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A Novel Approach to Delay-Variation-Dependent Stability Analysis of 2-D Discrete-Time Systems With Mixed Delays

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ABSTRACT In this paper, a stability analysis problem is studied for a class of two-dimensional (2-D) discrete-time systems with time-varying and distributed delays described by the second Fornasini-Marchesini (FM) model. First, new 2-D polynomials-based summation inequalities are proposed to estimate summation terms in the forward difference of Lyapunov-Krasovskii functional (LKF). The inequalities can reduce to 2-D Jensen inequalities and 2-Dfinite-sum inequalities by designing slack matrices and arbitrary vectors. Second, a new augment LKF is constructed, which makes full use of the delay changing information. By the Lyapunov stability theory, sufficient conditions for asymptotic stability of 2-D discrete-time systems are derived in the form of linear matrix inequalities. Finally, two simulation examples are given to demonstrate the effectiveness of the proposed methods.

INDEX TERMS Two-dimensional systems, time-varying delays, distributed delays, summation inequalities, Lyapunov-Krasovskii functional.

I. INTRODUCTION

Two-dimensional (2-D) systems are generally regarded as a kind of dynamic systems, which depend on two independent variables. Over the past decades, because of wide applications of 2-D systems in industrial field [1]- [3], great efforts from researchers have been devoted to the analysis and design of 2-D systems. In the study of 2-D discrete-time systems, Roesser model [4], the first Fornasini-Marchesini (FM) model [5], the second FM model [6], [7] and General model [8] have received extensive attention.

In practical industry, time delays commonly exist in the process of information transmission. Since time delays usually cause system performance degradation or even instability, stability analysis of time delay systems has become the focus of research. In the last few years, there have been many results for 2-D discrete-time systems with constant delays [9]–[11]. With the development of 2-D discrete-time systems and control theory, some researchers began to focus on 2-D discrete-time systems with time-varying delays [12]–[15]. Compared with the studies of 2-D constant delay systems before, 2-D systems with time-varying delays are closer to actual industrial model. During the above researches, there are two kinds of criteria on time-delay analysis of 2-D discrete-time systems, i.e., delay-independent [9]- [10] and delay-dependent [11]–[15] ones. By comparison, delay-dependent ones are less conservative due to delay information is fully utilized. It is worth noting that there is another kind of delays called distributed delays, which often exist in practice due to the existence of a large number of parallel paths in information transmission. Research on distributed delays has been developed in one-dimensional (1-D) systems [16]– [18]. Unfortunately, there is no research on stability analysis of 2-D discrete-time systems with distributed delays and time-varying delays, this motivates the present study.

In the study of Lyapunov asymptotic stability theory for 2-D discrete-time systems, the main purpose is to obtain less conservative stability conditions. To achieve this goal, many researchers follow two main directions: the construction of Lyapunov-Krasovskii functionals (LKFs) and the estimation of the forward difference of LKFs. For the construction

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of LKFs, those with simple form have been widely used in stability analysis of 2-D discrete-time systems [10]–[15]. In addition, stability analysis of 2-D discrete-time systems based on delay-partitioning technique has been considered in [19]. Recently, it has been found that augmented LKFs could help in reducing conservatism, because augmented matrices provide more room to be adjusted in stability criteria [20]. Augmented LKFs for 1-D systems have been developed to improve stability criteria in [21]–[24]. Furthermore, In [25], [26], a term of delay product type is included in LKFs for continuous-time systems, and the derivative of time-varying delay is introduced into the stability analysis. In [27], delay variation constraint has been taken into account for 1-D discrete-time systems, which is helpful to improve stability criteria. To the best knowledge of authors, up to now, the most constructions of LKFs are still simple forms and the delay changing information has not been fully utilized in 2-D discrete-time systems. Therefore, there is room for further study on the structure of LKFs for 2-D discrete-time systems.

For the bounds on difference of functionals, the crux is how to deal with the introduced summation terms $\sum_{l=\beta}^{\alpha} \Delta x(i+l, j+1)P\Delta x(i+l, j+1)$ and $\sum_{l=\beta}^{\alpha} \Delta x(i+1, j+l) Q\Delta x(i+1, j+l)$. Some methods have been proposed to solve the problems, such as, the free-weighting matrix approach [28], [29], the 2-D Jensen inequalities [30], the 2-D finite-sum inequalities [31]–[33]. 2-D Jensen inequalities and 2-D finite-sum inequalities are methods to estimate the difference items directly by boundary inequalities. 2-D finite-sum inequalities provide a more tighter lower bound than Jensen inequalities [31]. But there is room to improve the 2-D finite-sum inequalities as more general summation inequalities. In recent years, for 1-D systems, polynomials-based summation inequalities have been proposed in [34], which utilize slack matrices and arbitrary vectors. For systems with time-varying delays, researchers have proven that polynomials-based summation inequalities have more advantages than Jensen inequalities and Wirtinger-based inequalities [34]. The emergence of polynomials-based summation inequalities promotes the development of general summation inequalities. However, polynomials-based summation inequalities have not received adequate attention for 2-D discrete-time systems.

