IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received June 28, 2019, accepted July 15, 2019, date of publication July 18, 2019, date of current version August 5, 2019.

Digital Object Identifier 10.1109/ACCESS.2019.2929682

Two Different Systematic Techniques to Seek Analytical Solutions of the Higher-Order Modified Boussinesq Equation

YONGYI GU AND YINYING KONG
School of Statistics and Mathematics, Guangdong University of Finance and Economics, Guangzhou 510320, China
Corresponding author: Yongyi Gu (gdguyongyi@163.com)

This work was supported in part by the NSF of Guangdong Province under Grant 2018A030313954, in part by the Guangdong Universities (Basic Research and Applied Research) Major Project under Grant 2017KZDXM038, and in part by the Key Research Project of the Audit Office of Guangdong Province "Application of Big Data Technology in Auditing" and Antitrust Enforcement and Big Data Analysis Research Center Project of GDUFE under Grant 2019D04.

ABSTRACT In this paper, we seek analytical solutions of the higher-order modified Boussinesq equation by two different systematic techniques. Employing the $\exp(-\psi(z))$-expansion method, exact solutions of the mentioned equation, including hyperbolic, exponential, trigonometric, and rational function solutions, have been obtained. Based on the work of Yuan et al., we proposed the extended complex method to seek exact solutions of the higher-order modified Boussinesq equation. It shows that the extended complex method can solve more differential equations in mathematical physics than the complex method. The idea of this paper can be used to the complex nonlinear systems of electrical and electronics engineering.

INDEX TERMS Higher-order modified Boussinesq equation, exact solutions, $\exp(-\psi(z))$-expansion method, extended complex method.

I. INTRODUCTION

Nonlinear science is basic science to study the generality of nonlinear phenomena. It is a comprehensive discipline which has been gradually developed by various branch disciplines characterized by nonlinearity since the 1960s. It was known as the "Third Revolution" of Natural Science in the 20th century. The scientific community believes that the research of nonlinear science has not only great scientific significance but also broad application prospects. It involves almost all fields of natural science and social science, including engineering application, basic physical research, biological research, control theory, management, etc. The nonlinear science is changing people's traditional view of the real world.

The higher-order modified Boussinesq equation [1] is given by

$$u_{tt} + \alpha u_{txx} + \theta u_{xxxx} + \gamma(6uu_x^2 + 3u^2 u_{xx}) = 0, \tag{1}$$

which is a famous nonlinear differential equation (NLDE) and is used as a model to describe the water wave problem with surface tension.

The associate editor coordinating the review of this manuscript and approving it for publication was Bora Onat.

It is well known that NLDEs are universally applied in plasma physics, electrical engineering, nonlinear optics, fluid dynamics, biology, chemistry, etc. For example, the singular behaviors [2],[3] and impulsive phenomena [4],[5] often show some blow-up properties [6],[7] which occur in lots of complex physical processes. In order to solve various differential equations, symbolic calculation techniques as well as some analytical tools were established, such as sine-Gordon expansion method [8]-[10], modified simple equation method [11], modified extended tanh method [12]-[15], Kudryashov method [16]-[19], generalized (G'/G)-expansion method [20]-[23], improved F-expansion method [24], $\exp(-\psi(z))$-expansion method [25]-[29], complex method [30]-[35], fixed point method [36]-[39], and topological degree method [40]-[43].

The complex method, proposed by Yuan *et al.* [30],[31], is established via complex differential equations and complex analysis. It is a useful tool to find exact solutions of NLDEs which are Briot-Bouquet equations or satisfy $\langle p, q\rangle$ condition [32]. Based on their work, we introduce the extended complex method to seek exact solutions of NLDEs which are not Briot-Bouquet equations or do not satisfy $\langle p, q\rangle$ condition.

In this article, two different systematic methods which are the $\exp(-\psi(z))$-expansion method and extended complex method are applied to search analytical solutions of the higher-order modified Boussinesq equation. Computer simulations are given to illustrate our main results. Comparisons and conclusions are presented in the last two sections.

II. THE EXP($-\psi(Z)$)-EXPANSION METHOD

Consider a nonlinear PDE as follows:

$$F(u, u_x, u_t, u_{xx}, u_{tt}, \cdots) = 0, \tag{2}$$

where F is a polynomial consisting of the unknown function $u(x,t)$, the partial derivatives of $u(x,t)$, the higher order partial derivatives of $u(x,t)$ and some nonlinear terms.

Step 1. Substitute traveling wave transformation

$$u(x,t) = u(z), \quad z = x + \lambda t, \tag{3}$$

into Eq.(2) to convert it to the ODE,

$$P(u, u', u'', u''', \cdots) = 0, \tag{4}$$

where P is a polynomial of u and its derivatives.

Step 2. Suppose that Eq.(4) has the following exact solutions:

$$u(z) = \sum_{j=0}^{n} B_j(\exp(-\psi(z)))^j, \tag{5}$$

where B_j $(0 \leq j \leq n)$ are constants to be determined latter, such that $B_n \neq 0$ $(n \geq 1)$ and $\psi = \psi(z)$ satisfies the ODE as below:

$$\psi'(z) = \delta + \exp(-\psi(z)) + \nu \exp(\psi(z)). \tag{6}$$

Eq.(6) has the solutions as follows:

When $\delta^2 - 4\nu > 0$, $\nu \neq 0$,

$$\psi(z) = \ln\left(\frac{-\sqrt{(\delta^2-4\nu)}\tanh(\frac{\sqrt{\delta^2-4\nu}}{2}(z+a)) - \delta}{2\nu}\right), \tag{7}$$

$$\psi(z) = \ln\left(\frac{-\sqrt{(\delta^2-4\nu)}\coth(\frac{\sqrt{\delta^2-4\nu}}{2}(z+a)) - \delta}{2\nu}\right). \tag{8}$$

When $\delta^2 - 4\nu < 0$, $\nu \neq 0$,

$$\psi(z) = \ln\left(\frac{\sqrt{(4\nu-\delta^2)}\tan(\frac{\sqrt{(4\nu-\delta^2)}}{2}(z+a)) - \delta}{2\nu}\right), \tag{9}$$

$$\psi(z) = \ln\left(\frac{\sqrt{(4\nu-\delta^2)}\cot(\frac{\sqrt{(4\nu-\delta^2)}}{2}(z+a)) - \delta}{2\nu}\right). \tag{10}$$

When $\delta^2 - 4\nu > 0$, $\delta \neq 0$, $\nu = 0$,

$$\psi(z) = -\ln\left(\frac{\delta}{\exp(\delta(z+a)) - 1}\right). \tag{11}$$

When $\delta^2 - 4\nu = 0$, $\delta \neq 0$, $\nu \neq 0$,

$$\psi(z) = \ln\left(-\frac{2(\delta(z+a)+2)}{\delta^2(z+a)}\right). \tag{12}$$

When $\delta^2 - 4\nu = 0$, $\delta = 0$, $\nu = 0$,

$$\psi(z) = \ln(z+a), \tag{13}$$

where a is an arbitrary constant and $B_n \neq 0, \delta, \nu$ are constants in Eq.(7)-Eq.(13). We determine the positive integer n through considering the homogeneous balance between highest order derivatives and nonlinear terms of Eq.(4).

Step 3. Inserting Eq.(5) into Eq.(4) and then considering the function $\exp(-\psi(z))$ yields a polynomial of $\exp(-\psi(z))$. Let the coefficients of same power about $\exp(-\psi(z))$ equal to zero, then we get a set of algebraic equations. We solve the algebraic equations to obtain the values of $B_n \neq 0, \delta, \nu$ and then we put these values into Eq.(5) along with Eq.(7)-Eq.(13) to finish the determination of the solutions for the given PDE.

III. APPLICATION OF THE EXP($-\psi(Z)$)-EXPANSION METHOD TO THE HIGHER-ORDER MODIFIED BOUSSINESQ EQUATION

Substitute

$$u(x,t) = u(z), \quad z = x + \lambda t,$$

into Eq.(1), we get

$$\lambda^2 u'' - \alpha\lambda u''' + \theta u'''' + \gamma[6u(u')^2 + 3u^2u'] = 0. \tag{14}$$

Take the homogeneous balance between u'''' and $u(u')^2$ in Eq.(14) to yield

$$u(z) = B_0 + B_1\exp(-\psi(z)), \tag{15}$$

where $B_1 \neq 0$ and B_0 are constants.

