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Further Stability Analysis for Time-Delayed Neural Networks Based on an Augmented Lyapunov Functional

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ABSTRACT In this paper, the stability of time-delayed neural networks (DNN) is further analyzed. First, an augmented N-dependent Lyapunov–Krasovskii functional (LKF) is designed, where the non-integral terms are augmented with delay-dependent items and some additional state variables, and the integrated vector in the single-integral terms is also augmented by adding some integral interval-dependent items. The novel LKF complements some coupling information between the neuron activation function and other state variables. Second, a new delay-dependent stability criterion is proposed via the above LKF application. Third, in order to further demonstrate the advantages of the new LKF, two corollaries are also given under other simplistic LKFs. Finally, some common numerical examples are presented to show the effectiveness of the proposed approach.

INDEX TERMS Lyapunov–Krasovskii functional, neural networks, stability analysis, time delay.

I. INTRODUCTION

Neural network is very popular at present, because the application of neural network involves many aspects, such as image recognition, artificial intelligence, associative memory, algorithm optimization, signal processing and other scientific problems [1]. As we all know, the most important premise for the control design and other studies of the neural network is that it usually has to be stable. However, engineering practice has proved that time delays are inevitable, especially time-varying delays, for example, in the communication process between neurons, in the amplifier switching speed, and so on. Moreover, the existence of time delays has a major impact on the stability or other performances of neural networks [2]. And it is generally accepted that the higher the upper bound of the maximum permissible delay is, the higher endurability of neural network to transmission delay is. Therefore, the stability analysis of DNN is always a hot topic, and the main purpose is maximizing the delay upper bound as soon as possible for stable DNN. To analyze the stability problem of DNN based on the Lyapunov theorem, the main efforts are concentrated on the following several directions, one is finding an appropriate LKF, for example, LKF with augmented terms [3]–[7], LKF with delayed partitioning method [8], [9], LKF with triple-integral and quadruple-integral terms [10]–[14], LKF with delay-dependent matrices [15]–[17] and so on. The other is reducing the upper bounds of the time derivative of LKF as much as possible by developing various inequality techniques, such as Wirtinger-based inequality and reciprocally convex inequality [18]–[22], second order Bessel-Legendre inequality [23]–[25], integral inequality based on a nonorthogonal polynomial sequel [26], [27], and so on. Besides, further increasing the freedom of solving LMI, additional free-weighting-matrix technique is frequently introduced into the derivatives of LKF [28]–[32]. In conclusion, with the continuous augment of LKF and the development of integral inequality techniques, many good stability results are obtained and the conservatism of stability criteria becomes smaller and smaller.

Recently, based on the second order Bessel-Legendre inequality and reciprocally convex inequality techniques, two

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novel LKFs are constructed with some augmented vectors in [33], [34], where less conservative stability criteria than some previous ones were given for the stability of DNN. Zhang Zhang *et al.* [35] obtained a hierarchical type stability criteria for DNN via the N-dependent LKF and the affine Bessel-Legendre inequality, where some internal comparisons of [35] shown that the affine Bessel-Legendre inequality can further reduce the conservatism of stability criteria than Bessel-Legendre inequality. However, among [33], [34], there are still some shortcomings that can be improved, for example, using the affine Bessel-Legendre inequality instead of using Bessel-Legendre inequality inspired by [35], further augmenting their LKFs with delay-dependent matrices in non-integral terms, further augmenting integrated vectors in single-integral terms of LKFs via integral interval decomposition method, avoiding introducing the terms with $h^2(t)$, and so on. In addition, although the LKF of [35] is N-dependent, the non-integral terms and the integrated vectors in the single-integral terms are usual. So, we may combine the above potential improvement of [33], [34] and the idea of N-dependent LKF to further reduce the conservatism of stability criteria for DNN.

Inspired by the above analysis, this paper will construct an augmented LKF and the following ideas of deriving an enhanced stability criteria for DNN should be addressed as:

- Based on [22], [23], [34]–[36], some additional information has been added into the non-integral terms and single-integral terms, such as adding integrals of the neuron activation function over the delay interval into the non-integral terms, augmenting the usual integrated vectors of the single-integral term with $\int_a^b \dot{x}(u)du$-type integral terms. This not only increases some coupling information between state variables, but also increases the coupling information between some necessary variables of the affine Bessel-Legendre inequality.
- Inspired by [6], [26], [33], two different integrand vectors in the single-integral terms of the LKF are further augmented by supplementing $\int_{t-h(t)}^{s} \dot{x}(u)du$, $\int_s^t \dot{x}(u)du$ and $\int_s^{t-h(t)} \dot{x}(u)du$, $\int_{t-h}^{s} \dot{x}(u)du$ corresponding to different integral intervals, respectively. From the domain of integration, the whole integral domain of $\int_{t-h(t)}^{t} \dot{x}(u)du$ and $\int_{t-h}^{t-h(t)} \dot{x}(u)du$ are just the sum of the two integral domains of the integral items $\int_{t-h(t)}^{s} \dot{x}(u)du$, $\int_s^t \dot{x}(u)du$ and the sum of the two integral domains of the integral items $\int_s^{t-h(t)} \dot{x}(u)du$, $\int_{t-h}^{s} \dot{x}(u)du$, respectively. Thus, the two supplementary integral items can be seen as the decompositions and complements to the terms $\int_{t-h(t)}^{t} \dot{x}(u)du$ and $\int_{t-h}^{t-h(t)} \dot{x}(u)du$.
- To avoid introducing the terms with $h^2(t)$ involved in [33], [34], the $\int_a^b \dot{x}(u)du$-type integral items are chosen in two different integrand vectors of the single-integral terms, respectively, which leads to solve the corresponding linear matrix inequalities (LMIs) of stability results conveniently by using the convexity of LMI without introducing any additional inequality constraints. This may be another contribution to expand the feasible region of the corresponding LMIs.

This paper is organized as follows. *Section 2* represents a brief statement of the research problem and some necessary and important definitions, assumptions and lemmas. *Section 3* presents the main stability criteria for DNNs, including theorems and corollaries. *Section 4* shows some numerical examples based on the results of the previous section. See *Section 5* for conclusions.

Notation: The matrix P is a positive definite matrix if $P > 0$, vice versa. I and 0 represent corresponding dimension unit matrix and zero matrix. The diagonal matrix is represented by diag$\{\cdots\}$, and $e_i\,(i = 1, \ldots, m)$ are block entry matrices with $e_3^T = \begin{bmatrix} 0 & 0 & I & \underbrace{0\cdots0}_{m-3} \end{bmatrix}$, where m is the length of the vector $\xi(t)$ in theorems and corollaries. $*$ denotes the symmetric terms in a block matrix. $F[h(t), d(t)]$, $G[x(t)]$ denote F, G are the function of $h(t)$, $d(t)$ and $x(t)$, respectively. $Sym\{B\} = B + B^T$.

II. PROBLEM FORMULATION AND PRELIMINARY

The following DNN system descriptions are given:

$$\dot{y}(t) = -Ay(t) + W_0 f(W_2 y(t)) + W_1 f(W_2 y(t - h(t))) + J. \tag{1}$$

Here, $y(t) = [y_1(t)\ y_2(t)\ \cdots\ y_n(t)]^T$ is the neuron state vector of the above DNN. The positive definite diagonal matrix $A \in \mathbb{R}^{n\times n}$ denotes the feedback gain; W_0, W_1, $W_2 \in \mathbb{R}^{n\times n}$ represent the constant matrices of interconnection weight and $J \in \mathbb{R}^n$ represents a constant input vector. $f(y) = [f_1(y)\ f_2(y)\ \cdots\ f_n(y)]^T$ denotes the neuron activation function (NAF); $h(t)$ represents the corresponding time-varying delay and satisfies some constraints hypothesis described as follows:

$$0 \le h(t) \le h, \quad |\dot{h}(t)| \le \mu, \quad \forall t \ge 0 \tag{2}$$

with two nonnegative constant boundaries h and μ.

Suppose the nonlinear function, that is NAF, $f(y)$ satisfies the following hypothesis [37], [38].

$$k_i^- \le \frac{f_i(y_1) - f_i(y_2)}{y_1 - y_2} \le k_i^+, \quad \forall y_1 \ne y_2,\ i = 1, 2, \cdots, n. \tag{3}$$

The two real scalars k_i^- and k_i^+ could be arbitrary. As noted in [39], when $k_i^- = 0$ and $k_i^+ > 0$, assumption (3) describes a globally Lipschitz continuous and monotone nondecreasing case. And when $k_i^+ > k_i^- > 0$, the assumption represents the case of globally Lipschitz continuous and monotone increasing.

For the sake of simple mathematical analysis of the stability analysis for the DNN (1), we first let $e(\cdot) = y(\cdot) - y^*$ and $g(e) = f(e + y^*) - f(y^*)$, then the equilibrium point y^* can be shifted to the origin. Thus, the DNN (1) can be redescribed as the following system.

$$\dot{e}(t) = -Ae(t) + W_0 g(W_2 e(t)) + W_1 g(W_2 e(t - h(t))) \tag{4}$$

with the transformed NAF $g(\cdot)$ with $g_i(0) = 0$ satisfying

$$k_i^- \leq \frac{g_i(\alpha_1) - g_i(\alpha_2)}{\alpha_1 - \alpha_2} \leq k_i^+, \quad \forall \alpha_1 \neq \alpha_2, \quad i = 1, 2, \cdots, n, \tag{5}$$

which derive the following conditions for $\alpha_2 = 0$

$$k_i^- \leq \frac{g_i(\alpha)}{\alpha} \leq k_i^+, \quad i = 1, 2, \cdots, n. \tag{6}$$

The most important work to be dealt with in this paper is to obtain a less conservative stability condition than some recent results for the DNN (1) based on constraints (2)-(3) via Lyapunov stability theory. The following main lemmas are necessary for this purpose.

