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Negation of Basic Probability Assignment: Trends of Dissimilarity and Dispersion

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ABSTRACT In the field of knowledge representation, negation has been introduced so that practical issues can be modelled more effectively. The negation of probability was first formally determined by Zadeh, with its basic properties proposed by Yager. Recent studies have extended the negation of probability to that of basic probability assignment (BPA) by introducing Dempster-Shafer theory which is believed to perform well in dealing with uncertainty problems. Besides, the negation model has been proved to have the maximum entropy allocation, which attracts studies on uncertainty measures that can be applied in the negation process. In this paper, we have mainly investigated the trend of dissimilarity between two BPAs in the negation process. In particular, an evidence distance proposed by Jousselme et al. is used to serve as a dissimilarity measure to help quantify the variation trends. Moreover, standard deviation is used in this study to represent the dispersion in a BPA. Through our analysis, we obtained some interesting properties finally with their generalizations discussed in a proposed framework of negation methods.

INDEX TERMS Dempster-Shafer Theory, negation, evidence distance, belief function, knowledge representation.

I. INTRODUCTION

Knowledge representation has always been an attractive topic in history, and now it has become a crucial issue since the emergence of artificial intelligence. A considerable number of approaches have been proposed and applied to address issues of representing knowledge in multiple sources of information, such as belief function theory [1]–[3], D number theory [4]–[7], Z number theory [8], [9], soft set theory [10]–[12] and grey prediction model [13].

However, the inherent uncertainty attached to heterogenous sources of information increases the difficulty in handling with knowledge representation. Uncertainty has been taken into consideration in many fields like medical diagnosis [14], pattern classification [15]–[17], and management science [18]–[21]. With people's attention raised, many approaches have been subsequently developed specially for the inevitable uncertainty in the real world, such as belief entropy [22]–[24], evidential reasoning [25]–[28], intuitionistic fuzzy set theory [29]–[31], and ordered weighted aggregation (OWA) theory [32], [33].

Dempster-Shafer theory [34], [35], as an important and widely used reasoning method, assigns probabilities to the power set of events and performs well in dealing with epistemic uncertainty [36]–[38]. Dempster-Shafer theory has been applied in many fields, such as decision making [39]–[41], fault diagnosis [42], [43], and pattern classification [44].

Besides, how to represent the knowledge effectively is still an open issue [45]–[48]. In some circumstances, to answer *What it is* directly is likely more difficult than to say *What it is not*. Sometimes it is difficult to prove whether a theorem is correct or not, however, a state can be proved wrong easily by a counterexample. The concept of negation of events, first formally proposed by Zadeh, provides people with a new perspective on probability theory. The properties of negation of a probability distribution recently proposed by Yager are receiving increasing attention, with the final state proved to have the maximum entropy allocation [49].

Owing to its advantages, Dempster-Shafer theory has been introduced to generalize the theory of negation

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of probability [50]. Negation has thus been studied on power sets rather than sets with singleton elements only. Since the negation is proved to increase the entropy of the system, several uncertainty measures have been introduced to serve as math tools to quantify the effects caused by negation [49]–[52].

The uncertainty measures originating from quantifying the entropy provide different velocities of increase in entropy for different sensitivities [53]–[56]. However, what interests us is the trend of the dissimilarity between a BPA and its negation in the negation process. An evidence distance proposed by Jousselme et al. is an effective tool to detect similar sets and quantify the dissimilarity between two sets [57]–[59]. With the increase in dissimilarity between two sets, the corresponding evidence distance will also increase accordingly. Thus we believe that evidence distance will help quantify the trend of dissimilarity in the negation process. Also, we try to capture some relations between the dissimilarity of two BPAs and the disorder of a single BPA in the process, so the standard deviation has been introduced to represent the dispersion in a BPA.

The paper is structured as follows: in Section 2, some basic definitions associated with Dempster-Shafer theory, evidence distance, Yager's negation method and BPA's negation are presented. Section 3 details some new properties of negation viewed from the trend of dissimilarity between the current BPA and its negation, as well as the trend of dispersion in the current BPA, with a discussion about a general framework of negation methods. In Section 4, some numerical examples are provided to illustrate changes after negation and finally in section 5, we have a brief summary.

II. PRELIMINARIES

A. DEMPSTER-SHAFER THEORY

Uncertainty is inevitable in real applications [60]–[62]. Dempster-Shafer theory, serving as an efficient math tool to deal with uncertain information, was first proposed by Dempster and later developed by his student Shafer. Compared with Bayesian theory of probability, Dempster-Shafer theory satisfies weaker conditions and has a stronger ability to model uncertain knowledge [63]. Dempster-Shafer theory has played a significant role in many applications, such as decision making [64]–[66], fault diagnosis [67]–[69], target recognition and data fusion [70]–[72].

Let Θ be a set with mutually exclusive and exhaustive hypotheses. It is called the frame of discernment, namely,

$$\Theta = \{H_1, H_2, \cdots, H_n\} \tag{1}$$

The power set of Θ is denoted by 2^Θ, which contains all subsets of Θ, namely

$$2^\Theta = \{\phi, \{H_1\}, \{H_2\}, \cdots, \{H_1, H_2\}, \cdots, \Theta\} \tag{2}$$

The basic probability assignment (BPA), also called the mass function, is a mapping from 2^N to [0, 1], defined as

$$m : 2^N \to [0, 1] \tag{3}$$

which satisfies:

$$m(\phi) = 0$$
$$\sum_{A \subseteq \Theta} m(A) = 1$$

$m(A)$ measures the belief assigned to subset A and represents the strength to support A. A is called a focal element if $m(A) > 0$.

The belief function theory has been studied and applied in many fields [73]–[75]. While based on BPA, the plausibility function Pl and the belief function Bel are defined as follows:

$$Pl(A) = \sum_{B \cap A \neq \phi} m(B) \tag{4}$$

$$Bel(A) = \sum_{B \subseteq A} m(B) \tag{5}$$

The $Pl(A)$ represents the potential support to A while the $Bel(A)$ represents the justified total belief to A. $Bel(A)$and $Pl(A)$ serve as the lower and upper bounds of a limit interval, namely $[Bel(A), Pl(A)]$. The length of the interval $[Bel(A), Pl(A)]$ measures the degree of imprecision of A.