In this paper, a delay-variation-dependent stability problem for 2-D discrete-time systems described by the FM second model with time-varying delays and distributed delays is investigated. 2-D polynomials-based summation inequalities are proposed. Combining 2-D polynomials-based summation inequalities, a novel LKF is constructed to obtain improved stability criteria. This paper is organized in the following. Section II formulates the problem of 2-D discrete-time systems with mixed delays described by the second FM model and proposes 2-D polynomials-based summation inequalities. A delay-variation-dependent stability problem is studied in Section III. Two numerical examples are given in Section IV to illustrate effectiveness of the proposed methods. Finally, some conclusions are given in Section V.

Main contributions of this paper are summarized as below:

i. Distributed time delays and time-varying delays are considered simultaneously in the stability analysis problem for 2-D discrete-time systems.
ii. 2-D polynomials-based summation inequalities are proposed, which encompass 2-D Jensen inequalities and 2-D finite-sum inequalities as special cases.
iii. A new augmented LKF which takes more state information into account is constructed, and the delay changing information is introduced into the difference of the LKF.

Notation: Throughout the paper, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. $\mathbb{N}$ and $\mathbb{N}^+$ represent the sets of nonnegative and positive integers, respectively. For a real matrix P, P^T and P^{-1} represent the transpose and the inverse of P, respectively. A matrix $P > 0$, means that it is a symmetric, positive definite real matrix. The shorthand diag$\{\cdot\}$ denotes a block diagonal matrix. The symmetric terms in a symmetric matrix are denoted by $*$. The notation $\|\cdot\|$ refers to the Euclidean vector norm. $col\{x_1, x_2, \cdots, x_n\}$ means $\left[x_1^T \; x_2^T \; \cdots \; x_n^T\right]^T$. In a symmetric block matrix Z, Z_{ij} is the (i, j)th component. $sym\{M\} = M + M^T$. $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ for $0 \le k \le n$. $\mathbb{S}_n$ and $\mathbb{S}_n^+$ denote the set of symmetric definite matrices of $\mathbb{R}^{n\times n}$ and the set of symmetric positive definite matrices, respectively.

II. PROBLEM STATEMENT AND PRELIMINARIES

Consider a 2-D discrete-time system with mixed time delays as follows:

$$\begin{aligned} x(i+1, j+1) = {} & A_1x(i, j+1) + A_2x(i+1, j) \\ & + A_{1d}x(i-d_1(i), j+1) \\ & + A_{2d}x(i+1, j-d_2(j)) \\ & + A_3 \sum_{s_1=1}^{+\infty} \mu_{s_1} x(i-s_1, j+1) \\ & + A_4 \sum_{s_2=1}^{+\infty} \mu_{s_2} x(i+1, j-s_2), \end{aligned} \tag{1}$$

where $x(i, j) \in \mathbb{R}^n$ is the state vector, A_1, A_2, A_{1d}, A_{2d}, A_3 and A_4 are constant matrices with appropriate dimensions. $i, j \in \mathbb{N}$. $d_1(i)$ and $d_2(j)$ are time-varying delays along vertical direction and horizontal direction, respectively, satisfying:

$$0 < d_{1m} \le d_1(i) \le d_{1M}, \; 0 < d_{2m} \le d_2(j) \le d_{2M}, \tag{2}$$

$$\lambda_{1m} \le \Delta d_1(i) = d_1(i+1) - d_1(i) \le \lambda_{1M}, \tag{3}$$

$$\lambda_{2m} \le \Delta d_2(j) = d_2(j+1) - d_2(j) \le \lambda_{2M}, \tag{4}$$

where d_{1m}, d_{2m}, d_{1M} and d_{2M} are constant positive integers, denoting delay bounds. λ_{1m}, λ_{2m}, λ_{1M} and λ_{2M} are constant integers, denoting the delay variation bounds.