Substituting $u'''', u''', u'', u(u')^2, u^2u''$ into Eq.(14) and equating the coefficients about $\exp(-\psi(z))$ to zero, we get

$e^{0(-\psi(z))}$:

$$3\gamma B_0^2 B_1 \nu\delta + 6\gamma B_1^2 B_0 \nu^2 + \alpha\lambda B_1 \delta^2\nu + \theta B_1 \delta^3\nu + 2\alpha\lambda B_1 \nu^2 + \lambda^2 B_1 \nu\delta + 8\theta B_1 \delta\nu^2 = 0,$$

$e^{1(-\psi(z))}$:

$$3B_0^2 B_1 \delta^2\gamma + 18 B_0 B_1^2 \delta\gamma\nu + 6B_1^3\gamma\nu^2 + B_1\alpha\delta^3\lambda + B_1\delta^4\theta + B_1\delta^2\lambda^2 + 6B_0^2B_1\gamma\nu + 8B_1\alpha\delta\lambda\nu + 22B_1\delta^2\nu\theta + 2B_1\lambda^2\nu + 16B_1\nu^2\theta = 0,$$

$e^{2(-\psi(z))}$:

$$12B_0B_1^2\delta^2\gamma + 15B_1^3\delta\gamma\nu + 9B_0^2B_1\delta\gamma + 3B_1\delta\lambda^2 + 7B_1\alpha\delta^2\lambda + 15B_1\delta^3\theta + 8B_1\alpha\lambda\nu + 24B_0B_1^2\gamma\nu + 60B_1\delta\nu\theta = 0,$$

$e^{3(-\psi(z))}$:

$$9B_1^3\delta^2\gamma + 30B_0B_1^2\delta\gamma + 18B_1^3\gamma\nu + 6B_0^2B_1\gamma + 12B_1\alpha\delta\lambda + 50B_1\delta^2\theta + 2B_1\lambda^2 + 40B_1\nu\theta = 0,$$

$e^{4(-\psi(z))}$:

$$21B_1^3\delta\gamma + 18B_0B_1^2\gamma + 6B_1\alpha\lambda + 60B_1\delta\theta = 0,$$

$e^{5(-\psi(z))}$:

$$12B_1^3\gamma + 24B_1\theta = 0.$$

Solving the above algebraic equations yields

$$\nu = \frac{\alpha^2\lambda^2 + 3\,\delta^2\theta^2 - 6\,\lambda^2\theta}{12\theta^2},$$
$$B_1 = \frac{\sqrt{-2\gamma\theta}}{\gamma},$$
$$B_0 = \frac{\sqrt{-2\,\gamma\,\theta}\,(\alpha\,\lambda + 3\,\delta\,\theta)}{6\gamma\,\theta}, \tag{16}$$

where δ and ν are arbitrary constants.

We substitute Eqs.(16) into Eq.(15), then

$$u(z) = \frac{\sqrt{-2\,\gamma\,\theta}\,(\alpha\,\lambda + 3\,\delta\,\theta)}{6\gamma\,\theta} + \frac{\sqrt{-2\gamma\theta}}{\gamma}\exp(-\psi(z)). \tag{17}$$

Using Eq.(7) to Eq.(13) into Eq.(17) respectively, we obtain exact solutions of the higher-order modified Boussinesq equation as follows.

When $\delta^2 - 4\nu > 0$, $\nu \neq 0$,

$$u_1(z) = \frac{\sqrt{-2\,\gamma\,\theta}\,(\alpha\,\lambda + 3\,\delta\,\theta)}{6\gamma\,\theta} - \frac{\sqrt{-2\gamma\theta}}{\gamma} \cdot \frac{2\nu}{\sqrt{(\delta^2 - 4\nu)}\tanh\left(\frac{\sqrt{\delta^2-4\nu}}{2}(z+a)\right) + \delta},$$

$$u_2(z) = \frac{\sqrt{-2\,\gamma\,\theta}\,(\alpha\,\lambda + 3\,\delta\,\theta)}{6\gamma\,\theta} - \frac{\sqrt{-2\gamma\theta}}{\gamma} \cdot \frac{2\nu}{\sqrt{(\delta^2 - 4\nu)}\coth\left(\frac{\sqrt{\delta^2-4\nu}}{2}(z+a)\right) + \delta}.$$

When $\delta^2 - 4\nu < 0$, $\nu \neq 0$,

$$u_3(z) = \frac{\sqrt{-2\,\gamma\,\theta}\,(\alpha\,\lambda + 3\,\delta\,\theta)}{6\gamma\,\theta} + \frac{\sqrt{-2\gamma\theta}}{\gamma} \cdot \frac{2\nu}{\sqrt{(4\nu - \delta^2)}\tan\left(\frac{\sqrt{4\nu-\delta^2}}{2}(z+a)\right) - \delta},$$

$$u_4(z) = \frac{\sqrt{-2\,\gamma\,\theta}\,(\alpha\,\lambda + 3\,\delta\,\theta)}{6\gamma\,\theta} + \frac{\sqrt{-2\gamma\theta}}{\gamma} \cdot \frac{2\nu}{\sqrt{(4\nu - \delta^2)}\cot\left(\frac{\sqrt{4\nu-\delta^2}}{2}(z+a)\right) - \delta}.$$

When $\delta^2 - 4\nu > 0$, $\delta \neq 0$, $\nu = 0$,

$$u_5(z) = \frac{\sqrt{-2\,\gamma\,\theta}\,(\alpha\,\lambda + 3\,\delta\,\theta)}{6\gamma\,\theta} + \frac{\sqrt{-2\gamma\theta}}{\gamma}\,\frac{\delta}{\exp(\delta(z+a)) - 1}.$$

When $\delta^2 - 4\nu = 0$, $\delta \neq 0$, $\nu \neq 0$,

$$u_6(z) = \frac{\sqrt{-2\,\gamma\,\theta}\,(\alpha\,\lambda + 3\,\delta\,\theta)}{6\gamma\,\theta} - \frac{\sqrt{-2\gamma\theta}}{\gamma}\,\frac{\delta^2(z+a)}{2(\delta(z+a)+2)}.$$

When $\delta^2 - 4\nu = 0$, $\delta = 0$, $\nu = 0$,

$$u_7(z) = \frac{\sqrt{-2\,\gamma\,\theta}\alpha\,\lambda}{6\gamma\,\theta} + \frac{\sqrt{-2\gamma\theta}}{\gamma}\,\frac{1}{z+a}.$$

The properties of the solutions are shown in figures 1-6.

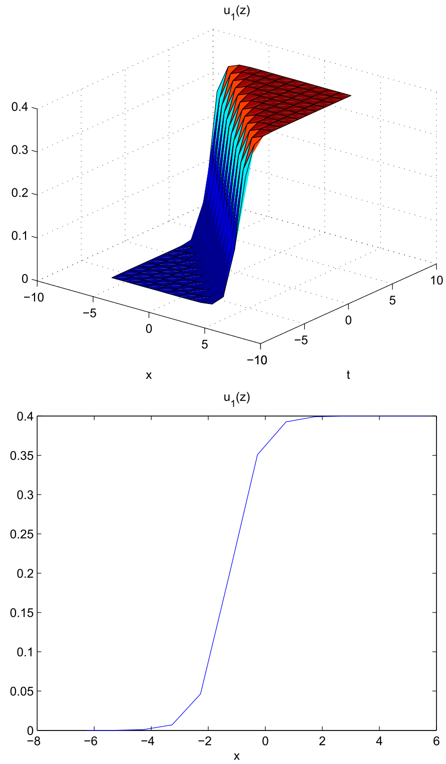


FIGURE 1. The 3D and 2D profiles of $u_1(z)$ by considering $\alpha = -1$, $\theta = \frac{1}{2}$, $\gamma = -1$, $\lambda = 3$, $\delta = 4$, $\nu = 3$, $a = 1$, and $t = 0$ for the 2D graphic.

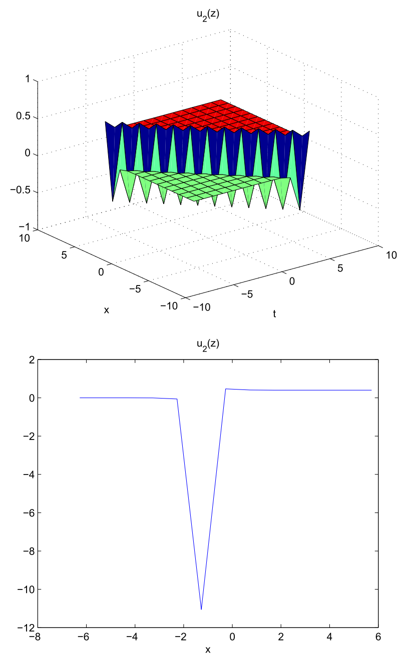


FIGURE 2. The 3D and 2D profiles of $u_2(z)$ by considering $\alpha = -1$, $\theta = \frac{1}{2}$, $\gamma = -1$, $\lambda = 3$, $\delta = 4$, $\nu = 3$, $a = 1$, and $t = 0$ for the 2D graphic.