B. EVIDENCE DISTANCE

An evidence distance was proposed by Anne-Laure Jousselme et al. aiming at quantifying the dissimilarity between two sets and particularly the conflict between two BPAs in Dempster-Shafer theory [57]–[59]. The associated definitions are as follows:

Let m_1 and m_2 be two BPAs on the same frame of discernment Θ, which contains N mutually exclusive and exhaustive hypotheses. The evidence distance between BPA m_1 and m_2 is defined as

$$d_{\mathrm{BPA}}(m_a, m_b) = \sqrt{\frac{1}{2}(\vec{m}_a - \vec{m}_b)^{\mathrm{T}} \underline{\underline{D}}(\vec{m}_a - \vec{m}_b)} \tag{6}$$

where $\vec{m}_1$ and $\vec{m}_2$ are BPAs defined above. Notice that BPAs here are written as vectors, i.e.,

$$\vec{m} = (m_1, m_2, \cdots, m_{2^N})^{\mathrm{T}}$$

The vector has exactly 2^N elements and $m_i = 0$ if the corresponding element A_i is a non-focal element.

$\underline{\underline{D}}$ is a $2^N \times 2^N$ matrix whose elements are

$$\underline{\underline{D}}(i, j) = \frac{|A_i \cap A_j|}{|A_i \cup A_j|}$$
$$A_i, A_j \in 2^\Theta$$

$\underline{\underline{D}}$ also called Jaccard index, is used for gauging the similarity and diversity of sample sets. Jousselme et al. make full use of Jaccard index here to serve as a similarity measure between sets. Obviously, the evidence distance will be exactly 0 if the two sets bear enough similarities. To quantify some trends in the negation process, evidence distance is introduced as a dissimilarity measure used to describe the degree of difference between two sets.

C. YAGER'S NEGATION

Assume a frame of reference

$$X = \{x_1, x_2, \cdots, x_n\}$$

Let

$$P = \{p_1, p_2, \cdots, p_n\}$$

be a probability distribution on X with

$$\sum_{i=1}^{n} p_i = 1$$
$$0 \leq p_i \leq 1$$

The negation of the probability distribution is defined as follows [49]:

$$\bar{P} = \{\bar{p}_1, \bar{p}_2, \cdots, \bar{p}_n\} \tag{7}$$

with

$$\sum_{i=1}^{n} \bar{p}_i = 1$$
$$0 \leq \bar{p}_i \leq 1$$

where

$$\bar{p}_i = \frac{1 - p_i}{\sum_{i=1}^{n}(1 - p_i)} = \frac{1 - p_i}{n - 1} \tag{8}$$

and n is the number of elements in X. The term $\sum_{i=1}^{n}(1 - p_i)$ is used to normalize the complementary probability (i.e., $1 - p_i$) in order to keep the basic assumptions of a probability distribution that the sum of the probabilities equals 1.

Viewed from Dempster-Shafer theory, for an single event A_i with probability p_i in a set containing only singleton focal elements as $X = \{A_1, A_2, \cdots, A_n\}$, the complement $\neg A_i$ is supposed to take probability $1 - p_i$ regardless of normalization. $\neg A_i$ is the whole event space less A_i, i.e., $\neg A_i = X - A_i$.

Yager has provided an interesting and intuitive deduction [49]. Let m be a BPA which contains n singleton focal elements only, i.e., $\{A_1, A_2, \cdots, A_n\}$ with probabilities $p_1, p_2, \cdots, p_n$. Assume that the negation yields a new BPA m^* which contains focal elements as $\{\neg A_1, \neg A_2, \cdots, \neg A_n\}$ with probabilities $p_1, p_2, \cdots, p_n$, i.e., $m^*(\neg A_i) = m(A_i)$. Let $\bar{m}$ denote the negation of m. The Bel and Pl for $\bar{m}$ have been derived by Yager with reference to m^*:

$$Bel(\bar{m}(A_i)) = \sum_{\substack{j=1 \\ \neg A_j \in \{A_i\}}}^{n} m^*(\neg A_j) = 0 \tag{9}$$

$$Pl(\bar{m}(A_i)) = \sum_{\substack{j=1 \\ A_i \in \neg A_j}}^{n} m^*(\neg A_j) = 1 - p_i \tag{10}$$

Then we have $0 \leq \bar{m}(A_i) \leq 1 - m(A_i)$.

Intuitively the probability of a focal element is assigned to its complement. For instance, the probability p_1 will be assigned to the set $\neg A_1$, i.e., $\{A_2, A_3, \cdots, A_n\}$. In essence, we are clear about the probabilities assigned to the complements after negation, but what can be provided here is just a probability interval for each original singleton element. Thus Yager concluded that the negation probabilities (i.e., $\bar{m}(A_i)$) are not uniquely determined, which indicates that different negation methods yield different negation results. However, the negation probabilities are constrained in some interval (i.e., $[0, 1 - m(A_i)]$).

D. NEGATION OF BPA

The negation of BPA has been derived earlier with Dempster-Shafer theory taken into consideration [50].

Let m_0 be the initial probability assigned to a focal element in a power set while m_i denotes the relative probability after ith negation. The general form of negation of a BPA is presented as

$$m_i = \frac{1 - m_{i-1}}{n - 1}$$

where n denotes the number of the focal element here instead of the number of elements in Yager's definition. From the formula above, $\frac{1}{n}$ can be easily identified as a fixed point. A little change to the form yields an obvious geometric progression relationship:

$$\begin{aligned} m_i - \frac{1}{n} &= \frac{1 - m_{i-1}}{n - 1} - \frac{1}{n} \\ &= \frac{1}{1 - n}(m_{i-1} - \frac{1}{n}) \end{aligned} \tag{11}$$

The general formula of the probability assigned to an element after ith negation can be derived according to the common ratio $\frac{1}{1-n}$ derived above:

$$\begin{aligned} m_i &= (\frac{1}{1 - n})^i (m_0 - \frac{1}{n}) + \frac{1}{n} \\ &= \frac{nm_0 - 1}{n(1 - n)^i} + \frac{1}{n} \end{aligned} \tag{12}$$

The convergence can be obtained by taking the limitation:

$$\lim_{i \to \infty} m_i = \lim_{i \to \infty} \frac{nm_0 - 1}{n(1 - n)^i} + \frac{1}{n} \tag{13}$$

We have

$$\lim_{i \to \infty} m_i = \frac{1}{n} \tag{14}$$

while $|n - 1| > 1$. The condition is held in the following context that n is more than 2 without specifying that again. The situation where $|n - 1| \leq 1$ will be discussed as a special case later.

III. TRENDS OF DISSIMILARITY AND DISPERSION IN BPA'S NEGATION

A. PROPERTIES OF BPA'S NEGATION

Several uncertainty measures have been applied to prove that negation will increase the entropy of the system [50], [76], [77]. A BPA will get to an evenly distributed state after negation finally, which raises our interest in exploring the regularity in the negation process, especially in the relation between a BPA and its negation. A dissimilarity measure is introduced here, namely Jousselme et al.'s evidence distance, to obtain some new properties of the dissimilarity of BPA in the negation process.

