A Simple Model for the Entropy of a System With Interacting Particles

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ABSTRACT We present a simple model to calculate the system entropy of interacting particles where the interaction is modeled by unusual collisions concerning initial and final states. This model is based on the concept of dissipators to describe dissipative interactions between particles. Starting from the entropy of Clausius, we obtain expressions for the entropy of Boltzmann, Shannon and Tsallis. For this, we use a few simple rules related to the dissipaters in a kinetic model (e.g., average energy and the concept of statistical temperature in molecular theory). This work shows a possible explanation about the physical interpretation of parameter q in the Tsallis theory and the connection between the entropy of Tsallis and the entropy of Boltzmann-Shannon; under the concept of the dissipaters this theory does not exclude them since the way in which the energy is dissipated is in a certain way shared, these entropies belong to the same family of dissipators, to the powers of energy. The usual form of entropy is recovered according to the probability of phase-space, in terms of the energy density of the system.

INDEX TERMS Interacting systems, generalized entropies, thermo-statistical model, tsallis entropy.

I. INTRODUCTION

Boltzmann's entropy is one of the most important pillars of statistical physics. Some decades ago, Shannon [1] proposed an alternative way of understanding classical entropy. Based on probability ideas, he defined the revolutionary concept of information and it redefined the entropy through information theory. This broadened the picture and, above all, allowed the concept of entropy to permeate other areas. Subsequently, Rényi [2] proposed an alternative method of defining the information that generated an unbridled explosion by proposing different ways of defining entropy, such as the trigonometric entropies proposed by Aczél and Daróczy [3] and the weighted entropies proposed by Belis and Guiasu [4]. Rényi's entropy (H_r) belongs to a group called parametric entropies; they are characterized by some parameters such as s, r or t. Rényi's entropy is defined as,

$$H_r = \left[\frac{1}{1-r} \ln \sum_{i=1}^{N} p_i^r \right], \quad r > 1, \tag{1}$$

where p_i is the probability concerning the outcome of an experiment i and r is the parameter. Sharma and Taneja [5]

define the entropy in terms of two parameters, expressed as

$$H_{r,s} = \frac{1}{2^{1-r} - 2^{1-s}} \sum_{i=1}^{N} \left(p_i^r - p_i^s \right), \quad r \neq s; r, \ s > 1, \tag{2}$$

where r and s are parameters. Sant'anna and Taneja [6] proposed several trigonometric entropies, such as the following,

$$H_s = \sum_{i=1}^{N} \frac{\sin(s p_i)}{2 \sin(\frac{s}{2})}, \quad 0 < s < \pi. \tag{3}$$

These entropies are only some of them; there are about 30 similar proposals, among them the entropy proposed by Tsallis [7] is the one that stands out. Although it is difficult to ensure that some entropy is better than another, the latter has had a recent boom, this has recently been used in different applications. Tsallis' entropy is defined as

$$H_T = \frac{1}{q-1} \left[1 - \sum_{i=1}^{N} (p_i)^q \right], \quad q \in \mathbb{R}, q \neq 1. \tag{4}$$

All these proposals are known as generalized entropies; from these entropies we obtain Shannon's as a particular case or as a limit. Shannon's entropy satisfies a large number of mathematical properties, as Taneja [8] pointed out. Almost all

alternative entropies comply with the same mathematical properties as Shannon's. There are several ways to obtain generalized entropies; one is by using a functional equation where the solution is the sought entropy, another is by an axiomatic procedure, that is, we give a set of axioms that must be satisfied. There is also an interesting proposal in the field of relativistic statistics, offered by Kaniadakis [9]. This is deduced by using exponential and logarithmic deformed functions in a similar way as those used by Tsallis. We can also use combinatorial arguments to obtain generalized combinatorial entropies, for example [10], or by using coding theorems such as [11], etc.

Tsallis' entropy [7] is the most widely accepted today, since it describes phenomena where experimental evidence shows that the principle of extensivity of classical entropy is not satisfied, but their application goes beyond the information or statistics. It has been used to solve non-linear differential equations [12], [13] and Tsallis himself proved that his statistics are reduced to the canonical ensemble when its parameter approach to one [7] and the analysis has been extended, by Parvan [14], [15], Tsallis has an excellent review about the evolution and applications of nonadditive entropy Sq [16]. The applications are copious and every day more appear in technical journals. For example, in social science and economy [17], in fluids [18], in time series [19], in pattern recognition [20], in image segmentation [21] and many more [22]–[25]. Something similar happens with the q-generalized functions which are products of such a theory [20], [26]–[28]. Many articles about Tsallis entropy are only concerned with the mathematical topics; after this, immediately the authors turn to applications. That research is leaving apart physics and statistic questions and we think they are also relevant to understanding Tsallis's theory. For this reason, the present work seeks to explore a little with highly correlated interacting particle systems where the use of such an entropy is justified. We try to explain a possible direct physical connection between Tsallis' entropy and the interactions by a simple mechanism of dissipation.

II. SIMPLIFIED INTERACTION MODEL

At this time we request to open our mind a little and move away from the traditional way of handling interaction and energy dissipation and try to locate the following model (although this model is also more an thought experiment than a well-defined physicist model, it explores and tries to give a physical meaning to the Tsallis parameter, as will be seen in the later development). Let us take N particles inside a container, these particles move with a velocity v_i, with $i = 1, 2, 3, \ldots, N$. It is exactly the same model as in the kinetic theory of gases; the collisions with the walls are elastic and between them there are elastic collisions too, except for what we will describe below. This is a totally out of equilibrium system; we have forced this system to this initial state, then we let it relax through the equilibrium. Obviously the initial state and all subsequently intermediate states are a non-equilibrium thermodynamic system and then, we cannot

assign a thermodynamic temperature, but it is possible to associate an initial *statistical temperature*. These particles can interact with each other; we will simplify that interaction by assuming that they only do it by collisions. In general in this system the particles lose and gain energy just by the collisions. In this example, the forced state to the equilibrium state, these particles will collide a very large number of times in such a way that they can lose energy. Then, those particles dissipate energy to the surroundings due to the shocks, but we will assume that the interaction is made in a peculiar way; when they collide with different energies there is no loss of energy, this is when the difference exceeds a certain amount ΔE_i. Then, all collisions are elastic, except when two particles collide with this difference in energy ΔE_i. In this case there is a loss of energy ΔQ which is proportional to the difference between particle energies and some quantity $g(E)$,

$$\Delta Q = g(E)\Delta E_i. \qquad (5)$$

The function $g(E)$, which we will call the dissipator, expresses the rate of energy dissipated in each shock by such difference ΔE_i. For the first example we will set $g(E) = -c$. Where $c \neq 0$ is a positive number with unities $[1/J]$. We will set the window of dissipation too narrow, in differential form, then the lost energy by the dissipator is

$$\delta Q = -c \, dE_i, \quad c > 0. \qquad (6)$$

With Boltzmann's background we can easily see and understand that the loss and gain energies has a *packed-like* behavior. This is, in the discrete case we see that the loss of energy depends on the difference between two consecutive states of energy $E_j - E_i$.

