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Explicit Iteration and Unique Positive Solution for a Caputo-Hadamard Fractional Turbulent Flow Model

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ABSTRACT Hadamard fractional calculus theory has made many scholars enthusiastic and excited because of its special logarithmic function integral kernel. In this paper, we focus on a class of Caputo-Hadamard-type fractional turbulent flow model involving $p(t)$-Laplacian operator and Erdélyi-Kober fractional integral operator. The $p(t)$-Laplacian operator involved in our model is the non-standard growth operator which arises in many fields such as elasticity theory, physics, nonlinear electrorheological fluids, ect. It is the first paper that studies a Caputo-Hadamard-type fractional turbulent flow model involving $p(t)$-Laplacian operator and Erdélyi-Kober fractional integral operator. Different from the constant growth operator, The non-standard growth characteristics of $p(t)$-Laplacian operator bring great difficulties and challenges. In order to achieve a good survey result, we take advantage of the popular mixed monotonic iterative technique. With the help of this approach, we obtain the uniqueness of positive solution for the new Caputo-Hadamard-type fractional turbulent flow model. In the end, an example is also given to illustrate the main results.

INDEX TERMS Caputo-Hadamard fractional turbulent flow model, Erdélyi-Kober fractional integral operator, mixed monotone operator, $p(t)$-Laplacian operator.

I. INTRODUCTION

Fractional differential equation models have been widespread in recent years in a number of fields such as traffic flow, blood flow phenomena, electrodynamics of a complex medium, rheology, viscoelasticity, and so on [1]–[5]. Thanks to the unremitting efforts of many researchers, many excellent results have been produced, see the literature [6]–[12] for the latest developments. Recently, Hadamard fractional calculus theory has made many scholars enthusiastic and excited because of its special logarithmic function integral kernel. Details and properties of Hadamard fractional calculus, see book [13] and papers [14]–[20].

The $p(t)$-Laplacian operator is the non-standard growth operator which arises in many fields such as elasticity theory, physics, nonlinear electrorheological fluids, ect. see [21]–[24]. When the variable $p(t)$ degenerates to a constant, the $p(t)$-Laplacian operator degenerates into the familiar p-Laplacian operator, which has been studied by many scholars and yields fruitful results. Some recent works on p-Laplacian fractional differential equations can be found in [25]–[31].

But up to now, no papers considered a Caputo-Hadamard-type fractional turbulent flow model involving $p(t)$-Laplacian operator and Erdélyi-Kober fractional integral operator. Different from the constant growth operator, the non-standard growth characteristics of $p(t)$-Laplacian operator bring great difficulties and challenges. In this survey, we focus our attention on the uniqueness of positive solution for a new Caputo-Hadamard-type fractional turbulent flow model involving $p(t)$-Laplacian operator and Erdélyi-Kober fractional integral operator

$$\begin{cases} {}^{cH}D^{\chi_2}\phi_{p(t)}({}^{cH}D^{\chi_1}x(t)) \\ \quad +f(x(t), I_\eta^{\gamma,\delta}x(t)) = 0, \quad t \in [1, e] \\ x'(1) = \lambda x'(e), \quad x(1) = x''(1) = 0, \\ {}^{cH}D^{\chi_1}x(1) = 0, \end{cases} \tag{1}$$

where ${}^{cH}D$ is Caputo-Hadamard fractional derivative, $2 < \chi_1 \leq 3$, $0 < \chi_2 \leq 1$, $\delta, \eta > 0$, $\frac{e}{2} < \lambda \leq \frac{2e}{4-e}$ and $\gamma \in \mathbb{R}$. $\phi_{p(t)}(\cdot)$ is the $p(t)$-Laplacian operator with $p(t) \in C^1[0, 1]$

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such that $p(t) > 1$. $I_\eta^{\gamma,\delta}$ is the generalized Erdélyi-Kober fractional integral operator (see [32]) of order $\delta > 0$ with $\eta > 0$ and $\gamma \in \mathbb{R}$, which is defined by

$$I_\eta^{\gamma,\delta} f(t) = \frac{\eta t^{-\eta(\delta+\gamma)}}{\Gamma(\delta)} \int_0^t \frac{s^{\eta\gamma+\eta-1} f(s)}{(t^\eta - s^\eta)^{1-\delta}} ds \tag{2}$$

if the integral exists.

When $\eta = 1$, the above operator turns into the Kober operator

$$I_1^{\gamma,\delta} f(t) = \frac{t^{-(\delta+\gamma)}}{\Gamma(\delta)} \int_0^t \frac{s^\gamma f(s)}{(t-s)^{1-\delta}} ds, \quad \gamma, \delta > 0, \tag{3}$$

that was first proposed and studied by Kober in 1940 [33]. Again as a special case, let $\eta = 1$ and $\gamma = 0$, Erdélyi-Kober operator degenerately degenerates into the well-known Riemann-Liouville integral operator that has recently been extensively studied.

$$I_1^{0,\delta} f(t) = \frac{t^{-\delta}}{\Gamma(\delta)} \int_0^t \frac{f(s)}{(t-s)^{1-\delta}} ds, \quad \delta > 0, \tag{4}$$

A natural question appears in the authors' mind: "how does one find out it if a positive solution exists?" The thought always kept going round and round in authors' head. In view of the thought and all that has been mentioned so far, we focus on the uniqueness of positive solution of the above Caputo-Hadamard-type fractional turbulent flow model involving $p(t)$-Laplacian operator and Erdélyi-Kober fractional integral operator. In order to achieve our goal, we will take advantage of the popular monotonic iterative technique, its importance is self-evident. Discover the new development of this method, the reader see the literature [34], [42].