Definition 1: If a BPA m_i has been converted into its negation m_{i+1} according to the negation rule of BPA specified above, m_{i+1} will be called the current BPA, and m_{i+2} will be the current BPA if m_{i+1} has been converted into m_{i+2}.

Essentially we focus on the trend of dissimilarity between the current BPA and its negation measured by evidence distance and have derived the theorem below:

Theorem 1: Let m_i be the BPA derived from the initial BPA m_0 after ith negation. The evidence distance between the current BPA and its negation will decrease with a constant ratio in the iterative negation process. Namely

$$\frac{d_{\mathrm{BPA}}(m_i, m_{i+1})}{d_{\mathrm{BPA}}(m_{i+1}, m_{i+2})} = n - 1 \tag{15}$$

where n is the number of focal elements in the power set associated to the BPA. Taking iterative negation, the evidence distance converges to 0 finally.

Proof: Assume a set Θ with exclusive and mutually exhaustive hypotheses $\{x_1, x_2, \cdots, x_N\}$ and the corresponding power set 2^N : $\{A_1, A_2, \cdots, A_{2^N}\}$. As the evidence distance between BPA m_A and m_B with frame of discernment is defined as

$$d_{\mathrm{BPA}}(m_A, m_B) = \sqrt{\frac{1}{2}(\vec{m}_A - \vec{m}_B)^{\mathrm{T}} \underline{\underline{D}}(\vec{m}_A - \vec{m}_B)}$$

where

$$\underline{\underline{D}} = \begin{pmatrix} \frac{|A_1 \cap A_1|}{|A_1 \cup A_1|} & \frac{|A_1 \cap A_2|}{|A_1 \cup A_2|} & \cdots & \frac{|A_1 \cap A_{2^N}|}{|A_1 \cup A_{2^N}|} \\ \frac{|A_2 \cap A_1|}{|A_2 \cup A_1|} & \frac{|A_2 \cap A_2|}{|A_2 \cup A_2|} & \cdots & \frac{|A_2 \cap A_{2^N}|}{|A_2 \cup A_{2^N}|} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{|A_{2^N} \cap A_1|}{|A_{2^N} \cup A_1|} & \frac{|A_{2^N} \cap A_2|}{|A_{2^N} \cup A_2|} & \cdots & \frac{|A_{2^N} \cap A_{2^N}|}{|A_{2^N} \cup A_{2^N}|} \end{pmatrix}$$

the formula can be expanded as follows:

$$d_{\mathrm{BPA}}(m_A, m_B) = \sqrt{\frac{1}{2}\sum_{p=1}^{2^N}\sum_{q=1}^{2^N}(m_A(A_p) - m_B(A_p))(m_A(A_q) - m_B(A_q))\frac{|A_p \cap A_q|}{|A_p \cup A_q|}} \tag{16}$$

where A_p and A_q are focal elements of 2^Θ.

Let m_0 be a BPA on the N-element set above. Then m_i denotes the current BPA derived from m_0 after ith negation and m_{i+1} denotes the negation of m_i, namely $m_{i+1} = \overline{m}_i$, while m_{i+2} denotes the negation of m_{i+1}, namely $m_{i+2} = \overline{m}_{i+1} = \overline{\overline{m}}_i$. Thus we have

$$m_{i+1}(A_p) = \frac{1 - m_i(A_p)}{n - 1} \tag{17}$$

and

$$\begin{aligned} m_{i+2}(A_p) &= \frac{1 - m_{i+1}(A_p)}{n - 1} \\ &= \frac{m_i(A_p) + n - 2}{(n-1)^2} \end{aligned} \tag{18}$$

For distance between m_i and m_{i+1},

$$\begin{aligned} m_i(A_p) - m_{i+1}(A_p) &= m_i(A_p) - \frac{1 - m_i(A_p)}{n - 1} \\ &= \frac{nm_i(A_p) - 1}{n - 1} \end{aligned} \tag{19}$$

Similarly,

$$m_i(A_q) - m_{i+1}(A_q) = \frac{nm_i(A_q) - 1}{n - 1} \tag{20}$$

Then we have

$$d_{\mathrm{BPA}}(m_i, m_{i+1}) = \sqrt{\frac{1}{2}\sum_{p=1}^{2^N}\sum_{q=1}^{2^N}\frac{(nm_i(A_p) - 1)(nm_i(A_q) - 1)}{(n-1)^2}\frac{|A_p \cap A_q|}{|A_p \cup A_q|}} \tag{21}$$

For distance between m_{i+1} and m_{i+2},

$$\begin{aligned} m_{i+1}(A_p) - m_{i+2}(A_p) &= m_{i+1}(A_p) - \frac{1 - m_{i+1}(A_p)}{n - 1} \\ &= \frac{1 - m_i(A_p)}{n - 1} - \frac{m_i(A_p) + n - 2}{(n-1)^2} \\ &= \frac{1 - nm_i(A_p)}{(n-1)^2} \end{aligned} \tag{22}$$

Similarly,

$$m_{i+1}(A_q) - m_{i+2}(A_q) = \frac{1 - nm_i(A_q)}{(n-1)^2} \tag{23}$$

Then we have

$$d_{\mathrm{BPA}}(m_{i+1}, m_{i+2}) = \sqrt{\frac{1}{2}\sum_{p=1}^{2^N}\sum_{q=1}^{2^N}\frac{(1 - nm_i(A_p))(1 - nm_i(A_q))}{(n-1)^4}\frac{|A_p \cap A_q|}{|A_p \cup A_q|}} \tag{24}$$

Obviously,

$$\frac{d_{\mathrm{BPA}}(m_i, m_{i+1})}{d_{\mathrm{BPA}}(m_{i+1}, m_{i+2})} = n - 1$$

Since $n-1$ is a common ratio, we have the evidence distance between m_i and m_{i+1} simplified as

$$Dis_i = \frac{Dis_0}{(n-1)^i} \tag{25}$$

where Dis_0 denotes the distance between m_0 and its negation m_1.

The convergence can be obtained by taking the limitation:

$$\lim_{i \to \infty} Dis_i = \lim_{i \to \infty} \frac{Dis_0}{(n-1)^i} \tag{26}$$

We have

$$\lim_{i\to\infty} Dis_i = 0 \tag{27}$$

□

Standard deviation is a widely used statistic which measures the dispersion of data. We have standard deviation to quantify the dispersion in the current BPA in the negation process. Based on the introduction of standard deviation, we have the theorem below.

Theorem II: Assume a set Θ with exclusive and mutually exhaustive hypotheses $\{x_1, x_2, \cdots, x_N\}$ and the corresponding power set $2^N : \{A_1, A_2, \cdots, A_{2^N}\}$.