IV. THE EXTENDED COMPLEX METHOD

Step 1. Substituting the transform $T : u(x,t) \to U(z)$, $(x,t) \to z$ into the given PDE yields

$$W(U, U', U'', U''', \cdots) = 0. \tag{18}$$

Step 2. Determine the weak $\langle p, q\rangle$ condition.

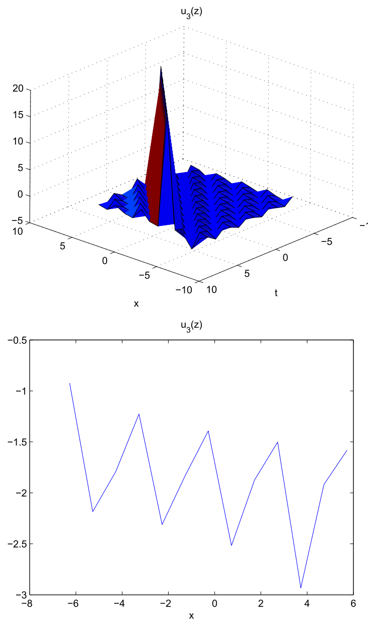


FIGURE 3. The 3D and 2D profiles of $u_3(z)$ by considering $\alpha = 1$, $\theta = \frac{1}{2}$, $\gamma = -1$, $\lambda = 3$, $\delta = 4$, $\nu = 5$, $a = 1$, and $t = 0$ for the 2D graphic.

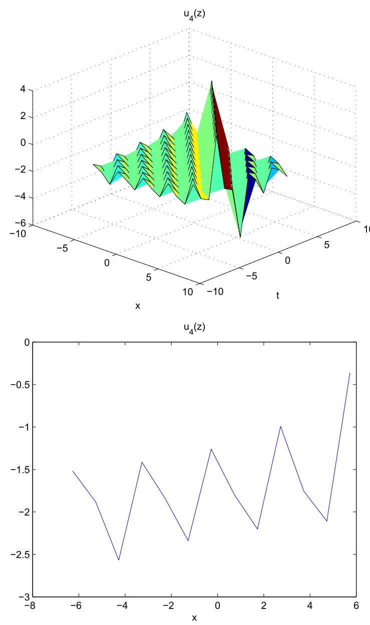


FIGURE 4. The 3D and 2D profiles of $u_4(z)$ by considering $\alpha = 1$, $\theta = \frac{1}{2}$, $\gamma = -1$, $\lambda = 3$, $\delta = 4$, $\nu = 5$, $a = 1$, and $t = 0$ for the 2D graphic.

Let $p,\ q \in \mathbb{N}$, and suppose that the meromorphic solutions U of Eq.(18) have at least one pole. Substituting the Laurent series

$$U(z) = \sum_{k=-q}^{\infty} A_k z^k, \quad q > 0,\ A_{-q} \neq 0, \tag{19}$$

into Eq.(18), if it is determined p distinct Laurent singular parts

$$\sum_{k=-q}^{-1} A_k z^k,$$

then the weak $\langle p, q\rangle$ condition of Eq.(18) holds.

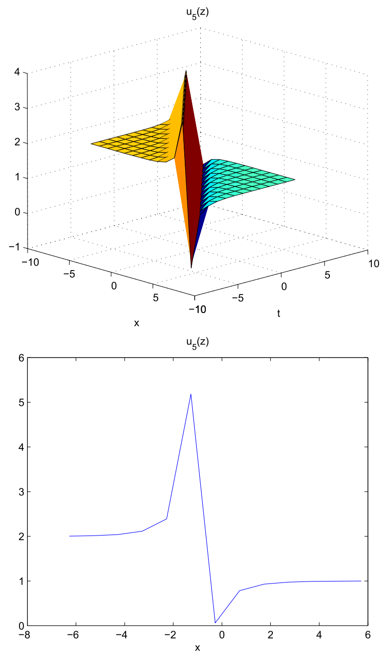


FIGURE 5. The 3D and 2D profiles of $u_5(z)$ by considering $\alpha = 9$, $\theta = \frac{1}{2}$, $\gamma = -1$, $\lambda = -\frac{1}{2}$, $\delta = 1$, $a = 1$, and $t = 0$ for the 2D graphic.

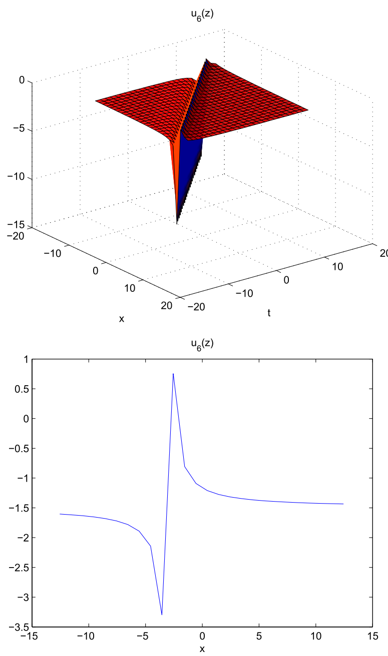


FIGURE 6. The 3D and 2D profiles of $u_6(z)$ by considering $\alpha = 1$, $\theta = \frac{3}{2}$, $\gamma = -3$, $\lambda = 1$, $\delta = 1$, $a = 1$, and $t = 0$ for the 2D graphic.

Weierstrass elliptic function $\wp(z) := \wp(z, g_2, g_3)$ with double periods satisfies the equation as follows:

$$(\wp'(z))^2 = 4\wp(z)^3 - g_2\wp(z) - g_3,$$

and has the following addition formula:

$$\wp(z - z_0) = -\wp(z) + \frac{1}{4}\left[\frac{\wp'(z) + \wp'(z_0)}{\wp(z) - \wp(z_0)}\right]^2 - \wp(z_0).$$

Step 3. Substitute the indeterminate forms

$$U(z) = \sum_{i=1}^{s-1}\sum_{j=2}^{q}\frac{(-1)^j\beta_{-ij}}{(j-1)!}\frac{d^{j-2}}{dz^{j-2}}(\frac{1}{4}[\frac{\wp'(z)+D_i}{\wp(z)-B_i}]^2 - \wp(z)) + \sum_{i=1}^{s-1}\frac{\beta_{-i1}}{2}\frac{\wp'(z)+D_i}{\wp(z)-B_i} + \sum_{j=2}^{q}\frac{(-1)^j\beta_{-sj}}{(j-1)!}\frac{d^{j-2}}{dz^{j-2}}\wp(z)+\beta_0, \quad (20)$$

$$U(z) = \sum_{i=1}^{s}\sum_{j=1}^{q}\frac{\beta_{ij}}{(z-z_i)^j}+\beta_0, \quad (21)$$

$$U(e^{\mu z}) = \sum_{i=1}^{s}\sum_{j=1}^{q}\frac{\beta_{ij}}{(e^{\mu z}-e^{\mu z_i})^j}+\beta_0, \quad (22)$$

into Eq.(18) respectively to yield the systems of algebraic equations, and solve the algebraic equations to obtain elliptic function solutions, rational function solutions and simply periodic solutions with the pole at $z = 0$, where β_{-ij} are determined by (19), $D_i^2 = 4B_i^3 - g_2B_i - g_3$ and $\sum_{i=1}^{s}\beta_{-i1} = 0$, and $U(z)$, $U(e^{\mu z})$ ($\mu \in \mathbb{C}$) have $s(\leq p)$ distinct poles of multiplicity q.

Step 4. Obtain the meromorphic solutions with arbitrary pole, and substitute the inverse transform T^{-1} into the meromorphic solutions to achieve the exact solutions to the mentioned PDE.

V. APPLICATION OF THE EXTENDED COMPLEX METHOD TO THE HIGHER-ORDER MODIFIED BOUSSINESQ EQUATION

Inserting (19) into Eq.(14) and equating the coefficients of the same powers of z to zero, we obtain $A_{-1} = \pm\frac{\sqrt{-2\gamma\theta}}{\gamma}$, $A_{-2} = A_{-3} = A_{-4} = \cdots = 0$, then we know that $p = 2$, $q = 1$. Therefore, the weak $\langle 2, 1\rangle$ condition of Eq.(14) holds.