II. PRE-PREPARATION AND LEMMAS

In paper [43], Jarad, Abdeljawad and Baleanu proposed the Caputo-Hadamard fractional derivatives:

$$^{cH}D^{\chi_1} x(t) = {}^H D^{\chi_1} \Big[x(s) - \sum_{k=0}^{n-1} \frac{\delta^k x(a)}{k!} \log(\frac{s}{a})^k \Big](t), \tag{5}$$

where $\delta = \frac{d}{dt}, s \in (a, t)$. Further, it was shown in Theorem 2.1 [43] that

$$^{cH}D^{\chi_1} x(t) = I^{n-\chi_1} \delta^n x(t).$$

For $0 < \chi_1 < 1$, it follows from (5) that

$$^{cH}D^{\chi_1} x(t) = {}^H D^{\chi_1} \Big[x(s) - x(a) \Big](t).$$

In addition, the following conclusions were also established in Lemma 2.4 and 2.5 of [43], respectively

$$^{cH}D^{\chi_1} (I^{\chi_1} x)(t) = x(t), \tag{6}$$

and

$$I^{\chi_1} (^{cH}D^{\chi_1}) x(t) = x(t) - \sum_{k=0}^{n-1} \frac{\delta^k x(a)}{k!} \log(\frac{t}{a})^k, \tag{7}$$

here, $^H D^{(\cdot)}$ and $I^{(\cdot)}$ are classic Hadamard fractional derivatives and integral operators [2].

Lemma 1: [32] If $\delta, \eta > 0$ and $\gamma, q \in \mathbb{R}$. the following conclusion holds

$$I_\eta^{\gamma,\delta} t^q = \frac{t^q \Gamma(\gamma + (\frac{q}{\eta}) + 1)}{\Gamma(\gamma + (\frac{q}{\eta}) + \delta + 1)}. \tag{8}$$

Lemma 2: [23] For any $(t, x) \in [0, 1] \times \mathbb{R}$, the inverse operator $\varphi_{p(t)}^{-1}(\cdot)$ of $p(t)$-Laplacian operator $\varphi_{p(t)}(x) = |x|^{p(t)-2} x$ is defined as

$$\begin{cases} \varphi_{p(t)}^{-1}(x) = |x|^{\frac{2-p(t)}{p(t)-1}} x, & x \in \mathbb{R} \setminus \{0\}, \\ \varphi_{p(t)}^{-1}(0) = 0, & x = 0. \end{cases} \tag{9}$$

Furthermore, $p(t)$-Laplacian operator $\varphi_{p(t)}(\cdot)$ and its inverse operator $\varphi_{p(t)}^{-1}(\cdot)$ have the following characteristic:
(i1) $\varphi_{p(t)}(\cdot)$ is a homeomorphism from $\mathbb{R}$ to $\mathbb{R}$.
(i2) For any fixed t, $\varphi_{p(t)}(\cdot)$ is strictly monotone increasing.
(i3) $\varphi_{p(t)}^{-1}(\cdot)$ is a map from bounded sets to bounded sets and continuous.

Lemma 3: If $h(t) \in C[0, 1]$, the unique solution of the following linear Caputo-Hadamard-type fractional turbulent flow model involving $p(t)$-Laplacian operator

$$\begin{cases} ^{cH}D^{\chi_2} \varphi_{p(t)}(^{cH}D^{\chi_1} x(t)) + h(t) = 0, \ t \in [1, e], \\ x'(1) = \lambda x'(e), \ x(1) = x''(1) = 0, \\ ^{cH}D^{\chi_1} x(1) = 0, \end{cases} \tag{10}$$

can be expressed as the integral equation

$$x(t) = \int_1^e G(t, s) \varphi_{p(s)}^{-1}(I^{\chi_2} h(s)) \frac{ds}{s}, \tag{11}$$

where

$$G(t, s) = \begin{cases} \dfrac{\mathcal{F}(t, s) - 2(2\lambda - e)(\log t - \log s)^{\chi_1 - 1}}{2(2\lambda - e)\Gamma(\chi_1)}, \\ \qquad\qquad 0 \le s \le t \le 1, \\ \dfrac{\mathcal{F}(t, s)}{2(2\lambda - e)\Gamma(\chi_1)}, \ 0 \le t \le s \le 1, \end{cases} \tag{12}$$

with

$$\mathcal{F}(t, s) = \lambda e(\chi_1 - 1)(2 \log t + \log^2 t)(1 - \log s)^{\chi_1 - 2}. \tag{13}$$

Proof: By (7), we have

$$\varphi_{p(t)}(^{cH}D^{\chi_1} x(t)) = -I^{\chi_2} h(t) + c. \tag{14}$$

Thus,

$$^{cH}D^{\chi_1} x(t) = \varphi_{p(t)}^{-1}(-I^{\chi_2} h(t) + c). \tag{15}$$

By the condition $^{cH}D^{\chi_1} x(1) = 0$, we have

$$\varphi_{p(t)}^{-1}(-I^{\chi_2} h(1) + c) = 0. \tag{16}$$

Combining Lemma 2, we get $c = 0$ and

$$x(t) = -I^{\chi_1}\varphi_{p(t)}^{-1}(I^{\chi_2}h(t)) + c_0 + c_1\log t + c_2\log^2 t \quad (17)$$

$$x'(t) = -I^{\chi_1-1}\varphi_{p(t)}^{-1}(I^{\chi_2}h(t)) + \frac{c_1}{t} + \frac{2c_2}{t}\log t \quad (18)$$

$$x''(t) = -I^{\chi_1-2}\varphi_{p(t)}^{-1}(I^{\chi_2}h(t)) - \frac{c_1}{t^2} + \frac{2c_2(1-\log t)}{t^2}. \quad (19)$$