The standard deviation of a BPA m is defined as follows:

$$Sd(m) = \sqrt{\frac{\sum_{p=1}^{n}(m(A_p) - \frac{1}{n})^2}{n}} \tag{28}$$

where n is the number of the focal elements in the BPA.

Let m_i be the BPA derived from m_0 after ith negation. The standard deviation of m_i is $(n-1)$ times that of its negation m_{i+1}, namely

$$\frac{Sd(m_i)}{Sd(m_{i+1})} = n - 1 \tag{29}$$

The standard deviation of the current BPA converges to 0 finally with iterative negation.

Proof: Since the standard deviation is defined as

$$Sd(m) = \sqrt{\frac{\sum_{p=1}^{n}(m(A_p) - \frac{1}{n})^2}{n}}$$

the formula can be expanded as

$$\begin{aligned} Sd(m_i) &= \sqrt{\frac{\sum_{p=1}^{n}(m_i(A_p) - \frac{1}{n})^2}{n}} \\ &= \sqrt{\frac{\sum_{p=1}^{n} m_i(A_p)^2 - \frac{2\times\sum_{p=1}^{n} m_i(A_p)}{n} + n \times \frac{1}{n^2}}{n}} \\ &= \sqrt{\frac{\sum_{p=1}^{n} m_i(A_p)^2 - \frac{1}{n}}{n}} \end{aligned} \tag{30}$$

Recall that the negation of m_i for each focal element is

$$m_{i+1}(A_p) = \frac{1 - m_i(A_p)}{n-1}$$

Similarly, we have

$$\begin{aligned} Sd(m_{i+1}) &= \sqrt{\frac{\sum_{p=1}^{n}(\frac{1-m_i(A_p)}{n-1} - \frac{1}{n})^2}{n}} \\ &= \sqrt{\frac{\sum_{p=1}^{n}(\frac{n-nm_i(A_p)-n+1}{n(n-1)})^2}{n}} \\ &= \sqrt{\frac{\frac{n^2\sum_{p=1}^{n} m_i(A_p)^2 - n}{n^2(n-1)^2}}{n}} \\ &= \sqrt{\frac{\sum_{p=1}^{n} m_i(A_p)^2 - \frac{1}{n}}{n(n-1)^2}} \end{aligned} \tag{31}$$

Obviously,

$$\frac{Sd(m_i)}{Sd(m_{i+1})} = n - 1$$

Thus the standard deviation of the current BPA can be simplified as follows:

$$Sd(m_i) = \frac{Sd_0}{(n-1)^i}$$

where Sd_0 denotes the standard deviation of the original BPA m_0.

The convergence can be obtained by taking the limitation:

$$\lim_{i\to\infty} Sd_i = \lim_{i\to\infty} \frac{Sd_0}{(n-1)^i} \tag{32}$$

We have

$$\lim_{i\to\infty} Sd_i = 0 \tag{33}$$

□

B. SPECIAL CASES

1) Assume that a BPA m_0 contains only one focal element (e.g., $m_0(A) = 1$ where A is the only focal element). Let m_i be the BPA derived from m_0 after ith negation. Then we have

$$\begin{cases} m_i(A) = 0, & m_i(\phi) = 1, \quad \text{if } i \text{ is odd} \\ m_i(A) = 1, & m_i(\phi) = 0, \quad \text{if } i \text{ is even} \end{cases} \tag{34}$$

Notice that ϕ is not a focal element, and we usually do not assign probabilities to such a non-focal element in the negation process. In this case, however, it is suggested that the empty set ϕ should also act as a focal element due to the lack of complete knowledge in the system. ϕ is often used to model the open-world system [78]–[80] and it represents a collection of other unknown focal elements.

In this case, the evidence distance between the current BPA and its negation will not change nor does the standard deviation of the current BPA.

2) Assume that a BPA m_0 contains two focal element, i.e., $m_0(A) = p$ and $m_0(B) = 1 - p$ where A and B are the only focal elements. Let m_i be the BPA derived from m_0 after ith negation. Then we have

$$\begin{cases} m_i(A) = 1 - p, & m_i(B) = p, \quad \text{if } i \text{ is odd} \\ m_i(A) = p, & m_i(B) = 1 - p, \quad \text{if } i \text{ is even} \end{cases} \tag{35}$$

Actually this case is less special, for it just suggests that the common ratio $(n-1)$ in theorem I and II becomes 1.

Also, the evidence distance between the current BPA and its negation will not change in this case nor does the standard deviation in the current BPA.

3) Assume a BPA m_0 with probabilities equally assigned, i.e., $m_0(A_p) = \frac{1}{n}$ where n is the number of the focal elements. Let m_i be the BPA derived from m_0 after ith negation. Then we have

$$m_i(A_p) = \frac{1}{n}$$

The evidence distance between the current BPA and its negation keeps 0 so does the standard deviation as the BPA has already reached its final state.

C. EXTENSION AND DISCUSSION

From the perspective of the dissimilarity between two sets, it can be concluded that BPA will be more and more similar to its negation in the negation process. Meanwhile, with the increasing similarity between the current BPA and its negation, the system suffers an information loss due to the irreversibility of negation [50]. Besides, the dispersion in the current BPA decreases with the increasing frequency of negation. Those trends end in a state where the probabilities are equally assigned.

Notice that the evidence distance between a BPA m and its negation $\bar{m}$ shares the same discount ratio with the standard deviation of m(i.e., $n-1$). We are interested in what $n-1$ is special about. What will happen if we take other negation methods? Some studies have already shown that other negation methods also apply well in some circumstances [51].

Motivated by questions raised above, we discuss a general framework of negation here. Assume a BPA m_0 : $\{m_0(A_1), m_0(A_2), \cdots, m_0(A_{2^N})\}$ on the frame of discernment $\{x_1, x_2, \cdots, x_N\}$. $m_0(A_p) = 0$ if the corresponding element A_p is a non-focal element. The constraint is released here that non-focal elements cannot take values in the whole negation process. Let m_i be the BPA derived from m_0 after ith negation. Then the negation formula will be

$$m_{i+1}(A_p) = \frac{1-m_i(A_p)}{2^N-1} \tag{36}$$

Particularly those non-focal elements will be assigned with

$$m_{i+1}(A_p) = \frac{1-0}{2^N-1} = \frac{1}{2^N-1} \tag{37}$$

Instead, if we take the negation method in [51], the negation formula will be

$$m_{i+1}(A_p) = \frac{1-m_i(A_p)}{2^N-2} \tag{38}$$

This method declares that the probability assigned to the empty set ϕ remains 0 in the whole negation process. In this case, all elements participate in the negation less ϕ.