Lemma 1 [36]: For given matrices $R(\in \mathbb{R}^{n\times n}) > 0$, X with appropriate dimensions, $\xi \in \mathbb{R}^m$ and a continuous and differentiable function $\{x(s) \in \mathbb{R}^n \mid s \in [a, b]\}$, the following integral inequality holds for all integer $N \geq 0$

$$-\int_a^b \dot{x}^T(s)R\dot{x}(s)ds \leq 2\zeta_N^T \Gamma_N^T X^T \xi + (b-a)\xi^T X R_N^{-1} X^T \xi, \tag{7}$$

where

$$\zeta_N = \begin{cases} \left[x^T(b)\, x^T(a)\right]^T, \\ \quad N = 0, \\ \left[x^T(b)\, x^T(a)\, \frac{1}{b-a}\Theta_0^T \cdots \frac{1}{b-a}\Theta_{N-1}^T\right]^T, \\ \quad N > 0, \end{cases}$$

$$\Gamma_N = \left[\pi_N^T(0)\; \pi_N^T(1)\; \cdots\; \pi_N^T(N)\right]^T,$$

$$\pi_N(k) = \begin{cases} [I \;\; -I], & N = 0, \\ \left[I \;\; (-1)^{k+1}I \;\; \theta_{Nk}^0 I \;\; \cdots \;\; \theta_{Nk}^{N-1} I\right], & N > 0, \end{cases}$$

$$\theta_{Nk}^j = \begin{cases} (2j+1)((-1)^{k+j} - 1), & j \leq k, \\ 0, & j > k, \end{cases}$$

$$\Theta_k = \int_a^b L_k(s)x(s)ds,$$

$$L_k(s) = (-1)^k \sum_{l=0}^k \left[(-1)^l \binom{k}{l}\binom{k+l}{l}\right]\left(\frac{s-a}{b-a}\right)^l,$$

$$R_N = \operatorname{diag}\{R,\; 3R,\; \cdots,\; (2N+1)R\}.$$

Remark 1: It is obvious that the inequality (7) will reduce to the Jensen's free-matrix-based inequality when $N = 0$, so to further reduce the conservatism of the stability criteria, the N is always assumed to be $N \geq 1$ in the following derivations. In addition, letting $\xi = \left[x^T(b)\; x^T(a)\; \lambda_1^T\; \lambda_2^T\; \cdots\; \lambda_N^T\right]$ with $\lambda_i = \int_a^b \frac{(s-a)^{i-1}}{(b-a)^i}x(s)ds$, $(i = 1, 2, \cdots, N)$, the inequality (7) can be rewritten as the following inequality.

$$-\int_a^b \dot{x}^T(s)R\dot{x}(s)ds \leq \xi^T\left[\Xi_N^T\Gamma_N^T X^T + X\Gamma_N\Xi_N\right]\xi + (b-a)\xi^T X R_N^{-1}X^T\xi, \tag{8}$$

where Ξ_N can be found at the bottom of this page.

III. MAIN RESULTS

In this section, a new stability criterion for the DNN (1) based on an augmented LKF is proposed at first. Besides, for detailed comparison, two stability criteria based on the same inequality techniques but different LKFs (including the LKF removed the nonlinear items in the non-integral terms and the LKF removed the augmented part of the integrated vectors in the single-integral terms) are also given.

To simplify the representation of vectors and matrices in the theorem and corollaries, use the following notations:

$$\begin{aligned}
h_d &= 1-\dot{h}(t), \quad h_t = h(t), \\
\bar{h}_t &= h - h(t), \quad i = 1, 2, \cdots, N, \\
\alpha_i(t) &= \int_{t-h_t}^t \frac{(s-t+h_t)^{i-1}}{h_t^i} e^T(s)ds, \\
\beta_i(t) &= \int_{t-h}^{t-h_t} \frac{(s-t+h)^{i-1}}{\bar{h}_t^i} e^T(s)ds, \\
v_1(t) &= \int_{t-h_t}^t g^T(W_2 e(s))ds, \\
v_2(t) &= \int_{t-h}^{t-h_t} g^T(W_2 e(s))ds, \\
\zeta_N^T(t) &= \left[e^T(t)\; e^T(t-h_t)\; e^T(t-h)\right. \\
&\quad \left. v_1(t)\; v_2(t)\; h_t\mathcal{C}_t\; \bar{h}_t\bar{\mathcal{C}}_t\right], \\
\zeta_{1N}^T(t) &= \left[e^T(t)\; e^T(t-h_t)\; e^T(t-h)\; v_1(t)\; \mathcal{C}_t\right], \\
\zeta_{2N}^T(t) &= \left[e^T(t)\; e^T(t-h_t)\; e^T(t-h)\; v_2(t)\; \bar{\mathcal{C}}_t\right], \\
\mathcal{C}_t &= [\alpha_1(t)\, \alpha_2(t) \cdots \alpha_N(t)], \\
\bar{\mathcal{C}}_t &= [\beta_1(t)\, \beta_2(t) \cdots \beta_N(t)],
\end{aligned}$$

$$\Xi_N = \begin{bmatrix}
I & 0 & 0 & 0 & \cdots & 0 \\
0 & I & 0 & 0 & \cdots & 0 \\
0 & 0 & (-1)^0\binom{1}{0}\binom{1+0}{0}I & 0 & \cdots & 0 \\
0 & 0 & (-1)^1\binom{1}{0}\binom{1+0}{1}I & (-1)^2\binom{1}{1}\binom{1+1}{1}I & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & (-1)^{N-1}\binom{1}{0}\binom{1+0}{0}I & (-1)^N\binom{N-1}{1}\binom{N-1+1}{1}I & \cdots & (-1)^{2N-2}\binom{N-1}{N-1}\binom{N-1+N-1}{N-1}I
\end{bmatrix}$$

$$\xi^T(t) = \Big[e^T(t)\ e^T(t-h_t)\ e^T(t-h)\ \dot{e}^T(t)\ \dot{e}^T(t-h_t)\ \dot{e}^T(t-h)\ g^T(W_2e(t))\ g^T(W_2e(t-h_t))\ g^T(W_2e(t-h))\ v_1(t)\ v_2(t)\ \mathcal{C}_t\ \bar{\mathcal{C}}_t\Big],$$

$$\eta_1^T(t,s) = \Big[e^T(s)\ \dot{e}^T(s)\ g^T(W_2e(s))\ \int_s^t \dot{e}^T(u)du\ \int_{t-h_t}^s \dot{e}^T(u)du\ \int_{t-h}^{t-h_t} \dot{e}^T(u)du\Big],$$

$$\eta_2^T(t,s) = \Big[e^T(s)\ \dot{e}^T(s)\ g^T(W_2e(s))\ \int_{t-h_t}^t \dot{e}^T(u)du\ \int_s^{t-h_t} \dot{e}^T(u)du\ \int_{t-h}^s \dot{e}^T(u)du\Big],$$

$$K_1 = \text{diag}\left\{k_1^+, k_2^+, \cdots, k_n^+\right\},$$
$$K_2 = \text{diag}\left\{k_1^-, k_2^-, \cdots, k_n^-\right\}.$$

A. AN AUGMENTED LKF

When analyzed the stability of the DNN, the main aim is reducing the conservatism of stability criteria via LKFs with broader form and much tighter integral inequality techniques application. It is foreseeable that broader LKFs or much tighter integral inequality techniques may reduce the conservatism. The purpose of this paper is to construct an augmented LKF to reduce the conservatism of stability criteria under the latest and much tighter inequality technique. So. firstly, we give this improved LKF described as follows:

$$V(t) = \sum_{j=1}^{5} V_j(t) \tag{9}$$

with

$$V_1(t) = \zeta_N^T(t)P_N\zeta_N(t) + h_t\zeta_{1N}^T(t)P_{aN}\zeta_{1N}(t) + \bar{h}_t\zeta_{2N}^T(t)P_{bN}\zeta_{2N}(t),$$

$$V_2(t) = \int_{t-h_t}^t \eta_1^T(t,s)Q_1\eta_1(t,s)ds + \int_{t-h}^{t-h_t} \eta_2^T(t,s)Q_2\eta_2(t,s)ds,$$

$$V_3(t) = 2\sum_{i=1}^n \int_0^{W_{2i}e(t)} \left[h_{1i}(g_i(s)-k_i^-s) + h_{2i}(k_i^+s-g_i(s))\right]ds + 2\sum_{i=1}^n \int_0^{W_{2i}e(t-h_t)} \left[h_{3i}(g_i(s)-k_i^-s) + h_{4i}(k_i^+s-g_i(s))\right]ds + 2\sum_{i=1}^n \int_0^{W_{2i}e(t-h)} \left[h_{5i}(g_i(s)-k_i^-s) + h_{6i}(k_i^+s-g_i(s))\right]ds,$$

$$V_4(t) = \int_{-h}^0 \int_{t+\theta}^t \dot{e}^T(s)R\dot{e}(s)dsd\theta,$$

$$V_5(t) = h\int_{-h}^0 \int_{t+\theta}^t g^T(W_2e(s))Zg(W_2e(s))dsd\theta,$$

where $P_N \in \mathbb{R}^{(5+2N)n\times(5+2N)n}$, $(P_{aN}, P_{bN} \in \mathbb{R}^{(4+N)n\times(4+N)n})$, $(Q_1, Q_2 \in \mathbb{R}^{6n\times 6n})$, $(R, Z \in \mathbb{R}^{n\times n})$ are positive definite matrices, $h_{pi} > 0\ (p = 1, 2, \cdots, 6)$.

Remark 2: The novelties of the LKF (9) lie in the following three aspects and these modified measures are a major contribution to reducing the conservatism of stability criteria for the DNN.

