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Families of Travelling Waves Solutions for Fractional-Order Extended Shallow Water Wave Equations, Using an Innovative Analytical Method

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ABSTRACT In the present research article, an efficient analytical technique is applied for travelling waves solutions of fractional partial differential equations. The investigated problems are reduced to ordinary differential equations, by a variable transformation. The solutions of the resultant ordinary differential equations are expressed in the term of some suitable polynomials, which provide trigonometric, hyperbolic and rational function solutions with some free parameters. To confirm the reliability and novelty of the current work, the proposed method is applied for the solutions of (2+1) and (3+1)-dimensional fractional-order extended shallow water wave equations.

INDEX TERMS Analytical method, shallow water wave equations, families of solutions.

I. INTRODUCTION

Fractional partial differential equations (FPDEs) are frequently used in different areas of applied sciences such as engineering, fluid mechanics, solid-state, propagation of waves, plasma physics, biology, the heat flow phenomena, quantum mechanics, optical fibers and so on. Due to applications mentioned above, the researchers have taken keen interest in the study of FPDEs. In this connection, several mathematicians have developed different techniques for the numerical and analytical solutions of FPDEs [1]–[8]. For example, Xu et al. have extended Homotopy perturbation method (HPM) to solve PDEs of fractal order [9], [10]. Wu and Baleanu have applied variational iteration method (VIM) for the fractional order Burgers's flow equation [11]. Duan and Baleanu have reviewed the ADM besides its modifications including the multistage ADM for FPDEs with given initial and boundary conditions [12]. To obtain travelling wave solutions, Raslan et al. have applied modified tanh method to EW and MEW equations of fractional-order [13]. To calculate the exact solutions of the fractional Sharma-Tasso-Olver (STO) equations and generalized reaction Duffing models, Jafari et al. have used the fractional sub-equation method [14]. Finite element method (FEM) is used to solve symmetric space-FPDEs with the Riesz fractional operator [15]. Besides these methods, some mathematicians have used other methods such as Jacobi elliptic expansion method [16], exp-function method [17], fractional reduced differential transform method (FRDTM) [18], Lie algebra method [19], finite difference method (FDM) [20], $(\frac{G'}{G})$-expansion method [21], [22] and many other numerical and analytical methods.

Among these methods, for constructing exact solutions of nonlinear FPDEs, many researchers have used $(\frac{G'}{G})$-expansion method [23]–[25]. This simple and efficient method uses a variable transformation which converts a

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FPDE into nonlinear integer order ordinary differential equation (ODE). The solution of the resultant ODE is expressed in the form of a polynomial in $\frac{G'}{G}$ consists of some free parameters. The function $G = G(\xi)$ satisfies an arbitrary ODE. By substituting this polynomial in resultant nonlinear ODE and by comparing the coefficients of $(\frac{G'}{G})$, a system of algebraic equations is obtained, which provide trigonometric, hyperbolic and rational function solutions.

The shallow water wave equations (SWWEs) govern fluid flow in the seas, oceans, estuaries, channels, rivers and coastal regions. These equations also arise in debris and atmospheric flows. SWWEs can also be used to predict storm surge levels, tides and coastline changes from ocean currents, hurricanes and to study dredging feasibility. By the variants of the SWWEs geophysical flows can also be modeled correctly. The SWWEs describe incompressible flow evaluation, ignoring the change in density along the flow depth. SWWEs are also applicable to cases where the depth of fluid is much smaller than horizontal scale of the flow. The $(2+1)$ and $(3+1)$-dimensional extended SWWEs are given as [26]:

$$v_{xt} + av_{xy} + v_{xxxy} - 2v_y v_{xx} - 4v_x v_{xy} = 0, \tag{1}$$
$$v_{yt} + av_{xy} + v_{xxxy} - 3v_y v_{xx} - 3v_x v_{xy} = 0, \tag{2}$$

and

$$v_{yzt} + av_{xyz} + v_{xxxyz} - 6v_{xy}v_{xz} - 6v_x v_{xyz} = 0, \tag{3}$$

where a is an arbitrary constant. These equations are obtained by adding the terms av_{xy} and av_{xyz} to SWWEs appear in them. Equations (1), (2) and (3) are important to study dispersive waves. These equations can also be reduced to KdV equations by supposing $y = x$ and $a = 0$.

The fractional order extended SWWES (FSWWEs) in $(2+1)$ and $(3+1)$-dimensions are obtained by replacing the integer-order derivatives in (1), (2) and (3) by fractional-order derivatives and get:

$$v_{xt}^{\beta\alpha} + av_{xy}^{\beta\gamma} + v_{xxxy}^{3\beta\gamma} - 2v_y^{\gamma} v_{xx}^{2\beta} - 4v_x^{\beta} v_{xy}^{\beta\gamma} = 0, \tag{4}$$
$$v_{yt}^{\gamma\alpha} + av_{xy}^{\beta\gamma} + v_{xxxy}^{3\beta\gamma} - 3v_y^{\gamma} v_{xx}^{2\beta} - 3v_x^{\beta} v_{xy}^{\beta\gamma} = 0, \tag{5}$$

Similarly, the $(3+1)$-dimensional extanded FSWWE is given as:

$$v_{yzt}^{\gamma\delta\alpha} + av_{xyz}^{\beta\gamma\delta} + v_{xxxyz}^{3\beta\gamma\delta} - 6v_{xy}^{\beta\gamma} v_{xz}^{\beta\delta} - 6v_x^{\beta} v_{xyz}^{\beta\gamma\delta} = 0, \tag{6}$$

where $0 < \alpha, \beta, \gamma, \delta \leq 1$. The fractional derivatives involved in equations (4-6) are defined in Riemann-liouville derivative's sense. The definition and some properties of this derivative of order α are listed as follow [27]:

$$D_t^{\alpha} v(t) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \dfrac{d}{dt} \int_0^t (t-\tau)^{-\alpha}(v(\tau)-v(0))d\tau, \\ \qquad\qquad 1 > \alpha > 0, \\ (v^{(n)}(t))^{(\alpha-n)}, \qquad n+1 > \alpha \geq n, \quad n \geq 1. \end{cases}$$

$$D_t^{\alpha}(m(t)n(t)) = n(t)D_t^{\alpha}m(t) + m(t)D_t^{\alpha}n(t)$$
$$D_t^{\alpha}t^{\gamma} = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)}t^{\gamma-\alpha}$$
$$D_t^{\alpha}m(n(t)) = m_n'[n(t)]D_t^{\alpha}n(t) = D_g^{\alpha}m[n(t)](g'(t))^{\alpha}.$$

Before this research work, mathematicians have solved equations (1-3) with the help of analytical and numerical methods. Bekir and Aksoy in [26] have used exp-function method to solve the equations (1-3) analytically. In [28] Faisal and Kumar have solved equation (1) and (2) by using extended form of $(\frac{G'}{G})$-expansion method. Similarly, Bekir and Aksoy in [29] have used simple $(\frac{G'}{G})$-expansion method and obtained six solutions of equations (1-3). In [30] Alkahtani and Atangana have solved the fractional order SWWEs. Seadawy et al. have suggested the same method for the solutions of a system of SWWE [31]. In [32] Ray has solved different FSWWEs by using a novel method.