No matter what exactly the normalization term is ($n-1$, 2^N-1, 2^N-2 or other terms), we can denote this term with an integer constant C, i.e.,

$$m_{i+1}(A_p) = \frac{1-m_i(A_p)}{C} \tag{39}$$

with $C+1$ denoting the number of elements participating in the negation process. It must be specified that we have $C \in [n-1, 2^N-1]$ where n is the number of focal elements. Thus the number of elements participating in the negation process ranges from n to 2^N, i.e., $C+1 \in [n, 2^N]$. The reason why we have such a constraint will be specified later.

Keeping the assumptions and notations in Theorem I and II, we have

$$d_{\mathrm{BPA}}(m_i, m_{i+1}) = \sqrt{\frac{1}{2}\sum_{p=1}^{2^N}\sum_{q=1}^{2^N} \frac{((C+1)m_i(A_p)-1)((C+1)m_i(A_q)-1)}{C^2} \frac{|A_p \cap A_q|}{|A_p \cup A_q|}} \tag{40}$$

and

$$d_{\mathrm{BPA}}(m_{i+1}, m_{i+2}) = \sqrt{\frac{1}{2}\sum_{p=1}^{2^N}\sum_{q=1}^{2^N} \frac{(1-(C+1)m_i(A_p))(1-(C+1)m_i(A_q))}{C^4} \frac{|A_p \cap A_q|}{|A_p \cup A_q|}} \tag{41}$$

Thus

$$\frac{d_{\mathrm{BPA}}(m_i, m_{i+1})}{d_{\mathrm{BPA}}(m_{i+1}, m_{i+2})} = C \tag{42}$$

For the standard deviation, we have

$$Sd(m_i) = \sqrt{\frac{\sum_{p=1}^{C+1} m_i(A_p)^2 - \frac{1}{C+1}}{C+1}} \tag{43}$$

and

$$Sd(m_{i+1}) = \sqrt{\frac{\sum_{p=1}^{C+1} m_i(A_p)^2 - \frac{1}{C+1}}{C^2(C+1)}} \tag{44}$$

Thus

$$\frac{Sd(m_i)}{Sd(m_{i+1})} = C \tag{45}$$

Here comes the reason why C cannot be less than $n-1$ or more than 2^N-1. Recall that the original definition of the negation of probability is as follows [49]:

$$\bar{p}_i = \frac{1-p_i}{\sum_{i=1}^{n}(1-p_i)}$$

where n denotes the number of focal elements. However, the '1' in the formula is essentially $\sum_{i=1}^{n} p_i$, which indicates that the collection of focal elements is the minimum subset of elements participating in the negation process.

Thus the negation formula for BPA m is essentially as follows:

$$\begin{aligned} \bar{m}_a &= \frac{1-m_a}{\sum_{i=1}^{x}(1-m_i) + \sum_{j=1}^{y}(1-m_j)} \\ &= \frac{1-m_a}{x-1+y} \end{aligned} \tag{46}$$

where x and y denote the number of focal elements and that of selected non-focal elements respectively, with $m_i \neq 0$ and $m_j = 0$. y ranges from 0 to $2^N - x$ here, i.e., $y \in [0, 2^N - x]$. Besides, a can be a focal element, or a non-focal element from those selected for the negation.

It is interesting that this general framework verifies what Yager has derived. From Yager's deduction, we have $0 \leq m_{i+1}(A_p) \leq 1 - m_i(A_p)$ according to Bel and Pl in Dempster-Shafer theory. While based on our general framework, for a certain BPA we have

$$\frac{1-m_i(A_p)}{2^N-1} \leq \frac{1-m_i(A_p)}{C} \leq \frac{1-m_i(A_p)}{n-1}$$

Obviously,

$$\left[\frac{1-m_i(A_p)}{2^N-1}, \frac{1-m_i(A_p)}{n-1}\right] \subseteq [0, 1-m_i(A_p)]$$

TABLE 1. Indices' trends in the negation process.

$Freq.$	$m(a)$	$m(b)$	$m(c)$	$m(\{a,b\})$	$Dis.$	$Dev.$
0	0.2	0.3	0.4	0.1	0.2108	0.1118
1	0.2667	0.2333	0.2	0.3	0.0703	0.0373
2	0.2444	0.2556	0.2667	0.2333	0.0234	0.0124
3	0.2519	0.2481	0.2444	0.2556	0.0078	0.0041
4	0.2494	0.2506	0.2519	0.2481	0.0026	0.0014
5	0.2502	0.2498	0.2494	0.2506	0.0009	0.0005
6	0.2499	0.2501	0.2502	0.2498	0.0003	0.0002
7	0.2500	0.2500	0.2499	0.2501	0.0001	0.0001
8	0.2500	0.2500	0.2500	0.2500	0.0000	0.0000
9	0.2500	0.2500	0.2500	0.2500	0.0000	0.0000
10	0.2500	0.2500	0.2500	0.2500	0.0000	0.0000

It can be concluded that the general negation term $m_{i+1}(A_p)$ is constrained in $[0, 1-m_i(A_p)]$ while more precisely, a certain negation term $m_{i+1}(A_p)$ is constrained in

$$\left[\frac{1-m_i(A_p)}{2^N-1}, \frac{1-m_i(A_p)}{n-1}\right]$$

Up to now, a brief conclusion can be drawn that for the same set, negation methods differ in the trends of evidence distance and standard deviation due to different numbers of elements selected for the negation. This indicates that if we use different negation methods, the evidence distance as a similarity measure, will show different sensitivities as well as the standard deviation.

IV. NUMERICAL EXAMPLES

Assume a set with mutually exclusive and exhaustive hypotheses $\Theta : \{a, b, c\}$. A BPA m_0 with reference to 2^Θ contains focal elements $\{m_0(a), m_0(b), m_0(c), m_0(\{a, b\})\}$ and we have $m_0(a) = 0.2$, $m_0(b) = 0.3$, $m_0(c) = 0.4$, and $m_0(\{a, b\}) = 0.1$ respectively. m_i denotes the BPA derived from m_0 after ith negation

Freq. denotes the frequency of negation, i.e., $Freq. = i$ if the current BPA is m_i. $m(a)$, $m(b)$, $m(c)$, and $m(\{a, b\})$ are the probabilities assigned to the focal elements in the current BPA. *Dis.* denotes the evidence distance between the current BPA and its negation while *Dev.* denotes the standard deviation of the current BPA.

We use the negation method in [50] and have the results in Table 1. It can be seen clearly that the probabilities approximately got to their mean while the evidence distance and the standard deviation reached 0 finally.

