (4): Using (1) and (2),

$$\begin{aligned}
f_j(\mathcal{Y}_k) &= P(\mathcal{Y}_k, X_k = j) \\
&= \sum_{i\in S_X} P(\mathcal{Y}_{k-1}, Y_k, X_{k-1} = i, X_k = j) \\
&= \sum_{i\in S_X} P(Y_k|\mathcal{Y}_{k-1}, X_{k-1} = i, X_k = j) \\
&\quad \times P(X_k = j|\mathcal{Y}_{k-1}, X_{k-1} = i) P(\mathcal{Y}_{k-1}, X_{k-1} = i) \\
&= \sum_{i\in S_X} P(Y_k|Y_{k-1}, X_{k-1} = i) \\
&\quad \times P(X_k = j|X_{k-1} = i) P(\mathcal{Y}_{k-1}, X_{k-1} = i) \\
&= \sum_{i\in S_X} e_i(Y_{k-1}, Y_k) a_{ij} f_i(\mathcal{Y}_{k-1})
\end{aligned}$$

□

Proof of (5): From the dependency structure of our model in Section 2, $P(\mathcal{Y}^{k+1}|Y_{k-1}, Y_k, X_{k-1} = j, X_k = i) = P(\mathcal{Y}^{k+1}|Y_k, X_k = i) = b_i(\mathcal{Y}^k)$.

Using (1) and (2),

$$\begin{aligned}
b_i(\mathcal{Y}^k) &= P(\mathcal{Y}^{k+1}|Y_k, X_k = i) \\
&= \sum_{j\in S_X} P(Y_{k+1}, \mathcal{Y}^{k+2}, X_{k+1} = j|Y_k, X_k = i) \\
&= \sum_{j\in S_X} P(\mathcal{Y}^{k+2}|Y_{k+1}, Y_k, X_{k+1} = j, X_k = i) \\
&\quad \times P(Y_{k+1}, X_{k+1} = j|Y_k, X_k = i) \\
&= \sum_{j\in S_X} b_j(\mathcal{Y}^{k+1}) P(X_{k+1} = j|Y_k, Y_{k+1}, X_k = i) \\
&\quad \times P(Y_{k+1}|Y_k, X_k = i) \\
&= e_i(Y_k, Y_{k+1}) \sum_{j\in S_X} a_{ij} b_j(\mathcal{Y}^{k+1}).
\end{aligned}$$

□

Proof of (8): The posterior probability of $X_k = i$ given $\mathcal{Y}_L$ is computed using the definitions of $f_i(\mathcal{Y}_k)$ and $b_i(\mathcal{Y}^k)$

$$\begin{aligned}
P(X_k = i|\mathcal{Y}_L) &= P(\mathcal{Y}_k, \mathcal{Y}^{k+1}, X_k = i)/P(\mathcal{Y}_L) \\
&= P(\mathcal{Y}^{k+1}|\mathcal{Y}_k, X_k = i) P(\mathcal{Y}_k, X_k = i)/P(\mathcal{Y}_L) \\
&= P(\mathcal{Y}^{k+1}|Y_k, X_k = i) P(\mathcal{Y}_k, X_k = i)/P(\mathcal{Y}_L) \\
&= b_i(\mathcal{Y}^k) f_i(\mathcal{Y}_k)/P(\mathcal{Y}_L)
\end{aligned}$$

□

Proof of (9): By using the dependency structure in Section 2 and the definitions of a_{ij}, $e_i(Y_k, Y_{k+1})$, and $b_i(\mathcal{Y}^k)$, we can show that

$$\begin{aligned}
\hat{n}_{ij} &= E\left[n_{ij}|\mathcal{Y}_L\right] \\
&= E\left[\sum_{k=0}^{L-1} \mathbf{1}_{(i,j)}\left((X_k, X_{k+1})\right) \big| \mathcal{Y}_L\right]
\end{aligned}$$

$$= \sum_{k=0}^{L-1} E\left[\mathbf{1}_{(i,j)}\left((X_k, X_{k+1})\right)|\mathcal{Y}_L\right]$$
$$= \sum_{k=0}^{L-1} P(X_k = i, X_{k+1} = j|\mathcal{Y}_L)$$
$$= \sum_{k=0}^{L-1} \frac{f_i(\mathcal{Y}_k)e_i(Y_k, Y_{k+1})a_{ij}b_j(\mathcal{Y}^{k+1})}{P(\mathcal{Y}_L)}$$

where the last equality is shown from the following equation. For $k = 0, \ldots, L-1$,

$$\begin{aligned} &P(X_k = i, X_{k+1} = j|\mathcal{Y}_L) \\ &= \frac{P(X_k = i, X_{k+1} = j, \mathcal{Y}_k, \mathcal{Y}^{k+1})}{P(\mathcal{Y}_L)} \\ &= \frac{P(X_k = i, \mathcal{Y}_k)P(X_{k+1} = j, \mathcal{Y}^{k+1}|X_k = i, \mathcal{Y}_k)}{P(\mathcal{Y}_L)} \\ &= \frac{f_i(\mathcal{Y}_k)P(X_{k+1} = j, \mathcal{Y}^{k+1}|X_k = i, Y_k)}{P(\mathcal{Y}_L)} \\ &= \frac{f_i(\mathcal{Y}_k)e_i(Y_k, Y_{k+1})a_{ij}b_j(\mathcal{Y}^{k+1})}{P(\mathcal{Y}_L)} \end{aligned}$$