By the condition $x'(1) = \lambda x'(e)$ and $x(1) = x''(1) = 0$, we get $c_0 = 0,\ c_1 = 2c_2$,

$$c_2 = \frac{\lambda e}{(4\lambda - 2e)\Gamma(\chi_1 - 1)} \times \int_1^e (1-\log s)^{\chi_1-2}\varphi_{p(s)}^{-1}(I^{\chi_2}h(s))\frac{ds}{s}, \quad (20)$$

thus, we have

$$\begin{aligned} x(t) &= -\frac{1}{\Gamma(\chi_1)}\int_1^t (\log t - \log s)^{\chi_1-1}\varphi_{p(s)}^{-1}(I^{\chi_2}h(s))\frac{ds}{s} \\ &\quad + \Big(\frac{\lambda e(2\log t + \log^2 t)}{2(2\lambda - e)\Gamma(\chi_1 - 1)} \\ &\quad \times \int_1^e (1-\log s)^{\chi_1-2}\varphi_{p(s)}^{-1}(I^{\chi_2}h(s))\frac{ds}{s}\Big) \\ &= \int_1^e G(t,s)\varphi_{p(s)}^{-1}(I^{\chi_2}h(s))\frac{ds}{s}. \end{aligned} \quad (21)$$

Therefore, Lemma 3 holds. □

Lemma 4: Let $\omega(t) = 2\log t + \log^2 t$, then Green function $G(t, s)$ has the following properties:

(D_1) $G(t, s) \in C([1, e] \times [1, e])$, $G(t, s) \geq 0$, for $t, s \in [1, e]$.

(D_2) $\frac{[\lambda e(\chi_1 - 1) - 2(2\lambda - e)](1-\log s)^{\chi_1-2}}{2(2\lambda - e)\Gamma(\chi_1)}\omega(t) \leq G(t, s) \leq \frac{\lambda e(\chi_1 - 1)(1-\log s)^{\chi_1-2}}{2(2\lambda - e)\Gamma(\chi_1)}\omega(t).$

Proof: Based on the definition of $G(t, s)$, (D_1) is satisfied. If $0 \leq s \leq t \leq 1$, then we have $0 \leq \log t - \log s \leq \log t - \log t \log s = \log t(1 - \log s)$,

$$\begin{aligned} &\lambda e(\chi_1 - 1)(2\log t + \log^2 t)(1-\log s)^{\chi_1-2} \\ &\quad -2(2\lambda - e)(\log t - \log s)^{\chi_1-1} \\ &\geq \lambda e(\chi_1 - 1)(2\log t + \log^2 t)(1-\log s)^{\chi_1-2} \\ &\quad -2(2\lambda - e)(\log t - \log t\log s)^{\chi_1-1} \\ &\geq \lambda e(\chi_1 - 1)(2\log t + \log^2 t)(1-\log s)^{\chi_1-2} \\ &\quad -\frac{2(2\lambda - e)(\log t - \log t\log s)^{\chi_1-1}}{1-\log s} \\ &= [\lambda e(\chi_1 - 1)(2\log t + \log^2 t) - 2(2\lambda - e)\log^{\chi_1-1} t] \\ &\quad \times(1-\log s)^{\chi_1-2} \\ &\geq [\lambda e(\chi_1 - 1)(2\log t + \log^2 t) - 2(2\lambda - e)\log t] \\ &\quad \times(1-\log s)^{\chi_1-2} \\ &\geq [\lambda e(\chi_1 - 1) - 2(2\lambda - e)](2\log t + \log^2 t) \\ &\quad \times(1-\log s)^{\chi_1-2} \end{aligned} \quad (22)$$

on the other hand,

$$\begin{aligned} &\lambda e(\chi_1 - 1)(2\log t + \log^2 t)(1-\log s)^{\chi_1-2} \\ &\quad -2(2\lambda - e)(\log t - \log s)^{\chi_1-1} \\ &\leq \lambda e(\chi_1 - 1)(2\log t + \log^2 t)(1-\log s)^{\chi_1-2}. \end{aligned} \quad (23)$$

If $0 \leq t \leq s \leq 1$, we have

$$\begin{aligned} &\lambda e(\chi_1 - 1)(2\log t + \log^2 t)(1-\log s)^{\chi_1-2} \\ &\geq [\lambda e(\chi_1 - 1) - 2(2\lambda - e)] \\ &\quad \times(2\log t + \log^2 t)(1-\log s)^{\chi_1-2}. \end{aligned} \quad (24)$$

Therefore, Lemma 4 holds. □

Lemma 5: [44] Let E be a real Banach space, $P \subset E$ be a normal solid cone. If mixed monotone operator U maps $\mathcal{P}$ into $\mathcal{P}$ and for a constant $0 < \varsigma < 1$, satisfies the following characteristic

$$U(cs, \frac{1}{c}t) \geq c^{\varsigma}U(s, t), \quad s, t \in \mathcal{P},\ 0 < c < 1. \quad (25)$$

Then the operator U has a unique fixed point $s^* \in \mathcal{P}$. Moreover, for any initial values $s_0, t_0 \in \mathcal{P}$, by constructing successively the sequences $s_n = U(s_{n-1}, t_{n-1})$, $t_n = U(t_{n-1}, s_{n-1})$, $n = 1, 2...$,we have $\|s_n - s^*\| \to 0$, and $\|t_n - t^*\| \to 0$ as $n \to +\infty$, where $\mathcal{P} = \{s \in P | s$ is an interior point of $P\}$.