By the weak $\langle 2, 1\rangle$ condition and (21), we have the indeterminate forms of rational solutions

$$U_r(z) = \frac{\beta_{11}}{z} + \frac{\beta_{21}}{z-z_1} + \beta_{10},$$

with pole at $z = 0$.

Inserting $U_r(z)$ into Eq.(14), we obtain

$$\sum_{i=1}^{8} c_{1i}z^{-i+3}(z-z_1)^{-5} = 0, \quad (23)$$

where

$$\begin{aligned}
c_{11} &= 3\gamma\beta_{10}{}^2\beta_{11} - 3\gamma\beta_{10}{}^2\beta_{21} + \lambda^2\beta_{11} - \lambda^2\beta_{21},\\
c_{12} &= 15\gamma\beta_{10}{}^2\beta_{11}z_1 + 6\gamma\beta_{10}{}^2\beta_{21}z_1 + 9\gamma\beta_{10}\beta_{11}{}^2\\
&\quad - 18\gamma\beta_{10}\beta_{11}\beta_{21} + 9\gamma\beta_{10}\beta_{21}{}^2 - 5\lambda^2\beta_{11}z_1\\
&\quad + 2\lambda^2\beta_{21}z_1 + 3\alpha\lambda\beta_{11} - 3\alpha\lambda\beta_{21},\\
c_{13} &= 30\gamma\beta_{10}{}^2\beta_{11}z_1{}^2 - 3\gamma\beta_{10}{}^2\beta_{21}z_1{}^2 - 45\gamma\beta_{10}\beta_{11}{}^2z_1\\
&\quad + 54\gamma\beta_{10}\beta_{11}\beta_{21}z_1 - 12\theta\beta_{21} - 9\gamma\beta_{10}\beta_{21}{}^2z_1\\
&\quad + 10\lambda^2\beta_{11}z_1{}^2\\
&\quad - \lambda^2\beta_{21}z_1{}^2 - 15\alpha\lambda\beta_{11}z_1 + 3\alpha\lambda\beta_{21}z_1 - 18\gamma\beta_{11}{}^2\beta_{21}\\
&\quad + 6\gamma\beta_{11}{}^3 + 18\gamma\beta_{11}\beta_{21}{}^2 - 6\gamma\beta_{21}{}^3 + 12\theta\beta_{11},\\
c_{14} &= -30\gamma\beta_{10}^2\beta_{11}z_1^3 + 90\gamma\beta_{10}\beta_{11}^2z_1^2 - 60\gamma\beta_{10}\beta_{11}\beta_{21}z_1^2\\
&\quad - 10\lambda^2\beta_{11}z_1^3 + 30\alpha\lambda\beta_{11}z_1^2 - 30\gamma\beta_{11}^3z_1\\
&\quad + 60\gamma\beta_{11}^2\beta_{21}z_1 - 30\gamma\beta_{11}\beta_{21}^2z_1 - 60\theta\beta_{11}z_1,\\
c_{15} &= 15\gamma\beta_{10}{}^2\beta_{11}z_1{}^4 - 90\gamma\beta_{10}\beta_{11}{}^2z_1{}^3 + 30\gamma\beta_{10}\beta_{11}\beta_{21}z_1{}^3\\
&\quad + 5\lambda^2\beta_{11}z_1{}^4 - 30\alpha\lambda\beta_{11}z_1{}^3 + 60\gamma\beta_{11}{}^3z_1{}^2\\
&\quad - 75\gamma\beta_{11}{}^2\beta_{21}z_1{}^2\\
&\quad + 15\gamma\beta_{11}\beta_{21}{}^2z_1{}^2 + 120\theta\beta_{11}z_1{}^2,\\
c_{16} &= -3\gamma\beta_{10}{}^2\beta_{11}z_1{}^5 + 45\gamma\beta_{10}\beta_{11}{}^2z_1{}^4 - 6\gamma\beta_{10}\beta_{11}\beta_{21}z_1{}^4\\
&\quad - \lambda^2\beta_{11}z_1{}^5 + 15\alpha\lambda\beta_{11}z_1{}^4 - 60\gamma\beta_{11}{}^3z_1{}^3\\
&\quad + 42\gamma\beta_{11}{}^2\beta_{21}z_1{}^3\\
&\quad - 3\gamma\beta_{11}\beta_{21}{}^2z_1{}^3 - 120\theta\beta_{11}z_1{}^3,\\
c_{17} &= -9\gamma\beta_{10}\beta_{11}{}^2z_1{}^5 - 3\alpha\lambda\beta_{11}z_1{}^5 + 30\gamma\beta_{11}{}^3z_1{}^4\\
&\quad - 9\gamma\beta_{11}{}^2\beta_{21}z_1{}^4 + 60\theta\beta_{11}z_1{}^4,\\
c_{18} &= -6\gamma\beta_{11}{}^3z_1{}^5 - 12\theta\beta_{11}z_1{}^5.
\end{aligned}$$

Equating the coefficients of the same powers about z in Eq.(24) to zero, we get a set of algebraic equations:

$$c_{1i} = 0, \ (i = 1, 2, \cdots, 8). \quad (24)$$

Solving the above equations, we obtain

$$\beta_{11} = \frac{\sqrt{-2\gamma\theta}}{\gamma}, \beta_{21} = 0, \beta_{10} = \frac{\alpha\lambda\sqrt{-2\gamma\theta}}{6\gamma\theta},$$

and

$$\beta_{11} = \frac{\sqrt{-2\gamma\theta}}{\gamma}, \beta_{21} = \frac{\sqrt{-2\gamma\theta}}{\gamma},$$

$$\beta_{10} = -\frac{\sqrt{-2\gamma\theta}}{\gamma z_1} + \frac{\alpha\lambda\sqrt{-2\gamma\theta}}{6\gamma\theta}, \alpha = \frac{6\theta+\sqrt{6\theta}\lambda z_1}{z_1\lambda},$$

then

$$U_{r10}(z) = \frac{\sqrt{-2\gamma\theta}}{\gamma z} + \frac{\alpha\lambda\sqrt{-2\gamma\theta}}{6\gamma\theta},$$

and

$$U_{r20}(z) = \frac{\sqrt{-2\gamma\theta}}{\gamma z} + \frac{\sqrt{-2\gamma\theta}}{\gamma}\frac{1}{z-z_1} - \frac{\sqrt{-2\gamma\theta}}{\gamma z_1} + \frac{\alpha\lambda\sqrt{-2\gamma\theta}}{6\gamma\theta},$$

where $\theta = \frac{\alpha^2}{6}$ in the first case, $\theta = \frac{\lambda^2 z_1^2}{6}$ in the second case.

Substitute $U(z) = R(\eta)$ into Eq.(14), then

$$\begin{aligned}
&\lambda^2\mu^2(\eta R' + \eta^2R'') - \alpha\lambda\mu^3(\eta^3R''' + \eta R' + 3\eta^2R'')\\
&\quad + \theta\mu^4(R^{(4)}\eta^4 + 6R'''\eta^3 + 7R''\eta^2 + R'\eta) + \gamma(6R(\mu R'\eta)^2\\
&\quad + 3\mu^2\eta^2(\eta R' + \eta^2R'')) = 0,
\end{aligned} \quad (25)$$

where $\eta = e^{\mu z}$ ($\mu \in \mathbb{C}$).