If we concentrate on a particle i, with speed v_i and energy E_i we are sure that at some point in time, this particle will hit another with an energy inside the speed window dE_i and at that time the energy δQ will be transmitted to the surroundings. During its whole trajectory until reaching the diluted state, this particle may collide elastically with another and its speed will be modified without lost of energy, but at some point another particle will collide with another; both with the appropriate energies such that energy δQ will be dissipated. Thus, this process will be repeated once again with another particle with appropriate speed and so on, although we have to wait an infinite time, in this modeled world for this to happen. The same thing will happen to a particle j with initial speed v_j, and thus for all N particles. Until what moment? until the system has diluted itself, so that dissipative shocks are no longer possible. At that moment, the system will have reached equilibrium, and in its final state the system will have again a common temperature T_f. The same thing will happen with the other particles once the equilibrium is reached. All processes involved in our model can be related to the summary that Ponmurugan [29] presents about the theorem of fluctuations of Crooks [30] and Jarzynski's Equality [31]. These theorems are an alternative to the thermodynamics and the nonequilibrium statistical mechanics of Prigogine [32], [33]. Following Ponmurugan ideas, if you take an irreversible process

between two equilibrium states A and B at the same inverse temperature $1/kT$, obviously this process generated by some external force, the probability of phase-space $P_F[\gamma^F]$ from the forward path γ^F and the probability of the phase-space $P_F[\gamma^R]$ of the return path γ^R satisfies the relation

$$\frac{P_F[\gamma^F]}{P_R[\gamma^R]} = e^{(\beta W - \Delta F)}, \tag{7}$$

where W is the work done by the system and ΔF is the change in Helmholtz's free energy between the two equilibrium states. We must be careful, because such a change in the energy is based on the fact that heat does not contribute because the temperature is the same in the two equilibrium points. The change is only related to the work and the change in chemistry energy due to the change in the number of particles remembering $dF = -SdT + PdV + \sum \mu_i dN_i$. In our case there is no variation in the number of particles and we have changes due to dissipation. Returning to Ponmurugan's arguments, the integrated version of this equation is Jarzynski's equality.

$$\langle P_F[\gamma^F] \rangle = P_R[\gamma^R]. \tag{8}$$

We can see the direct relationship between the average of the statistical ensembles and the work dissipation. We arrive at a similar conclusion at the end of the following analysis. Ponmurugan's q-generalization exactly ensures the same, even though it is emphasized that the processes are at the same temperature. Returning to our process, the objective that we want to achieve is to try to write the entropy of an interacting particles system as a function of Phase-Space in some possible way; after this we will be able to write the entropy in function of its probabilities. In this case we will not use classic ensemble statistics. We are only interested in the final outcome of the process and in its initial conditions because the model allows it. We are not interested in the details of how the final state was achieved, nor the time taken by the system to obtain it. Thus, the change in entropy is thermodynamically calculated by the Clausius entropy. Now, Returning to our example, by using (6),

$$\Delta S = -\sum_{i=1}^{l} \left[\int_{E_{i,o}}^{E_{i,f}} \frac{c}{T_i} dE_i \right]. \tag{9}$$

where l is the number of possible particles that could collide in the initial state. The question now is what temperature will we use? We will use a local *temperature-like* quantity. During the process of the system between the extreme equilibrium states we have many elastic collisions and also many dissipative collisions that are responsible for lost energy. We can think up a virtual space formed by $N_m/2$ disjointed sets of pairs of particles with the same speed v_m, then there will be m different virtual sets; note that $\sum_i N_i = l$. In a virtual space the pairs of dissipative colliding particles are so many such that we can calculate an average kinetic energy, which will be that speed v_m and therefore we can associate them with a temperature-like quantity. We can take several of these

virtual sets where their speed is between $v_m \pm v$. In this way, we ensure a greater number of particles involved. At this point we will calculate the average of their kinetic energy and then assign them a temperature. After this, return to the original pair because we have already calculated the intensive property required, a kind of temperature is associated to v_m by means of a kinetic energy, this is $T_i = \lambda \xi_i$ instead of $T_i = \frac{2}{3k}\xi_i$ as usual, where k is the Boltzmann constant. In this case constant λ has unities $[K/J]$. Note that we can use the Clausius entropy because there is a local equilibrium in each virtual set. Then, (9) is

$$\Delta S = -\sum_{i=1}^{l} \left[\frac{c}{\lambda} \int_{\xi_{i,o}}^{\xi_{i,f}} \frac{1}{\xi_i} d\xi_i \right].$$

Now, integrating and evaluating for all particles, we have

$$\Delta S = -\frac{c}{\lambda} \left[\ln\left(\frac{\xi_{1,f}}{\xi_{1,o}}\right)^2 + \ln\left(\frac{\xi_{2,f}}{\xi_{2,o}}\right)^2 + \ldots + \ln\left(\frac{\xi_{l,f}}{\xi_{l,o}}\right)^2 \right].$$

Remembering there are $N_m/2$ pairs of particles which have the same energy ξ_m,

$$\Delta S = -\frac{c}{\lambda} \left[\frac{N_1}{2} \ln\left(\frac{\xi_{1,f}}{\xi_{1,o}}\right)^2 + \frac{N_2}{2} \ln\left(\frac{\xi_{2,f}}{\xi_{2,o}}\right)^2 + \ldots \right.$$
$$\left. + \frac{N_m}{2} \ln\left(\frac{\xi_{m,f}}{\xi_{m,o}}\right)^2 \right].$$

In an equivalent way in terms of probabilities,

$$\Delta S = -\frac{cN}{2\lambda} \left[P_1 \ln\left(\frac{\xi_{1,f}}{\xi_{1,o}}\right)^2 + P_2 \ln\left(\frac{\xi_{2,f}}{\xi_{2,o}}\right)^2 + \ldots \right.$$
$$\left. + P_m \ln\left(\frac{\xi_{m,f}}{\xi_{m,o}}\right)^2 \right]. \tag{10}$$

This entropy is associated to dissipation, loss of energy depends only on the initial and final states of the system which are written in terms of the energy by corresponding speeds. We can even declare as Jarzynski [31] ensures that the change only depends on the final and initial Hamiltonians. It does not depend on the switching process time, and then Jarzynski's equality is satisfied when time tends to infinity, exactly as we propose.

Immersed in this last part of the section there is an **intermediate step as a link bridge in the processes**, which will be described in the following lines.