μ_{s_1} and μ_{s_2} are constants. $\mu_{s_1} \ge 0, \mu_{s_2} \ge 0 (s_1, s_2 = 1, 2, \cdots)$, in the same time, satisfying the following restrictions:

$$\sum_{s_1=1}^{+\infty} s_1\mu_{s_1} < +\infty, \sum_{s_2=1}^{+\infty} s_2\mu_{s_2} < +\infty, \quad (5)$$

$$\overline{\mu}_{s_1} = \sum_{s_1=1}^{+\infty} \mu_{s_1} < +\infty, \overline{\mu}_{s_2} = \sum_{s_2=1}^{+\infty} \mu_{s_2} < +\infty. \quad (6)$$

The boundary conditions are assumed as:

$$\begin{cases} x(i,j) = \varphi_{i,j}, & \forall 0 \le i \le r_1,\ j = -d_{2M},\ -d_{2M}+1, \cdots, 0, \\ x(i,j) = 0, & \forall i > r_1,\ j = -d_{2M},\ -d_{2M}+1, \cdots, 0, \\ x(i,j) = \psi_{i,j}, & \forall 0 \le j \le r_2,\ i = -d_{1M}, -d_{1M}+1, \cdots, 0, \\ x(i,j) = 0, & \forall j > r_2,\ i = -d_{1M},\ -d_{1M}+1, \cdots, 0, \\ \varphi_{0,0} = \psi_{0,0}, \end{cases} \quad (7)$$

where r_1 and r_2 are positive integers.

Definition 1: The system (1) is asymptotically stable if $\lim_{r\to\infty} X_r = 0$ *under any bounded boundary conditions of (7), where* $X_r = \sup\{||x(i,j)|| : i+j = r, i, j \in \mathbb{N}\}$.

Before presenting the main results of the paper, the following lemmas are introduced first, which will be important for subsequent derivation.

Lemma 1: For a matrix $R \in \mathbb{S}_n^+$, *constants* $a \in \mathbb{Z}$, $h \in \mathbb{N}^+$, *and a function* $x : \mathbb{Z}[a, a+h-1] \times \mathbb{Z}[a, a+h-1] \to \mathbb{R}^n$, *the following inequalities hold:*

(1) 2-D discrete Jensen inequalities in [29]

$$\sum_{i=a}^{a+h-1} \Delta x_1^T(i,j) R \Delta x_1(i,j) \ge \frac{1}{h}\Phi_1^T R \Phi_1, \quad (8)$$

$$\sum_{j=a}^{a+h-1} \Delta x_2^T(i,j) R \Delta x_2(i,j) \ge \frac{1}{h}\Psi_1^T R \Psi_1, \quad (9)$$

(2) 2-D finite-sum inequalities in [31]

$$\sum_{i=a}^{a+h-1} \Delta x_1^T(i,j) R \Delta x_1(i,j) \ge \frac{1}{h}\Phi_1^T R \Phi_1 + \frac{3}{h}\Phi_2^T R \Phi_2, \quad (10)$$

$$\sum_{j=a}^{a+h-1} \Delta x_2^T(i,j) R \Delta x_2(i,j) \ge \frac{1}{h}\Psi_1^T R \Psi_1 + \frac{3}{h}\Psi_2^T R \Psi_2, \quad (11)$$

(3) 2-D finite-sum inequalities in [33]

$$\sum_{i=a}^{a+h-1} \Delta x_1^T(i,j) R \Delta x_1(i,j) \ge \frac{1}{h}\Phi_1^T R \Phi_1 + \frac{3}{h}\Phi_2^T R \Phi_2 + \frac{5}{h}\Phi_3^T R \Phi_3, \quad (12)$$

$$\sum_{j=a}^{a+h-1} \Delta x_2^T(i,j) R \Delta x_2(i,j) \ge \frac{1}{h}\Psi_1^T R \Psi_1 + \frac{3}{h}\Psi_2^T R \Psi_2 + \frac{5}{h}\Psi_3^T R \Psi_3, \quad (13)$$

where

$$\Delta x_1(i,j) = x(i+1,j) - x(i,j),$$
$$\Delta x_2(i,j) = x(i,j+1) - x(i,j),$$
$$\Phi_1 = x(a+h,j) - x(a,j),$$
$$\Psi_1 = x(i,a+h) - x(i,a),$$
$$\Phi_2 = x(a+h,j) + x(a,j) - \frac{2}{h+1}\sum_{i=a}^{a+h} x(i,j),$$
$$\Psi_2 = x(i,a+h) + x(i,a) - \frac{2}{h+1}\sum_{j=a}^{a+h} x(i,j),$$
$$\Phi_3 = x(a+h,j) - x(a,j) + \frac{6}{h+1}\sum_{i=a}^{a+h} x(i,j) - \frac{12}{(h+1)(h+2)}\sum_{s=a}^{a+h}\sum_{i=s}^{a+h} x(i,j),$$
$$\Psi_3 = x(i,a+h) - x(i,a) + \frac{6}{h+1}\sum_{j=a}^{a+h} x(i,j) - \frac{12}{(h+1)(h+2)}\sum_{s=a}^{a+h}\sum_{j=s}^{a+h} x(i,j).$$