Substituting

$$U_s(e^{\mu z}) = \frac{b_{11}}{e^{\mu z}-1} + \frac{b_{21}}{e^{\mu z}-e^{\mu z_1}} + b_{10},$$

into the Eq.(25), we obtain that

$$\sum_{i=1}^{9}\frac{c_{2i}\mu^2 e^{\mu i z}}{(e^{\mu z}-1)^5(e^{\mu z}-e^{\mu z_1})^5}=0, \tag{26}$$

where

$$\begin{aligned}
c_{21} &= -3\,\gamma\,{b_{11}}^3{z_1}^5-\lambda^2 b_{11}{z_1}^5+\alpha\,\lambda\,\mu\,b_{21}{z_1}^3-\lambda^2 b_{21}{z_1}^3\\
&\quad -3\,\gamma\,{b_{21}}^3 z_1+6\,\gamma\,b_{10}b_{11}b_{21}{z_1}^4+\alpha\,\lambda\,\mu\,b_{11}{z_1}^5\\
&\quad -3\,\gamma\,{b_{10}}^2 b_{11}{z_1}^5+6\,\gamma\,b_{10}{b_{11}}^2{z_1}^5-\mu^2\theta\,b_{11}{z_1}^5\\
&\quad -6\,\gamma\,{b_{11}}^2 b_{21}{z_1}^4-\mu^2\theta\,b_{21}{z_1}^3+6\,\gamma\,b_{10}b_{11}b_{21}{z_1}^3\\
&\quad +6\,\gamma\,b_{10}{b_{21}}^2{z_1}^2-6\,\gamma\,b_{11}{b_{21}}^2{z_1}^2-3\,\gamma\,{b_{11}}^2 b_{21}{z_1}^3\\
&\quad -3\,\gamma\,b_{11}{b_{21}}^2{z_1}^3-3\,\gamma\,{b_{10}}^2 b_{21}{z_1}^3,\\
c_{22} &= c_{29}=\alpha\,\lambda\,\mu\,b_{11}+\alpha\,\lambda\,\mu\,b_{21}+3\,\gamma\,{b_{10}}^2 b_{11}+3\,\gamma\,{b_{10}}^2 b_{21}\\
&\quad +\mu^2\theta\,b_{11}+\mu^2\theta\,b_{21}+\lambda^2 b_{11}+\lambda^2 b_{21},\\
c_{23} &= c_{28}=-5\,\alpha\,\lambda\,\mu\,b_{11}z_1+3\,\alpha\,\lambda\,\mu\,b_{21}z_1-15\,\gamma\,{b_{10}}^2 b_{11}z_1\\
&\quad +12\,\gamma\,b_{10}{b_{11}}^2+24\,\gamma\,b_{10}b_{11}b_{21}+12\,\gamma\,b_{10}{b_{21}}^2\\
&\quad -5\,\lambda^2 b_{11}z_1-3\,\gamma\,{b_{10}}^2 b_{21}z_1-5\,\mu^2\theta\,b_{11}z_1\\
&\quad -15\,\gamma\,{b_{10}}^2 b_{21}-\lambda^2 b_{21}z_1+11\,\mu^2\theta\,b_{21}z_1+3\,\alpha\,\lambda\,\mu\,b_{11}\\
&\quad -5\,\alpha\,\lambda\,\mu\,b_{21}-3\,\gamma\,{b_{10}}^2 b_{11}+11\,\mu^2\theta\,b_{11}-5\,\mu^2\theta\,b_{21}\\
&\quad -\lambda^2 b_{11}-5\,\lambda^2 b_{21},\\
c_{24} &= c_{27}=10\,\alpha\,\lambda\,\mu\,b_{11}{z_1}^2-3\,\alpha\,\lambda\,\mu\,b_{21}{z_1}^2+9\,\gamma\,{b_{11}}^3\\
&\quad -55\,\mu^2\theta\,b_{11}z_1-55\,\mu^2\theta\,b_{21}z_1-3\,\alpha\,\lambda\,\mu\,b_{11}\\
&\quad +10\,\alpha\,\lambda\,\mu\,b_{21}+30\,\gamma\,{b_{10}}^2 b_{21}-6\,\gamma\,b_{10}{b_{11}}^2\\
&\quad -66\,\gamma\,b_{10}b_{11}b_{21}-60\,\gamma\,b_{10}{b_{21}}^2\\
&\quad +30\,\gamma\,{b_{10}}^2 b_{11}{z_1}^2-3\,\gamma\,{b_{10}}^2 b_{21}{z_1}^2+15\,\gamma\,{b_{10}}^2 b_{11}z_1\\
&\quad +11\,\mu^2\theta\,b_{21}{z_1}^2-15\,\alpha\,\lambda\,\mu\,b_{11}z_1-15\,\alpha\,\lambda\,\mu\,b_{21}z_1\\
&\quad +27\,\gamma\,{b_{11}}^2 b_{21}+10\,\lambda^2 b_{21}+5\,\lambda^2 b_{21}z_1+11\,\mu^2\theta\,b_{11}\\
&\quad -60\,\gamma\,b_{10}{b_{11}}^2 z_1-66\,\gamma\,b_{10}b_{11}b_{21}z_1-6\,\gamma\,b_{10}{b_{21}}^2 z_1\\
&\quad +10\,\mu^2\theta\,b_{11}{z_1}^2+27\,\gamma\,b_{11}{b_{21}}^2+9\,\gamma\,{b_{21}}^3+5\,\lambda^2 b_{11}z_1\\
&\quad -\lambda^2 b_{21}{z_1}^2-\lambda^2 b_{11}-3\,\gamma\,{b_{10}}^2 b_{11}+10\,\mu^2\theta\,b_{21}\\
&\quad +15\,\gamma\,{b_{10}}^2 b_{21}z_1+10\,\lambda^2 b_{11}{z_1}^2,\\
c_{25} &= c_{26}=3\,\alpha\,\lambda\,\mu\,b_{11}{z_1}^5+3\,\gamma\,{b_{10}}^2 b_{11}{z_1}^5+6\,\gamma\,b_{10}{b_{11}}^2{z_1}^5\\
&\quad +21\,\gamma\,b_{11}{b_{21}}^2{z_1}^2+5\,\lambda^2 b_{21}{z_1}^3-11\,\mu^2\theta\,b_{21}{z_1}^2\\
&\quad +6\,\gamma\,b_{10}{b_{21}}^2 z_1-9\,\gamma\,{b_{11}}^3{z_1}^5-11\,\mu^2\theta\,b_{11}{z_1}^5\\
&\quad -6\,\gamma\,b_{11}{b_{21}}^2 z_1+5\,\lambda^2 b_{11}{z_1}^4+15\,\gamma\,{b_{11}}^3{z_1}^4\\
&\quad +5\,\mu^2\theta\,b_{21}{z_1}^3+3\,\gamma\,{b_{11}}^2 b_{21}{z_1}^2+15\,\gamma\,{b_{21}}^3 z_1\\
&\quad -6\,\gamma\,{b_{11}}^2 b_{21}{z_1}^4+\lambda^2 b_{11}{z_1}^5+5\,\mu^2\theta\,b_{11}{z_1}^4\\
&\quad -5\,\alpha\,\lambda\,\mu\,b_{21}{z_1}^3-36\,\gamma\,b_{10}b_{11}b_{21}{z_1}^3+21\,\gamma\,{b_{11}}^2 b_{21}{z_1}^3\\
&\quad +3\,\gamma\,b_{11}{b_{21}}^2{z_1}^3+15\,\gamma\,{b_{10}}^2 b_{21}{z_1}^3-6\,\gamma\,b_{10}b_{11}b_{21}{z_1}^4\\
&\quad -30\,\gamma\,b_{10}{b_{21}}^2{z_1}^2-5\,\alpha\,\lambda\,\mu\,b_{11}{z_1}^4+15\,\gamma\,{b_{10}}^2 b_{11}{z_1}^4\\
&\quad -30\,\gamma\,b_{10}{b_{11}}^2{z_1}^4+3\,\alpha\,\lambda\,\mu\,b_{21}{z_1}^2+3\,\gamma\,{b_{10}}^2 b_{21}{z_1}^2\\
&\quad -6\,\gamma\,b_{10}b_{11}b_{21}{z_1}^2-9\,\gamma\,{b_{21}}^3+\lambda^2 b_{21}{z_1}^2.
\end{aligned}$$