III. UNIQUENESS OF POSITIVE SOLUTION

We work throughout in the Banach space $C[1, e]$ endowed with the max-norm $\|x\| = \max_{t\in[1,e]}|x(t)|$. Let $P = \{x | x \in C[1, e] : x(t) \geq 0\}$, $P_L = \min_{t\in[1,e]} p(t)$, $P_M = \max_{t\in[1,e]} p(t)$ and $\omega(t) = 2\log t + \log^2 t$, and define a cone

$$Q_\omega = \{x \in P : \frac{1}{M}\omega(t) \leq x(t) \leq M\omega(t), t \in [1, e]\} \quad (26)$$

where M is a constant and satisfies

$$\begin{aligned} M > &\Big\{\Big[\frac{(\lambda e)^{p_L-1}[2^{\varsigma}a^{\varsigma}\emptyset(1,1) + b^{-\varsigma}\wp(1,1)]}{\Gamma(\chi_2+1)[(\Gamma(\chi_1)2(2\lambda - e)]^{P_L-1}}\Big]^{\frac{1}{P_L-\varsigma-1}}, \\ &\Big[\Big(\frac{[2(2\lambda - 1)\Gamma(\chi_1)]^{P_M-1}}{[\lambda e(\chi_1 - 1) - 2(2\lambda - e)]^{P_M-1}}\Big) \\ &\times\Big(\frac{\Gamma(\chi_2+1)}{2^{-\varsigma}a^{-\varsigma}\wp(1,1)(B(\frac{\chi_2+p_M-1}{p_M-1}, \chi_1 - 1))^{p_M-1}}\Big)\Big]^{\frac{\varsigma}{\varsigma(P_M-1)+1}}\Big\} \end{aligned} \quad (27)$$

here

$$\begin{aligned} a &= \max\{\omega(t), I_\eta^{\gamma,\delta}\omega(t)\}, \\ b &= \min\{\omega(t), I_\eta^{\gamma,\delta}\omega(t)\}, \\ 0 &< \varsigma < P_M - 1. \end{aligned} \quad (28)$$

For convenience, the following conditions are listed:

(H_1) $f(u, v) = \emptyset(u, v) + \wp(u, v)$, where $\emptyset : [0, +\infty)^2 \to [0, +\infty)$, $\wp : [0, +\infty)^2 \to [0, +\infty)$ are continuous, and $\emptyset(u, v)$ is non-decreasing and $\wp(u, v)$ is non-increasing in $u, v > 0$ respectively.

(H_2) there exists $0 < \varsigma < P_M - 1$, such that, for $u, v > 0$ and for any $0 < c < 1$,

$$\emptyset(cu, cv) \geq c^{\varsigma}\emptyset(u, v), \quad \wp(c^{-1}u, c^{-1}v) \geq c^{\varsigma}\wp(u, v), \tag{29}$$

for $c \geq 1$,

$$\emptyset(cu, cv) \leq c^{\varsigma}\emptyset(u, v), \quad \wp(c^{-1}u, c^{-1}v) \leq c^{\varsigma}\wp(u, v), \tag{30}$$

Theorem 6: Assume that (H_1), (H_2) hold, then the Caputo-Hadamard-type nonlinear fractional turbulent flow model involving $p(t)$-Laplacian and Erdélyi-Kober operator (1) has a unique positive solution x^*, and there exist two constants $0 < \rho < \mu$ such that

$$\rho\omega(t) \leq x^* \leq \mu\omega(t). \tag{31}$$

Moreover, for any initial value $u_0, v_0 \in Q_\omega$, one can construct two iterative sequences by

$$\begin{aligned} u_m &= \int_1^e G(t,s)\varphi_{p(s)}^{-1}\Big(\frac{1}{\Gamma(\chi_2)} \int_1^s (\log s - \log\tau)^{\chi_2 - 1}\Big[\emptyset\Big(u_{m-1}(\tau), I_\eta^{\gamma,\delta}u_{m-1}(\tau)\Big) \\ &\quad + \wp\Big(v_{m-1}(\tau), I_\eta^{\gamma,\delta}v_{m-1}(\tau)\Big)\Big]d\tau\Big)\frac{ds}{s} \\ v_m &= \int_1^e G(t,s)\varphi_{p(s)}^{-1}\Big(\frac{1}{\Gamma(\chi_2)} \int_1^s (\log s - \log\tau)^{\chi_2 - 1}\Big[\emptyset\Big(v_{m-1}(\tau), I_\eta^{\gamma,\delta}v_{m-1}(\tau)\Big) \\ &\quad + \wp\Big(u_{m-1}(\tau), I_\eta^{\gamma,\delta}u_{m-1}(\tau)\Big)\Big]d\tau\Big)\frac{ds}{s} \end{aligned} \tag{32}$$

and the sequences $u_m(t), v_m(t)$ converge uniformly to $x^*(t)$ on $[1, e]$ as $m \to \infty$, *i.e.*, $\|u_m - x^*\| \to 0$, $\|v_m - x^*\| \to 0$ as $m \to \infty$.