Lemma 2: [35] Given linearly independent functions $\{p_s(i), s \in [0, m] \bigcap \mathbb{Z} | p_0(i) = 1\}$, *where* $m \in \mathbb{N}$, *the orthogonal function of* $p_s(i)$ *based on* $\{p_k(i), k \in [0, s-1] \bigcap \mathbb{Z}\}$, *say* $\widetilde{p}_s(i)$, *can be generated by*

$$\widetilde{p}_s(i) = p_s(i) - \sum_{k=0}^{s-1}\left(\sum_{i=a}^{a+h-1} p_s(i)\widetilde{p}_k(i)\right)\left(\sum_{i=a}^{a+h-1} \widetilde{p}_k^2(i)\right)^{-1}\widetilde{p}_k(i),$$
$$\widetilde{p}_0(i) = p_0(i).$$

Then, the following properties are satisfied

$$\sum_{i=a}^{a+h-1} \widetilde{p}_s(i) = 0, \quad 1 \le s \le m,$$

$$\sum_{i=a}^{a+h-1} \widetilde{p}_s(i)\widetilde{p}_k(i) = 0, \quad 0 \le s,\ k \le m,\ s \ne k.$$

Lemma 3: [34] For $r \in \mathbb{N}$, $a \in \mathbb{Z}$, $h \in \mathbb{N}^+$, *let* $x : [a, a+h-1] \bigcap \mathbb{Z} \to \mathbb{R}^n$ *be a vector function. Then, we have*

$$\sum_{i=a}^{a+h-1} \binom{i-a+r}{r} x(i) = \sum_{i_{r+1}=a}^{a+h-1} \cdots \sum_{i_2=i_3}^{a+h-1}\sum_{i_1=i_2}^{a+h-1} x(i_1).$$

The following rth-order polynomial functions are chosen when deriving 2-D polynomials-based summation inequality.

$$p_r(i) = \frac{1}{r!}\prod_{u=1}^{r}\frac{(i-a+u)}{(n+u)}(r = 0, \cdots, m)(m \in \mathbb{N}).$$

Lemma 4: (2-D polynomials-based summation inequality)

For $a \in \mathbb{Z}$, $m \in \mathbb{N}$ *and* $h, q \in \mathbb{N}^+$, *a vector function* $x : \mathbb{Z}[a, a+h-1] \times \mathbb{Z}[a, a+h-1] \to \mathbb{R}^n$, *a matrix* $M \in \mathbb{S}^+_{((m+1)q+1)n}$, *an arbitrary vector function* $\eta_1(i,j) \in R^{qn}$, *and kth-order polynomial functions* $p_k (k = 0, \cdots, m)$, *the following inequality holds*:

$$\begin{aligned} &-\sum_{i=a}^{a+h-1}\Delta x_1^T(i,j)M_{(m+2)(m+2)}\Delta x_1(i,j) \\ &\quad\le \sum_{s=1}^{m+1}\Big(\sum_{i=a}^{a+h-1}P_{s-1}^2(i)\Big)\eta_1^T(i,j)M_{ss}\eta_1(i,j) \\ &\qquad+\sum_{k=1}^{m}\sum_{s=k+1}^{m+1} sym\{\rho\eta_1^T(i,j)M_{ks}\eta_1(i,j)\} \\ &\qquad+\sum_{s=1}^{m+1} sym\Big\{\eta_1^T(i,j)M_{s(m+2)}\sum_{i=a}^{a+h-1}\Delta x_1(i,j)\Big\}, \quad (14) \end{aligned}$$

where

$$\rho=\sum_{i=a}^{a+h-1}P_{k-1}(i)P_{s-1}(i).$$

Proof: Choose $\xi_1(i,j) = col\{P_0(i)\eta_1(i,j),\cdots,P_m(i)\eta_1(i,j),\Delta x_1(i,j)\}$. $\Delta x_1(i,j)$ is defined in Lemma 1.