The aim of this paper is to solve extended FSWWEs given in equations (4-6) by using $(\frac{G'}{G})$-expansion method in extended form. The direct implementation of the present method to the given problems, makes it superior over other methods present in literature. That is, problems are solved without any discretization or linearization and avoid any unrealistic assumption in providing a number of exact solutions in various families.

The onward representation of the paper is structured as: Section.2 describes the $(\frac{G'}{G})$-expansion Method. Section.3 consist exact solutions for the three targeted problems. In Section.4 the results and graphs are discussed while Section.5 concludes our whole study.

II. THE $(\frac{G'}{G})$-expansion Method

In this part of the paper, the methodology of $(\frac{G'}{G})$-expansion method for the solutions of FPDEs of the form:

$$F(v, D_t^{\alpha}v, D_{x_1}^{\beta}v, D_{x_2}^{\gamma}v, vD_{x_1}^{\beta}v, \ldots) = 0, \quad 0 < \alpha, \beta, \gamma \leq 1, \tag{7}$$

is presented. Where v is a function of x_1, x_2, x_3, x_n and t.

The procedure of the present method is as follows:

Step.1: First, we transform equation (7) into a nonlinear ODE of the form:

$$T(V, V', V'', VV', \ldots) = 0, \tag{8}$$

the derivatives of V are w.r.t ξ. The complex transformation is an appropriate transformation to use for FPDEs. The general form of this transformation is given as:

$$\xi = a\frac{t^{\alpha}}{\Gamma(\alpha+1)} + b\frac{x_1^{\beta}}{\Gamma(\beta+1)} + \ldots + \xi_0, \tag{9}$$

where $a, b \ldots \xi_0$ are unknown constants.

Step.2: The series form solution, $V(\xi)$ of equation (8) is expressed in term of $(\frac{G'}{G})$ polynomial as [21]:

$$V(\xi) = \sum_{i=-m}^{m} a_i(\frac{G'(\xi)}{G(\xi)})^i, \tag{10}$$

where $a_i's$ are unknowns and function $G(\xi)$ is satisfying the ODE given bellow:

$$G''(\xi) + AG'(\xi) + BG(\xi) = 0, \tag{11}$$

where A and B are unknown constants.

By the general solution of equation (11) [21]:

$$(\frac{G'}{G}) = \begin{cases} \frac{\sqrt{Z}}{2}\frac{D_1 sinh(\frac{\sqrt{Z}}{2}\xi)+D_2 cosh(\frac{\sqrt{Z}}{2}\xi)}{D_1 cosh(\frac{\sqrt{Z}}{2}\xi)+D_2 sinh(\frac{\sqrt{Z}}{2}\xi)}-\frac{A}{2} & Z>0 \\ \frac{\sqrt{-Z}}{2}\frac{-D_1 sin(\frac{\sqrt{-Z}}{2}\xi)+D_2 cos(\frac{\sqrt{-Z}}{2}\xi)}{D_1 cos(\frac{\sqrt{-Z}}{2}\xi)+D_2 sin(\frac{\sqrt{-Z}}{2}\xi)}-\frac{A}{2} & Z<0 \\ \frac{D_2}{D_1+D_2\xi}-\frac{A}{2} & Z=0 \end{cases}, \tag{12}$$

where D_1 and D_2 are unknown constants and $Z = A^2 - 4B$.

The number m present in equation (10) called balance number and can be obtained by taking the homogeneous balance between derivative of the highest order and nonlinear term involve in equation (8). To find m, we use following two formulae [21]:

$$D(\frac{d^qV}{d\xi^q}) = m+q \ \ and \ \ D(V^p(\frac{d^qV}{d\xi^q})^s) = mp+s(q+m), \tag{13}$$

where D means degree and m is the degree of $V(\xi)$ i.e. $D(V(\xi)) = m$.

Step.3: By putting (10) into (8) a polynomial in $(\frac{G'}{G})$ is formed and after coefficients comparison a system of non-linear algebraic equations in free parameters is obtained.

Step.4: In last step the obtained system of algebraic equations is solved by using a mathematical software, which provides the unknown coefficients given in equation (10). Thus, some exact solutions/families for equation (7) are obtained.

III. PROBLEMS

In present section, the proposed method is applied to the following examples:

Problem 1: Consider the first extended FSWWE given in equation (4):

$$v_{xt}^{\beta\alpha} + av_{xy}^{\beta\gamma} + v_{xxxy}^{3\beta\gamma} - 2v_y^{\gamma}v_{xx}^{2\beta} - 4v_x^{\beta}v_{xy}^{\beta\gamma} = 0, \tag{14}$$

applying the transformation

$$v(x,t) = v(\xi), \quad where$$
$$\xi = k_1\frac{x^\beta}{\Gamma(1+\beta)} + k_2\frac{y^\gamma}{\Gamma(1+\gamma)} - c\frac{t^\alpha}{\Gamma(1+\alpha)}, \tag{15}$$

we get an ODE, integrating ODE and taking constant of integration zero, we have

$$k_1^2k_2V''' + (k_2a-c)V' - 3k_1k_2(V')^2 = 0, \tag{16}$$

where $V' = \frac{dV}{d\xi}$, c represents the speed of the traveling wave while k_1 and k_2 are constants.

For homogenous number m, we consider the homogeneous balance between highest derivative V''' and $(V')^2$ given in equation (16). So, by using equation (13) we get $m = 1$. Now putting $m = 1$ in (10) we get the following $(\frac{G'}{G})$ solution for (14):

$$V(\xi)=\sum_{i=-1}^{1} a_i(\frac{G'(\xi)}{G(\xi)})^i = a_{-1}(\frac{G'(\xi)}{G(\xi)})^{-1}+a_0+a_1(\frac{G'(\xi)}{G(\xi)})^1, \tag{17}$$

where a_1, a_0 and a_{-1} are constants to be calculated later.

Now putting equation (17) in equation (16) and using equation (11), we get a polynomial in $\frac{G'(\xi)}{G(\xi)}$, equating the coefficients of same power of $(\frac{G'}{G})$, a system of algebraic equations is obtained and after solving it by using Maple, we get the following two solutions.