Although we know it is impossible to obtain the form of the entropy using the Clausius definition, we want to show that under certain assumptions it is really possible, because the initial and final entropies are contained in the last equation. We will calculate the entropy of the final state of the system in our example, that is, the entropy of the *diluted state* of the system by rewriting the initial energies using the arithmetic mean energy. We know that it does not represent the best scenario, that it is perhaps more convenient to use the most probable energy ξ_o^* which satisfies $\frac{d f(\xi_o)}{d\xi_o}|_{\xi_o^*} = 0$, or the average energy $\langle \xi_o \rangle = \int_0^\infty \xi_o f(\xi_o) d\xi_o$ but unfortunately

we do not know the system's distribution function in the initial state $f(\xi_o)$; for this reason we will simply take this approach, we will use mean energy. Let us assume the next aproximation. The total energies can be written as $\xi_o = \xi_{1,o} + \xi_{2,o} + \xi_{3,o} + \ldots + \xi_{N,o}$ and $\xi_f = \xi_{1,f} + \xi_{2,f} + \xi_{3,f} + \ldots + \xi_{N,f}$. We will take the fact that the total final and initial energies are proportional, this is $a\xi_f = \xi_o$. Where $a \geq 1$, because in our example the initial total energy is greater than the final total energy.

Then each $\xi_{i,o}$ will be taken as $\bar{\xi}_o$ where $\bar{\xi}_o = \frac{\xi_o}{N}$. The initial total kinetic energy is $\xi_o = \sum_{i=1}^{N} \xi_{io}$. We can consider the previous assertion because we will assume that the process of relaxation from the totally out of equilibrium system to the equilibrium system is carried out through processes that we will call *prope chaotic* processes; a process of moving from a chaotic system state to a very close chaotic one, where a small amount of energy was dissipated. Since they are very close systems, although we do not know their function of energy distribution, the two distributions are very close and similar. When a couple of particles dissipate energy, and then collide elastically with others, their energy will change. Obviously this particle's energy will never coincide with the final proposed energy. But since the distributions are very similar, we assure that there will be a pair of particles with the corresponding energies to continue carrying out the process until reaching equilibrium. These *prope chaotic* processes are equivalent to quasi statistic processes, prope is synonymous with quasi, this means near, close or very close. Those processes are the way to connect highly dissipative processes step by step until reaching equilibrium.

We insist, nothing assures that the initial particle i-th is exactly the final particle i-th; it can fall anywhere in the final energy distribution but there will actually be a particle with that characteristic, so for simplicity we will take this fact. The following simplified diagram shown in figure 1 sketches the process from a totally out of equilibrium system to the equilibrium system in graph form by means of what happens to the distribution of particles with certain energy. Thus, for this case (10) can be written,

$$\Delta S = -\frac{cN}{2\lambda m} \left[P_1 \ln \left(\frac{N\xi_{1,f}}{\xi_o} \right) + P_2 \ln \left(\frac{N\xi_{2,f}}{\xi_o} \right) + \ldots \right.$$
$$\left. + P_m \ln \left(\frac{N\xi_{m,f}}{\xi_o} \right) \right].$$
$$\Delta S = -\frac{cN}{2\lambda m} \left[P_1 \ln \left(\frac{\xi_{1,f}}{a\,\xi_{1,f}} \right) + P_2 \ln \left(\frac{\xi_{2,f}}{a\,\xi_{2,f}} \right) + \ldots \right.$$
$$\left. + P_m \ln \left(\frac{\xi_{m,f}}{a\,\xi_{m,f}} \right) \right],$$

Note that we can write the relation between the total and initial energy thanks to the properties of the logarithm; if we separate the integral into a very large number of integrals which model the *prope chaotic* processes, the small variations are simplified giving rise only to the relation between the final and initial energies. If we define $\kappa = \frac{cN}{2\lambda}$, we obtain in the
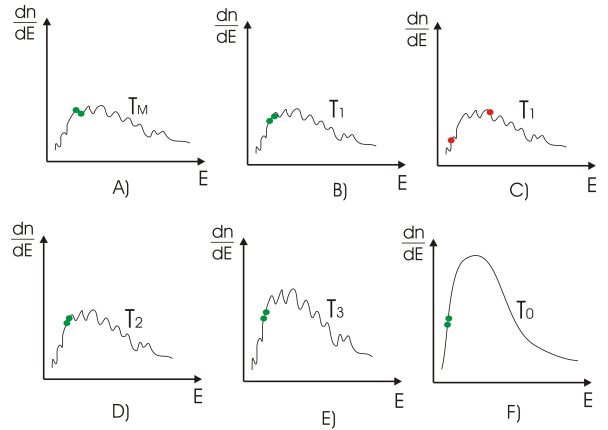

FIGURE 1. Outline of the process through which the system goes through a state out of equilibrium system to a state of equilibrium represented in terms of the distribution of particles with certain energy.
Graph A) represents the distribution of particles according to their energy. The irregular shape states that it is an unknown function, in turn it presents irregularities because there are hotter and colder regions due to the interaction itself. This graph describes the state of the initial system, a system with a very high kinetic temperature T_M, the green dots represent a pair of particles that will help to understand the process.
Graph B) represents the distribution of a system in a state very close to that represented by A), but once a dissipation has been carried out due to the colosion of these particles by means of the disipator. Graph C) is a more real representation of the process described by B). That is, once the particles interact, an amount of energy that is due to the disipator is lost. The remaining energy adheres to the principle of energy conservation and therefore the particles can have different energies, velocities and directions as usual. For this reason, said red particles are separated and in different places in the distribution. But reality, with a system very close to A) there must be some particles within this distribution that are located at the points shown in B), obviously they are different particles, but that conception helps us to carry out the process. This is our statistical version (in terms of distribution) of the ergodic hypothesis. For this reason we will use the conception of B) instead of C). Graphs D) and E) represent intermediate dissipation processes but at lower temperatures. Processes exactly the same as described above. Each time a process is carried out, the distribution of the system changes in a very small way, step by step. while this happens, the distribution becomes smoother, until finally it reaches equilibrium, state described by the distribution F) at temperature T_0.

last equation,

$$\Delta S = \kappa \left[(P_1 + P_2 + \ldots + P_m) \ln a \right].$$

Obviously this depends on the marginal probabilities p_m, but nothing prohibits that there is an initial continuous energy distribution such that $m \to N$ and $(N_1 + N_2 + \ldots + N_m) = l \to N$, and $(P_1 + P_2 + \ldots + P_m) \to (P_1 + P_2 + \ldots + P_N) = 1$ because eventually all particles will take part in the process. Therefore we ensure

$$\Delta S = \kappa \sum_{i=1}^{N} P_i \ln a = k \ln a, \quad a \geq 1. \qquad (11)$$

This quantity represents the change entropy for the system and only depends on a; $\Delta S = h(a)$, this equation is equivalent to (10). When the final and initial states have the same energy, they have the same distribution, and then this quantity will be equal to zero; $h(1) = 0$.