$$\theta(i,j)=\xi_1^T(i,j)M\xi_1(i,j)\ge 0.$$

Summing $\theta(i,j)$ over $i \in [a, a+h-1]\bigcap \mathbb{Z}$, it can be shown that:

$$\begin{aligned} 0 &\le \sum_{i=a}^{a+h-1}\theta(i,j) \\ &= \sum_{s=1}^{m+1}\sum_{k=1}^{m+1}\sum_{i=a}^{a+h-1}(P_{s-1}(i)P_{k-1}(i))\eta_1^T(i,j)M\eta_1(i,j) \\ &\quad+\sum_{s=1}^{m+1} sym\Big\{\eta_1^T(i,j)M_{s(m+2)}\sum_{i=a}^{a+h-1}P_{s-1}(i)\Delta x_1(i,j)\Big\} \\ &\quad+\sum_{i=a}^{a+h-1}\{\Delta x_1^T(i,j)M_{(m+2)(m+2)}\Delta x_1(i,j)\}. \end{aligned}$$

Due to $M_{ij}\neq M_{ji}^T (i\neq j)$, the inequality (14) is obtained. □

Remark 1: For $a \in \mathbb{Z}$, $m \in \mathbb{N}$ and $h, q \in \mathbb{N}^+$, a vector function $x : \mathbb{Z}[a, a+h-1]\times\mathbb{Z}[a, a+h-1] \to \mathbb{R}^n$, a matrix $M \in \mathbb{S}^+_{((m+1)q+1)n}$, an arbitrary vector function $\eta_2(i,j) \in R^{qn}$, and kth-order polynomial functions $p_k(k = 0,\cdots,m)$, the following inequality holds:

$$\begin{aligned} &-\sum_{j=a}^{a+h-1}\Delta x_2^T(i,j)M_{(m+2)(m+2)}\Delta x_2(i,j) \\ &\quad\le \sum_{s=1}^{m+1}\Big(\sum_{j=a}^{a+h-1}P_{s-1}^2(i)\Big)\eta_2^T(i,j)M_{ss}\eta_2(i,j) \\ &\qquad+\sum_{k=1}^{m}\sum_{s=k+1}^{m+1} sym\{\rho\eta_2^T(i,j)M_{ks}\eta_2(i,j)\} \\ &\qquad+\sum_{s=1}^{m+1} sym\Big\{\eta_2^T(i,j)M_{s(m+2)}\sum_{j=a}^{a+h-1}\Delta x_2(i,j)\Big\}. \quad (15) \end{aligned}$$

where ρ is defined in Lemma 4.

In order to be applied to 2-D discrete-time systems with time-varying delays, the following Lemma is proposed.

Lemma 5: For $a \in \mathbb{Z}$, $h, q \in \mathbb{N}^+$, a vector function $x : \mathbb{Z}[a, a+h-1]\times\mathbb{Z}[a, a+h-1] \to \mathbb{R}^n$, a matrix $M \in \mathbb{S}^+_{(3q+1)n}$, an arbitrary vector function $\eta_1(i,j) \in R^{qn}$, the following inequality holds:

$$\begin{aligned} &-\sum_{i=a}^{a+h-1}\Delta x_1^T(i,j)M_{44}\Delta x_1(i,j) \\ &\quad\le \eta_1^T(i,j)\Big(hM_{11}+\frac{h}{12}M_{22}+\frac{h}{720}M_{33}\Big)\eta_1(i,j) \\ &\qquad+sym\Big\{\eta_1^T(i,j)M_{14}\chi_1+\eta_1^T(i,j)M_{24}\chi_2 \\ &\qquad+\eta_1^T(i,j)M_{34}\chi_3\Big\}, \quad (16) \end{aligned}$$

where

$$\begin{aligned} \chi_1 &= x(a+h,j)-x(a,j), \\ \chi_2 &= \frac{1}{2}x(a+h,j)+\frac{1}{2}x(a,j)-\frac{1}{h+1}\sum_{i=a}^{a+h}x(i,j), \\ \chi_3 &= \frac{1}{12}x(a+h,j)-\frac{1}{12}x(a,j)+\frac{1}{2(h+1)}\sum_{i=a}^{a+h}x(i,j) \\ &\quad-\frac{1}{(h+1)(h+2)}\sum_{s=a}^{a+h}\sum_{i=s}^{a+h}x(i,j). \end{aligned}$$