Case 1:

$$a_0 = a_0, a_{-1} = 2k_1B, a_1 = 0 \quad and$$
$$c = -4k_1^2k_2B + k_2a + k_1^2k_2A^2 \tag{18}$$

Case 2:

$$a_0 = a_0, a_{-1} = 0, a_1 = -2k_1 \quad and$$
$$c = -4k_1^2k_2B + k_2a + k_1^2k_2A^2 \tag{19}$$

Considering case.1 and suppose that $Z = A^2 - 4B$ for simplicity in calculation, we get the following families of solutions

Family1: When $Z < 0$ then equations (17) and (12) implies the following periodic solitary wave solutions:

(i) When $D_1 \neq 0$ and $D_2 = 0$

$$V_1(x,y,t)=a_0 - 2k_1B\left(\frac{\sqrt{-Z}}{2}tan(\frac{\sqrt{-Z}}{2}\xi)+\frac{A}{2}\right)^{-1}, \tag{20}$$

(ii) When $D_1 = 0$ and $D_2 \neq 0$

$$V_2(x,y,t)=a_0+ 2k_1B\left(\frac{\sqrt{-Z}}{2}cot(\frac{\sqrt{-Z}}{2}\xi)-\frac{A}{2}\right)^{-1}, \tag{21}$$

where $\xi = (k_1\frac{x^\beta}{\Gamma(\beta+1)}+k_2\frac{y^\gamma}{\Gamma(\gamma+1)}-\frac{(-4\ k_1^2\ k_2\ B+k_2\ a+k_1^2\ k_2\ A^2)t^\alpha}{\Gamma(\alpha+1)})$

Family2: When $Z > 0$ then equations (17) and (12) implies the following hyperbolic solitary wave solutions:

(i) When $D_1 \neq 0$ and $D_2 = 0$

$$V_3(x,y,t)=a_0+2k_1B\left(\frac{\sqrt{Z}}{2}tanh(\frac{\sqrt{Z}}{2}\xi)-\frac{A}{2}\right)^{-1}, \tag{22}$$

(ii) When $D_1 = 0$ and $D_2 \neq 0$

$$V_4(x,y,t)=a_0+2k_1B\left(\frac{\sqrt{Z}}{2}coth(\frac{\sqrt{Z}}{2}\xi)-\frac{A}{2}\right)^{-1}, \tag{23}$$

where $\xi = (k_1\frac{x^\beta}{\Gamma(\beta+1)}+k_2\frac{y^\gamma}{\Gamma(\gamma+1)}-\frac{(-4\ k_1^2\ k_2\ B+k_2\ a+k_1^2\ k_2\ A^2)t^\alpha}{\Gamma(\alpha+1)})$

Family3: When $Z = 0$ then equations (17) and (12) implies the following rational solitary wave solutions:

(i) When $D_1 \neq 0$ and $D_2 = 0$

$$V_5(x,y,t) = a_0 + 2k_1 B\left(-\frac{A}{2}\right)^{-1}, \tag{24}$$

(ii) When $D_1 = 0$ and $D_2 \neq 0$

$$V_6(x,y,t) = a_0 + 2k_1 B\left(\frac{1}{\xi} - \frac{A}{2}\right)^{-1}, \tag{25}$$

where $\xi = (k_1 \frac{x^\beta}{\Gamma(\beta+1)} + k_2 \frac{y^\gamma}{\Gamma(\gamma+1)} - \frac{(-4\ k_1^2\ k_2\ B + k_2\ a + k_1^2\ k_2\ A^2)t^\alpha}{\Gamma(\alpha+1)})$.

Now considering case.2 and suppose that $Z = A^2 - 4B$ for simplicity in calculation, we get further two families of solutions given as:

Family4: When $Z < 0$ then equations (17) and (12) implies the following periodic solitary wave solutions:

(i) When $D_1 \neq 0$ and $D_2 = 0$

$$V_7(x,y,t) = a_0 + 2k_1\left(\frac{\sqrt{-Z}}{2} tan(\frac{\sqrt{-Z}}{2}\xi) + \frac{A}{2}\right), \tag{26}$$

(ii) When $D_1 = 0$ and $D_2 \neq 0$

$$V_8(x,y,t) = a_0 - 2k_1\left(\frac{\sqrt{-Z}}{2} cot(\frac{\sqrt{-Z}}{2}\xi) - \frac{A}{2}\right), \tag{27}$$

where $\xi = (k_1 \frac{x^\beta}{\Gamma(\beta+1)} + k_2 \frac{y^\gamma}{\Gamma(\gamma+1)} - \frac{(-4\ k_1^2\ k_2\ B + k_2\ a + k_1^2\ k_2\ A^2)t^\alpha}{\Gamma(\alpha+1)})$

Family5: When $Z > 0$ then equations (17) and (12) implies the following hyperbolic solitary wave solutions:

(i) When $D_1 \neq 0$ and $D_2 = 0$

$$V_9(x,y,t) = a_0 - 2k_1\left(\frac{\sqrt{Z}}{2} tanh(\frac{\sqrt{Z}}{2}\xi) - \frac{A}{2}\right), \tag{28}$$

(ii) When $D_1 = 0$ and $D_2 \neq 0$

$$V_{10}(x,y,t) = a_0 - 2k_1\left(\frac{\sqrt{Z}}{2} coth(\frac{\sqrt{Z}}{2}\xi) - \frac{A}{2}\right), \tag{29}$$

where $\xi = (k_1 \frac{x^\beta}{\Gamma(\beta+1)} + k_2 \frac{y^\gamma}{\Gamma(\gamma+1)} - \frac{(-4\ k_1^2\ k_2\ B + k_2\ a + k_1^2\ k_2\ A^2)t^\alpha}{\Gamma(\alpha+1)})$

Family6: When $Z = 0$ then equations (17) and (12) implies the following rational solitary wave solutions:

(i) When $D_1 \neq 0$ and $D_2 = 0$

$$V_{11}(x,y,t) = a_0 + 2k_1\left(\frac{A}{2}\right), \tag{30}$$

(ii) When $D_1 = 0$ and $D_2 \neq 0$

$$V_{12}(x,y,t) = a_0 - 2k_1\left(\frac{1}{\xi} - \frac{A}{2}\right), \tag{31}$$

where $\xi = (k_1 \frac{x^\beta}{\Gamma(\beta+1)} + k_2 \frac{y^\gamma}{\Gamma(\gamma+1)} - \frac{(-4\ k_1^2\ k_2\ B + k_2\ a + k_1^2\ k_2\ A^2)t^\alpha}{\Gamma(\alpha+1)})$

Problem 2: Consider the second extended FSWWE given in equation (5):

$$v_{yt}^{\gamma\alpha} + a v_{xy}^{\beta\gamma} + v_{xxxy}^{3\beta\gamma} - 3v_y^\gamma v_{xx}^{2\beta} - 3v_x^\beta v_{xy}^{\beta\gamma} = 0, \tag{32}$$

applying transformation of the form:

$$v(x,t) = v(\xi), \quad where$$
$$\xi = k_1 \frac{x^\beta}{\Gamma(1+\beta)} + k_2 \frac{y^\gamma}{\Gamma(1+\gamma)} - c\frac{t^\alpha}{\Gamma(1+\alpha)}, \tag{33}$$

we get an ODE, integrating ODE and considering constant of integration zero, we have

$$k_1^3 V''' + (k_1 a - c)V' - 3k_1^2 (V')^2 = 0, \tag{34}$$

where $V' = \frac{dV}{d\xi}$, c represents the speed of the traveling wave while k_1 and k_2 are constants.