Equating the coefficients of the same powers about $e^{\mu z}$ in Eq.(26) to zero, we get a set of algebraic equations:

$$c_{2i}=0,\ (i=1,2,\cdots,9).$$

Solving the above equations, we obtain

$$\begin{aligned}
b_{11} &= \frac{\mu\sqrt{-2\gamma\theta}}{\gamma},\quad b_{21}=0,\\
b_{10} &= \frac{\mu\sqrt{-2\gamma\theta}}{2\gamma}+\frac{\alpha\lambda\sqrt{-2\gamma\theta}}{6\gamma\theta},
\end{aligned}$$

and

$$\begin{aligned}
b_{11} &= \frac{\mu\sqrt{-2\gamma\theta}}{\gamma},\quad b_{21}=-\frac{\mu\sqrt{-2\gamma\theta}z_1}{\gamma},\\
b_{10} &= -\frac{\mu\sqrt{-2\gamma\theta}(z_1+1)}{2\gamma(z_1-1)}+\frac{\alpha\lambda\sqrt{-2\gamma\theta}}{6\gamma\theta},\\
\lambda &= \frac{\sqrt{2\,\theta\,{z_1}^2+20\,\theta\,z_1+2\,\theta}\mu}{2(z_1-1)},\\
\alpha &= \frac{6\sqrt{2}\left(\theta\,\mu\,z_1+\sqrt{\theta^2\mu^2{z_1}^2}\right)}{\sqrt{\theta\,\left({z_1}^2+10\,z_1+1\right)}\mu},
\end{aligned}$$

then

$$U_{s10}(z)=\frac{\mu\sqrt{-2\gamma\theta}}{\gamma(e^{\mu z}-1)}+\frac{\mu\sqrt{-2\gamma\theta}}{2\gamma}+\frac{\alpha\lambda\sqrt{-2\gamma\theta}}{6\gamma\theta},$$

and

$$\begin{aligned}
U_{s20}(z)=\frac{\mu\sqrt{-2\gamma\theta}}{\gamma(e^{\mu z}-1)}-\frac{\sqrt{-2\gamma\theta}z_1}{\gamma}\frac{1}{e^{\mu z}-e^{\mu z_1}}\\
-\frac{\mu\sqrt{-2\gamma\theta}(z_1+1)}{2\gamma(z_1-1)}+\frac{\alpha\lambda\sqrt{-2\gamma\theta}}{6\gamma\theta}.
\end{aligned}$$

Therefore, simply periodic solutions to Eq.(14) with pole at $z=0$ are

$$U_{s10}(z)=\frac{\sqrt{-2\gamma\theta}}{2\gamma}\mu\coth\frac{\mu}{2}z+\frac{\alpha\lambda\sqrt{-2\gamma\theta}}{6\gamma\theta},$$

and

$$\begin{aligned}
U_{s20}(z)=\frac{\sqrt{-2\gamma\theta}}{2\gamma}\mu(\coth\frac{\mu}{2}z-\coth\frac{\mu}{2}(z-z_1)-\coth\frac{\mu}{2}z_1)\\
+\frac{\alpha\lambda\sqrt{-2\gamma\theta}}{6\gamma\theta},
\end{aligned}$$

where $\alpha^2\lambda^2=6\theta\lambda^2-3\theta^2\mu^2$ in the first case, $\lambda=\frac{\sqrt{2\,\theta\,{z_1}^2+20\,\theta\,z_1+2\,\theta}\mu}{2(z_1-1)}$, $\alpha=\frac{6\sqrt{2}\left(\theta\,\mu\,z_1+\sqrt{\theta^2\mu^2{z_1}^2}\right)}{\sqrt{\theta\,({z_1}^2+10\,z_1+1)}\mu}$ in the second case.

By the weak $\langle 2,1\rangle$ condition and (20), we have the form of the elliptic solutions of Eq.(14) with pole at $z=0$

$$U_{d0}(z)=\frac{\beta_{-11}}{2}\frac{\wp'(z)+D_1}{\wp(z)-B_1}+\beta_{30},$$

where $D_1^2=4B_1^3-g_2B_1-g_3$. By the results obtained above, we deduce that $\beta_{30}=\frac{\alpha\lambda\sqrt{-2\gamma\theta}}{6\gamma\theta}$, $D_1=B_1=0$, then

$$U_{d0}(z)=\pm\frac{\sqrt{-2\gamma\theta}}{\gamma}\frac{\wp'(z)}{\wp(z)}+\frac{\alpha\lambda\sqrt{-2\gamma\theta}}{6\gamma\theta},$$

in which $g_3=0$.

Therefore, the elliptic function solutions of Eq.(14) are

$$U_d(z) = \pm\frac{\sqrt{-2\gamma\theta}}{\gamma}\frac{\wp'(z-z_0, g_2, 0)}{\wp(z-z_0, g_2, 0)} + \frac{\alpha\lambda\sqrt{-2\gamma\theta}}{6\gamma\theta},$$

where $z_0 \in \mathbb{C}$, $g_3 = 0$, g_2 is arbitrary.

By above procedures, we collect meromorphic solutions of Eq.(14) with arbitrary pole as follows:

$$(1) U_{r1}(z) = \frac{\sqrt{-2\gamma\theta}}{\gamma(z-z_0)} + \frac{\alpha\lambda\sqrt{-2\gamma\theta}}{6\gamma\theta},$$

$$(2) U_{r2}(z) = \frac{\sqrt{-2\gamma\theta}}{\gamma(z-z_0)} + \frac{\sqrt{-2\gamma\theta}}{\gamma}\frac{1}{z-z_1-z_0} - \frac{\sqrt{-2\gamma\theta}}{\gamma z_1} + \frac{\alpha\lambda\sqrt{-2\gamma\theta}}{6\gamma\theta},$$

where $\theta = \frac{\alpha^2}{6}$ in the first case, $\theta = \frac{\lambda^2 z_1^2}{6}$ in the second case;

$$(3) U_{s1}(z) = \frac{\sqrt{-2\gamma\theta}}{2\gamma}\mu \coth\frac{\mu}{2}(z-z_0) + \frac{\alpha\lambda\sqrt{-2\gamma\theta}}{6\gamma\theta},$$

$$(4) U_{s2}(z) = \frac{\sqrt{-2\gamma\theta}}{2\gamma}\mu\left(\coth\frac{\mu}{2}(z-z_0) - \coth\frac{\mu}{2}(z-z_1-z_0) - \coth\frac{\mu}{2}z_1\right) + \frac{\alpha\lambda\sqrt{-2\gamma\theta}}{6\gamma\theta},$$

where $\alpha^2\lambda^2 = 6\theta\lambda^2 - 3\theta^2\mu^2$ in the first case, $\lambda = \frac{\sqrt{2\theta z_1^2+20\theta z_1+2\theta\mu}}{2(z_1-1)}$, $\alpha = \frac{6\sqrt{2}\left(\theta\mu z_1+\sqrt{\theta^2\mu^2 z_1^2}\right)}{\sqrt{\theta(z_1^2+10z_1+1)}\mu}$ in the second case;

$$(5) U_d(z) = \pm\frac{\sqrt{-2\gamma\theta}}{\gamma}\frac{\wp'(z-z_0, g_2, 0)}{\wp(z-z_0, g_2, 0)} + \frac{\alpha\lambda\sqrt{-2\gamma\theta}}{6\gamma\theta},$$

where $z_0 \in \mathbb{C}$, $g_3 = 0$, g_2 is arbitrary. The properties of the solutions are shown in figures 7-10.

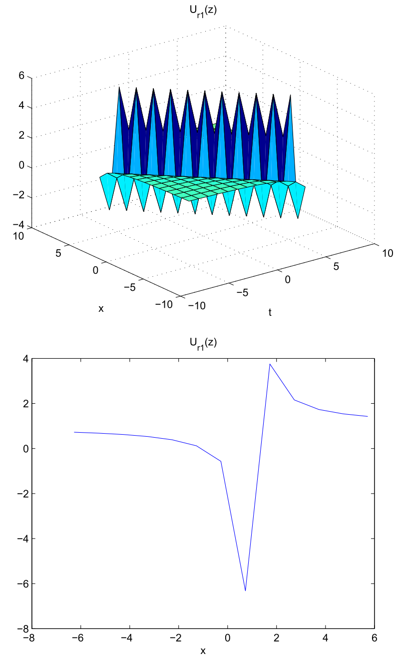


FIGURE 7. The 3D and 2D profiles of $U_{r1}(z)$ by considering $\alpha = 3$, $\theta = -2$, $\gamma = 1$, $\lambda = -2$, $z_0 = 1$, and $t = 0$ for the 2D graphic.

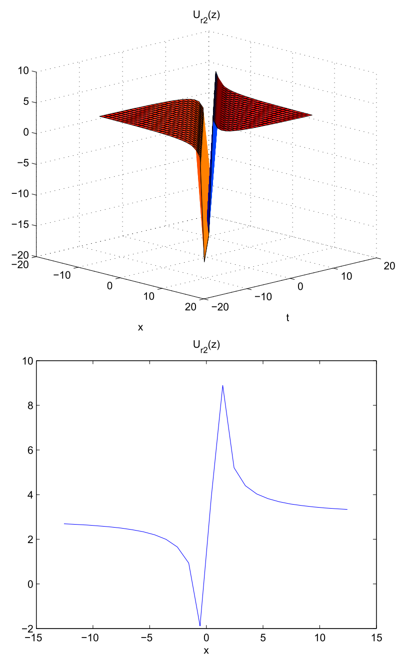


FIGURE 8. The 3D and 2D profiles of $U_{r2}(z)$ by considering $\alpha = 3$, $\theta = -2$, $\gamma = 1$, $\lambda = -2$, $z_1 = -1$, $z_0 = 1$, and $t = 0$ for the 2D graphic.