Now, we will return to (10) and change again the equation, taking the fact $a\xi_f = \xi_o$, then we can write ΔS_{alt} (the entropy

of a very simplified alternative system) as,

$$\Delta S_{alt} = -\frac{cN}{2\lambda} \left[P_1 \ln\left(\frac{N\xi_{1,f}}{a\xi_f}\right) + P_2 \ln\left(\frac{N\xi_{2,f}}{a\xi_f}\right) + \ldots \right. $$
$$\left. + P_m \ln\left(\frac{N\xi_{m,f}}{a\xi_f}\right) \right].$$

In another way,

$$\Delta S_{alt}$$
$$= -\frac{cN}{2\lambda} \left\{ \left[P_1 \ln\left(\frac{\xi_{1,f}}{\xi_f}\right) + P_2 \ln\left(\frac{\xi_{2,f}}{\xi_f}\right) + \ldots \right. \right.$$
$$\left. + P_m \ln\left(\frac{\xi_{m,f}}{\xi_f}\right) \right] + \ldots$$
$$\left. + \left[P_1 \ln\left(\frac{N}{a}\right) + P_2 \ln\left(\frac{N}{a}\right) + \ldots + P_m \ln\left(\frac{N}{a}\right) \right] \right\}.$$

But nothing prohibits that there is an initial continuous energy distribution such that $m \to N$ and $(N_1 + N_2 + \ldots + N_m) = l \to N$, and $(P_1 + P_2 + \ldots + P_m) \to (P_1 + P_2 + \ldots + P_N) = 1$ because eventually all particles will take part in the process. Let us write,

$$\Delta S_{alt} = -\frac{cN}{2\lambda} \left[\sum_{i=1}^{N} P_i \ln\left(p_{i,f}\right) + \sum_{i=1}^{N} P_i \ln\left(\frac{N}{a}\right) \right]. \quad (12)$$

which is equivalent to,

$$\Delta S_{alt} = -\frac{cN}{2\lambda} \left[\sum_{i=1}^{N} P_i \ln\left(\frac{N}{a} p_{i,f}\right) \right]. \quad (13)$$

The next step is to obtain the *instant entropy* in the final state, the greatest entropy for the system, the diluted case in equilibrium through that alternative system by using $\Delta S_{alt} = S_f - S_{o(alt)}$ and (12), we are going to associate

$$S_f = -\frac{cN}{2\lambda} \sum_{i=1}^{N} P_i \ln\left(p_{i,f}\right),$$

and

$$S_{o(alt)} = \frac{cN}{2\lambda} \sum_{i=1}^{N} P_i \ln\left(\frac{N}{a}\right).$$

This means that if we *move* the initial state as much as possible only through the growth of a close to N, keeping the final state fixed, we guarantee $S_{o(alt)}$ will be zero. Then, we can obtain the final state entropy if we require the same condition, such that a grows so much as N in (13). We should note that P_i is common in both points of the system. Now, we can ensure in an equivalent way,

$$S_f = \lim_{a \to N} \Delta S_{alt} \left(\frac{N}{a} p_{i,f}\right). \quad (14)$$

If we define $\alpha = \frac{N}{a}$, then we obtain,

$$S_f = \lim_{\alpha \to 1} \Delta S_{alt}(\alpha p_{i,f}). \quad (15)$$

Now, for the final entropy, with $\kappa = \frac{cN}{2\lambda}$,

$$S_f = -\kappa \sum_{i=1}^{N} P_i \ln p_{i,f}. \quad (16)$$

This equation is just the Boltzmann-Gibbs entropy or Shannon entropy, at least in the form, κ is not the Boltzmann constant, although we can always argue the value of the constants to obtain that numerical value. We must also note that the probabilities are different, that is, one represents the probability of particles per micelle with energy ξ_i and the other represents the probability of the energies ξ_i. Which coincide in the case of systems with a large number of particles, when the energy is discretized exactly the same way, this is $N\xi_i = \xi$, then we can write $P_i = p_{i,f}$. This equation represents the entropy of a diluted state system. Note that we could use the Boltzmann-Gibbs entropy directly because fortunately this example has the Boltzmann-Gibbs entropy-form; but in another case the last development will be very useful.

Now we are able to calculate the initial entropy of the interacting particle system, using (16) and first line in (11), this is,

$$S_o = -\kappa \sum_{i=1}^{N} P_{i,f} \ln p_{i,f} - \kappa \sum_{i=1}^{N} P_i \ln a,$$
$$S_o = -\kappa \sum_{i=1}^{N} P_{i,f} \ln(a\, p_{i,f}).$$

Or simply,

$$S_o = -\kappa \sum_{i=1}^{N} p_i \ln(a\, p_i), \quad (17)$$

In the case $P_i = p_i$ or $P_{i,f} = p_{i,f}$. Where $a \geq 1$. We can see that the entropy equation of the system of interacting particles is similar to the Boltzmann entropy of the diluted case, except the constant a, which relates the final and initial total energies.

The last equation is Shannon's entropy and is related with the Boltzmann-Gibbs entropy through $S = \langle -k \ln \rho \rangle = k \ln W$, where ρ is the phase-space density, remembering that the Boltzmann-Gibbs entropy represents the maximum entropy, the diluted case. Equation (17) shows that the entropy of a system of particles highly correlated is also compatible with the Boltzmann entropy. We have inverted the roles of the orthodox process, that is, we should generally propose a statistical model that reproduces the thermodynamic system properties. In this case, based on the thermodynamic entropy we have not obtained the statistical model, but we have obtained the probabilistic behavior of the Phase-Space and its direct relationship with entropy.

Through the process developed to obtain the last equation we can appreciate Jarzynski equality; the average of the probabilities of the system is directly associated with dissipated work, heat or change in entropy in our case. And Crooks' work fluctuation theorem; that is, the probabilities of

non-equilibrium forward and reversed trajectories are related by taking the initial conditions from the Boltzmann-Gibbs equilibrium distribution.

Note that nothing has really been mentioned about the mass, because the energy governs the model. Therefore the system accepts equal masses, different masses, average mass or reduced mass, and again the model works. But even then we can ensure that the behavior of the entropy remains proportional and compatible with the Boltzmann-Gibbs entropy. It was not clarified during the previous discussion because finally such an argument may be contained in the constant k and the goal was to show that the entropy of the interacting particles obey the Boltmann-Gibbs statistics for this dissipator. We should note that this exact value of entropy is different depending on the dissipator type used; this issue will be better understood in the next section. Again, what we are looking for in this moment is to show that the entropy of a dissipative case is compatible with the general entropy or entropy of Shannon if we are talking about information theory. One could say that the final temperature is proportional to the initial one. Also, one could say that the final average speed and the initial average speed are proportional, but this cannot be said about entropy; it is not evident to anticipate, much less to assimilate that the initial entropy has an identical behavior to a diluted system, which is the case in the final state.