For homogenous number m, we consider the homogeneous balance between highest derivative V''' and $(V')^2$ given in equation (34). So, by using equation (13) we get $m = 1$. Now putting $m = 1$ in (10) we get the following $(\frac{G'}{G})$ solution for (32):

$$V(\xi) = \sum_{i=-1}^{1} a_i (\frac{G'(\xi)}{G(\xi)})^i = a_{-1}(\frac{G'(\xi)}{G(\xi)})^{-1} + a_0 + a_1(\frac{G'(\xi)}{G(\xi)})^1, \tag{35}$$

where a_1, a_0 and a_{-1} are unknown constants.

By putting equation (35) with the help of (11) in equation (34) a polynomial in $\frac{G'(\xi)}{G(\xi)}$ is formed, equating the coefficients of same power of $(\frac{G'}{G})$ to 0 the system of non-linear equations is formed. By using Maple, this system provides two solutions given as:

Case 1:

$$a_0 = a_0,\ a_{-1} = 2k_1 B,\ a_1 = 0 \quad and\ c = -4k_1^3 B + k_1 a + k_1^3 A^2 \tag{36}$$

Case 2:

$$a_0 = a_0,\ a_{-1} = 0,\ a_1 = -2k_1 \quad and\ c = -4k_1^3 B + k_1 a + k_1^3 A^2 \tag{37}$$

Considering case.1 and suppose that $Z = A^2 - 4B$ for simplicity in calculation, we get the following families of solutions

Family1: When $Z < 0$ then equation (35) and (12) implies the following periodic solitary wave solutions:

(i) When $D_1 \neq 0$ and $D_2 = 0$

$$V_1(x,y,t) = a_0 - 2k_1 B\left(\frac{\sqrt{-Z}}{2} tan(\frac{\sqrt{-Z}}{2}\xi) + \frac{A}{2}\right)^{-1}, \tag{38}$$

(ii) When $D_1 = 0$ and $D_2 \neq 0$

$$V_2(x,y,t) = a_0 + 2k_1 B\left(\frac{\sqrt{-Z}}{2} cot(\frac{\sqrt{-Z}}{2}\xi) - \frac{A}{2}\right)^{-1}, \tag{39}$$

where $\xi = (k_1 \frac{x^\beta}{\Gamma(\beta+1)} + k_2 \frac{y^\gamma}{\Gamma(\gamma+1)} - \frac{(-4\ k_1^3\ B + k_1\ a + k_1^3\ A^2)t^\alpha}{\Gamma(\alpha+1)})$

Family2: When $Z > 0$ then equation (35) and (12) implies the following hyperbolic solitary wave solutions:

(i) When $D_1 \neq 0$ and $D_2 = 0$

$$V_3(x,y,t) = a_0 + 2k_1 B\left(\frac{\sqrt{Z}}{2} tanh(\frac{\sqrt{Z}}{2}\xi) - \frac{A}{2}\right)^{-1}, \tag{40}$$

(ii) When $D_1 = 0$ and $D_2 \neq 0$

$$V_4(x,y,t) = a_0 + 2k_1 B\left(\frac{\sqrt{Z}}{2} coth(\frac{\sqrt{Z}}{2}\xi) - \frac{A}{2}\right)^{-1}, \tag{41}$$

where $\xi = (k_1 \frac{x^\beta}{\Gamma(\beta+1)} + k_2 \frac{y^\gamma}{\Gamma(\gamma+1)} - \frac{(-4\ k_1^3\ B + k_1\ a + k_1^3\ A^2)t^\alpha}{\Gamma(\alpha+1)})$

Family3: When $Z = 0$ then equation (35) and (12) implies the following rational solitary wave solutions:

(i) When $D_1 \neq 0$ and $D_2 = 0$

$$V_5(x, y, t) = a_0 + 2k_1 B\left(-\frac{A}{2}\right)^{-1}, \tag{42}$$

(ii) When $D_1 = 0$ and $D_2 \neq 0$

$$V_6(x, y, t) = a_0 + 2k_1 B\left(\frac{1}{\xi} - \frac{A}{2}\right)^{-1}, \tag{43}$$

where $\xi = (k_1 \frac{x^\beta}{\Gamma(\beta+1)} + k_2 \frac{y^\gamma}{\Gamma(\gamma+1)} - \frac{(-4\,k_1^3\,B+k_1\,a+k_1^3\,A^2)t^\alpha}{\Gamma(\alpha+1)})$.

Now considering case.2 and suppose that $Z = A^2 - 4B$ for simplicity in calculation, we get further two families of solutions given as:

Family4: When $Z < 0$ then equation (35) and (12) implies the following periodic solitary wave solutions:

(i) When $D_1 \neq 0$ and $D_2 = 0$

$$V_7(x, y, t) = a_0 + 2k_1\left(\frac{\sqrt{-Z}}{2}tan(\frac{\sqrt{-Z}}{2}\xi) + \frac{A}{2}\right), \tag{44}$$

(ii) When $D_1 = 0$ and $D_2 \neq 0$

$$V_8(x, y, t) = a_0 - 2k_1\left(\frac{\sqrt{-Z}}{2}cot(\frac{\sqrt{-Z}}{2}\xi) - \frac{A}{2}\right), \tag{45}$$

where $\xi = (k_1 \frac{x^\beta}{\Gamma(\beta+1)} + k_2 \frac{y^\gamma}{\Gamma(\gamma+1)} - \frac{(-4\,k_1^3\,B+k_1\,a+k_1^3\,A^2)t^\alpha}{\Gamma(\alpha+1)})$

Family5: When $Z > 0$ then equation (35) and (12) implies the following hyperbolic solitary wave solutions:

(i) When $D_1 \neq 0$ and $D_2 = 0$

$$V_9(x, y, t) = a_0 - 2k_1\left(\frac{\sqrt{Z}}{2}tanh(\frac{\sqrt{Z}}{2}\xi) - \frac{A}{2}\right), \tag{46}$$

(ii) When $D_1 = 0$ and $D_2 \neq 0$

$$V_{10}(x, y, t) = a_0 - 2k_1\left(\frac{\sqrt{Z}}{2}coth(\frac{\sqrt{Z}}{2}\xi) - \frac{A}{2}\right), \tag{47}$$

where $\xi = (k_1 \frac{x^\beta}{\Gamma(\beta+1)} + k_2 \frac{y^\gamma}{\Gamma(\gamma+1)} - \frac{(-4\,k_1^3\,B+k_1\,a+k_1^3\,A^2)t^\alpha}{\Gamma(\alpha+1)})$

Family6: When $Z = 0$ then equation (35) and (12) implies the following rational solitary wave solutions:

(i) When $D_1 \neq 0$ and $D_2 = 0$

$$V_{11}(x, y, t) = a_0 + 2k_1\left(\frac{A}{2}\right), \tag{48}$$

(ii) When $D_1 = 0$ and $D_2 \neq 0$

$$V_{12}(x, y, t) = a_0 - 2k_1\left(\frac{1}{\xi} - \frac{A}{2}\right), \tag{49}$$

where $\xi = (k_1 \frac{x^\beta}{\Gamma(\beta+1)} + k_2 \frac{y^\gamma}{\Gamma(\gamma+1)} - \frac{(-4\,k_1^3\,B+k_1\,a+k_1^3\,A^2)t^\alpha}{\Gamma(\alpha+1)})$

Problem 3: Consider the third extended FSWWE given in equation (6):

$$v_{yzt}^{\gamma\delta\alpha} + av_{xyz}^{\beta\gamma\delta} + v_{xxxyz}^{3\beta\gamma\delta} - 6v_{xy}^{\beta\gamma}v_{xz}^{\beta\delta} - 6v_x^{\beta}v_{xyz}^{\beta\gamma\delta} = 0, \tag{50}$$

applying transformation of the form:

$$v(x, t) = v(\xi), \quad where\ \xi = k_1\frac{x^\beta}{\Gamma(1+\beta)} + k_2\frac{y^\gamma}{\Gamma(1+\gamma)} + k_3\frac{z^\delta}{\Gamma(1+\delta)} + c\frac{t^\alpha}{\Gamma(1+\alpha)} \tag{51}$$

we get the following ODE:

$$(k_2k_3c + ak_1k_2k_3)V''' + k_1^3k_2k_3V''''' - 6k_1^2k_2k_3(V'V''' + (V'')^2) = 0, \tag{52}$$

where $V' = \frac{dV}{d\xi}$, c represents the speed of the traveling wave while k_1, k_2 and k_3 are constants.

For homogeneous number m, we consider the homogenous balance between highest derivative V''''' and $(V'')^2$ given in equation (52). So, by using equation (13) we get $m = 1$. Now putting $m = 1$ in (10) we get the following $(\frac{G'}{G})$ solution for (50):

$$V(\xi) = \sum_{i=-1}^{1} a_i(\frac{G'(\xi)}{G(\xi)})^i = a_0 + a_{-1}(\frac{G'(\xi)}{G(\xi)})^{-1} + a_1(\frac{G'(\xi)}{G(\xi)})^1, \tag{53}$$

where a_1, a_0 and a_{-1} are unknown constants. By putting equation (53) with the help of (11) in equation (52) a polynomial in $\frac{G'(\xi)}{G(\xi)}$ is formed, equating the coefficients of same power of $(\frac{G'}{G})$ to 0 the system of non-linear equations is formed. By using Maple, this system provides two solutions given as:

Case 1:

$$a_{-1} = 2k_1B, \quad a_0 = a_0,\ a_1 = 0,\ k_1 = k_1,\ k_2 = k_2,\ k_3 = k_3 \quad and\ c = 4k_1^3B - k_1a - k_1^3A^2 \tag{54}$$

Case 2:

$$a_{-1} = 0, \quad a_0 = a_0,\ a_1 = -2k_1,\ k_1 = k_1,\ k_2 = k_2,\ k_3 = k_3 \quad and\ c = 4k_1^3B - k_1a - k_1^3A^2 \tag{55}$$

Case 3:

$$a_0 = a_0, \quad k_1 = k_1,\ a_{-1} = a_{-1},\ k_2 = k_2,\ a_1 = a_1,\ k_3 = 0 \quad and\ c = c \tag{56}$$

Case 4:

$$a_0 = a_0, \quad k_1 = k_1,\ a_{-1} = a_{-1},\ k_2 = 0,\ a_1 = a_1,\ k_3 = k_3 \quad and\ c = c \tag{57}$$

Case 5:

$$a_0 = a_0, \quad k_1 = 0,\ a_{-1} = a_{-1},\ k_2 = k_2,\ a_1 = a_1,\ k_3 = k_3 \quad and\ c = 0 \tag{58}$$

Considering case.1 and suppose that $Z = A^2 - 4B$ for simplicity in calculation, we get the following families of solutions

Family1: When $Z < 0$ then equation (53) and (12) implies the following periodic solitary wave solutions:

(i) When $D_1 \neq 0$ and $D_2 = 0$

$$V_1(x, y, t) = a_0 - 2k_1B\left(\frac{\sqrt{-Z}}{2}tan(\frac{\sqrt{-Z}}{2}\xi) + \frac{A}{2}\right)^{-1}, \tag{59}$$

(ii) When $D_1 = 0$ and $D_2 \neq 0$

$$V_2(x, y, t) = a_0 + 2k_1B\left(\frac{\sqrt{-Z}}{2}cot(\frac{\sqrt{-Z}}{2}\xi) - \frac{A}{2}\right)^{-1}, \tag{60}$$

where $\xi = k_1\frac{x^\beta}{\Gamma(\beta+1)} + k_2\frac{y^\gamma}{\Gamma(\gamma+1)} + k_3\frac{z^\delta}{\Gamma(\delta+1)} + \frac{(4\,k_1^3\,B-k_1\,a-k_1^3\,A^2)t^\alpha}{\Gamma(\alpha+1)}$

Family2: When $Z > 0$ then equation (53) and (12) implies the following hyperbolic solitary wave solutions:

(i) When $D_1 \neq 0$ and $D_2 = 0$

$$V_3(x,y,t) = a_0 + 2k_1B\left(\frac{\sqrt{Z}}{2}tanh(\frac{\sqrt{Z}}{2}\xi) - \frac{A}{2}\right)^{-1}, \tag{61}$$

(ii) When $D_1 = 0$ and $D_2 \neq 0$

$$V_4(x,y,t) = a_0 + 2k_1B\left(\frac{\sqrt{Z}}{2}coth(\frac{\sqrt{Z}}{2}\xi) - \frac{A}{2}\right)^{-1}, \tag{62}$$

where $\xi = k_1\frac{x^\beta}{\Gamma(\beta+1)} + k_2\frac{y^\gamma}{\Gamma(\gamma+1)} + k_3\frac{z^\delta}{\Gamma(\delta+1)} + \frac{(4\,k_1^3\,B-k_1\,a-k_1^3\,A^2)t^\alpha}{\Gamma(\alpha+1)}$

Family3: When $Z = 0$ then equation (53) and equation (12) implies the following rational solitary wave solutions:

(i)When $D_1 \neq 0$ and $D_2 = 0$

$$V_5(x,y,t) = a_0 + 2k_1B\left(-\frac{A}{2}\right)^{-1}, \tag{63}$$

(ii) When $D_1 = 0$ and $D_2 \neq 0$

$$V_6(x,y,t) = a_0 + 2k_1B\left(\frac{1}{\xi} - \frac{A}{2}\right)^{-1}, \tag{64}$$

where $\xi = k_1\frac{x^\beta}{\Gamma(\beta+1)} + k_2\frac{y^\gamma}{\Gamma(\gamma+1)} + k_3\frac{z^\delta}{\Gamma(\delta+1)} + \frac{(4\,k_1^3\,B-k_1\,a-k_1^3\,A^2)t^\alpha}{\Gamma(\alpha+1)}$.