Proof: With the aim of achieving the unique positive solution of the Caputo-Hadamard-type new fractional turbulent flow model involving $p(t)$-Laplacian operator and Erdélyi-Kober fractional integral operator (1), we first investigate an auxiliary problem:

$$\begin{cases} {}^{cH}D^{\chi_2}\phi_{p(t)}\big({}^{cH}D^{\chi_1}x(t)\big) \\ \quad + f\big(x(t) + \frac{1}{n}, I_\eta^{\gamma,\delta}x(t) + \frac{1}{n}\big) = 0, \ t \in [1, e] \\ x'(1) = \lambda x'(e), \quad x(1) = x''(1) = 0, \\ {}^{cH}D^{\chi_1}x(1) = 0, \end{cases} \tag{33}$$

where $n \in 2, 3 \ldots$. With the help of Lemma 3, one can easily know that x is a solution of the boundary value problem (33) if and only if x solves the following integral equation

$$x(t) = \int_1^e G(t,s)\varphi_{p(s)}^{-1}\Big(I^{\chi_2}f(x(s) + \frac{1}{n}, I_\eta^{\gamma,\delta}x(s) + \frac{1}{n}\Big)\frac{ds}{s}. \tag{34}$$

This, together with the condition (H_1), we define an operator T by

$$\begin{aligned} T(u,v)(t) &= \int_1^e G(t,s)\varphi_{p(s)}^{-1}\Big(\frac{1}{\Gamma(\chi_2)}\int_1^s (\log s - \log\tau)^{\chi_2-1} \\ &\quad \Big[\emptyset\Big(u(\tau) + \frac{1}{n}, I_\eta^{\gamma,\delta}u(\tau) + \frac{1}{n}\Big) \\ &\quad + \wp\Big(v(\tau) + \frac{1}{n}, I_\eta^{\gamma,\delta}v(\tau) + \frac{1}{n}\Big)\Big]d\tau\Big)\frac{ds}{s}. \end{aligned} \tag{35}$$

Firstly, we prove that $T : Q_\omega \times Q_\omega \to Q_\omega$. In fact,

$$\begin{aligned} \emptyset\Big(u(t)+\frac{1}{n}, I_\eta^{\gamma,\delta}u(t)+\frac{1}{n}\Big) &\leq \emptyset\Big(M\omega(t) + 1, MI_\eta^{\gamma,\delta}\omega(t) + 1\Big) \\ &\leq (Ma + 1, Ma + 1) \\ &\leq (Ma + 1)^{\varsigma}\emptyset(1, 1) \\ &\leq 2^{\varsigma}a^{\varsigma}M^{\varsigma}\emptyset(1, 1), \end{aligned} \tag{36}$$

and

$$\begin{aligned} \wp\Big(v(t)+\frac{1}{n}, I_\eta^{\gamma,\delta}v(t)+\frac{1}{n}\Big) &\leq \wp\Big(\frac{1}{M}\omega(t)+\frac{1}{n}, \frac{1}{M}I_\eta^{\gamma,\delta}\omega(t)+\frac{1}{n}\Big) \\ &\leq \wp\Big(\frac{b}{M} + \frac{1}{n}, \frac{b}{M} + \frac{1}{n}\Big) \\ &\leq \Big(\frac{b}{M} + \frac{1}{n}\Big)^{-\varsigma}\wp(1, 1) \\ &\leq b^{-\varsigma}M^{\varsigma}\wp(1, 1), \end{aligned} \tag{37}$$

in which

$$\begin{aligned} a &= \max\{\omega(t), I_\eta^{\gamma,\delta}\omega(t)\}, \\ b &= \min\{\omega(t), I_\eta^{\gamma,\delta}\omega(t)\}. \end{aligned} \tag{38}$$

Since $0 < \varsigma < P_M - 1$, then

$$\begin{aligned} &\frac{1}{\Gamma(\chi_2)}\int_1^s (\log s - \log\tau)^{\chi_2-1} \\ &\quad \times\Big[\emptyset\Big(u(\tau) + \frac{1}{n}, I_\eta^{\gamma,\delta}u(\tau) + \frac{1}{n}\Big) \\ &\quad + \wp\Big(v(\tau) + \frac{1}{n}, I_\eta^{\gamma,\delta}v(\tau) + \frac{1}{n}\Big)\Big]\frac{d\tau}{\tau} \\ &\leq \frac{1}{\Gamma(\chi_2)}\int_1^s (\log s - \log\tau)^{\chi_2-1} \\ &\quad \times[2^{\varsigma}a^{\varsigma}M^{\varsigma}\emptyset(1, 1) + b^{-\varsigma}M^{\varsigma}\wp(1, 1)]\frac{d\tau}{\tau} \\ &= \frac{\log^{\chi_2} s[2^{\varsigma}a^{\varsigma}M^{\varsigma}\emptyset(1, 1) + b^{-\varsigma}M^{\varsigma}\wp(1, 1)]}{\Gamma(\chi_2 + 1)}. \end{aligned} \tag{39}$$

As a result,

$$\begin{aligned} &T(u,v)(t) \\ &\leq \frac{\lambda e(\chi_1 - 1)\omega(t)}{2(2\lambda - e)\Gamma(\chi_1)}\int_1^e (1 - \log s)^{\chi_1-2} \\ &\quad \times\Big[\frac{\log^{\chi_2} s[2^{\varsigma}a^{\varsigma}M^{\varsigma}\emptyset(1, 1) + b^{-\varsigma}M^{\varsigma}\wp(1, 1)]}{\Gamma(\chi_2 + 1)}\Big]^{\frac{1}{p(s)-1}}\frac{ds}{s} \\ &\leq \frac{\lambda e\omega(t)}{2(2\lambda - e)\Gamma(\chi_1)} \\ &\quad \times\Big[\frac{2^{\varsigma}a^{\varsigma}M^{\varsigma}\emptyset(1, 1) + b^{-\varsigma}M^{\varsigma}\wp(1, 1)}{\Gamma(\chi_2 + 1)}\Big]^{\frac{1}{p_L-1}} \\ &\leq M\omega(t). \end{aligned} \tag{40}$$