In any book on statistical mechanics we can find that the Boltzmann's and Shannon's entropy are equivalent when we deal with equiprobable systems. However, in the context of information, we can say that (17) defines the system information entropy; the average of the entropy by energy levels is known as Shannon's entropy,

$$H_s \equiv \langle -S_o \rangle = \langle I_o \rangle = -\kappa \sum_{i=1}^{N} p_i \ln(ap_i).$$

Thus, we can identify the system information for each different level of interacting particles as

$$I_{i,o} = -\kappa \ln(ap_i).$$

In information theory, we always choose $\kappa = 1$. we have to mention that in the analysis, this is a system that is highly out of balance, it is relaxed, then $a > 1$ is restricted, but in reality it accepts values less than 1, that is, if a system is in equilibrium and gains energy, < 1, and also satisfies the laws of thermodynamics.

III. OTHER POSSIBLE SCENARIOS FOR THE DISSIPATOR
With the motivation of the previous example, we want to extend the process to another type of "dissipators". We note that to be able to imitate the previous development the dissipator must have a particular way; once the integral factor $1/T$ is taken into account, it must allow us to write the change in the entropy in terms of quotients; let us abuse the notation, this is $\Delta s = \sum_{i=1}^{N} \left[h(x_{i,f}) - h(x_{i,o}) \right] = \sum_{i=1}^{l} h(\frac{x_f}{x_o})$. We can do this because even though the sum total of the variable also

appears, that sum is measured at the end point associated with a constant at the end of the development. Obviously, it must also meet $h(x_{i,f}) \geq h(x_{i,o})$ to satisfy the Clausius entropy. Now we generalize the previous dissipator by means of the following proposition, we assume that,

$$g(\xi) = -c\xi^a, \tag{18}$$

then we get,

$$\delta Q = -cE_i^a dE_i, \tag{19}$$

where c is a positive constant with unities $[J^{1-a}]$. Thus, the change in the entropy is determined by,

$$\Delta S = -\sum_{i=1}^{l} \left[\frac{c}{\lambda} \int_{\xi_o}^{\xi_f} \xi_i^{a-1} d\xi_i \right],$$

where we have used as in the previous example $T_i = \lambda \xi_i$ with $[\lambda] = K/J$. Integrating the above equation, we obtain,

$$\Delta S = -\sum_{i=1}^{l} \left[\frac{c}{\lambda a} \left(\xi_{i,f}^a - \xi_{i,o}^a \right) \right].$$

If we expand it, we get,

$$\Delta S = -\frac{c}{\lambda a} \left[\left(\xi_{1,f}^a - \xi_{1,o}^a \right) + \left(\xi_{2,f}^a - \xi_{2,o}^a \right) + \ldots \right.$$
$$\left. + \left(\xi_{l,f}^a - \xi_{l,o}^a \right) \right],$$

remembering that there are $N_m/2$ pairs of particles with the same energy ξ_k in each virtual set, then,

$$\Delta S = -\frac{Nc}{\lambda 2a} \left[\frac{N_1}{N} \left(\xi_{1,f}^a - \xi_{1,o}^a \right) + \frac{N_2}{N} \left(\xi_{2,f}^a - \xi_{2,o}^a \right) + \right.$$
$$\left. \ldots + \frac{N_m}{N} \left(\xi_{m,f}^a - \xi_{m,o}^a \right) \right],$$

which can be written in terms of the probability as,

$$\Delta S = -\frac{Nc}{\lambda 2a} \left[P_1 \left(\xi_{1,f}^a - \xi_{1,o}^a \right) + \right.$$
$$\left. + P_2 \left(\xi_{2,f}^a - \xi_{2,o}^a \right) + \ldots + + P_m \left(\xi_{m,f}^a - \xi_{m,o}^a \right) \right]. \tag{20}$$

Now we arrive at **intermediate step**, as in the first example, then we have to procedure in a similar way to that of the previous section; the initial energies $\xi_{i,o}$ will be changed by final energies because there is a corresponding energy associated to the initial energy, this is $\xi_{i,o} = b\xi_{i,f}$, with $b \geq 1$, then we have,

$$\Delta S = -\frac{Nc}{\lambda 2a} \left[P_1 \xi_{1,f}^a \left(1 - b^a \right) + P_2 \xi_{2,f}^a \left(1 - b^a \right) + \ldots \right.$$
$$\left. + P_m \xi_{m,f}^a \left(1 - b^a \right) \right],$$
$$\Delta S = -\frac{Nc\xi_f^a}{\lambda 2a} \left[P_1 p_1^a \left(1 - b^a \right) + P_2 p_2^a \left(1 - b^a \right) + \right.$$
$$\left. \ldots + P_m p_m^a \left(1 - b^a \right) \right],$$

where ξ_f is the total final kinetic energy. Obviously this depends on the marginal probabilities p_m, but nothing prohibits that there is an initial continuous energy distribution

such that $m \to N$ and $(N_1 + N_2 + \ldots + N_m) = l \to N$, and $(p_1 + p_2 + \ldots + p_m) \to (p_1 + p_2 + \ldots + p_N) = 1$ because eventually all particles will take part in the process. Therefore we ensure,

$$\Delta S = -\frac{\kappa}{a} \sum_{i=1}^{N} \left[P_i p_{i,f}^{a+1} \left(1 - b^a \right) \right],$$

or in equivalent way,

$$\Delta S = -\frac{\kappa}{\alpha - 1} \sum_{i=1}^{N} \left[P_i p_{i,f}^{\alpha-1} \left(1 - b^{\alpha-1} \right) \right], \quad (21)$$

with $\kappa = \frac{Nc\xi_f^a}{2\lambda}$ and $\alpha = a + 1$. This equation represents the change in entropy for a system of interacting particles with a particular dissipator described at the beginning of this section.