- It is different from the ones in [22], [23], [34], [35] due to $V_1(t)$ and $V_2(t)$. Some additional information has been added into $V_1(t)$, such as $v_1(t)$, $v_2(t)$ and delay-dependent terms, where more coupling information between some states and NAF is described by the Lyapunov matrices P_N, P_{aN} and P_{bN}. Especially, the usual integrated vectors of the single-integral term, $\eta^T(s) = \left[e^T(s)\ \dot{e}^T(s)\ g^T(W_2e(s))\right]$, are augmented as two different integrated vectors dependent on the different integral ranges, that is $\eta_1(t,s)$ and $\eta_2(t,s)$. This not only increases some coupling information between some system variables, but also increases the coupling information between some necessary variables of the inequality lemma 1.
- When $N = 2$, compared with the ones in [26], where the quadratic term $e^T(t)Pe(t)$ is merged into the single integral term, that is, there were no non-integral terms. Indeed, Chen *et al.* [40] pointed out that the conservatism can be further reduced by combining an extended non-integral terms with the LKF of [26]. Therefore, the augmented non-integral terms $V_1(t)$ is added into the new LKF proposed in (9). Moreover, based on [6], [26], [33], two different vectors $\eta_1(t,s)$ and $\eta_2(t,s)$ are further augmented by supplementing $\int_{t-h_t}^s \dot{e}(u)du$, $\int_s^t \dot{e}(u)du$ and $\int_s^{t-h_t} \dot{e}(u)du$, $\int_{t-h}^s \dot{e}(u)du$ corresponding to different integral intervals, respectively. From the domain of integration, the whole integral domain of $\int_{t-h_t}^t \dot{e}(u)du$ and $\int_{t-h}^{t-h_t} \dot{e}(u)du$ are just the sum of the two integral domains of the integral items $\int_{t-h_t}^s \dot{e}(u)du$ and $\int_s^t \dot{e}(u)du$ and the sum of the two integral domains of the integral items $\int_{t-h}^s \dot{e}(u)du$ and $\int_s^{t-h_t} \dot{e}(u)du$, respectively. Thus, the two supplementary integral items can be seen as the decompositions and complements to the terms $\int_{t-h_t}^t \dot{e}(u)du$ and $\int_{t-h}^{t-h_t} \dot{e}(u)du$.
- Based on the N-dependent generalized free-matrix-based integral inequality lemma 1, the LKF (9) is also N-dependent, so the stability criteria derived via the LKF are also hierarchy of LMI conditions, that is, the conservatism of the stability criteria decreases as N increases.

B. STABILITY CRITERIA

The following theorem and corollaries will give some stability criteria for the DNN (1) satisfying the conditions (2)-(3).

Theorem 1: The DNN (1) with the conditions (2)-(3) is globally asymptotically stable for given non-negative scalars of h, μ, if there exist positive definite matrices $P_N \in \mathbb{R}^{(5+2N)n\times(5+2N)n}$, $(P_{aN}, P_{bN} \in \mathbb{R}^{(4+N)n\times(4+N)n})$,

$(Q_1, Q_2 \in \mathbb{R}^{6n\times 6n})$, $(R, Z \in \mathbb{R}^{n\times n})$, positive definite diagonal matrices $H_p = \mathrm{diag}\{h_{p1}, h_{p2}, \cdots, h_{pn}\}$, $(\Lambda_{jk}, \Theta_{jr} \in \mathbb{R}^{n\times n})$ and any matrices $(S_1, S_2, U_l \in \mathbb{R}^{n\times n})$, $(X, Y \in \mathbb{R}^{(N+2)n\times(N+1)n})$, $(p = 1, \cdots, 6;\ j, r = 1, 2;\ k = 1, 2, 3;\ l = 1, \cdots, 4)$ such that LMIs (10)-(11) hold for all $N \in \mathbb{N}$:

$$\begin{bmatrix} \Omega_{[0,\dot{h}_t]} & e_{10}S_2 & hE_2Y \\ * & -Z & 0 \\ * & * & -h\bar{R}_N \end{bmatrix} < 0, \tag{10}$$

$$\begin{bmatrix} \Omega_{[h,\dot{h}_t]} & e_{11}S_1^T & hE_1X \\ * & -Z & 0 \\ * & * & -h\bar{R}_N \end{bmatrix} < 0, \tag{11}$$

where

$$\begin{aligned}
\Omega_{[h_t,\dot{h}_t]} &= Sym\left\{\Pi_{1[h_t,\dot{h}_t]}\right\} + \Pi_{2[h_t,\dot{h}_t]} + \Pi_{3[h_t,\dot{h}_t]},\\
\Pi_{1[h_t,\dot{h}_t]} &= \Delta_1 P_N \Delta_{1i}^T + \Delta_2 P_{aN}\Delta_{2i}^T + \Delta_3 P_{bN}\Delta_{3i}^T\\
&\quad + \Delta_4 Q_1 \Delta_{4d}^T + \Delta_5 Q_2 \Delta_{5d}^T + \Omega_{1[h(t)]} + \Omega_{2[\dot{h}(t)]},\\
\Pi_{2[h_t,\dot{h}_t]} &= \dot{h}_t \Delta_2 P_{aN}\Delta_2^T - \dot{h}_t\Delta_3 P_{bN}\Delta_3^T + he_4Re_4^T\\
&\quad + h^2 e_7 Z e_7^T + \Delta_6 Q_1 \Delta_6^T + h_d\Delta_7(Q_2 - Q_1)\Delta_7^T\\
&\quad - \Delta_8 Q_2 \Delta_8^T,\\
\Pi_{3[h_t,\dot{h}_t]} &= E_1\left[XH_N + H_N^T X^T\right]E_1^T\\
&\quad E_2\left[YH_N + H_N^T Y^T\right]E_2^T - E_3\Pi_4 E_3^T,\\
E_1 &= [e_1 \quad e_2 \quad e_{12} \quad \cdots \quad e_{11+N}],\\
E_2 &= [e_2 \quad e_3 \quad e_{12+N} \quad \cdots \quad e_{11+2N}],\\
E_3 &= [e_{10} \quad e_{11}],\\
\Pi_4 &= \begin{bmatrix} (2-\rho)Z & (1-\rho)S_1 + \rho S_2 \\ * & (1+\rho)Z \end{bmatrix}, \quad \rho = \frac{h_t}{h},\\
\Delta_1 &= [e_1 \quad e_2 \quad e_3 \quad e_{10} \quad e_{11} \quad h_t e_{12} \quad \cdots \quad h_t e_{11+N}\\
&\quad \bar{h}_t e_{12+N} \quad \cdots \quad \bar{h}_t e_{11+2N}],\\
\Delta_2 &= [e_1 \quad e_2 \quad e_3 \quad e_{10} \quad e_{12} \quad \cdots \quad e_{11+N}],\\
, \Delta_3 &= [e_1 \quad e_2 \quad e_3 \quad e_{11} \quad e_{12+N} \quad \cdots \quad e_{11+2N}],\\
\Delta_{1i} &= [e_4\ h_d e_5\ e_6\ e_7 - h_d e_8\ h_d e_8 - e_9\ \Delta_{11i}\ \Delta_{12i}],\\
\Delta_{2i} &= [h_t e_4 \quad h_t h_d e_5 \quad h_t e_6 \quad h_t(e_7 - h_d e_8)\\
&\quad \Delta_{011} \quad \cdots \quad \Delta_{01N}],\\
\Delta_{3i} &= [\bar{h}_t e_4 \quad \bar{h}_t h_d e_5 \quad \bar{h}_t e_6 \quad \bar{h}_t(h_d e_8 - e_9)\\
&\quad \Delta_{021} \quad \cdots \quad \Delta_{02N}],\\
\Delta_{11i} &= \dot{h}_t e_{11+i} + \Delta_{01i},\\
\Delta_{12i} &= -\dot{h}_t e_{11+i+N} + \Delta_{02i},\\
\Delta_{01i} &= \begin{cases} e_1 - h_d e_2 - \dot{h}_t e_{12}, & i = 1,\\ e_1 - (i-1)h_d e_{10+i} - i\dot{h}_t e_{11+i}, & i > 1,\end{cases}\\
\Delta_{02i} &= \begin{cases} h_d e_2 - e_3 + \dot{h}_t e_{12+N}, & i = 1,\\ h_d e_2 - (i-1)e_{10+N+i} & \\ +i\dot{h}_t e_{11+N+i}, & i > 1,\end{cases}\\
\Delta_4 &= [h_t e_{12} \quad e_1 - e_2 \quad e_{10}\\
&\quad h_t(e_1 - e_{12}) \quad h_t(e_{12} - e_2) \quad h_t(e_2 - e_3)],\\
\Delta_{4d} &= [0 \quad 0 \quad 0 \quad e_4 \quad -h_d e_5 \quad h_d e_5 - e_6],
\end{aligned}$$

$$\begin{aligned}
\Delta_5 &= \left[\bar{h}_t e_{12+N} \quad e_2 - e_3 \quad e_{11}\right.\\
&\quad \left.\bar{h}_t(e_1 - e_2) \quad \bar{h}_t(e_2 - e_{12+N}) \quad \bar{h}_t(e_{12+N} - e_3)\right],\\
\Delta_{5d} &= [0 \quad 0 \quad 0 \quad e_4 - h_d e_5 \quad h_d e_5 \quad -e_6],\\
\Delta_6 &= [e_1 \quad e_4 \quad e_7 \quad 0 \quad e_1 - e_2 \quad e_2 - e_3],\\
\Delta_7 &= [e_2 \quad e_5 \quad e_8 \quad e_1 - e_2 \quad 0 \quad e_2 - e_3],\\
\Delta_8 &= [e_3 \quad e_6 \quad e_9 \quad e_1 - e_2 \quad e_2 - e_3 \quad 0].\\
\bar{R}_N &= \mathrm{diag}\{R, 3R, \cdots, (2N+1)R\},\\
H_N &= \Gamma_N \Xi_N,\\
\Omega_{1[h(t)]} &= \sum_{j=1}^{3}\left[e_{6+j} - e_j W_2^T K_2^T\right]\cdot\\
&\quad \left(\frac{h_t}{h}\Lambda_{1j} + \frac{\bar{h}_t}{h}\Lambda_{2j}\right)\left[K_1 W_2 e_j^T - e_{6+j}\right]\\
&\quad + \sum_{j=1}^{2}\left[(e_{6+j} - e_{7+j}) - (e_j - e_{1+j})W_2^T K_2^T\right]\cdot\\
&\quad \left(\frac{h_t}{h}\Theta_{1j} + \frac{\bar{h}_t}{h}\Theta_{2j}\right)\left[K_1 W_2(e_j - e_{1+j})^T\right.\\
&\quad \left.-(e_{6+j} - e_{7+j})^T\right],\\
\Omega_{2[\dot{h}(t)]} &= \left[e_7 - e_1 W_2^T K_2^T\right]H_1 W_2 e_4^T\\
&\quad + \left[e_1 W_2^T K_1^T - e_7\right]H_2 W_2 e_4^T\\
&\quad + h_d\left[e_8 - e_2 W_2^T K_2^T\right]H_3 W_2 e_5^T\\
&\quad + h_d\left[e_2 W_2^T K_1^T - e_8\right]H_4 W_2 e_5^T\\
&\quad + \left[e_9 - e_3 W_2^T K_2^T\right]H_5 W_2 e_6^T\\
&\quad + \left[e_3 W_2^T K_1^T - e_9\right]H_6 W_2 e_6^T.
\end{aligned}$$