On the other hand,

$$\begin{aligned}
&\wp(v(t)+\frac{1}{n}, I_\eta^{\gamma,\delta}v(t)+\frac{1}{n})\\
&\geq \wp(M\omega(t)+\frac{1}{n}, MI_\eta^{\gamma,\delta}\omega(t)+\frac{1}{n})\\
&\geq (Ma+1, Ma+1)\\
&\geq (Ma+1)^{-\varsigma}\wp(1,1)\\
&\geq 2^{-\varsigma}a^{-\varsigma}M^{-\varsigma}\wp(1,1), \qquad (41)
\end{aligned}$$

$$\begin{aligned}
&T(u,v)(t)\\
&=\int_1^e G(t,s)\varphi_{p(s)}^{-1}\Big(\frac{1}{\Gamma(\chi_2)}\\
&\quad\times\int_1^s(\log s-\log\tau)^{\chi_2-1}\Big[\emptyset\Big(u(\tau)+\frac{1}{n}, I_\eta^{\gamma,\delta}u(\tau)+\frac{1}{n}\Big)\\
&\quad+\wp\Big(v(\tau)+\frac{1}{n}, I_\eta^{\gamma,\delta}v(\tau)+\frac{1}{n}\Big)\Big]d\tau\Big)\frac{ds}{s}\\
&\geq\int_1^e G(t,s)\varphi_{p(s)}^{-1}\Big(\frac{1}{\Gamma(\chi_2)}\times\int_1^s(\log s-\log\tau)^{\chi_2-1}\\
&\quad\wp\Big(v(\tau)+\frac{1}{n}, I_\eta^{\gamma,\delta}v(\tau)+\frac{1}{n}\Big)\Big]d\tau\Big)\frac{ds}{s}\\
&\geq\frac{[\lambda e(\chi_1-1)-2(2\lambda-e)]\omega(t)}{2(2\lambda-e)\Gamma(\chi_1)}\\
&\quad\times\int_1^e(1-\log s)^{\chi_1-2}\varphi_{p(s)}^{-1}\Big(\frac{1}{\Gamma(\chi_2)}\\
&\quad\times\int_1^s(\log s-\log\tau)^{\chi_2-1}[2^{-\varsigma}a^{-\varsigma}M^{-\varsigma}\wp(1,1)]d\tau\Big)\frac{ds}{s}\\
&=\frac{[\lambda e(\chi_1-1)-2(2\lambda-e)]\omega(t)}{2(2\lambda-e)\Gamma(\chi_1)}\int_1^e(1-\log s)^{\chi_1-2}\\
&\quad\times\Big[\frac{2^{-\varsigma}a^{-\varsigma}M^{-\varsigma}\wp(1,1)\log^{\chi_2}s}{\Gamma(\chi_2+1)}\Big]^{\frac{1}{p(s)-1}}\frac{ds}{s}\\
&\geq\frac{[\lambda e(\chi_1-1)-2(2\lambda-e)]\omega(t)}{2(2\lambda-e)\Gamma(\chi_1)}\Big[\frac{2^{-\varsigma}a^{-\varsigma}M^{-\varsigma}\wp(1,1)}{\Gamma(\chi_2+1)}\Big]^{\frac{1}{p_M-1}}\\
&\quad\times\int_1^e(1-\log s)^{\chi_1-2}(\log s)^{\frac{\chi_2}{p_M-1}}\frac{ds}{s}\\
&=\frac{[\lambda e(\chi_1-1)-2(2\lambda-e)][2^{-\varsigma}a^{-\varsigma}M^{-\varsigma}\wp(1,1)]^{\frac{1}{p_M-1}}}{(\Gamma(\chi_2+1))^{\frac{1}{p_M-1}}}\\
&\quad\times\frac{B(\frac{\chi_2+p_M-1}{p_M-1},\chi_1-1)}{2(2\lambda-e)\Gamma(\chi_1)}\omega(t)\\
&\geq\frac{1}{M}\omega(t). \qquad (42)
\end{aligned}$$

Therefore, the operator $T:Q_\omega\times Q_\omega\to Q_\omega$ holds.

Next, we prove T is mixed monotone. For any $u_1,u_2\in Q_\omega$ and $u_1\leq u_2$, we have

$$\begin{aligned}
T(u_1,v)(t)&=\int_1^e G(t,s)\varphi_{p(s)}^{-1}\Big(\frac{1}{\Gamma(\chi_2)}\int_1^s(s-\tau)^{\chi_2-1}\\
&\quad\times\Big[\emptyset\Big(u_1(\tau)+\frac{1}{n}, I_\eta^{\gamma,\delta}u_1(\tau)+\frac{1}{n}\Big)\\
&\quad+\wp\Big(v(\tau)+\frac{1}{n}, I_\eta^{\gamma,\delta}v(\tau)+\frac{1}{n}\Big)\Big]d\tau\Big)\frac{ds}{s}\\
&\leq\int_1^e G(t,s)\varphi_{p(s)}^{-1}\Big(\frac{1}{\Gamma(\chi_2)}\int_1^s(s-\tau)^{\chi_2-1}\\
&\quad\times\Big[\emptyset\Big(u_2(\tau)+\frac{1}{n}, I_\eta^{\gamma,\delta}u_2(\tau)+\frac{1}{n}\Big)\\
&\quad+\wp\Big(v(\tau)+\frac{1}{n}, I_\eta^{\gamma,\delta}v(\tau)+\frac{1}{n}\Big)\Big]d\tau\Big)\frac{ds}{s}\\
&=T(u_2,v)(t), \qquad (43)
\end{aligned}$$