Now considering case.2 and suppose that $Z = A^2 - 4B$ for simplicity in calculation, we get further two families of solutions given as:

Family4: When $Z < 0$ then equation (53) and (12) implies the following periodic solitary wave solutions:

(i) When $D_1 \neq 0$ and $D_2 = 0$

$$V_7(x,y,t) = a_0 + 2k_1\left(\frac{\sqrt{-Z}}{2}tan(\frac{\sqrt{-Z}}{2}\xi) + \frac{A}{2}\right), \tag{65}$$

(ii) When $D_1 = 0$ and $D_2 \neq 0$

$$V_8(x,y,t) = a_0 - 2k_1\left(\frac{\sqrt{-Z}}{2}cot(\frac{\sqrt{-Z}}{2}\xi) - \frac{A}{2}\right), \tag{66}$$

where $\xi = k_1\frac{x^\beta}{\Gamma(\beta+1)} + k_2\frac{y^\gamma}{\Gamma(\gamma+1)} + k_3\frac{z^\delta}{\Gamma(\delta+1)} + \frac{(4\,k_1^3\,B-k_1\,a-k_1^3\,A^2)t^\alpha}{\Gamma(\alpha+1)}$

Family5: When $Z > 0$ then equation (53) and (12) implies the following hyperbolic solitary wave solutions:

(i) When $D_1 \neq 0$ and $D_2 = 0$

$$V_9(x,y,t) = a_0 - 2k_1\left(\frac{\sqrt{Z}}{2}tanh(\frac{\sqrt{Z}}{2}\xi) - \frac{A}{2}\right), \tag{67}$$

(ii) When $D_1 = 0$ and $D_2 \neq 0$

$$V_{10}(x,y,t) = a_0 - 2k_1\left(\frac{\sqrt{Z}}{2}coth(\frac{\sqrt{Z}}{2}\xi) - \frac{A}{2}\right), \tag{68}$$

where $\xi = k_1\frac{x^\beta}{\Gamma(\beta+1)} + k_2\frac{y^\gamma}{\Gamma(\gamma+1)} + k_3\frac{z^\delta}{\Gamma(\delta+1)} + \frac{(4\,k_1^3\,B-k_1\,a-k_1^3\,A^2)t^\alpha}{\Gamma(\alpha+1)}$

Family6: When $Z = 0$ then equation (53) and (12) implies the following rational solitary wave solutions:

(i) When $D_1 \neq 0$ and $D_2 = 0$

$$V_{11}(x,y,t) = a_0 + 2k_1\left(\frac{A}{2}\right), \tag{69}$$

(ii) When $D_1 = 0$ and $D_2 \neq 0$

$$V_{12}(x,y,t) = a_0 - 2k_1\left(\frac{1}{\xi} - \frac{A}{2}\right), \tag{70}$$

where $\xi = k_1\frac{x^\beta}{\Gamma(\beta+1)} + k_2\frac{y^\gamma}{\Gamma(\gamma+1)} + k_3\frac{z^\delta}{\Gamma(\delta+1)} + \frac{(4\,k_1^3\,B-k_1\,a-k_1^3\,A^2)t^\alpha}{\Gamma(\alpha+1)}$.

Now considering case.3 and suppose that $Z = A^2 - 4B$ for simplicity in calculation, we get further two families of solutions given as:

Family7: When $Z < 0$ then equation (53) and (12) implies the following periodic solitary wave solutions:

(i) When $D_1 \neq 0$ and $D_2 = 0$

$$\begin{aligned}V_{13}(x,y,t) &= a_{-1}\left(\frac{\sqrt{-Z}}{2}tan(\frac{\sqrt{-Z}}{2}\xi) + \frac{A}{2}\right)^{-1}\\ &\quad + a_0 + a_1\left(\frac{\sqrt{-Z}}{2}tan(\frac{\sqrt{-Z}}{2}\xi) + \frac{A}{2}\right),\end{aligned} \tag{71}$$

(ii) When $D_1 = 0$ and $D_2 \neq 0$

$$\begin{aligned}V_{14}(x,y,t) &= a_{-1}\left(\frac{\sqrt{-Z}}{2}cot(\frac{\sqrt{-Z}}{2}\xi) + \frac{A}{2}\right)^{-1}\\ &\quad + a_0 + a_1\left(\frac{\sqrt{-Z}}{2}cot(\frac{\sqrt{-Z}}{2}\xi) + \frac{A}{2}\right),\end{aligned} \tag{72}$$

where $\xi = k_1\frac{x^\beta}{\Gamma(\beta+1)} + k_2\frac{y^\gamma}{\Gamma(\gamma+1)} + c\frac{t^\alpha}{\Gamma(\alpha+1)}$

Family8: When $Z > 0$ then equation (53) and (12) implies the following hyperbolic solitary wave solutions: (i) When $D_1 \neq 0$ and $D_2 = 0$

$$\begin{aligned}V_{15}(x,y,t) &= a_{-1}\left(\frac{\sqrt{Z}}{2}tanh(\frac{\sqrt{Z}}{2}\xi) + \frac{A}{2}\right)^{-1}\\ &\quad + a_0 + a_1\left(\frac{\sqrt{-Z}}{2}tanh(\frac{\sqrt{Z}}{2}(\xi)) + \frac{A}{2}\right),\end{aligned} \tag{73}$$

(ii) When $D_1 = 0$ and $D_2 \neq 0$

$$\begin{aligned}V_{16}(x,y,t) &= a_{-1}\left(\frac{\sqrt{Z}}{2}coth(\frac{\sqrt{Z}}{2}\xi) + \frac{A}{2}\right)^{-1}\\ &\quad + a_0 + a_1\left(\frac{\sqrt{Z}}{2}coth(\frac{\sqrt{Z}}{2}(\xi)) + \frac{A}{2}\right),\end{aligned} \tag{74}$$

where $\xi = k_1\frac{x^\beta}{\Gamma(\beta+1)} + k_2\frac{y^\gamma}{\Gamma(\gamma+1)} + c\frac{t^\alpha}{\Gamma(\alpha+1)}$

Family9: When $Z = 0$ then equation (53) and (12) implies the following rational solitary wave solutions:
(i) When $D_1 \neq 0$ and $D_2 = 0$

$$V_{17}(x, y, t) = a_{-1}\left(\frac{A}{2}\right)^{-1} + a_0 + a_1\left(\frac{A}{2}\right), \quad (75)$$

(ii) When $D_1 = 0$ and $D_2 \neq 0$

$$V_{18}(x, y, t) = a_{-1}\left(\frac{1}{\xi} - \frac{A}{2}\right)^{-1} + a_0 + a_1\left(\frac{1}{\xi} - \frac{A}{2}\right), \quad (76)$$

where $\xi = k_1\frac{x^\beta}{\Gamma(\beta+1)} + k_2\frac{y^\gamma}{\Gamma(\gamma+1)} + c\frac{t^\alpha}{\Gamma(\alpha+1)}$.