VI. COMPARISONS

Mothibi *et al.* [1] utilized the (G'/G)-expansion method to study the higher-order modified Boussinesq equation. We can observe that some important results to higher-order modified Boussinesq equation have been obtained by using the (G'/G)-expansion method. However, when we compare the results of this paper by applying two different systematic methods with the results of [1] the exponential function solutions and elliptic function solutions as a new aspect have been proposed to the literature.

The $\exp(-\psi(z))$-expansion method allows us to express the exact solutions of NLDEs as a polynomial of $\exp(-\psi(z))$, in which $\psi(z)$ satisfies the ODE (6). We can determine the degree of the polynomial through the homogeneous balance and obtain the values of the undetermined coefficients of the polynomial via the calculations of computer software, and then we obtain the exact solutions. With this method, we obtained seven solutions to the mentioned equation in which rational solution $u_7(z)$ is equivalent to $U_{r1}(z)$ if we consider $z_0 = -a$.

With the extended complex method, we can derive meromorphic solutions of the differential equations which do not satisfy $\langle p, q\rangle$ condition or are not Briot-Bouquet equation [32]. Therefore, more NLDEs in mathematical physics can be solved by the extended complex method. Using the indeterminate forms of the solutions, we are able to seek meromorphic solutions $U(z)$ for the differential equation with a pole at $z = 0$, then we can derive meromorphic solutions $U(z-z_0)$, $z_0 \in \mathbb{C}$ for the differential equation with an arbitrary pole. By implementing this method, we obtained

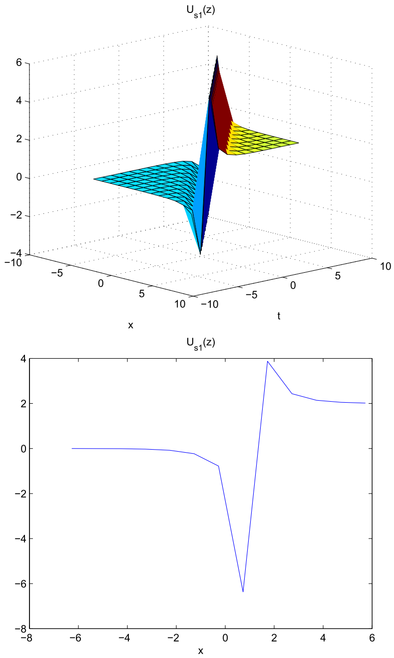


FIGURE 9. **The 3D and 2D profiles of $U_{s1}(z)$ by considering $\alpha = 3$, $\theta = -2$, $\gamma = 1$, $\lambda = -2$, $\mu = 1$, $z_0 = 1$, and $t = 0$ for the 2D graphic.**

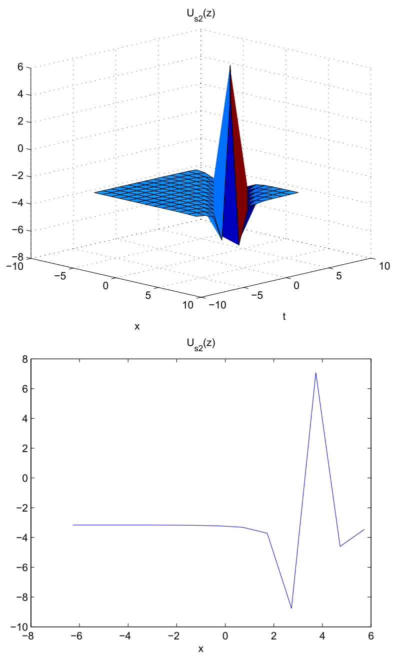


FIGURE 10. **The 3D and 2D profiles of $U_{s2}(z)$ by considering $\alpha = -3$, $\theta = -2$, $\gamma = 1$, $\lambda = -2$, $\mu = 1$, $z_1 = 1$, $z_0 = 3$, and $t = 0$ for the 2D graphic.**

five solutions to the mentioned equation in which simply periodic solutions $U_{s1}(z)$ and $U_{s2}(z)$ and rational solutions $U_{r2}(z)$ are new and can not be degenerated successively by the elliptic function solutions.

VII. CONCLUSIONS

In this article, we search for analytical solutions of the higher-order modified Boussinesq equation by two different systematic methods. By the $\exp(-\psi(z))$-expansion method and extended complex method, we obtain five kinds of exact solutions including trigonometric, hyperbolic, rational, exponential and elliptic function solutions. All the computations of this paper are done with the aid of Matlab. More solutions are obtained in this paper comparing to the solutions derived in [1]. We also give some computer simulations to illustrate our results by using Matlab. The extended complex method which is proposed in this paper is a improvement of the complex method so that we can solve more NLDEs arising in the mathematical physics. The results demonstrate that these two efficient and direct methods allow us to do complicated and tedious algebraic calculation.

REFERENCES

[1] D. M. Mothibi and C. M. Khalique, "On the exact solutions of a modified Kortweg de Vries type equation and higher-order modified Boussinesq equation with damping term," *Adv. Difference Equ.*, vol. 2013, no. 1, p. 166, 2013.

[2] J. Jiang, L. Liu, and Y. Wu, "Positive solutions to singular fractional differential system with coupled boundary conditions," *Commun. Nonlinear Sci. Numer. Simul.*, vol. 18, no. 11, pp. 3061–3074, 2013.

[3] J. Liu and Z. Zhao, "Existence of positive solutions to a singular boundary-value problem using variational methods," *Electron. J. Differ. Equ.*, vol. 2014, no. 135, pp. 1–9, 2014.

[4] Y. Xu and H. Zhang, "Positive solutions of an infinite boundary value problem for *n*th-order nonlinear impulsive singular integro-differential equations in Banach spaces," *Appl. Math. Comput.*, vol. 218, no. 9, pp. 5806–5818, 2012.

[5] J. Liu and Z. Zhao, "Multiple solutions for impulsive problems with non-autonomous perturbations," *Appl. Math. Lett.*, vol. 64, pp. 143–149, Feb. 2017.

[6] F. Sun, L. Liu, and Y. Wu, "Finite time blow-up for a class of parabolic or pseudo-parabolic equations," *Comput. Math. Appl.*, vol. 75, pp. 3685–3701, May 2018.

[7] X. Zheng, Y. Shang, and X. Peng, "Orbital stability of solitary waves of the coupled Klein–Gordon–Zakharov equations," *Math. Methods Appl. Sci.*, vol. 40, no. 7, pp. 2623–2633, 2017.

[8] H. M. Baskonus, T. A. Sulaiman, and H. Bulut, "New solitary wave solutions to the (2+1)-dimensional Calogero–Bogoyavlenskii–Schiff and the Kadomtsev–Petviashvili hierarchy equations," *Indian J. Phys.*, vol. 135, pp. 327–336, Oct. 2017.

[9] H. M. Baskonus, T. A. Sulaiman, and H. Bulut, "On the novel wave behaviors to the coupled nonlinear Maccari's system with complex structure," *Optik*, vol. 131, pp. 1036–1043, Feb. 2017.

[10] H. Bulut, T. A. Sulaiman, H. M. Baskonus, and T. Aktürk, "On the bright and singular optical solitons to the (2+1)-dimensional NLS and the Hirota equations," *Opt. Quantum Electron.*, vol. 50, p. 134, Mar. 2018.

[11] A. J. M. Jawad, M. D. Petković, and A. Biswas, "Modified simple equation method for nonlinear evolution equations," *Appl. Math. Comput.*, vol. 217, pp. 869–877, Sep. 2010.

[12] K. R. Raslan, K. K. Ali, and M. A. Shallal, "The modified extended tanh method with the Riccati equation for solving the space-time fractional EW and MEW equations," *Chaos, Solitons Fractals*, vol. 103, pp. 404–409, Oct. 2017.

[13] M. A. Shallal, H. N. Jabbar, and K. K. Ali, "Analytic solution for the space-time fractional Klein-Gordon and coupled conformable Boussinesq equations," *Results Phys.*, vol. 8, pp. 372–378, Mar. 2018.

[14] R. I. Nuruddeen, K. S. Aboodh, and K. K. Ali, "Analytical investigation of soliton solutions to three quantum Zakharov-Kuznetsov equations," *Commun. Theor. Phys.*, vol. 70, no. 4, pp. 405–412, 2018.