Proof: Calculating the derivative of $V(t)$ along the solution of the DNN (1) yields

$$\begin{aligned}
\dot{V}_1(t) &= 2\zeta_N^T(t)P_N\dot{\zeta}_N(t) + \dot{h}_t\zeta_{1N}^T(t)P_{aN}\zeta_{1N}(t)\\
&\quad - \dot{h}_t\zeta_{2N}^T(t)P_{bN}\zeta_{2N}(t) + 2h_t\zeta_{1N}^T(t)P_{aN}\dot{\zeta}_{1N}(t)\\
&\quad + 2\bar{h}_t\zeta_{2N}^T(t)P_{bN}\dot{\zeta}_{2N}(t),\\
\dot{V}_2(t) &= \eta_1^T(t,t)Q_1\eta_1(t,t)\\
&\quad - h_d\eta_1^T(t, t-h_t)Q_1\eta_1(t, t-h_t)\\
&\quad + h_d\eta_2^T(t, t-h_t)Q_2\eta_2(t, t-h_t)\\
&\quad - \eta_2^T(t, t-h)Q_2\eta_2(t, t-h)\\
&\quad + 2\int_{t-h_t}^{t}\eta_1^T(t,s)Q_1\frac{\partial\eta_1(t,s)}{\partial t}ds\\
&\quad + 2\int_{t-h}^{t-h_t}\eta_2^T(t,s)Q_2\frac{\partial\eta_2(t,s)}{\partial t}ds,\\
\dot{V}_3(t) &= 2g^T(W_2 e(t))(H_1 - H_2)W_2\dot{e}(t)\\
&\quad + 2e^T(t)W_2^T(K_1H_2 - K_2H_1)W_2\dot{e}(t)\\
&\quad + 2h_d g^T(W_2 e(t - h(t)))(H_3\\
&\quad - H_4)W_2\dot{e}(t - h(t))\\
&\quad + 2h_d e^T(t - h(t))W_2^T(K_1H_4
\end{aligned}$$

$$
\begin{aligned}
&- K_2H_3)W_2\dot{e}(t-h(t)) \\
&+ 2g^T(W_2e(t-h))(H_5 \\
&- H_6)W_2\dot{e}(t-h) \\
&+ 2e^T(t-h)W_2^T(K_1H_6 \\
&- K_2H_5)W_2\dot{e}(t-h), \\
\dot{V}_4(t) &= h\dot{e}^T(t)R\dot{e}(t) - \int_{t-h}^{t}\dot{e}^T(s)R\dot{e}(s)ds, \\
\dot{V}_5(t) &= h^2g^T(W_2e(t))Zg(W_2e(t)) \\
&- h\int_{t-h}^{t}g^T(W_2e(s))Zg(W_2e(s))ds.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
\dot{h}_t\mathcal{C}_t &= \dot{h}_t\xi^T(t)\left[e_{12}\; e_{13}\;\cdots\; e_{11+N}\right], \\
\dot{\bar{h}}_t\mathcal{C}_t &= -\dot{h}_t\xi^T(t)\left[e_{12+N}\; e_{13+N}\;\cdots\; e_{11+2N}\right], \\
h_t\dot{\mathcal{C}}_t &= h_t\left[\dot{\alpha}_1(t)\;\dot{\alpha}_2(t)\;\cdots\;\dot{\alpha}_N(t)\right], \\
\bar{h}_t\dot{\bar{\mathcal{C}}}_t &= \bar{h}_t\left[\dot{\beta}_1(t)\;\dot{\beta}_2(t)\;\cdots\;\dot{\beta}_N(t)\right], \\
h_t\dot{\alpha}_i(t) &= \begin{cases}\xi^T(t)[e_1 - h_de_2 - \dot{h}_te_{12}], & i=1,\\ \xi^T(t)\left[e_1-(i-1)h_de_{10+i}\right.\\ \left.-i\dot{h}_te_{11+i}\right], & i>1,\end{cases} \\
\bar{h}_t\dot{\beta}_i(t) &= \begin{cases}\xi^T(t)[h_de_2 - e_3 + \dot{h}_te_{12+N}], & i=1,\\ \xi^T(t)\left[h_de_2-(i-1)e_{10+N+i}\right.\\ \left.+i\dot{h}_te_{11+N+i}\right], & i>1,\end{cases} \\
\frac{d}{dt}(h_t\mathcal{C}_t) &= \dot{h}_t\mathcal{C}_t + h_t\dot{\mathcal{C}}_t, \\
\frac{d}{dt}(\bar{h}_t\bar{\mathcal{C}}_t) &= -\dot{h}_t\mathcal{C}_t + \bar{h}_t\dot{\bar{\mathcal{C}}}_t, \\
\dot{\zeta}_N^T(t) &= \Big[\xi^T(t)\left[e_4\; h_de_5\; e_6\; e_7 - h_de_8\right. \\
&\quad\left. h_de_8 - e_9\right]\; \frac{d}{dt}(h_t\mathcal{C}_t)\; \frac{d}{dt}(\bar{h}_t\bar{\mathcal{C}}_t)\Big], \\
\dot{\zeta}_{1N}^T(t) &= \left[\xi^T(t)\left[e_4\; h_de_5\; e_6\; e_7 - h_de_8\right]\; \dot{\mathcal{C}}_t\right], \\
\dot{\zeta}_{2N}^T(t) &= \left[\xi^T(t)\left[e_4\; h_de_5\; e_6\; h_de_8 - e_9\right]\; \dot{\bar{\mathcal{C}}}_t\right], \\
\eta_1^T(t,t) &= \xi^T(t)\left[e_1\; e_4\; e_7\; 0\; e_1-e_2\; e_2-e_3\right], \\
\eta_1^T(t,t-h_t) &= \eta_2^T(t,t-h_t) = \xi^T(t)\left[e_2\; e_5\; e_8\right. \\
&\quad\left. e_1-e_2\; 0\; e_2-e_3\right], \\
\eta_2^T(t,t-h) &= \xi^T(t)\left[e_3\; e_6\; e_9\; e_1-e_2\; e_2-e_3\; 0\right], \\
\int_{t-h_t}^{t}\eta_1^T(t,s)ds &= \xi^T(t)\left[h_te_{12}\; e_1-e_2\; e_{10}\; h_t(e_1-e_{12})\right. \\
&\quad\left. h_t(e_{12}-e_2)\; h_t(e_2-e_3)\right], \\
\int_{t-h}^{t-h_t}\eta_2^T(t,s)ds &= \xi^T(t)\left[\bar{h}_te_{12+N}\; e_2-e_3\; e_{11}\right. \\
&\quad \bar{h}_t(e_1-e_2)\; \bar{h}_t(e_2-e_{12+N}) \\
&\quad\left.\bar{h}_t(e_{12+N}-e_3)\right], \\
\frac{\partial\eta_1(t,s)}{\partial t} &= \xi^T(t)\left[0\;0\;0\;e_4\; -h_de_5\; h_de_5-e_6\right], \\
\frac{\partial\eta_2(t,s)}{\partial t} &= \xi^T(t)\left[0\;0\;0\;e_4 - h_de_5\; h_de_5\; -e_6\right].
\end{aligned}
$$

It can be obtained from lemmas 1 and 2 that

$$
\begin{aligned}
-\int_{t-h}^{t}\dot{e}^T(s)R\dot{e}(s)ds &\le \xi(t)^TE_1\Big[XH_N + H_N^TX^T \\
&\quad + h_tX\bar{R}_N^{-1}X^T\Big]E_1^T\xi(t) \\
&\quad + \xi(t)^TE_2\Big[YH_N + H_N^TY^T \\
&\quad + \bar{h}_tY\bar{R}_N^{-1}Y^T\Big]E_2^T\xi(t),
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
&-h\int_{t-h}^{t}g^T(W_2e(s))Zg(W_2e(s))ds \\
&\quad = -h\int_{t-h_t}^{t}g^T(W_2e(s))Zg(W_2e(s))ds \\
&\qquad - h\int_{t-h}^{t-h_t}g^T(W_2e(s))Zg(W_2e(s))ds \\
&\quad \le -\frac{h}{h_t}\xi^T(t)e_{10}Ze_{10}^T\xi(t) - \frac{h}{\bar{h}_t}\xi^T(t)e_{11}Ze_{11}^T\xi(t) \\
&\quad \le -\xi^T(t)\begin{bmatrix}e_{10}^T\\ e_{11}^T\end{bmatrix}^T\begin{bmatrix}(2-\rho)Z & (1-\rho)S_1+\rho S_2\\ * & (1+\rho)Z\end{bmatrix} \\
&\qquad \times\begin{bmatrix}e_{10}^T\\ e_{11}^T\end{bmatrix} + (1-\rho)\xi^T(t)e_{10}S_2Z^{-1}S_2^Te_{10}^T\xi(t) \\
&\qquad + \rho\xi^T(t)e_{11}S_1^TZ^{-1}S_1e_{11}^T\xi(t).
\end{aligned} \tag{13}
$$