For the entropy of the final state of the system, that is, the entropy of the *quasi diluted state*, the entropy of the equilibrium state system, we will return again to (20) and concentrate on $\xi_{i,o}$ ($i = 1, \ldots, m$), exactly as in the previous section. Thus, we will use the mean initial kinetic energy $\bar{\xi}_o = \frac{\xi_o}{N}$ for each $\xi_{i,o}$, where $\xi_o = \sum_{i=1}^{N} \xi_{io}$. Then equation (20) for an alternative system can be written as,

$$\Delta S_{alt} = -\frac{Nc}{2\lambda a} \left[P_1 \left(\xi_{1,f}^a - \frac{\xi_o^a}{N^a} \right) + P_2 \left(\xi_{2,f}^a - \frac{\xi_o^a}{N^a} \right) + \ldots + P_m \left(\xi_{m,f}^a - \frac{\xi_o^a}{N^a} \right) \right],$$

taking the fact $\xi_o = b\xi_f$, with $b \geq 1$, we can write,

$$\Delta S_{alt} = -\frac{Nc}{2\lambda a} \left[P_1 \left(\xi_{1,f}^a - \frac{b^a \xi_f^a}{N^a} \right) + \right.$$
$$\left. + P_2 \left(\xi_{2,f}^a - \frac{b^a \xi_f^a}{N^a} \right) + \ldots + P_m \left(\xi_{m,f}^a - \frac{b^a \xi_f^a}{N^a} \right) \right],$$

in an equivalent way,

$$\Delta S_{alt}$$
$$= -\frac{Nc\xi_f^a}{2\lambda a} \left\{ P_1 \left[\left(\frac{\xi_{1,f}}{\xi_f} \right)^a - \frac{b^a}{N^a} \right] + \right.$$
$$\left. + P_2 \left[\left(\frac{\xi_{2,f}}{\xi_f} \right)^a - \frac{b^a}{N^a} \right] + \ldots + P_m \left[\left(\frac{\xi_{m,f}}{\xi_f} \right)^a - \frac{b^a}{N^a} \right] \right\},$$

writing the last equation in terms of probabilities and defining $\gamma = \frac{N}{b}$,

$$\Delta S_{alt} = -\frac{Nc\xi_f^a}{2\lambda a} \left[P_1 \left(p_1^a - \frac{1}{\gamma^a} \right) + P_2 \left(p_2^a - \frac{1}{\gamma^a} \right) + \ldots + P_m \left(p_m^a - \frac{1}{\gamma^a} \right) \right].$$

Finally, we write

$$\Delta S_{alt} = \frac{\kappa}{a} [(P_1 + P_2 + \ldots + P_m) \frac{1}{\gamma^a} - \sum_{i=1}^{m} P_i p_i^a], \quad (22)$$

where $\kappa = \frac{Nc\xi_f^a}{2\lambda}$. This equation represents the change of entropy in terms of the probability of the different m final energy levels. As in the previous section, we can assume that there is an initial continuous energy distribution such that $m \to N$ and $(N_1 + N_2 + \ldots + N_m) = l \to N$, $(p_1 + p_2 + \ldots + p_m) \to 1$ and $(P_1 + P_2 + \ldots + P_m) \to 1$, because eventually all particles will take part in the process. Therefore we ensure,

$$\Delta S_{alt} = \frac{\kappa}{a} \left[\frac{1}{\gamma^a} - \sum_{i=1}^{N} P_i p_i^a \right].$$

If we rename the exponents $\alpha = a + 1$ then we have,

$$\Delta S_{alt} = \frac{\kappa}{\alpha - 1} \left[\frac{1}{\gamma^{\alpha-1}} - \sum_{i=1}^{N} P_i (p_i)^{\alpha-1} \right]. \quad (23)$$

Exactly as in the first example, the equation above involves the final and initial entropy, although we are tempted to make an immediate separation, this will undoubtedly be erroneous, because it is obvious that they must be similar functions. Therefore, before proceeding we will make a brief summary of the procedure in a simple way.

A. SUMMARY AND GENERALIZATION OF THE ALTERNATIVE PROCEDURE

When we have a thermodynamic system with an initial state totally out of equilibrium composed of N particles, if said system relaxes even a state of equilibrium, the change in entropy through *prope chaotic* processes by using Clausius' entropy only depends on the **ratio** of the **final and initial energies** that characterizies the process (we must remember that P_i is common in both points of the system). If we denote as h the function that characterizes ΔS can be written as $h(\xi_{i,f}) - h(\xi_{i,o}) = h(\frac{\xi_{i,f}}{\xi_{i,o}})$, then as $\xi_{i,o} = a\xi_{i,f}$ with $\xi_{i,o} \geq \xi_{i,f}$ we acquire,

$$\Delta S = -h(a), \quad a \geq 1. \quad (24)$$

We also get,

$$h(1) = 0, \quad (25)$$

because change in entropy is zero for $\xi_{i,o} = \xi_{i,f}$.

For an alternative system from $\Delta s = \sum_{i=1}^{N} h \left(\frac{\xi_{i,f}}{\xi_{i,o}} \right)$, if we characterize each $\xi_{i,o}$ by the mean $\bar{\xi}_o$, with $\bar{\xi}_o = \frac{1}{N} \sum_{i=1}^{N} \xi_{i,o}$, we get,

$$\Delta s_{alt} = \sum_{i=1}^{N} h \left(\frac{\xi_{i,f}}{\bar{\xi}_o} \right). \quad (26)$$

In terms of $\bar{\xi}_f$ we obtain $\Delta S_{alt} = \sum_{i=1}^{N} h \left(\frac{\xi_{i,f}}{a\bar{\xi}_f} \right)$,

$$\Delta S_{alt} = \sum_{i=1}^{N} h \left(\frac{N\xi_{i,f}}{a\xi_f} \right) \text{ or in terms of probabilities}$$

$\Delta S_{alt} = \sum_{i=1}^{N} h\left(\frac{N}{a} p_{i,f}\right)$, using $\Delta S_{alt} = S_f - S_{o(alt)}$ with $S_{o(alt)} = 0$ by a substantially large as N, this is $S_{o(alt)}$ as far from S_f as it will be possible. Before applying some limit, we are going to analyze the behavior of this change entropy. Now, by using only one property of h function, we can expand again this variation as $h\left(\frac{N}{a} p_{if}\right) = h(p_{if}) + h\left(\frac{N}{a}\right)$, from which we can see that this quantity is measured from final state fixed at p_{if}, and it is clear that the variation will be really given only by N/a. Using $h(\frac{N}{a} p_{if}) = S_f - S_{o(alt)}$ we obtain $S_f = -h(p_i)$ and $S_{o(alt)} = -h(\frac{N}{a})$ (We want to highlight that this equation can also depend on p_{if} but as we have fixed the final state, we can assume that it only depends on the quotient). Comparing $S_{o(alt)}$ with (25) we see that $\frac{N}{a} = 1$ ensures $S_{o(alt)} = 0$ besides, the argument in 24 cannot be less than one which is the term associated to the variation. We have to underline that $h(p_i)$ is fixed and common for all variations and $h(\xi)$ with $\xi \in [0, 1]$ is expressed in $h(p_i)$. Then, we can get $a \le N$. From this analysis, we have to note that a will be bounded, $a \in [1, N]$; if we define $\alpha = \frac{N}{a}$ we see that this quantity cannot be less than one, because it has the same propose as "a" in 24, $a \ge 1$, therefore $\alpha \in [1, N]$ too but in inverse form. Then, the frontier $a \to N$ or $\alpha \to 1$ gives the maximum initial entropy, at the same time, ensures $S_{o(alt)} = 0$ and this frontier guarantees that the maximum entropy will be just a function of one variable. We must clarify that in this development the concept of maximum entropy has been used differently, but it has been contemplated. It has been demanded that there must be a point of reference where entropy is zero by $S_{o(alt)} = 0$. This taking into account the widely known results from Boltzmann on maximum entropy but not in the way that Boltzmann did. Finally, we propose in general $S_f = \lim_{\alpha \to 1} \Delta S_{alt}$. In explicit form $S_f = \lim_{\alpha \to 1} \Delta S_{alt}(\alpha \frac{\xi_{i,f}}{\xi_f}) = \sum_{i=1}^{N} h\left(\frac{\xi_{i,f}}{\xi_f}\right)$ or in terms of probabilities