where the fourth equality follows from the following equation. For $k = 0, \ldots, L-1$

$$\begin{aligned} &P(X_{k+1} = j, \mathcal{Y}^{k+1}|X_k = i, Y_k) \\ &= P(\mathcal{Y}^{k+2}, Y_{k+1}|X_{k+1} = j, X_k = i, Y_k) \\ &\quad \times P(X_{k+1} = j|X_k = i, Y_k) \\ &= P(\mathcal{Y}^{k+2}, Y_{k+1}|X_{k+1} = j, X_k = i, Y_k) \\ &\quad \times P(X_{k+1} = j|X_k = i) \\ &= P(\mathcal{Y}^{k+2}|Y_{k+1}, X_{k+1} = j, X_k = i, Y_k) \\ &\quad \times P(Y_{k+1}|X_{k+1} = j, X_k = i, Y_k)a_{ij} \\ &= P(\mathcal{Y}^{k+2}|Y_{k+1}, X_{k+1} = j)P(Y_{k+1}|X_k = i, Y_k)a_{ij} \\ &= b_j(\mathcal{Y}^{k+1})e_i(Y_k, Y_{k+1})a_{ij} \end{aligned}$$

□

Proof of (10)*:* The conditional expectation of $n_i(r, s)$ given $\mathcal{Y}_L$ is evaluated by

$$\begin{aligned} \hat{n}_i(r, s) &:= E\left[\sum_{k=0}^{L-1} \mathbf{1}_{(r,s)}\left((Y_k, Y_{k+1})\right) \cdot \mathbf{1}_i(X_k)\Big|\mathcal{Y}_L\right] \\ &= \sum_{k=0}^{L-1} E\left[\mathbf{1}_{(r,s)}\left((Y_k, Y_{k+1})\right) \cdot \mathbf{1}_i(X_k)|\mathcal{Y}_L\right] \\ &= \sum_{k \in I_Y(r,s)} E[\mathbf{1}_i(X_k)|\mathcal{Y}_L] \\ &= \sum_{k \in I_Y(r,s)} P(X_k = i|\mathcal{Y}_L) \\ &= \sum_{k \in I_Y(r,s)} \frac{f_i(\mathcal{Y}_k)b_i(\mathcal{Y}^k)}{P(\mathcal{Y}_L)}. \end{aligned}$$

where $I_Y(r, s) := \{k : (Y_k, Y_{k+1}) = (r, s), k = 0, \ldots, L-1\}$ and the last equality follows from (8). □

APPENDIX B
INITIAL PARAMETERS FOR EM ALGORITHM

We set $\boldsymbol{\theta}^{(0)} = \{a_{ij}^{(0)}, e_i^{(0)}(r, s), a_0^{(0)}(i), e_0^{(0)}(r)\ i, j \in S_X,\ r, s \in S_Y\}$ as follows. First, we make $a_{ij}^{(0)} = (1-\epsilon)\mathbf{1}_{\{i=j\}} + \epsilon\mathbf{1}_{\{i\neq j\}}$ and use a very small positive value for ϵ ($\epsilon \approx 0$). The reason why we set $a_{ii}^{(0)} \approx 1$ and $a_{ij}^{(0)} \approx 0$ for $i \neq j$ is due to the null hypothesis of our model selection criterion (LR test) claiming that the fitting performance of the single regime (homogeneous Markov chain) is better than that of multiple regimes (regime switching Markov chain). Second, we set all state transition probabilities, $e_i^{(0)}(r, s)$, to be identical as $e_i^{(0)}(r, s) = 1/|S_Y|$ for all $r, s \in S_Y$. Note that $|S_Y|$ is the number of elements in S_Y. Third, we let $a_0(1) = 1$ and $a_0(i) = 0$ for $i \neq 1$ as we mentioned in section 2. Lastly, the value of $e_0^{(0)}(r)$ is set to be the frequency of initial state r in our observed data sets.

APPENDIX C
MODEL SELECTION CRITERION BETWEEN MARKOV CHAIN AND RSMC MODEL

When the number of regime $N = 1$, the proposed RSMC model embraces homogeneous Markov chain (benchmark Markov chain). To compare fitting performance of multiple regimes, we construct a likelihood ratio (LR) test where the null hypothesis is that the observation process evolves according to a homogeneous Markov chain (MC) and the alternative that the process evolves according to RSMC model. Using the log likelihood function $\ln \mathcal{L}(\boldsymbol{\theta})$ in Section 3.B, we define the log-likelihood ratio

$$\Lambda := \ln\left(\frac{\mathcal{L}(\boldsymbol{\theta}_{MC})}{\mathcal{L}(\boldsymbol{\theta}_{RSMC})}\right) = \ln \mathcal{L}(\boldsymbol{\theta}_{MC}) - \ln \mathcal{L}(\boldsymbol{\theta}_{RSMC})$$

where $\boldsymbol{\theta}_{MC}$ and $\boldsymbol{\theta}_{RSMC}$ are parameter sets for MC and RSMC models respectively, and the parameter set is defined in equation (3).

Standard theories indicate that the asymptotic distribution of -2Λ is chi-squared,

$$-2\Lambda \overset{a}{\sim} \chi^2(\mathrm{df}(\boldsymbol{\theta}_{RSMC}) - \mathrm{df}(\boldsymbol{\theta}_{MC})) \tag{15}$$

where $\mathrm{df}(\boldsymbol{\theta})$ is the degrees of freedom of a model with parameter set $\boldsymbol{\theta}$.

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