For any appropriately dimensioned matrices $\bar{U} = [U_1^T\; U_2^T\; U_3^T\; U_4^T]^T \in \mathbb{R}^{4n\times n}$, it is true that

$$
\begin{aligned}
0 = 2\xi^T(t)\left[e_1\; e_4\; e_7\; e_8\right]\bar{U}[-Ae_1^T + W_0e_7^T \\
+ W_1e_8^T - e_4^T]\xi(t).
\end{aligned} \tag{14}
$$

If the nonlinear constraint conditions (5) and (6) are partitioned according to time-varying delay, we can obtain

$$
\begin{aligned}
\lambda_k(s) &\triangleq 2\left[g(W_2e(s)) - K_2W_2e(s)\right]^T\left(\frac{h_t}{h}\Lambda_{1k}\right. \\
&\quad \left.+\frac{\bar{h}_t}{h}\Lambda_{2k}\right)\left[K_1W_2e(s) - g(W_2e(s))\right] \ge 0,
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
\delta_i(s_1,s_2) &\triangleq 2\left[g(W_2e(s_1)) - g(W_2e(s_2))\right. \\
&\quad \left.- K_2W_2(e(s_1)-e(s_2))\right]^T\left(\frac{h_t}{h}\Theta_{1j} + \frac{\bar{h}_t}{h}\Theta_{2j}\right) \\
&\quad \times\left[K_1W_2(e(s_1)-e(s_2)) - g(W_2e(s_1))\right. \\
&\quad \left.+ g(W_2e(s_2))\right] \ge 0,
\end{aligned} \tag{16}
$$

where Λ_{rk} and Θ_{rj} $(j, r = 1, 2;\; k = 1, 2, 3)$ are positive definite diagonal matrices.

Thus, the following inequalities hold

$$
\lambda_1(t) + \lambda_2(t-h_t) + \lambda_3(t-h) \ge 0, \tag{17}
$$

$$
\delta_1(t, t-h_t) + \delta_2(t-h_t, t-h) \ge 0. \tag{18}
$$

Finally, from the above derivation, we have

$$
\begin{aligned}
\dot{V}(t) &\le \xi^T(t)\Big\{\Omega_{[h_t,\dot{h}_t]} + h_tE_1X\bar{R}_N^{-1}X^TE_1^T \\
&\quad + \bar{h}_tE_2Y\bar{R}_N^{-1}Y^TE_2^T\Big\}\xi(t) \\
&\quad + \xi^T(t)\Big\{(1-\rho)e_{10}S_2Z^{-1}S_2^Te_{10}^T \\
&\quad + \rho e_{11}S_1^TZ^{-1}S_1e_{11}^T\Big\}\xi(t).
\end{aligned} \tag{19}
$$

Therefore, LMIs (10)-(11) hold, which implies that $\dot{V}(t) < 0$ by the transformation of **Schur complement equivalence**. This shows that the DNN (1) is stable from Lyapunov stability theory, which completes the proof.

Remark 3: Recently, some improved stability criteria for DNNs via a new augmented LKF were given in [33], [34], where the LKFs with $\int_a^b e(u)du$-type integral in $\eta_1(t,s)$ and $\eta_2(t,s)$ were constructed, which was aimed at coordinating with the second-order B-L integral inequality with the $\int_a^b e(s)ds$-type and $\int_a^b \int_\theta^b e(s)dsd\theta$-type integral items. However, they had to introduce the terms with $h^2(t)$ when bounding the derivative of the LKFs, which led to adding some additional inequality constraints into the main results. This narrowed the feasible region of the corresponding LMIs. In this paper, the coupling relationship between the necessary integral items for lemma 1 are already included in the derivative of $V_1(t)$. Thus, to avoid introducing the terms with $h^2(t)$, the $\int_a^b \dot{e}(u)du$-type integral items are chosen in $\eta_1(t,s)$ and $\eta_2(t,s)$, respectively, which can be solved conveniently by using the convexity of LMI without introducing any additional inequality constraints, which is also another contribution to expand the feasible region of the corresponding LMIs.

Remark 4: The second-order B-L integral inequality was used in [22], [33] to bound the integral item $\int_a^b \dot{e}^T(s)R\dot{e}(s)ds$. However, an affine version of the B-L integral inequality was widely used in a lot of literature, such as [23], [35], [36], [41], and inspired by them, the affine version is not only affine with respect to the length of the integral interval but also can further reduce the conservatism of stability criteria. Thus, the generalized free-matrix-based integral inequality lemma 1 is used to bound the derivative of $\dot{V}_4(t)$ instead of using the B-L integral inequality in this paper.

Remark 5: As described in the literature [35], the stability criteria proposed in this paper form a hierarchy of LMI conditions, that is the conservatism of the stability criteria decreases as N increases, which can also be seen from the comparison of the numerical examples in the next section. However, the number of decision variables will increase as N increases. So, we just solve the corresponding LMIs with N from 1 to 3 in the numerical examples.

Remark 6: To illustrate the effectiveness of augmenting the LKF with augmented $\eta_1(t,s)$ and $\eta_2(t,s)$, the following corollary can be obtained by removing the $\int_a^b \dot{e}(u)du$-type integral items in $\eta_1(t,s)$ and $\eta_2(t,s)$, that is, letting $\eta_1(t,s) = \eta_2(t,s) = [e^T(s)\ \dot{e}^T(s)\ g^T(W_2e(s))]$. And the proof of Corollary 1 is omitted here.

Corollary 1: The DNN (1) with the conditions (2)-(3) is globally asymptotically stable for given non-negative scalars of h, μ, if there exist positive definite matrices $P \in \mathbb{R}^{(5+2N)n\times(5+2N)n}$, ($P_a$, $P_b \in \mathbb{R}^{(4+N)n\times(4+N)n}$), ($Q_1$, $Q_2 \in \mathbb{R}^{3n\times 3n}$), ($R$, $Z \in \mathbb{R}^{n\times n}$), positive definite diagonal matrices $H_p = \mathrm{diag}\{h_{p1}, h_{p2}, \cdots, h_{pn}\}$, ($\Lambda_{jk}$, $\Theta_{jr} \in \mathbb{R}^{n\times n}$) and any matrices ($S_1$, S_2, $U_l \in \mathbb{R}^{n\times n}$), ($X$, $Y \in \mathbb{R}^{(N+2)n\times(N+1)n}$), ($p = 1, \cdots, 6$; $j, r = 1, 2$; $k = 1, 2, 3$; $l = 1, \cdots, 4$) such that LMIs (20)-(21) hold:

$$\begin{bmatrix} \tilde{\Omega}_{[0,\dot{h}_t]} & e_{10}S_2 & hE_2Y \\ * & -Z & 0 \\ * & * & -h\bar{R}_N \end{bmatrix} < 0, \tag{20}$$

$$\begin{bmatrix} \tilde{\Omega}_{[h,\dot{h}_t]} & e_{11}S_1^T & hE_1X \\ * & -Z & 0 \\ * & * & -h\bar{R}_N \end{bmatrix} < 0, \tag{21}$$

where

$$\begin{aligned}
\tilde{\Omega}_{[h_t,\dot{h}_t]} &= Sym\left\{\tilde{\Pi}_{1[h_t,\dot{h}_t]}\right\} + \tilde{\Pi}_{2[h_t,\dot{h}_t]} + \Pi_{3[h_t,\dot{h}_t]},\\
\tilde{\Pi}_{2[h_t,\dot{h}_t]} &= \dot{h}_t\Delta_2 P_{aN}\Delta_2^T - \dot{h}_t\Delta_3 P_{bN}\Delta_3^T + he_4Re_4^T \\
&\quad + h^2e_7Ze_7^T + \tilde{\Delta}_6Q_1\tilde{\Delta}_6^T + h_d\tilde{\Delta}_7(Q_2 - Q_1)\tilde{\Delta}_7^T \\
&\quad - \tilde{\Delta}_8Q_2\tilde{\Delta}_8^T,\\
\tilde{\Pi}_{1[h_t,\dot{h}_t]} &= \Delta_1P_N\Delta_{1i}^T + \Delta_2P_{aN}\Delta_{2i}^T + \Delta_3P_{bN}\Delta_{3i}^T \\
&\quad + \Omega_{1[h(t)]} + \Omega_{2[\dot{h}(t)]},\\
\tilde{\Delta}_6 &= [e_1\ e_4\ e_7],\ \tilde{\Delta}_7 = [e_2\ e_5\ e_8],\ \tilde{\Delta}_8 = [e_3\ e_6\ e_9].
\end{aligned}$$

Remark 7: To illustrate the effectiveness of augmented $V_1(t)$ in the LKF proposed in this paper, the following corollary can be obtained by choosing the LKF removed $v_1(t)$ and $v_2(t)$ in $V_1(t)$. And the proof of Corollary 2 is omitted here.