$$S_f = \lim_{\alpha \to 1} \Delta S_{alt}(\alpha p_{i,f}) = \sum_{i=1}^{N} h(p_{if}). \quad (27)$$

We conclude this summary with the following equation,

$$S_o = \sum_{i=1}^{N} \left[h(p_{i,f}) + h(a)\right], \quad a \ge 1. \quad (28)$$

The last equation represents the entropy of the initial state totally out of equilibrium in that system.

Now, for the final entropy in our example, we know that there is an extreme in $b \to N$ or $\gamma \to 1$ as has been explained in the summary. Then, the maximum entropy for the system, the *quasi-diluted* case will be,

$$S_f = \lim_{\gamma \to 1} \Delta S. \quad (29)$$

Then we write,

$$S_f = \frac{\kappa}{\alpha - 1} \left[1 - \sum_{i=1}^{N} P_i \left(p_{i,f}\right)^{\alpha-1}\right]. \quad (30)$$

This equation represents the entropy for a quasi-diluted final state of the system. We note that this entropy is only generated by the dissipator described in this section; this final state of the system *may* not be characterized by the Maxwell distribution, in this case really allowed collisions by particles but their conditions do not dissipate energy.

Now we are able to calculate the real entropy for an interacting and dissipative state of the system using (21) and (33), with $p_{i,f} = p_i$ for simplicity we get,

$$S_o = \frac{\kappa}{\alpha - 1} \left[1 - b^{\alpha-1} \sum_{i=1}^{N} P_i (p_i)^{\alpha-1}\right]. \quad (31)$$

Or simply,

$$S_o = \frac{\kappa}{\alpha - 1} \left[1 - \beta \sum_{i=1}^{N} P_i (p_i)^{\alpha-1}\right]. \quad (32)$$

where $\beta = b^{\alpha-1}$ is taken just as a parameter and $\beta \ge 1$.

Again, when the energy is discretized in usual way, this is $N\xi_i = \xi$, then we can write $P_i = p_{i,f} = p_i$, then we get,

$$S_f = \frac{\kappa}{\alpha - 1} \left[1 - \sum_{i=1}^{N} (p_i)^{\alpha}\right]. \quad (33)$$

This equation is known as the Tsallis' entropy. the maximum entropy for this system, exactly as Tsallis pointed when he calculated the extreme entropy [7].

On the other hand,

$$S_o = \frac{\kappa}{\alpha - 1} \left[1 - \beta \sum_{i=1}^{N} (p_i)^{\alpha}\right]. \quad (34)$$

This equation represents the entropy for an interacting system. We see that its similar to Tsallis entropy except by β.

In the field of information, the last equation represents the system information entropy. If we interpret the number one as $1 = \sum_{i=1}^{N} p_i$, or we use (21) and (33) with marginal probabilities and $P_i = p_{i,f} = p_i$, we get,

$$S_o = \frac{\kappa}{a} \left[(p_1 + p_2 + \ldots + p_m) - b^a \sum_{i=1}^{m} (p_i)^{a+1}\right],$$

or in an equivalent way,

$$S_o = \frac{\kappa}{\alpha - 1} \Big[p_1(1 - \beta p_1^{\alpha-1}) + p_2(1 - \beta p_2^{\alpha-1}) + \ldots + p_m(1 - \beta p_m^{\alpha-1})\Big], \quad (35)$$

If we take (33), for the case $\alpha = \beta = 1$, and force it to have the Boltzmann form, $k \ln W$, we will arrive to the interpretation of Tsallis [7]. But using equation (35), we can perfectly reinterpret the entropy *additivity* of interacting

particles, because there is a partial contribution by each subset of interacting virtual sets as

$$S_{i,o} = \kappa \frac{1}{\alpha - 1} p_i (1 - \beta p_i^{\alpha - 1}). \qquad (36)$$

In this case, we can identify information of each energy level as,

$$I_{i,o} = -\kappa \frac{1}{\alpha - 1} (1 - \beta p_i^{\alpha - 1}). \qquad (37)$$

If we take again (33), we recover the interpretation of Tsallis if we force Boltzmann-like form $S = k \ln_q W$ for the case of equiprobable energy levels and we can obtain the q-deformed functions, obtained by (35) written for total N particles; that is,

$$S_o = \kappa \frac{1}{\alpha - 1} \sum_{i=1}^{N} p_i (1 - \beta p_i^{\alpha - 1}). \qquad (38)$$

Exactly as it appears in the formulation of Tsallis [7], with $\kappa = 1$, $\beta = 1$ and $q = \alpha$.

We must note that the dissipation of energy and therefore the change in entropy, is not equal for different dissipation cases. This and the previous example have different expressions of entropy. In addition, these expressions are affected by the number of particles N, the used exponent a and ξ in the last case. We are incorporating this fact within the constant k, but strictly speaking it could have different values. We are enclosing them in the constant k because the objective of this article is to deduce the entropy of Tsallis and to show the direct relation between that entropy, the statistical entropy and Shannon's entropy, because they are particular cases of the dissipator expressed in powers of the energy.