V. CONCLUSION

In this paper we presented some new properties of the negation of BPA. Instead of entropy-based measures used in other studies [50], [77], we view the negation from a perspective of dissimilarity and dispersion, which has driven us to introduce evidence distance and standard deviation to quantify the variation trends in the negation process. Other than increases in entropy, the degree of similarity between a BPA and its negation is now also proved to increase. More specifically, the dissimilarity between a BPA and its negation decreases with a constant ratio, also shared by the decrease of the dispersion of the current BPA. Besides, we discussed a general framework of negation methods which proves to be consistent with Yager's theorem. However, how to make full use of different negation methods based on the framework becomes an open issue, which calls for further study as well as applications of the proposed properties.

REFERENCES

[1] R. R. Yager, "Entailment for measure based belief structures," *Inf. Fusion*, vol. 47, pp. 111–116, May 2019.

[2] W. Jiang, "A correlation coefficient for belief functions," *Int. J. Approx. Reasoning*, vol. 103, pp. 94–106, Dec. 2018.

[3] F. Xiao, "Multi-sensor data fusion based on the belief divergence measure of evidences and the belief entropy," *Inf. Fusion*, vol. 46, pp. 23–32, Mar. 2019.

[4] H. Seiti, A. Hafezalkotob, S. E. Najaf, and M. Khalaj, "Developing a novel risk-based MCDM approach based on D numbers and fuzzy information axiom and its applications in preventive maintenance planning," *Appl. Soft Comput.*, vol. 82, Sep. 2019, Art. no. 105559. doi: 10.1016/j.asoc.2019.105559.

[5] X. Deng and W. Jiang, "Evaluating green supply chain management practices under fuzzy environment: A novel method based on D number theory," *Int. J. Fuzzy Syst.*, vol. 21, no. 5, pp. 1389–1402, Jul. 2019. doi: 10.1007/s40815-019-00639-5.

[6] J. Zhao and Y. Deng, "Performer selection in human reliability analysis: D numbers approach," *Int. J. Comput. Commun. Control*, vol. 14, no. 4, pp. 521–536, 2019.

[7] F. Xiao, "A multiple-criteria decision-making method based on D numbers and belief entropy," *Int. J. Fuzzy Syst.*, vol. 21, no. 4, pp. 1144–1153, Jun. 2019.

[8] W. Jiang, Y. Cao, and X. Deng, "A novel Z-network model based on Bayesian network and Z-number," *IEEE Trans. Fuzzy Syst.*, to be published. doi: 10.1109/TFUZZ.2019.2918999.

[9] B. Kang, P. Zhang, Z. Gao, G. Chhipi-Shrestha, K. Hewage, and R. Sadiq, "Environmental assessment under uncertainty using Dempster–Shafer theory and Z-numbers," *J. Ambient Intell. Hum. Comput.*, Feb. 2019. doi: 10.1007/s12652-019-01228-y.

[10] F. Feng, J. Cho, W. Pedryczc, H. Fujita, and T. Herawan, "Soft set based association rule mining," *Knowl.-Based Syst.*, vol. 111, pp. 268–282, Nov. 2016.

[11] F. Xiao, "A hybrid fuzzy soft sets decision making method in medical diagnosis," *IEEE Access*, vol. 6, pp. 25300–25312, 2018.

[12] F. Feng and Y. Li, "Soft subsets and soft product operations," *Inf. Sci.*, vol. 232, pp. 44–57, May 2013.

[13] D. Zhou, A. Al-Durra, K. Zhang, A. Ravey, and F. Gao, "A robust prognostic indicator for renewable energy technologies: A novel error correction grey prediction model," *IEEE Trans. Ind. Electron.*, vol. 66, no. 12, pp. 9312–9325, Dec. 2019. doi: 10.1109/TIE.2019.2893867.

[14] Z. Cao, C.-H. Chuang, J.-K. King, and C.-T. Lin, "Multi-channel EEG recordings during a sustained-attention driving task," *Sci. Data*, vol. 6, Apr. 2019, Art. no. 19. doi: 10.1038/s41597-019-0027-4.

[15] Z. Liu, Q. Pan, J. Dezert, and A. Martin, "Combination of classifiers with optimal weight based on evidential reasoning," *IEEE Trans. Fuzzy Syst.*, vol. 26, no. 3, pp. 1217–1230, Jun. 2018.

[16] J. Geng, X. Ma, X. Zhou, and H. Wang, "Saliency-guided deep neural networks for SAR image change detection," *IEEE Trans. Geosci. Remote Sens.*, to be published.

[17] Z. Liu, Y. Liu, J. Dezert, and F. Cuzzolin, "Evidence combination based on credal belief redistribution for pattern classification," *IEEE Trans. Fuzzy Syst.*, to be published. doi: 10.1109/TFUZZ.2019.2911915.