[15] A. Souleymanou, K. K. Ali, H. Rezazadeh, M. Eslami, M. Mirzazadeh, and A. Korkmaz, "The propagation of waves in thin-film ferroelectric materials," *Pramana*, vol. 93, p. 27, Aug. 2019.

[16] K. K. Ali and R. I. Nuruddeen, "Analytical treatment for the conformable space-time fractional Benney-Luke equation via two reliable methods," *Int. J. Phys. Res.*, vol. 5, no. 2, pp. 109–114, 2017.

[17] K. R. Raslan, T. S. El-Danaf, and K. K. Ali, ''Exact solution of the space-time fractional coupled EW and coupled MEW equations,'' *Eur. Phys. J. Plus*, vol. 132, p. 319, Jul. 2017.

[18] F. Mahmud, M. Samsuzzoha, and M. A. Akbar, ''The generalized Kudryashov method to obtain exact traveling wave solutions of the PHI-four equation and the Fisher equation,'' *Results Phys.*, vol. 7, pp. 4296–4302, Oct. 2017.

[19] K. Khan and M. A. Akbar, ''Solving unsteady Korteweg–de Vries equation and its two alternatives,'' *Math. Methods Appl. Sci.*, vol. 39, pp. 2752–2760, Jul. 2016.

[20] M. N. Alam and M. A. Akbar, ''Some new exact traveling wave solutions to the simplified MCH equation and the (1+1)-dimensional combined KdV–mKdV equations,'' *J. Assoc. Arab Universities Basic Appl. Sci.*, vol. 17, pp. 6–13, Apr. 2015.

[21] M. H. Islam, K. Khan, M. A. Akbar, and M. A. Salam, ''Exact traveling wave solutions of modified KdV–Zakharov–Kuznetsov equation and viscous Burgers equation,'' *SpringerPlus*, vol. 3, p. 105, Dec. 2014.

[22] K. Khan, M. A. Akbar, and H. Koppelaar, ''Study of coupled nonlinear partial differential equations for finding exact analytical solutions,'' *Roy. Soc. Open Sci.*, vol. 2, Jul. 2015, Art. no. 140406.

[23] M. F. Hoque and M. A. Akbar, ''New extended (G/G)-expansion method for traveling wave solutions of nonlinear partial differential equations, (NPDEs) in mathematical physics,'' *Italian J. Pure Appl. Math.*, vol. 33, pp. 175–190, Jan. 2014.

[24] M. S. Islam, K. Khan, M. A. Akbar, and A. Mastroberardino, ''A note on improved F-expansion method combined with Riccati equation applied to nonlinear evolution equations,'' *Roy. Soc. Open Sci.*, vol. 1, no. 2, 2014, Art. no. 140038.

[25] S. M. R. Islam, K. Khan, and M. A. Akbar, ''Exact solutions of unsteady Korteweg-de Vries and time regularized long wave equations,'' *SpringerPlus*, vol. 4, p. 124, Mar. 2015.

[26] Y. Gu and J. Qi, ''Symmetry reduction and exact solutions of two higher-dimensional nonlinear evolution equations,'' *J. Inequalities Appl.*, vol. 2017, p. 314, Dec. 2017.

[27] M. G. Hafez and M. A. Akbar, ''New exact traveling wave solutions to the (1+1)-dimensional Klein-Gordon-Zakharov equation for wave propagation in plasma using the $\exp(-\Phi(\xi))$-expansion method,'' *Propuls. Power Res.*, vol. 4, no. 1, pp. 31–39, 2015.

[28] H.-O. Roshid and M. A. Rahman, ''The $\exp(-\Phi(\eta))$-expansion method with application in the (1+1)-dimensional classical Boussinesq equations,'' *Results Phys.*, vol. 4, pp. 150–155, Aug. 2014.

[29] N. Kadkhoda and H. Jafari, ''Analytical solutions of the Gerdjikov–Ivanov equation by using $\exp(-\varphi(\xi))$-expansion method,'' *Optik*, vol. 139, pp. 72–76, Jun. 2017.

[30] W. Yuan, Y. Li, and J. Lin, ''Meromorphic solutions of an auxiliary ordinary differential equation using complex method,'' *Math. Methods Appl. Sci.*, vol. 36, no. 13, pp. 1776–1782, 2013.

[31] W. Yuan, Y. Shang, Y. Huang, and H. Wang, ''The representation of meromorphic solutions to certain ordinary differential equations and its applications,'' *Scientia Sinica Math.*, vol. 43, no. 6, pp. 563–575, 2013.

[32] W. Yuan, Z. Huang, M. Fu, and J. Lai, ''The general solutions of an auxiliary ordinary differential equation using complex method and its applications,'' *Adv. Difference Equ.*, vol. 2014, p. 147, Dec. 2014.

[33] W. Yuan, B. Xiao, Y. Wu, and J. Qi, ''The general traveling wave solutions of the Fisher type equations and some related problems,'' *J. Inequalities Appl.*, vol. 2014, p. 500, Dec. 2014.

[34] Y. Gu, W. Yuan, N. Aminakbari, and Q. Jiang, ''Exact solutions of the Vakhnenko-Parkes equation with complex method,'' *J. Function Spaces*, vol. 2017, Nov. 2017, Art. no. 6521357.

[35] Y. Gu, W. Yuan, N. Aminakbari, and J. Lin, ''Meromorphic solutions of some algebraic differential equations related Painlevé equation IV and its applications,'' *Math. Meth. Appl. Sci.*, vol. 41, no. 10, pp. 3832–3840, 2018.

[36] X. Hao, H. Wang, L. Liu, and Y. Cui, ''Positive solutions for a system of nonlinear fractional nonlocal boundary value problems with parameters and p-Laplacian operator,'' *Boundary Value Problems*, vol. 2017, p. 182, Dec. 2017.

[37] J. Jiang, L. Liu, and Y. Wu, ''Symmetric positive solutions to singular system with multi-point coupled boundary conditions,'' *Appl. Math. Comput.*, vol. 220, pp. 536–548, Sep. 2013.

[38] Y. Wang and J. Jiang, ''Existence and nonexistence of positive solutions for the fractional coupled system involving generalized p-Laplacian,'' *Adv. Difference Equ.*, vol. 2017, p. 337, Oct. 2017.

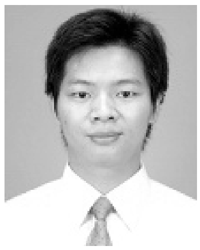
[39] M. Zou, X. Hao, L. Liu, and Y. Cui, ''Existence results for impulsive fractional integro-differential equation of mixed type with constant coefficient and antiperiodic boundary conditions,'' *Boundary Value Problems*, vol. 2017, p. 161, Nov. 2017.

[40] L. Liu, F. Sun, X. Zhang, and Y. Wu, ''Bifurcation analysis for a singular differential system with two parameters via to topological degree theory,'' *Nonlinear Anal., Model. Control*, vol. 22, no. 1, pp. 31–50, 2017.

[41] F. Sun, L. Liu, X. Zhang, and Y. Wu, ''Spectral analysis for a singular differential system with integral boundary conditions,'' *Medit. J. Math.*, vol. 13, pp. 4763–4782, Dec. 2016.

[42] Y. Wang, L. Liu, X. Zhang, and Y. Wu, ''Positive solutions of an abstract fractional semipositone differential system model for bioprocesses of HIV infection,'' *Appl. Math. Comput.*, vol. 258, pp. 312–324, May 2015.

[43] X. Zhang, L. Liu, Y. Wu, and B. Wiwatanapataphee, ''The spectral analysis for a singular fractional differential equation with a signed measure,'' *Appl. Math. Comput.*, vol. 257, pp. 252–263, Apr. 2015.



YONGYI GU was born in Guangdong, China, in 1985. He received the B.S. and M.S. degrees from Wuyi University, China, in 2009 and 2013, respectively, and the Ph.D. degree from Guangzhou University, in 2018. He is currently a Lecturer with the Guangdong University of Finance and Economics. His research interests include complex nonlinear systems and differential equation.

YINYING KONG was born in Guangdong, China, in 1979. He received the B.S. degree from Guangzhou University, China, in 2001, and the M.S. and Ph.D. degrees from South China Normal University, in 2004 and 2007, respectively. He is currently a Professor with the Guangdong University of Finance and Economics. His research interests include Hadoop architecture of large data and super data processing, and complex analysis.