Corollary 2: The DNN (1) with the conditions (2)-(3) is globally asymptotically stable for given non-negative scalars of h, μ, if there exist positive definite matrices $P \in \mathbb{R}^{(3+2N)n\times(3+2N)n}$, ($P_a$, $P_b \in \mathbb{R}^{(3+N)n\times(3+N)n}$), ($Q_1$, $Q_2 \in \mathbb{R}^{6n\times 6n}$), ($R$, $Z \in \mathbb{R}^{n\times n}$), positive definite diagonal matrices $H_p = \mathrm{diag}\{h_{p1}, h_{p2}, \cdots, h_{pn}\}$, ($\Lambda_{jk}$, $\Theta_{jr} \in \mathbb{R}^{n\times n}$) and any matrices ($S_1$, S_2, $U_l \in \mathbb{R}^{n\times n}$), ($X$, $Y \in \mathbb{R}^{(N+2)n\times(N+1)n}$), ($p = 1, \cdots, 6$; $j, r = 1, 2$; $k = 1, 2, 3$; $l = 1, \cdots, 4$) such that LMIs (22)-(23) hold:

$$\begin{bmatrix} \bar{\Omega}_{[0,\dot{h}_t]} & e_{10}S_2 & hE_2Y \\ * & -Z & 0 \\ * & * & -h\bar{R}_N \end{bmatrix} < 0, \tag{22}$$

$$\begin{bmatrix} \bar{\Omega}_{[h,\dot{h}_t]} & e_{11}S_1^T & hE_1X \\ * & -Z & 0 \\ * & * & -h\bar{R}_N \end{bmatrix} < 0, \tag{23}$$

$$\begin{aligned}
\bar{\Omega}_{[h_t,\dot{h}_t]} &= Sym\left\{\bar{\Pi}_{1[h_t,\dot{h}_t]}\right\} + \bar{\Pi}_{2[h_t,\dot{h}_t]} + \Pi_{3[h_t,\dot{h}_t]},\\
\Pi_{1[h_t,\dot{h}_t]} &= \bar{\Delta}_1P_N\bar{\Delta}_{1i}^T + \bar{\Delta}_2P_{aN}\bar{\Delta}_{2i}^T + \bar{\Delta}_3P_{bN}\bar{\Delta}_{3i}^T \\
&\quad + \Delta_4Q_1\Delta_{4d}^T + \Delta_5Q_2\Delta_{5d}^T \\
&\quad + \Omega_{1[h(t)]} + \Omega_{2[\dot{h}(t)]},\\
\Pi_{2[h_t,\dot{h}_t]} &= \dot{h}_t\bar{\Delta}_2P_{aN}\bar{\Delta}_2^T - \dot{h}_t\bar{\Delta}_3P_{bN}\bar{\Delta}_3^T + he_4Re_4^T \\
&\quad + h^2e_7Ze_7^T + \Delta_6Q_1\Delta_6^T \\
&\quad + h_d\Delta_7(Q_2 - Q_1)\Delta_7^T - \Delta_8Q_2\Delta_8^T,\\
\bar{\Delta}_1 &= [e_1\ e_2\ e_3\ h_te_{12}\ \cdots\ h_te_{11+N} \\
&\quad \bar{h}_te_{12+N}\ \cdots\ \bar{h}_te_{11+2N}],\\
\bar{\Delta}_2 &= [e_1\ e_2\ e_3\ e_{12}\ \cdots\ e_{11+N}],\\
\bar{\Delta}_3 &= [e_1\ e_2\ e_3\ e_{12+N}\ \cdots\ e_{11+2N}],\\
\bar{\Delta}_{1i} &= [e_4\ h_de_5\ e_6\ \Delta_{11i}\ \Delta_{12i}],\\
\bar{\Delta}_{2i} &= [h_te_4\ h_th_de_5\ h_te_6\ \Delta_{011}\ \cdots\ \Delta_{01N}],\\
\bar{\Delta}_{3i} &= [\bar{h}_te_4\ \bar{h}_th_de_5\ \bar{h}_te_6\ \Delta_{021}\ \cdots\ \Delta_{02N}].
\end{aligned}$$

Remark 8: It is noteworthy that Seuret and Gouaisbaut [42] pointed that the set of upper and lower bound of delays and upper and lower bound of delay derivatives constitutes a polyhedron set, and proposed two main characterizations of the allowable delay sets, that is a usual set $[h_t, \dot{h}_t] \in \mathcal{H}_1 = [0, h] \times [-\mu, \mu]$ and another new allowable delay set $[h_t, \dot{h}_t] \in \mathcal{H}_2 = \{(0, 0), (0, \mu), (h, 0), (h, -\mu)\}$, as depicted in FIGURE 1. From the figure, we can find that once the values of h, μ are given, $\mathcal{H}_2$ is included in $\mathcal{H}_1$. In the next section of this paper, the two allowable delay sets $\mathcal{H}_1$ and $\mathcal{H}_2$ will be used to show the effectiveness of Theorem 1.

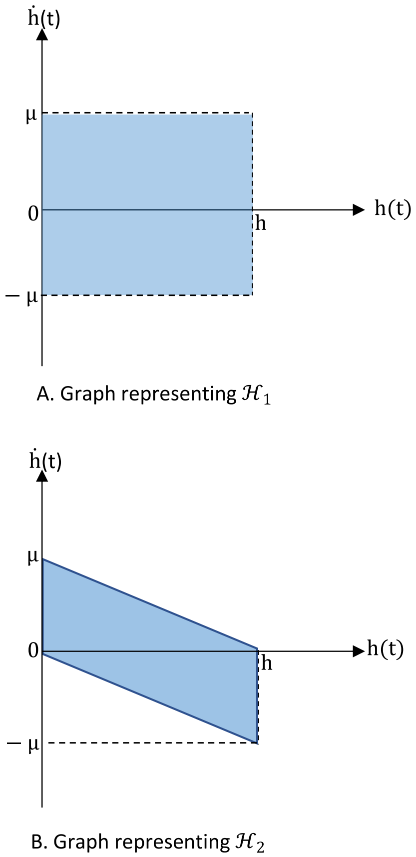

FIGURE 1. Graphical interpretation of $\mathcal{H}_1$ and $\mathcal{H}_2$.

Remark 9: Besides, the original forms of inequalities (10)-(11), (20)-(21) and (22)-(23) are not LMIs due to their dependence on the two time-varying delay parameters $h(t)$ and $\dot{h}(t)$. Indeed, the matrix inequalities in the conditions can be rewritten as the following form:

$$\varXi_1 + \dot{h}_t[\varXi_2 + h_t\varXi_3] < 0, \tag{24}$$

where $\varXi_i$, $i = 1, 2, 3$ are time-independent matrix functions. In the light of the convex combination technique proposed in [43], the original forms of inequalities (10)-(11), (20)-(21) and (22)-(23) hold if the following LMIs hold for the above two allowable delay sets $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively,

$$\mathcal{H}_1 : \varXi_1 + \dot{h}_t[\varXi_2 + h_t\varXi_3]_{\{[h_t,\dot{h}_t]=[0,h]\times[-\mu,\mu]\}} < 0, \tag{25}$$

$$\mathcal{H}_2 : \varXi_1 + \dot{h}_t[\varXi_2 + h_t\varXi_3]_{\{(h_t,\dot{h}_t)=\{(0,0),(0,\mu),(h,0),(h,-\mu)\}\}} < 0, \tag{26}$$

which implies that the solutions of inequalities (10)-(11), (20)-(21) and (22)-(23) become the feasibility-checking of the LMIs.

IV. NUMERICAL EXAMPLES

In this section, we give three examples to show the effectiveness of the criteria proposed in this paper for the two allowable delay sets $\mathcal{H}_1$ and $\mathcal{H}_2$. Moreover, by comparing maximal admissible delay upper bounds (MADUBs), the conservatism of the criteria is checked. And the index of the number of decision variables (NoVs) is applied to show the complexity of criteria. '–' in tables denotes the data are not given in the corresponding papers.

Remark 10: At present, Lyapunov stability theory is the main method to analyze the stability of systems with time-varying delays. With the development of the theory, the construction of Lyapunov functional mainly focuses on augmenting vectors and introducing various time delay-dependent terms. As a result, the dimension of Lyapunov matrices and the variable of LMI are increasing. In addition, in order to further improve the previous stability results by reducing the upper bounds of the time derivative of Lyapunov functionals as much as possible, various inequality techniques with more and more free-weight matrices are used to increase the freedom for checking the feasibility of stable conditions based on LMI. And the main purpose of this paper is further to improve the stability criterion based on conventional ideas, and obtain some larger maximum upper bound of the time delays with an unavoidable cost of increasing matrix variables and computational complexity. Indeed, the NoVs involved in our stability criteria can be calculated as $(66.5 + 21N + 5N^2)n^2 + (13.5 + 2N)n$ for theorem 1, $(39.5 + 21N + 5N^2)n^2 + (10.5 + 2N)n$ for corollary 1 and $(51.5+21N+5N^2)n^2+(11.5+2N)n$ for corollary 2. Obviously, theorem 1 requires the most decision variables than the corollaries. However, it is shown from the numerical examples that the upper bounds of the delays obtained by theorem 1 are the largest than the corollaries.

How we can remove some redundant matrix variables and try to reduce the decision variables such that it becomes simple computation, which will be an important future topic.

A. CONSERVATISM COMPARISON

The system parameters of the examples are as follows.

Example 1:

$$A = \begin{bmatrix} 1.5 & 0 \\ 0 & 0.7 \end{bmatrix}, \quad W_0 = \begin{bmatrix} 0.0503 & 0.0454 \\ 0.0987 & 0.2075 \end{bmatrix},$$

$$W_1 = \begin{bmatrix} 0.2381 & 0.9320 \\ 0.0388 & 0.5062 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$K_1 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.8 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad J = \begin{bmatrix} 0.4 \\ 0.2 \end{bmatrix}.$$

Example 2:

$$A = \begin{bmatrix} 7.3458 & 0 & 0 \\ 0 & 6.9987 & 0 \\ 0 & 0 & 5.5949 \end{bmatrix}, \quad W_0 = 0_3,$$

TABLE 1. MADUBs h for different μ and delay sets (Example 1).