Now considering case.4 and suppose that $Z = A^2 - 4B$ for simplicity in calculation, we get further two families of solutions given as:

Family10: When $Z < 0$ then equation (53) and (12) implies the following periodic solitary wave solutions:

(i) When $D_1 \neq 0$ and $D_2 = 0$

$$V_{19}(x, y, t) = a_{-1}\left(\frac{\sqrt{-Z}}{2}tan(\frac{\sqrt{-Z}}{2}\xi) + \frac{A}{2}\right)^{-1} + a_0 + a_1\left(\frac{\sqrt{-Z}}{2}tan(\frac{\sqrt{-Z}}{2}\xi) + \frac{A}{2}\right), \quad (77)$$

(ii) When $D_1 = 0$ and $D_2 \neq 0$

$$V_{20}(x, y, t) = a_{-1}\left(\frac{\sqrt{-Z}}{2}cot(\frac{\sqrt{-Z}}{2}\xi) + \frac{A}{2}\right)^{-1} + a_0 + a_1\left(\frac{\sqrt{-Z}}{2}cot(\frac{\sqrt{-Z}}{2}\xi) + \frac{A}{2}\right), \quad (78)$$

where $\xi = k_1\frac{x^\beta}{\Gamma(\beta+1)} + k_3\frac{z^\delta}{\Gamma(\delta+1)} + c\frac{t^\alpha}{\Gamma(\alpha+1)}$

Family11: When $Z > 0$ then equation (53) and (12) implies the following hyperbolic solitary wave solutions:
(i) When $D_1 \neq 0$ and $D_2 = 0$

$$V_{21}(x, y, t) = a_{-1}\left(\frac{\sqrt{Z}}{2}tanh(\frac{\sqrt{Z}}{2}\xi) + \frac{A}{2}\right)^{-1} + a_0 + a_1\left(\frac{\sqrt{Z}}{2}tanh(\frac{\sqrt{Z}}{2}\xi) + \frac{A}{2}\right), \quad (79)$$

(ii) When $D_1 = 0$ and $D_2 \neq 0$

$$V_{22}(x, y, t) = a_{-1}\left(\frac{\sqrt{Z}}{2}coth(\frac{\sqrt{Z}}{2}\xi) + \frac{A}{2}\right)^{-1} + a_0 + a_1\left(\frac{\sqrt{Z}}{2}coth(\frac{\sqrt{Z}}{2}\xi) + \frac{A}{2}\right), \quad (80)$$

where $\xi = k_1\frac{x^\beta}{\Gamma(\beta+1)} + k_3\frac{z^\delta}{\Gamma(\delta+1)} + c\frac{t^\alpha}{\Gamma(\alpha+1)}$

Family12: When $Z = 0$ then equation (53) and (12) implies the following rational solitary wave solutions:
(i) When $D_1 \neq 0$ and $D_2 = 0$

$$V_{23}(x, y, t) = a_{-1}\left(\frac{A}{2}\right)^{-1} + a_0 + a_1\left(\frac{A}{2}\right), \quad (81)$$

(ii) When $D_1 = 0$ and $D_2 \neq 0$

$$V_{24}(x, y, t) = a_{-1}\left(\frac{1}{\xi} - \frac{A}{2}\right)^{-1} + a_0 + a_1\left(\frac{1}{\xi} - \frac{A}{2}\right), \quad (82)$$

where $\xi = k_1\frac{x^\beta}{\Gamma(\beta+1)} + k_3\frac{z^\delta}{\Gamma(\delta+1)} + c\frac{t^\alpha}{\Gamma(\alpha+1)}$.

Now considering case.5 and suppose that $Z = A^2 - 4B$ for simplicity in calculation, we get further two families of solutions given as:

Family13: When $Z < 0$ then equation (53) and (12) implies the following periodic solitary wave solutions:

(i) When $D_1 \neq 0$ and $D_2 = 0$

$$V_{25}(x, y, t) = a_{-1}\left(\frac{\sqrt{-Z}}{2}tan(\frac{\sqrt{-Z}}{2}\xi) + \frac{A}{2}\right)^{-1} + a_0 + a_1\left(\frac{\sqrt{-Z}}{2}tan(\frac{\sqrt{-Z}}{2}\xi) + \frac{A}{2}\right), \quad (83)$$

(ii) When $D_1 = 0$ and $D_2 \neq 0$

$$V_{26}(x, y, t) = a_{-1}\left(\frac{\sqrt{-Z}}{2}cot(\frac{\sqrt{-Z}}{2}\xi) + \frac{A}{2}\right)^{-1} + a_0 + a_1\left(\frac{\sqrt{-Z}}{2}cot(\frac{\sqrt{-Z}}{2}\xi) + \frac{A}{2}\right), \quad (84)$$

where $\xi = k_2\frac{y^\gamma}{\Gamma(\gamma+1)} + k_3\frac{z^\delta}{\Gamma(\delta+1)}$

Family14: When $Z > 0$ then equation (53) and (12) implies the following hyperbolic solitary wave solutions:
(i) When $D_1 \neq 0$ and $D_2 = 0$

$$V_{27}(x, y, t) = a_{-1}\left(\frac{\sqrt{Z}}{2}tanh(\frac{\sqrt{Z}}{2}\xi) + \frac{A}{2}\right)^{-1} + a_0 + a_1\left(\frac{\sqrt{Z}}{2}tanh(\frac{\sqrt{Z}}{2}\xi) + \frac{A}{2}\right), \quad (85)$$

(ii) When $D_1 = 0$ and $D_2 \neq 0$

$$V_{28}(x, y, t) = a_{-1}\left(\frac{\sqrt{Z}}{2}coth(\frac{\sqrt{Z}}{2}\xi) + \frac{A}{2}\right)^{-1} + a_0 + a_1\left(\frac{\sqrt{Z}}{2}coth(\frac{\sqrt{Z}}{2}\xi) + \frac{A}{2}\right), \quad (86)$$

where $\xi = k_2\frac{y^\gamma}{\Gamma(\gamma+1)} + k_3\frac{z^\delta}{\Gamma(\delta+1)}$

Family15: When $Z = 0$ then equation (53) and (12) implies the following rational solitary wave solutions:
(i) When $D_1 \neq 0$ and $D_2 = 0$

$$V_{29}(x, y, t) = a_{-1}\left(\frac{A}{2}\right)^{-1} + a_0 + a_1\left(\frac{A}{2}\right), \quad (87)$$

(ii) When $D_1 = 0$ and $D_2 \neq 0$

$$V_{30}(x, y, t) = a_{-1}\left(\frac{1}{\xi} - \frac{A}{2}\right)^{-1} + a_0 + a_1\left(\frac{1}{\xi} - \frac{A}{2}\right), \quad (88)$$

where $\xi = k_2\frac{y^\gamma}{\Gamma(\gamma+1)} + k_3\frac{z^\delta}{\Gamma(\delta+1)}$ and $Z = A^2 - 4B$.