which means that

$$T(u_1,v)(t)\leq T(u_2,v)(t),\qquad v\in Q_\omega. \qquad (44)$$

Thus, the operator $T(u,v)$ is non-decreasing in u for any $v\in Q_\omega$. Similarly, one can easily prove that, for any $v_1\geq v_2, v_1, v_2\in Q_\omega$,

$$T(u,v_1)(t)\leq T(u,v_2)(t),\qquad u\in Q_\omega. \qquad (45)$$

Therefore, the operator $T:Q_\omega\times Q_\omega\to Q_\omega$ is mixed monotone.

Finally, we show T satisfies (25) of Lemma 5. For any $u,v\in Q_\omega$ and $0<c<1$, by means of the condition (H_2), we achieve

$$\begin{aligned}
T(cu,\frac{1}{c}v)(t)&=\int_1^e G(t,s)\varphi_{p(s)}^{-1}\Big(\frac{1}{\Gamma(\chi_2)}\int_1^s(s-\tau)^{\chi_2-1}\\
&\quad\times\Big[\emptyset\Big(cu(\tau)+\frac{1}{n}, cI_\eta^{\gamma,\delta}u(\tau)+\frac{1}{n}\Big)\\
&\quad+\wp\Big(\frac{1}{c}v(\tau)+\frac{1}{n},\frac{1}{c}I_\eta^{\gamma,\delta}v(\tau)+\frac{1}{n}\Big)\Big]d\tau\Big)\frac{ds}{s}\\
&\geq\int_1^e G(t,s)\varphi_{p(s)}^{-1}\Big(\frac{1}{\Gamma(\chi_2)}\int_1^s(s-\tau)^{\chi_2-1}\\
&\quad\times\Big[c^\varsigma\emptyset\Big(u(\tau)+\frac{1}{n}, I_\eta^{\gamma,\delta}u(\tau)+\frac{1}{n}\Big)\\
&\quad+\wp\Big(v(\tau)+\frac{1}{n}, I_\eta^{\gamma,\delta}v(\tau)+\frac{1}{n}\Big)\Big]d\tau\Big)\frac{ds}{s}\\
&=c^{\frac{\varsigma}{p(s)-1}}\int_1^e G(t,s)\varphi_{p(s)}^{-1}\Big(\frac{1}{\Gamma(\chi_2)}\\
&\quad\int_1^s(s-\tau)^{\chi_2-1}\Big[\emptyset\Big(u(\tau)+\frac{1}{n}, I_\eta^{\gamma,\delta}u(\tau)+\frac{1}{n}\Big)\\
&\quad+\wp\Big(v(\tau)+\frac{1}{n}, I_\eta^{\gamma,\delta}v(\tau)+\frac{1}{n}\Big)\Big]d\tau\Big)\frac{ds}{s}\\
&\geq c^{\frac{\varsigma}{p_M-1}}T(u,v)(t). \qquad (46)
\end{aligned}$$

In view of $0<\varsigma<p_M-1$, it follows from Lemma 5 that the operator T has a unique fixed point $x_n^*\in Q_\omega$, such that $T(x_n^*,x_n^*)=x_n^*$. That is, the problem (33) has a unique positive solution for every $n\in 2,3,\ldots$. It follows from a standard argument that $\{x_n^*\}_{n\geq 2}$ being an equicontinuous family on $[1,e]$. Let $x^*=\lim_{n\to\infty}x_n^*$, by using of Lemma 5, we know x^* is a unique positive solution of (1).

It follows from $x_n^*\in Q_\omega$ that x_n^* has uniform lower and upper bounds. Let $x_n^*\to x^*$ as $n\to\infty$, by a standard process, we know from Lemma 5 that the mixed monotone operator T has a unique fixed point $x^*\in Q_\omega$. That is, the Hadamard type fractional turbulent flow model (1) has a unique solution x^* and $\rho\omega(t)=\frac{1}{M}\omega(t)\leq x(t)\leq M\omega(t)=\mu\omega(t)$. Furthermore, for any initial value $u_0, v_0 \in Q_\omega$, one can develop two mixed monotone iterative sequences by

$$\begin{aligned} u_m =& \int_1^e G(t,s)\varphi_{p(s)}^{-1}\Big(\frac{1}{\Gamma(\chi_2)}\int_1^s(\log s-\log\tau)^{\chi_2-1} \\ &\times\Big[\emptyset\Big(u_{m-1}(\tau), I_\eta^{\gamma,\delta}u_{m-1}(\tau)\Big) \\ &+\wp\Big(v_{m-1}(\tau), I_\eta^{\gamma,\delta}v_{m-1}(\tau)\Big)\Big]d\tau\Big)\frac{ds}{s} \\ v_m =& \int_1^e G(t,s)\varphi_{p(s)}^{-1}\Big(\frac{1}{\Gamma(\chi_2)}\int_1^s(\log s-\log\tau)^{\chi_2-1} \\ &\times\Big[\emptyset\Big(v_{m-1}(\tau), I_\eta^{\gamma,\delta}v_{m-1}(\tau)\Big) \\ &+\wp\Big(u_{m-1}(\tau), I_\eta^{\gamma,\delta}u_{m-1}(\tau)\Big)\Big]d\tau\Big)\frac{ds}{s} \end{aligned} \tag{47}$$

and the sequences $u_m(t), v_m(t)$ satisfy $\|u_m - x^*\| \to 0$, $\|v_m - x^*\| \to 0$ as $m \to \infty$.