[18] E. K. Zavadskas, Z. Turskis, T. Vilutienė, and N. Lepkova, ''Integrated group fuzzy multi-criteria model: Case of facilities management strategy selection,'' *Expert Syst. Appl.*, vol. 82, pp. 317–331, Oct. 2017.
[19] D. Zhou, A. Al-Durra, F. Gao, A. Ravey, I. Matraji, and M. G. Simões, ''Online energy management strategy of fuel cell hybrid electric vehicles based on data fusion approach,'' *J. Power Sources*, vol. 366, pp. 278–291, Oct. 2017.
[20] H. Seiti and A. Hafezalkotob, ''Developing the R-TOPSIS methodology for risk-based preventive maintenance planning: A case study in rolling mill company,'' *Comput. Ind. Eng.*, vol. 128, pp. 622–636, Feb. 2019.
[21] E. K. Zavadskas, R. Bausys, A. Kaklauskas, I. Ubarte, A. Kuzminske, and N. Gudiene, ''Sustainable market valuation of buildings by the single-valued neutrosophic MAMVA method,'' *Appl. Soft Comput.*, vol. 57, pp. 74–87, Aug. 2017.
[22] Y. Li and Y. Deng, ''Generalized ordered propositions fusion based on belief entropy,'' *Int. J. Comput. Commun. Control*, vol. 13, no. 5, pp. 792–807, 2018.
[23] H. Cui, Q. Liu, J. Zhang, and B. Kang, ''An improved deng entropy and its application in pattern recognition,'' *IEEE Access*, vol. 7, pp. 18284–18292, 2019.
[24] Z. Cao, W. Ding, Y.-K. Wang, F. K. Hussain, A. Al-Jumaily, and C.-T. Lin, ''Effects of repetitive SSVEPs on EEG complexity using multiscale inherent fuzzy entropy,'' *Neurocomputing*, to be published. doi: 10.1016/j.neucom.2018.08.091.
[25] M. Zhou, X. Liu, and J. Yang, ''Evidential reasoning approach for MADM based on incomplete interval value,'' *J. Intell. Fuzzy Syst.*, vol. 33, no. 6, pp. 3707–3721, 2017.
[26] H. Xu and Y. Deng, ''Dependent evidence combination based on DEMATEL method,'' *Int. J. Intell. Syst.*, vol. 34, no. 7, pp. 1555–1571, 2019.
[27] C. Fu, J.-B. Yang, and S.-L. Yang, ''A group evidential reasoning approach based on expert reliability,'' *Eur. J. Oper. Res.*, vol. 246, no. 3, pp. 886–893, 2015.
[28] M. Zhou, X.-B. Liu, Y.-W. Chen, and J.-B. Yang, ''Evidential reasoning rule for MADM with both weights and reliabilities in group decision making,'' *Knowl.-Based Syst.*, vol. 143, pp. 142–161, Mar. 2018.
[29] F. Feng, H. Fujita, M. I. Ali, R. R. Yager, and X. Liu, ''Another view on generalized intuitionistic fuzzy soft sets and related multiattribute decision making methods,'' *IEEE Trans. Fuzzy Syst.*, vol. 27, no. 3, pp. 474–488, Mar. 2019.
[30] Y. Song, X. Wang, J. Zhu, and L. Lei, ''Sensor dynamic reliability evaluation based on evidence theory and intuitionistic fuzzy sets,'' *Appl. Intell.*, vol. 48, no. 11, pp. 3950–3962, Nov. 2018.
[31] C.-L. Fan, Y. Song, Q. Fu, L. Lei, and X. Wang, ''New operators for aggregating intuitionistic fuzzy information with their application in decision making,'' *IEEE Access*, vol. 6, pp. 27214–27238, 2018.
[32] M. T. Lamata and E. C. Pérez, ''Obtaining OWA operators starting from a linear order and preference quantifiers,'' *Int. J. Intell. Syst.*, vol. 27, no. 3, pp. 242–258, Mar. 2012.
[33] Z. Xu, ''An overview of methods for determining OWA weights,'' *Int. J. Intell. Syst.*, vol. 20, no. 8, pp. 843–865, Aug. 2005.
[34] A. P. Dempster, ''Upper and lower probabilities induced by a multivalued mapping,'' *Ann. Math. Statist.*, vol. 38, no. 2, pp. 325–339, 1967.
[35] G. Shafer, *A Mathematical Theory of Evidence*, vol. 42. Princeton, NJ, USA: Princeton Univ. Press, 1976.
[36] X. Su, L. Li, H. Qian, M. Sankaran, and Y. Deng, ''A new rule to combine dependent bodies of evidence,'' *Soft Comput.*, pp. 1–7, Feb. 2019. doi: 10.1007/s00500-019-03804-y.
[37] X. Su, L. Li, F. Shi, and H. Qian, ''Research on the fusion of dependent evidence based on mutual information,'' *IEEE Access*, vol. 6, pp. 71839–71845, 2018.
[38] X. Wang and Y. Song, ''Uncertainty measure in evidence theory with its applications,'' *Appl. Intell.*, vol. 48, no. 7, pp. 1672–1688, 2017.
[39] H. Seiti, A. Hafezalkotob, S. E. Najaf, and M. Khalaj, ''A risk-based fuzzy evidential framework for FMEA analysis under uncertainty: An interval-valued DS approach,'' *J. Intell. Fuzzy Syst.*, vol. 35, no. 2, pp. 1419–1430, Aug. 2018.
[40] W. Zhang and Y. Deng, ''Combining conflicting evidence using the DEMATEL method,'' *Soft Comput.*, vol. 23, no. 17, pp. 8207–8216, Sep. 2019. doi: 10.1007/s00500-018-3455-8.
[41] Z. He and W. Jiang, ''An evidential dynamical model to predict the interference effect of categorization on decision making,'' *Knowl.-Based Syst.*, vol. 150, pp. 139–149, Jun. 2018.
[42] H. Zhang and Y. Deng, ''Engine fault diagnosis based on sensor data fusion considering information quality and evidence theory,'' *Adv. Mech. Eng.*, vol. 10, no. 11, Nov. 2018, Art. no. 1687814018809184.
[43] Y. Dong, J. Zhang, Z. Li, Y. Hu, and Y. Deng, ''Combination of evidential sensor reports with distance function and belief entropy in fault diagnosis,'' *Int. J. Comput. Commun. Control*, vol. 14, no. 3, pp. 293–307, 2019.
[44] Z. Liu, Q. Pan, J. Dezert, J.-W. Han, and Y. He, ''Classifier fusion with contextual reliability evaluation,'' *IEEE Trans. Cybern.*, vol. 48, no. 5, pp. 1605–1618, May 2018.
[45] X. Deng and W. Jiang, ''D number theory based game-theoretic framework in adversarial decision making under a fuzzy environment,'' *Int. J. Approx. Reasoning*, vol. 106, pp. 194–213, Mar. 2019.
[46] F. Xiao, ''A novel multi-criteria decision making method for assessing health-care waste treatment technologies based on D numbers,'' *Eng. Appl. Artif. Intell.*, vol. 71, pp. 216–225, May 2018.
[47] B. K. Tripathy and K. R. Arun, ''A new approach to soft sets, soft multisets and their properties,'' *Int. J. Reasoning-Based Intell. Syst.*, vol. 7, nos. 3–4, pp. 244–253, 2015.
[48] L. Fei, ''On interval-valued fuzzy decision-making using soft likelihood functions,'' *Int. J. Intell. Syst.*, vol. 34, no. 7, pp. 1631–1652, Jul. 2019. doi: 10.1002/int.22110.
[49] R. R. Yager, ''On the maximum entropy negation of a probability distribution,'' *IEEE Trans. Fuzzy Syst.*, vol. 23, no. 5, pp. 1899–1902, Oct. 2015.
[50] L. Yin, X. Deng, and Y. Deng, ''The negation of a basic probability assignment,'' *IEEE Trans. Fuzzy Syst.*, vol. 27, no. 1, pp. 135–143, Jan. 2019.
[51] X. Gao and Y. Deng, ''The negation of basic probability assignment,'' *IEEE Access*, to be published. doi: 10.1109/ACCESS.2019.2901932.
[52] X. Gao and Y. Deng, ''The generalization negation of probability distribution and its application in target recognition based on sensor fusion,'' *Int. J. Distrib. Sensor Netw.*, vol. 15, no. 5, May 2019, Art. no. 1550147719849381.
[53] Z. Cao and C.-T. Lin, ''Inherent fuzzy entropy for the improvement of EEG complexity evaluation,'' *IEEE Trans. Fuzzy Syst.*, vol. 26, no. 2, pp. 1032–1035, Apr. 2018.
[54] Q. Wang, Y. Li, and X. Liu, ''Analysis of feature fatigue EEG signals based on wavelet entropy,'' *Int. J. Pattern Recognit. Artif. Intell.*, vol. 32, no. 8, 2018, Art. no. 1854023.
[55] M. Zhou, X.-B. Liu, J.-B. Yang, Y.-W. Chen, and J. Wu, ''Evidential reasoning approach with multiple kinds of attributes and entropy-based weight assignment,'' *Knowl.-Based Syst.*, vol. 163, pp. 358–375, Jan. 2018.
[56] Z. Cao, C.-T. Lin, K.-L. Lai, L.-W. Ko, J.-T. King, K.-K. Liao, J.-L. Fuh, and S.-J. Wang, ''Extraction of SSVEPs-based inherent fuzzy entropy using a wearable headband EEG in migraine patients,'' *IEEE Trans. Fuzzy Syst.*, to be published. doi: 10.1109/TFUZZ.2019.2905823.
[57] A.-L. Jousselme, D. Grenier, and É. Bossé, ''A new distance between two bodies of evidence,'' *Inf. Fusion*, vol. 2, no. 2, pp. 91–101, Jun. 2001.
[58] W. Liu, ''Analyzing the degree of conflict among belief functions,'' *Artif. Intell.*, vol. 170, no. 11, pp. 909–924, Aug. 2006.
[59] A.-L. Jousselme and P. Maupin, ''Distances in evidence theory: Comprehensive survey and generalizations,'' *Int. J. Approx. Reasoning*, vol. 53, no. 2, pp. 118–145, Feb. 2012.
[60] F. Feng, M. Liang, H. Fujita, R. R. Yager, and X. Liu, ''Lexicographic orders of intuitionistic fuzzy values and their relationships,'' *Mathematics*, vol. 7, no. 2, p. 166, Feb. 2019.
[61] Y. Song, X. Wang, W. Quan, and W. Huang, ''A new approach to construct similarity measure for intuitionistic fuzzy sets,'' *Soft Comput.*, vol. 23, no. 6, pp. 1985–1998, Mar. 2019.
[62] C. Fu, D.-L. Xu, and S.-L. Yang, ''Distributed preference relations for multiple attribute decision analysis,'' *J. Oper. Res. Soc.*, vol. 67, no. 3, pp. 457–473, 2016.
[63] R. R. Yager, ''Generalized Dempster–Shafer structures,'' *IEEE Trans. Fuzzy Syst.*, vol. 27, no. 3, pp. 428–435, Mar. 2019.
[64] H. Seiti, A. Hafezalkotob, and L. Martínez, ''R-numbers, a new risk modeling associated with fuzzy numbers and its application to decision making,'' *Inf. Sci.*, vol. 483, pp. 206–231, May 2019.
[65] L. Fei, Y. Deng, and Y. Hu, ''DS-VIKOR: A new multi-criteria decision-making method for supplier selection,'' *Int. J. Fuzzy Syst.*, vol. 21, no. 1, pp. 157–175, Feb. 2019. doi: 10.1007/s40815-018-0543-y.
[66] E. K. Zavadskas, A. Mardani, Z. Turskis, A. Jusoh, and K. M. Nor, ''Development of TOPSIS method to solve complicated decision-making problems—An overview on developments from 2000 to 2015,'' *Int. J. Inf. Technol. Decis. Making*, vol. 15, no. 3, pp. 645–682, 2016.