We have chosen to use the initial energy to make the average energy more meaningful, but in fact we can use any intermediate step; that is, any state pertaining to that process. But not only that, we can also assume that the system is composed of other small subsystems which obey certain particular rules of dissipation, and then, the entropy will involve more than one particular power, including a mixture of the different powers. Thus, we can even get an entropy as an expansion of various entropies of Tsallis (34). Even so, we could have a good approach to the total entropy, for example, using (35) or (38) with marginal probabilities. We have mentioned in this section that at the beginning of the development the probability distribution function for the power dissipator is unknown. Now let's analyze the closed form of distribution function for the non-interacting case. First we take limit cases, in (38) if $\beta \to 1$ we recover the entropy of Tsallis, therefore in the case of a system with non-interacting particles (a system in equilibrium where the interactions do not reach to dissipate energy) we can affirm that the probability distribution function is a q-deformed Maxwell distribution [34]. Besides when $a \to 0$ (in Tsallis' theory it is equivalent to $q \to 1$) we return to Boltzmann's entropy and for this case we know that Maxewll's distribution describes the system and this behavior confirms the third law of thermodynamics. For the whole

interactive case, now a new variable b comes into play but the analysis requires an additional work that possibly involves a particular example that illustrates the use of this theory, work that will be analyzed in later paper since it exceeds the objective of the present work. That future work must be in a similar way for example as Khordad did [35], [36] taking a concrete example, and then calculating the distribution function for the non-interactive case by maximizing entropy, verifying that the system really satisfies the third law of thermodynamics in the limit $q \to 1$ and finally determining its thermodynamic properties.

One of the immediate applications of (38), easy to note, is that this generalization of entropy can be applied in non-equilibrium systems that are **not too far from** it. Another possible immediate and simple use is; in those widely known examples where the value of q in Tsallis entropy is calculated that best models the system. In this case we also have a new β parameter in (38) that can be manipulated to better adjust the entropy that best models the system.

B. NOTATION AND SYMBOLS
We want to finish this section with a summary of the symbols and the notation used in the main part of this work, the remainder simbols are conventional symbology. In case of aforementioned symbology, there is a better description in its corresponding reference.

- N - Number of particles inside the container.
- i - i-th particle inside the container.
- v_i - Speed of i-th particle inside the container.
- $\Delta E_i = E_j - E_i$ - Change in energy referred to the i-th particle.
- ΔQ - Heat, energy lost.
- $g(E)$ - Dissipator, a function of E.
- δQ - Energy lost in differential form.
- ξ_i - Kinetic energy of i-th particle.
- $\xi_{i,o}$ - Kinetic energy of i-th particle at initial state.
- $\xi_{i,f}$ - Kinetic energy of i-th particle at final state.
- P_i - Probability of particles per micelle with energy ξ_i.
- p_i - Probability of phase-space, probability of the energies ξ_i.
- ΔS_{alt} - Entropy of a very simplified alternative system.
- $\Delta S_{o(alt)}$ - Entropy of a very simplified alternative system at initial state ($\Delta S_{alt} = S_f - \Delta S_{o(alt)}$).
- $p_{i,f}$ - Probability of the energies ξ_i at final state.
- $I_{i,o}$ - Information of each i-th level of energy for interacting particles.

IV. CONCLUDING REMARKS
In summary, we have proposed a model to obtain the entropies of Shannon and Tsallis, with a simple kinetic approach rather than a statistical one. With this we do not mean that our model reflects the actual physical mechanism of the interaction between particles, but it is capable of generating generalized entropies through the concept of dissipators. As we have illustrated, this concept can be managed in a kind of virtual space where it is possible to define a

temperature-like quantity that permits using the entropy of Clausius as a starting point towards generalized entropies; in particular it permits arriving at Shannon and Tsallis entropies. We should note a simple interpretation for the dissipator; that is the gain and loss of energy in the form of packages and the discretization of energy. The first example is directly associated with the one already known, that the loss or gain of energy in the discrete case only depends by difference between two consecutive energy states $\Delta E_i = E_j - E_i$. But the second suggests energy packages in the form of powers $E_i^a \Delta E_i$. Exactly as Boltzmann did at the time, the most we can say about it is that they are physically unacceptable; it is only a mathematical resource that allows us to obtain a different form of entropy, that of Tsallis. Additional experiments are needed to verify or disprove this proposal of dissipator. It will be necessary to explore another form of dissipators $g(\xi)$ to get other entropies and verify their possible connection with acceptable examples. We must emphasize that this approach does not replace the use of molecular statistical physics, since we are only interested in the initial and final equilibrium states of the system and the simple rules of the dissipator, unlike the conventional approach of molecular statistical physics involving a sophisticated mathematical formalism.

We must also say that using the proposed procedure, the dissipators will always generate an entropy in terms of the probabilities or the quantity $(1 - p_i)$ then it will be possible to rewrite any form of "entropy" in terms of these probabilities as a simple Taylor expansion or in terms of powers of $(1 - p_i)$, because these are also smaller than one. But the simplest explanation for this development is that the Boltzmann entropy is associated with the simple dissipator E_i^0 (equation (6)), therefore any expansion in entropy in the linear case when the Taylor series is cut to the first order, the Shannon's entropy is recovered. Perhaps this is the reason why there was a great variety of proposals for the entropy taking the Shannons' entropy as a limit. This can be seen clearly in the second example; a dissipator of the form E_i^a generates the entropy of Tsallis, and in the famous limit, when $q \to 1$ or $\alpha \to 1$, is equivalent to demand $a \to 0$ (remember $\alpha = a + 1$), which goes back to the most simple dissipator; it is a particular case since they belong to the same family of powers. The next simple schem shows the relation between that entropies.

$$\text{Disipator } g(E) \to \begin{cases} E_i^a \to \begin{cases} S_{B-G}, S_{Sh} & \text{if } a = 0 \\ S_T & \text{if } a \neq 0. \end{cases} \\ \dots \end{cases}$$

where S_{B-G} is Boltzmann-Gibbs Entropy, S_{Sh} is Shannon's Entropy and S_T is Tsallis' Entropy.

As you can see, this work proposes future research, for example using other functions in the dissipator, using it in well-defined problems, analyzing the interaction in triplets or in greater order, investigating if other proposed entropies can be obtained in this way, etc. But this will be developed in later works.

This approach is a different path from the traditional one, the maximization of the entropy of the system. The advantage is that it starts with classical thermodynamic concepts and uses the average of certain physical quantities to obtain the same result in a simpler way. Unfortunately, as in the Boltzman theory, it is very general, it only describes the entropy of the system, it does not specifically contemplate the form of the interaction, only in a certain particular way; It tells us about dissipation. That is, it assures us that all the systems that interact in a particular way obey Boltzman's entropy but if the interaction is another, it will obey Tsallis' entropy, for example. This opens a new gap; Each form of interaction generates a different form of entropy. Hopefully another form in the disipator in last equation $g(E)$ can generate some of the entropies already known, such as Renyis' entropy. This is an improvement, which involves novel concepts for this theory to work, needs the concept of disipator which is directly linked to the way in which energy is lost or gained, which in turn suggests that there are other possible ways to quantify it.

Finally, we can emphasize that the entropy of Tsallis is the most versatile because we can adjust the parameter q depending on the decribed system and this feature has motivated its wide applicability.

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