□

Example 7: Consider the following Caputo-Hadamard-type $p(t)$-Laplacian fractional turbulent flow model with Erdélyi-Kober operator:

$$\begin{cases} {}^{cH}D^{\frac{1}{2}}\phi_{t^2+1}({}^{cH}D^{\frac{5}{2}}x(t)) + x^{\frac{1}{6}}(t) \\ +[I_{\frac{1}{2}}^{1,\frac{1}{2}}x(t)]^{\frac{1}{8}} + x^{-\frac{1}{5}}(t) + [I_{\frac{1}{2}}^{1,\frac{1}{2}}x(t)]^{-\frac{1}{4}} = 0, t \in [1,e] \\ x'(1) = 2x'(e), \quad x(1) = x''(1) = 0, \\ {}^{cH}D^{\frac{5}{2}}x(1) = 0. \end{cases} \tag{48}$$

where $\chi_1 = \frac{5}{2}, \chi_2 = \frac{1}{2}, \gamma = 1, \eta = \frac{1}{2}, \delta = \frac{1}{2}, \lambda = 2$, and

$$p(t)=t^2+1, \emptyset(u,v)=u^{\frac{1}{6}}+v^{\frac{1}{8}}, \wp(u,v) = u^{-\frac{1}{5}}+v^{-\frac{1}{4}}. \tag{49}$$

Thus, $P_L = 1, P_M = 2$, and we can choose $\varsigma = \frac{1}{2} < 1$, then

$$\begin{aligned} \emptyset(cu,cv) &= c^{\frac{1}{6}}u^{\frac{1}{6}} + c^{\frac{1}{8}}v^{\frac{1}{8}} \geq c^{\frac{1}{5}}\emptyset(u,v), \\ \wp(c^{-1}u, c^{-1}v) &= (c^{-1}u)^{-\frac{1}{5}} + (c^{-1}v)^{-\frac{1}{4}} \geq c^{\frac{1}{5}}\wp(u,v), \end{aligned}$$

for any $u, v > 0$ and $0 < c < 1$, the conditon $(H2)$ holds. It is obviously that $(H1)$ holds. Thus, with the help of Theorem 6, we can safely conclude that the problem (48) has a unique positive solution x^*, and there exists a constant $M = 330 > \max\{324.78, 1.2\}$ such that

$$\frac{t}{330} \leq x^*(t) \leq 330t. \tag{50}$$

Moreover, for any initial $u_0, v_0 \in Q_\omega$, we can construct successively two sequences $\{u_m\}$ and $\{v_m\}$ by

$$\begin{aligned} u_m =& \int_1^e G(t,s)\varphi_{s^2+1}^{-1}\Big(\frac{1}{\Gamma(\frac{1}{2})}\int_0^s(s-\tau)^{-\frac{1}{2}} \\ &\times\Big[\emptyset\Big(u_{m-1}(\tau), I_{\frac{1}{2}}^{1,\frac{1}{2}}u_{m-1}(\tau)\Big) \\ &+\wp\Big(v_{m-1}(\tau), I_{\frac{1}{2}}^{1,\frac{1}{2}}v_{m-1}(\tau)\Big)\Big]d\tau\Big)\frac{ds}{s} \\ v_m =& \int_1^e G(t,s)\varphi_{s^2+1}^{-1}\Big(\frac{1}{\Gamma(\frac{1}{2})}\int_0^s(s-\tau)^{-\frac{1}{2}} \\ &\times\Big[\emptyset\Big(v_{m-1}(\tau), I_{\frac{1}{2}}^{1,\frac{1}{2}}v_{m-1}(\tau)\Big) \\ &+\wp\Big(u_{m-1}(\tau), I_{\frac{1}{2}}^{1,\frac{1}{2}}u_{m-1}(\tau)\Big)\Big]d\tau\Big)\frac{ds}{s} \end{aligned} \tag{51}$$

and the iterative sequences $u_m(t), v_m(t)$ converge uniformly to $x^*(t)$ on $[1,e]$ as $n \to \infty$, *i.e.*, $\|u_m - x^*\| \to 0$, $\|v_m - x^*\| \to 0$ as $m \to \infty$.

IV. CONCLUSION

In this paper, by employing mixed monotone iterative technique, we have developed the existence theory for a class of Caputo-Hadamard-type fractional turbulent flow model involving $p(t)$-Laplacian operator as well as Erdélyi-Kober fractional integral operator. Our results are new and contribute significantly to the literature on the topic. By fixing the parameters involved in the given problem, we can obtain some new fractional turbulent flow model and results as special cases of the present work. For example, if we take $\eta = 1$, the above fractional turbulent flow model turns into fractional turbulent flow model involving the Kober operator. Letting $\gamma = 0$ in fractional turbulent flow model involving the Kober operator, our model degenerates into fractional turbulent flow model involving the well-known Riemann-Liouville integral operator. Moreover, if we put $p(t) = p$ (constant). our fractional turbulent flow model also contains a series of Caputo-Hadamard-type fractional turbulent flow model involving standard p-Laplacian operator.

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