Delay sets	Methods	Constraint of $\dot{h}(t)$	μ 0.4	0.45	0.5	0.55	NoVs
$\mathcal{H}_1$	[21, Th. 1]	$\dot{h}(t) \le \mu$	7.6697	6.7287	6.4126	6.2569	$15n^2 + 16n$
	[30, Th. 1]	$\dot{h}(t) \le \mu$	8.3498	7.3817	7.0219	6.8156	$73n^2 + 13n$
	[13, Th. 1]	$\dot{h}(t) \le \mu$	8.5669	7.6260	7.2809	7.0683	$90n^2 + 14n$
	[18, Th. 9]	$\dot{h}(t) \le \mu$	9.6800	8.5192	8.0535	7.7707	$142.5n^2 + 16.5n$
	[10, Th. 3]	$\lvert\dot{h}(t)\rvert \le \mu$	9.7094	7.7523	6.8570	6.2977	$42n^2 + 27n$
	[5, Th. 3]	$\dot{h}(t) \le \mu$	10.1095	8.6732	8.1733	7.8993	$46n^2 + 42n$
	[15, Th. 1]	$\dot{h}(t) \le \mu$	10.2367	9.0586	8.5986	8.3181	$79.5n^2 + 15.5n$
	[3, Th. 1]	$\dot{h}(t) \le \mu$	10.4371	9.1910	8.6957	8.3806	$128n^2 + 20n$
	[34, Th. 2]	$\lvert\dot{h}(t)\rvert \le \mu$	10.5730	9.3566	8.8467	8.5176	$244.5n^2 + 9.5n$
	[4, Th. 3]	$\lvert\dot{h}(t)\rvert \le \mu$	16.8020	11.6745	9.9098	9.0062	$194.5n^2 + 10.5n$
	[30, Th. 3]	$\lvert\dot{h}(t)\rvert \le \mu$	13.8671	11.1174	10.0050	9.4157	$79.5n^2 + 15.5n$
	[35, N=2]	$\lvert\dot{h}(t)\rvert \le \mu$	23.8409	17.6941	14.8593	–	$131n^2 + 24n$
	[35, N=3]	$\lvert\dot{h}(t)\rvert \le \mu$	25.3564	18.7688	15.7256	–	$191n^2 + 26n$
	Co. 1 N=2	$\lvert\dot{h}(t)\rvert \le \mu$	20.8811	14.5321	10.4212	9.4618	$101.5n^2 + 14.5n$
	Co. 2 N=2	$\lvert\dot{h}(t)\rvert \le \mu$	25.4616	19.5819	18.8974	16.3418	$113.5n^2 + 15.5n$
	Th. 1 N=2	$\lvert\dot{h}(t)\rvert \le \mu$	26.0397	21.7601	19.1309	16.5619	$128.5n^2 + 17.5n$
	Th. 1 N=3	$\lvert\dot{h}(t)\rvert \le \mu$	28.9952	23.5578	21.1130	17.0019	$174.5n^2 + 19.5n$
$\mathcal{H}_2$	[35, N=2]	$\lvert\dot{h}(t)\rvert \le \mu$	77.4833	46.6448	46.6448	–	$131n^2 + 24n$
	[35, N=3]	$\lvert\dot{h}(t)\rvert \le \mu$	120.120	67.9672	49.8307	–	$191n^2 + 26n$
	Co. 1 N=2	$\lvert\dot{h}(t)\rvert \le \mu$	45.5518	43.1109	43.1108	43.1108	$101.5n^2 + 14.5n$
	Co. 2 N=2	$\lvert\dot{h}(t)\rvert \le \mu$	50.9984	45.6574	45.6573	45.6570	$113.5n^2 + 15.5n$
	Th. 1 N=2	$\lvert\dot{h}(t)\rvert \le \mu$	53.5441	47.5632	47.5631	47.5630	$128.5n^2 + 17.5n$
	Th. 1 N=3	$\lvert\dot{h}(t)\rvert \le \mu$	70.6579	68.9996	68.8888	68.8885	$174.5n^2 + 19.5n$

$$W_1 = I_3, \quad W_2 = \begin{bmatrix} 13.6014 & -2.9616 & -0.6936 \\ 7.4736 & 21.6810 & 3.2100 \\ 0.7290 & -2.6344 & -20.1300 \end{bmatrix},$$

$$K_1 = \begin{bmatrix} 0.3680 & 0 & 0 \\ 0 & 0.1795 & 0 \\ 0 & 0 & 0.2876 \end{bmatrix}, \quad K_2 = 0_3,$$

$$J = [0.4 \quad 0.2 \quad 0.3]^T .$$

Example 3:

$$A = \begin{bmatrix} 1.2769 & 0 & 0 & 0 \\ 0 & 0.6231 & 0 & 0 \\ 0 & 0 & 0.9230 & 0 \\ 0 & 0 & 0 & 0.4480 \end{bmatrix},$$

$$W_0 = \begin{bmatrix} -0.0373 & 0.4852 & -0.3351 & 0.2336 \\ -1.6033 & 0.5988 & -0.3224 & 1.2352 \\ 0.3394 & -0.0860 & -0.3824 & -0.5785 \\ -0.1311 & 0.3253 & -0.9534 & -0.5015 \end{bmatrix},$$

$$W_1 = \begin{bmatrix} 0.8674 & -1.2405 & -0.5325 & 0.0220 \\ 0.0474 & -0.9164 & 0.0360 & 0.9816 \\ 1.8495 & 2.6117 & -0.3788 & 0.8428 \\ -2.0413 & 0.5179 & 1.1734 & -0.2775 \end{bmatrix},$$

$$K_1 = \begin{bmatrix} 0.1137 & 0 & 0 & 0 \\ 0 & 0.1279 & 0 & 0 \\ 0 & 0 & 0.7994 & 0 \\ 0 & 0 & 0 & 0.2368 \end{bmatrix}, \quad K_2 = 0_4,$$

$$W_2 = I_4, \ J = [0.1 \ 0.2 \ 0.3 \ 0.5]^T .$$

In Tables 1-3, the MADUBs obtained by Theorem 1 and Corollaries 1-2 are listed and compared with some recent results for different constraints of $\dot{h}(t)$. The following is a summary of the results.

1) Obviously, the MADUBs calculated by Theorem 1 are lager for all of the different constraints of $\dot{h}(t)$ than some existing literature, which shows that the augmented LKF (9) proposed in this paper can reduce the conservatism of some recent stability results.
2) Theorem 1 and Corollary 1 are derived by choosing a related LKF with or without the $\int_a^b \dot{e}(u)du$-type integral items in $\eta_1(t, s)$ and $\eta_2(t, s)$. It shows that the MADUBs based on Theorem 1 are larger than those based on Corollary 1, which shows that the augmented LKF with $\int_a^b \dot{e}(u)du$-type integral items in $\eta_1(t, s)$ and $\eta_2(t, s)$ is important to improving the stability results.
3) Theorem 1 and Corollary 2 are derived by choosing a related LKF with and without $v_1(t)$ and $v_2(t)$ in $V_1(t)$ based on the same inequality technique. However, the MADUBs calculated by Theorem 1 are larger than the ones calculated by Corollary 2, so the LKF with additional integral information on NAF is more effective than the one without the relevant information on NAF in $V_1(t)$, which matches the explanation in Remark 6.
4) The MADUBs calculated by Corollary 2 are larger than the ones calculated by Corollary 1, which shows that the separately augmented $V_2(t)$ with the $\int_a^b \dot{e}(u)du$-type integral items is more effective than the separately augmented $V_1(t)$ with $v_1(t)$ and $v_2(t)$ in reducing the conservatism of stability criteria.
5) The MADUBs calculated by Theorem 1 are larger and larger with the increasing of N, which shows that the

TABLE 2. MADUBs *h* for different μ and delay sets (Example 2).

Delay sets	Methods	Constraint of $\dot{h}(t)$	μ 0.0	0.1	0.5	0.9	NoVs
$\mathcal{H}_1$	[5, Th. 3]	$\dot{h}(t) \le \mu$	1.8899	1.1115	0.4807	–	$46n^2+42n$
	[15, Th. 1]	$\dot{h}(t) \le \mu$	1.9261	1.1205	0.4614	0.3963	$185.5n^2+21.5n$
	[30, Th. 3]	$\lvert\dot{h}(t)\rvert \le \mu$	1.8899	1.1135	0.4922	0.4701	$79.5n^2+15.5n$
	[4, Th. 1]	$\lvert\dot{h}(t)\rvert \le \mu$	1.8899	1.1193	0.4590	0.3945	$194.5n^2+10.5n$
	[26, Pro. 1]	$\lvert\dot{h}(t)\rvert \le \mu$	1.9349	1.1454	0.5806	–	$115n^2+22n$
	[35, N=2]	$\lvert\dot{h}(t)\rvert \le \mu$	1.9349	1.1511	0.5836	–	$131n^2+24n$
	[35, N=3]	$\lvert\dot{h}(t)\rvert \le \mu$	1.9365	1.1554	0.5967	–	$191n^2+26n$
	[33, Th. 2]	$\lvert\dot{h}(t)\rvert \le \mu$	–	1.3248	0.6323	0.5237	$172.5n^2+15.5n$
	[6, Th. 1]	$\lvert\dot{h}(t)\rvert \le \mu$	–	2.0497	0.9860	0.5831	$172.5n^2+15.5n$
	Co. 1 N=2	$\lvert\dot{h}(t)\rvert \le \mu$	1.8998	1.1516	0.5688	0.4776	$101.5n^2+14.5n$
	Co. 2 N=2	$\lvert\dot{h}(t)\rvert \le \mu$	2.0006	1.4746	0.6007	0.4901	$113.5n^2+15.5n$
	Th. 1 N=2	$\lvert\dot{h}(t)\rvert \le \mu$	3.3518	1.9518	0.7259	0.5579	$128.5n^2+17.5n$
	Th. 1 N=3	$\lvert\dot{h}(t)\rvert \le \mu$	3.9109	2.2676	0.9914	0.5927	$174.5n^2+19.5n$
$\mathcal{H}_2$	[35, N=2]	$\lvert\dot{h}(t)\rvert \le \mu$	1.9349	1.9202	1.3348	–	$131n^2+24n$
	[35, N=3]	$\lvert\dot{h}(t)\rvert \le \mu$	1.9365	1.9341	1.5005	–	$191n^2+26n$
	Co. 1 N=2	$\lvert\dot{h}(t)\rvert \le \mu$	1.9202	1.9123	1.3503	1.3046	$101.5n^2+14.5n$
	Co. 2 N=2	$\lvert\dot{h}(t)\rvert \le \mu$	2.3559	2.1129	1.7726	1.7702	$113.5n^2+15.5n$
	Th. 1 N=2	$\lvert\dot{h}(t)\rvert \le \mu$	3.7893	3.1059	2.8793	2.3125	$128.5n^2+17.5n$
	Th. 1 N=3	$\lvert\dot{h}(t)\rvert \le \mu$	3.9947	3.4257	3.0001	3.0000	$174.5n^2+19.5n$

TABLE 3. MADUBs *h* for different μ and delay sets (Example 3).