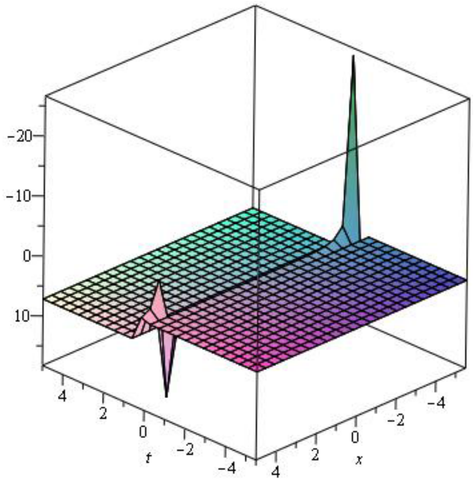

FIGURE 1. Graph of (29) for $a_0 = 3, k_1 = 0.8, k_2 = 2, a = 5, A = 3, B = 1, \alpha = \beta = \gamma = 1$ *and* $y = 0$.

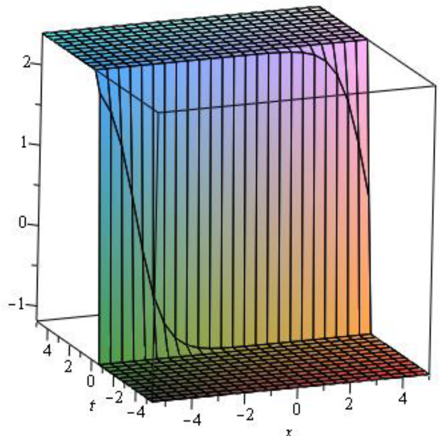

FIGURE 2. Graph of (22) for $a_0 = 3, k_1 = 0.8, k_2 = 2, a = 2, A = 3, B = 1, \alpha = \beta = \gamma = 1$ *and* $y = 0$.

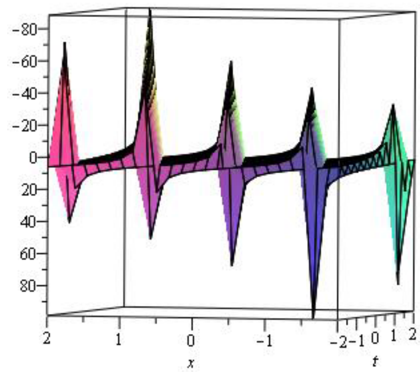

FIGURE 3. Graph of (26) for $a_0 = 5, k_1 = -0.8, k_2 = -1, a = 2, A = 0.2, B = 3, \alpha = \beta = \gamma = 1$ *and* $y = 0$.

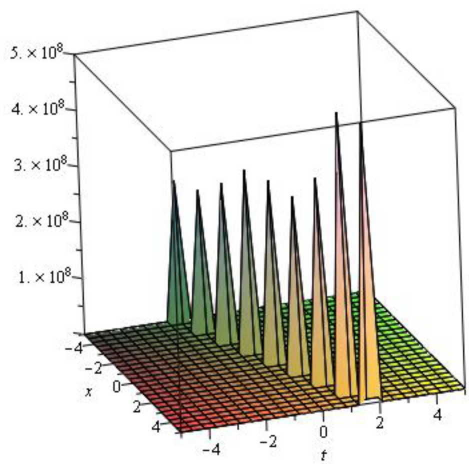

FIGURE 4. Graph of (49) for $a_0 = 3, k_1 = 1.5, k_2 = -0.5, a = 3, A = 2, B = 1, \alpha = \beta = \gamma = 1$ *and* $y = 0$.

IV. DISCUSSION AND GRAPHS

The aim of the present paper is to solve extanded SWWEs in fractional order by using extended $(\frac{G'}{G})$-expansion method. Thus we solved and found various families of solutions. It is concluded that our work is the generalization of the works present in literature. For example, Faisal and Kumar have solved equation (1-2) with the help of improved $(\frac{G'}{G})$-expansion method [28]. The solutions given in [28] are the special cases of the solutions obtained in the present research work, that is for $\alpha = \beta = \gamma = 1$ in the current solutions given in equations (20-31) and (38-49) we get the solutions given in [28]. Similarly, Bekir in [29] have used simple $(\frac{G'}{G})$-expansion method and obtained only six solutions. If, we ignore the portion $\sum_{i=-n}^{0} a_i(\frac{G'(\xi)}{G(\xi)})^i$ in our proposed method then our solutions coincide with Bekir's solutions. Hence, it is concluded that simple $(\frac{G'}{G})$-expansion the special case of the current method. If, we consider our solutions in fractional order then the results obtained are new so far. As compare to differential equations in integer order, the fractional ordered differential equations can model any physical phenomena correctly. Therefore, the proposed method provided distinct exact solutions to physical problems in non-linear science which explain the inner part mechanism of any physical problem. Moreover, the present method doesn't need any linearization process as compare to some other methods in literature for solving non-linear problems.

The suggested method provide three types of travelling wave solutions for the problems, periodic, hyperbolic and rational function solutions. The solutions are further classified by using different values to parameters in the above solitary wave solutions. The parameter c in the solutions obtained the speed of these waves. The balance between linear and non-linearity effect delivers solitary wave solutions. By using a Maple software and assigning particular values to parameters present in the exact solutions, we have shown the nature of some periodic and solitary solutions in following figures.

Remarks 1: The Fig.1 shows soliton profile which is similar to the figure of equation (21) in [28].

Remarks 2: The Fig.2 shows kink profile which is similar to the figure of equation (24) in [28].

Remarks 3: The Fig.3 shows periodic profile which is similar to the figure of equation (28) in [28].

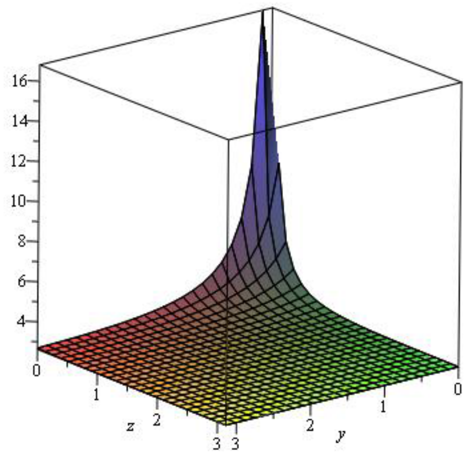

FIGURE 5. Graph of (86) for $a_0 = 3, a_1 = 1, a_{-1} = 0, k_1 = 1.5, k_2 = 0.5, k_3 = 1, a = 3, A = 3, B = 1, \delta = \gamma = 1$ *and* $y = 0$.

Remarks 4: The Fig.4 shows soliton profile which is similar to the figure of equation (55) in [28].

V. CONCLUSION

The analytical solutions of extended FSWWEs are obtained by using extended $(\frac{G'}{G})$-expansion method. These solutions include hyperbolic, rational and trigonometric function solution. The free parameters in various solutions, provide the behaviour of solutions. Moreover, with the help of these parameters we constructed solutions that may be related to physical problems in real phenomena.

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