[67] Y. Gong, X. Su, H. Qian, and N. Yang, "Research on fault diagnosis methods for the reactor coolant system of nuclear power plant based on D-S evidence theory," *Ann. Nucl. Energy*, vol. 112, pp. 395–399, Feb. 2018.

[68] F. Xiao and W. Ding, "Divergence measure of Pythagorean fuzzy sets and its application in medical diagnosis," *Appl. Soft Comput.*, vol. 79, pp. 254–267, Jun. 2019.

[69] H. Zhang and Y. Deng, "Weighted belief function of sensor data fusion in engine fault diagnosis," *Soft Comput.*, pp. 1–11, May 2019. doi: 10.1007/s00500-019-04063-7.

[70] X. Deng, W. Jiang, and Z. Wang, "Zero-sum polymatrix games with link uncertainty: A Dempster–Shafer theory solution," *Appl. Math. Comput.*, vol. 340, pp. 101–112, Jan. 2019.

[71] L. Fei and Y. Deng, "A new divergence measure for basic probability assignment and its applications in extremely uncertain environments," *Int. J. Intell. Syst.*, vol. 34, no. 4, pp. 584–600, 2018.

[72] Y. Song and Y. Deng, "A new method to measure the divergence in evidential sensor data fusion," *Int. J. Distrib. Sensor Netw.*, vol. 15, no. 4, Apr. 2019, Art. no. 1550147719841295. doi: 10.1177/1550147719841295.

[73] Y. Li and Y. Deng, "TDBF: Two-dimensional belief function," *Int. J. Intell. Syst.*, vol. 34, no. 8, pp. 1968–1982, Aug. 2019. doi: 10.1002/int.22135.

[74] W. Jiang, C. Huang, and X. Deng, "A new probability transformation method based on a correlation coefficient of belief functions," *Int. J. Intell. Syst.*, vol. 34, no. 6, pp. 1337–1347, Jun. 2019. doi: 10.1002/int.22098.

[75] Z. Huang, L. Yang, and W. Jiang, "Uncertainty measurement with belief entropy on the interference effect in the quantum-like Bayesian networks," *Appl. Math. Comput.*, vol. 347, pp. 417–428, Apr. 2019.

[76] A. Srivastava and S. Maheshwari, "Some new properties of negation of a probability distribution," *Int. J. Intell. Syst.*, vol. 33, no. 6, pp. 1133–1145, Jun. 2018.

[77] A. Srivastava and L. Kaur, "Uncertainty and negation—Information theoretic applications," *Int. J. Intell. Syst.*, vol. 34, no. 6, pp. 1248–1260, Jun. 2019.

[78] Y. Deng, "Generalized evidence theory," *Appl. Intell.*, vol. 43, no. 3, pp. 530–543, Oct. 2015.

[79] R. Sun and Y. Deng, "A new method to determine generalized basic probability assignment in the open world," *IEEE Access*, vol. 7, no. 1, pp. 52827–52835, 2019.

[80] R. Sun and Y. Deng, "A new method to identify incomplete frame of discernment in evidence theory," *IEEE Access*, vol. 7, pp. 15547–15555, 2019.

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