Delay sets	Methods	Constraint of $\dot{h}(t)$	μ 0.1	0.5	0.9	NoVs
$\mathcal{H}_1$	[22, Th. 5]	$\lvert\dot{h}(t)\rvert \le \mu$	4.370	3.187	2.907	$34.5n^2+20.5n$
	[13, Th. 1]	$\dot{h}(t) \le \mu$	4.3231	3.2592	2.9846	$90n^2+14n$
	[5, Th. 1]	$\dot{h}(t) \le \mu$	4.3919	3.4273	3.2516	$51n^2+42n$
	[3, Th. 2]	$\dot{h}(t) \le \mu$	4.4530	3.4929	3.0726	$128n^2+11n$
	[15, Th. 1]	$\dot{h}(t) \le \mu$	4.4873	3.5330	3.0894	$184.5n^2+11.5n$
	[12, Th. 1]	$\dot{h}(t) \le \mu$	4.9989	3.8038	3.1855	$38n^2+11n$
	[44, Pro. 1]	$\lvert\dot{h}(t)\rvert \le \mu$	4.5382	3.9313	3.4763	$60n^2+22n$
	[35, N=3]	$\lvert\dot{h}(t)\rvert \le \mu$	4.5468	4.0253	3.6246	$191n^2+26n$
	[6, Th. 1]	$\lvert\dot{h}(t)\rvert \le \mu$	5.3384	5.0201	4.9745	$25.5n^2+8.5n$
	Co. 1 N=2	$\lvert\dot{h}(t)\rvert \le \mu$	4.4189	3.8755	3.3476	$101.5n^2+14.5n$
	Co. 2 N=2	$\lvert\dot{h}(t)\rvert \le \mu$	4.6489	4.3477	3.8778	$113.5n^2+15.5n$
	Th. 1 N=2	$\lvert\dot{h}(t)\rvert \le \mu$	5.0485	4.9348	4.5612	$128.5n^2+17.5n$
	Th. 1 N=3	$\lvert\dot{h}(t)\rvert \le \mu$	5.3441	5.0943	4.9951	$174.5n^2+19.5n$
$\mathcal{H}_2$	[35, N=2]	$\lvert\dot{h}(t)\rvert \le \mu$	4.8116	4.8116	4.8116	$131n^2+24n$
	[35, N=3]	$\lvert\dot{h}(t)\rvert \le \mu$	4.8123	4.8123	4.8123	$191n^2+26n$
	Co. 1 N=2	$\lvert\dot{h}(t)\rvert \le \mu$	4.7789	4.7664	4.7664	$101.5n^2+14.5n$
	Co. 2 N=2	$\lvert\dot{h}(t)\rvert \le \mu$	4.8002	4.8001	4.8000	$113.5n^2+15.5n$
	Th. 1 N=2	$\lvert\dot{h}(t)\rvert \le \mu$	5.3568	5.2259	5.2258	$128.5n^2+17.5n$
	Th. 1 N=3	$\lvert\dot{h}(t)\rvert \le \mu$	5.5577	5.5569	5.5569	$174.5n^2+19.5n$

N-dependent stability criteria proposed in this paper are also hierarchy of LMI conditions as described in [35]. This matches the explanation in Remarks 1 and 4.

6) The appropriate selection of the delay set, such as $\mathcal{H}_2$, makes a big difference on increasing the MADUBs, which matches the description in [42] and Remark 7.
7) It can be known from the comparative analysis of NoVs in the tables that the conservatism of our criteria is reduced at the cost of increasing decision variables compared with some relevant references. However, when $N = 2$, the NoVs of our stability criteria are less than those of [4], [6], [15], [18], [33]–[35].

To confirm the obtained result from Tables 1-3, the simulation result is shown in the following section. As you can see from FIGUREs 2-4 that the state responses of the DNN (4) converge to zero, which verifies the DNN (1) is stable at the equilibrium points.

B. SIMULATION VERIFICATION

DNNs should be stable for the following conditions according to Tables 1-3.

Example 1: $g(e) = \begin{bmatrix} 0.3tanh(e_1) \\ 0.8tanh(e_2) \end{bmatrix}$, $\quad e(t) = [2 \quad -3]^T$,

$$t \in [-70.6579, \quad 0],$$

$$h(t) = \frac{70.6579}{2} + \frac{70.6579}{2} sin\left(\frac{0.8t}{70.6579}\right);$$

Example 2: $g(e) = \begin{bmatrix} 0.3680tanh(e_1) \\ 0.1795tanh(e_2) \\ 0.2876tanh(e_3) \end{bmatrix}$,

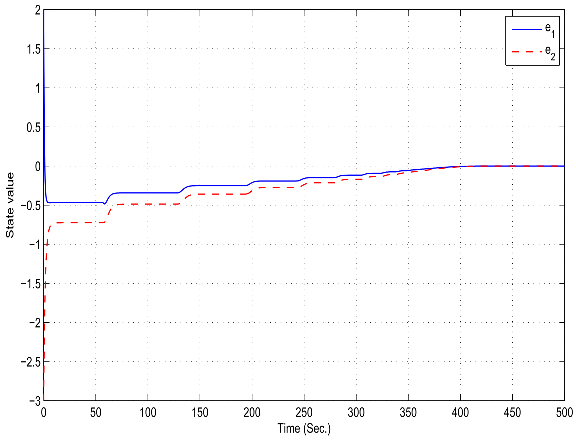

FIGURE 2. The state responses for Example 1.

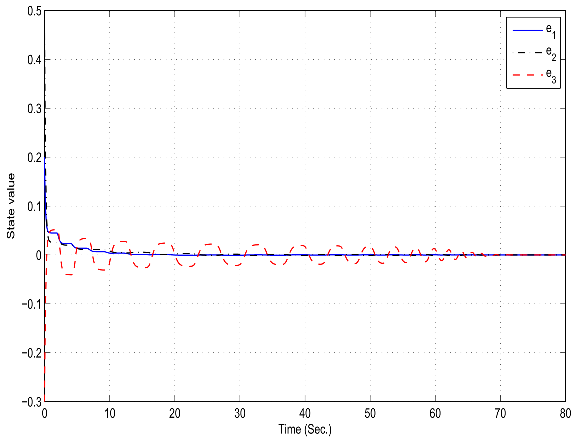

FIGURE 3. The error state responses for Example 2.

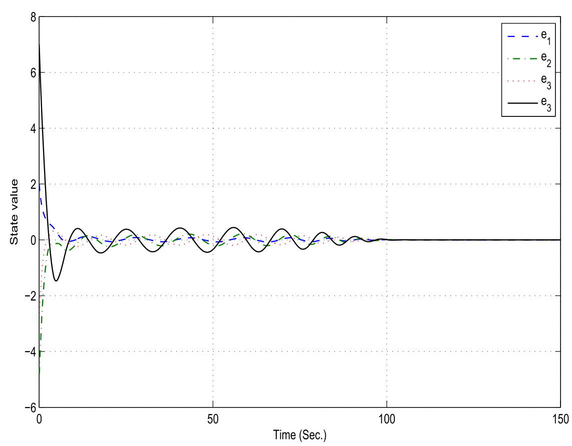

FIGURE 4. The state responses for Example 3.

$$e(t) = [0.2\ 0.5\ \ -0.3]^T, \quad t \in [-3.4257,\ 0],$$
$$h(t) = \frac{3.4257}{2} + \frac{3.4257}{2} sin\left(\frac{0.2t}{3.4257}\right).$$

$$\text{Example 3: } g(e) = \begin{bmatrix} 0.1137tanh(e_1) \\ 0.1279tanh(e_2) \\ 0.7994tanh(e_3) \\ 0.2368tanh(e_4) \end{bmatrix},$$

$$e(t) = [2\ -5\ -3\ 7]^T, \quad t \in [-5.5577,\ 0],$$
$$h(t) = \frac{5.5577}{2} + \frac{5.5577}{2} sin\left(\frac{0.2t}{5.5577}\right).$$

V. CONCLUSION

In this paper, based on the previous researches, the relevant LKFs are augmented, and a less conservative stability criterion for a class of DNN than the ones of some existing literature is obtained according to the modified LKF and two effective integral inequality techniques. Finally, the effectiveness of the proposed method is illustrated by comparison and discussion in numerical examples.

Only the DNN is considered in this paper via a novel LKF application. Certainly, the novel LKF with delay-dependent terms and augmented variables proposed in this paper can be also applied to stability analysis of other time-delayed systems, for example, delayed Lur'e systems [45], delayed neutral-type systems [46], networked control systems [47], multi-agent systems [48], and so on, which may be also the future topics.

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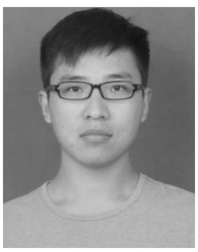

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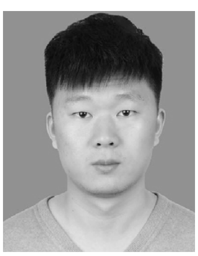

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