Proof: Design $P_r(i)$ as $\widetilde{P}_r(i)(r = 0, 1, 2)$ in lemma 4, according to Lemma 2, it is obtained that:

$$\begin{aligned} &\widetilde{p}_0(i)=1, \quad \widetilde{p}_1(i)=\frac{i-a+1}{h+1}-\frac{1}{2}, \\ &\widetilde{p}_2(i)=\frac{(i-a+2)(i-a+1)}{2(h+1)(h+2)}-\frac{i-a+1}{2(h+1)}+\frac{1}{12}, \end{aligned}$$

where

$$\begin{aligned} \sum_{i=a}^{a+h-1}\widetilde{p}_1^2(i) &= \frac{h(h+1)}{12(h+1)}, \\ \sum_{i=a}^{a+h-1}\widetilde{p}_2^2(i) &= \frac{h(h-1)(h-2)}{720(h+1)(h+2)}. \end{aligned}$$

According to Lemma 3, several summation terms are obtained as follows:

$$\begin{aligned} &\sum_{i=a}^{a+h-1}\widetilde{p}_1(i)x(i,j) \\ &\quad=\frac{1}{h+1}\sum_{s=a}^{a+h-1}\sum_{i=s}^{a+h-1}x(i,j)-\frac{1}{2}\sum_{i=a}^{a+h-1}x(i,j), \\ &\sum_{i=a}^{a+h-1}\widetilde{p}_2(i)x(i,j) \\ &\quad=\frac{1}{(h+1)(h+2)}\sum_{k=a}^{a+h-1}\sum_{s=k}^{a+h-1}\sum_{i=s}^{a+h-1}x(i,j) \end{aligned}$$

$$-\frac{1}{2(h+1)}\sum_{s=a}^{a+h-1}\sum_{i=s}^{a+h-1}x(i,j)+\frac{1}{12}\sum_{i=a}^{a+h-1}x(i,j),$$

$$\sum_{s=a}^{a+h-1}\sum_{i=s}^{a+h-1}\Delta x(i,j)$$
$$=(h+1)x(a+h,j)-\sum_{i=a}^{a+h}x(i,j),$$

$$\sum_{k=a}^{a+h-1}\sum_{s=k}^{a+h-1}\sum_{i=s}^{a+h-1}\Delta x(i,j)$$
$$=\frac{(h+1)(h+2)}{2}x(a+h,j)-\sum_{s=a}^{a+h}\sum_{i=s}^{a+h}x(i,j).$$

Let $m=2$ in lemma 4, it can be shown that:

$$\begin{aligned}-\sum_{i=a}^{a+h-1}\Delta x_1^T(i,j)M_{44}\Delta x_1(i,j)&\\ \le \eta_1^T(i,j)\Big(hM_{11}+\frac{h(h-1)}{12(h+1)}M_{22}&\\ +\frac{h(h-1)(h-2)}{720(h+1)(h+2)}M_{33}\Big)\eta_1(i,j)&\\ +sym\Big\{\eta_1^T(i,j)M_{14}\chi_1+\eta_1^T(i,j)M_{24}\chi_2&\\ +\eta_1^T(i,j)M_{34}\chi_3\Big\}.&\end{aligned}$$

Due to $\frac{h-1}{h+1}\le 1$, $\frac{(h-1)(h-2)}{(h+1)(h+2)}\le 1$, the inequality (16) is obtained. □

Remark 2: For $a\in\mathbb{Z}$, $h,q\in\mathbb{N}^+$, a vector function $x:\mathbb{Z}[a,a+h-1]\times\mathbb{Z}[a,a+h-1]\to\mathbb{R}^n$, a matrix $M\in\mathbb{S}^+_{(3q+1)n}$, an arbitrary vector function $\eta_1(i,j)\in R^{qn}$, the following inequality holds:

$$\begin{aligned}-\sum_{j=a}^{a+h-1}\Delta x_2^T(i,j)M_{44}\Delta x_2(i,j)&\\ \le \eta_2^T(i,j)\Big(hM_{11}+\frac{h}{12}M_{22}+\frac{h}{720}M_{33}\Big)\eta_2(i,j)&\\ +sym\Big\{\eta_2^T(i,j)M_{14}\alpha_1+\eta_2^T(i,j)M_{24}\alpha_2&\\ +\eta_2^T(i,j)M_{34}\alpha_3\Big\},&\end{aligned}\tag{17}$$

where

$$\alpha_1=x(i,a+h)-x(i,a),$$
$$\alpha_2=\frac{1}{2}x(i,a+h)+\frac{1}{2}x(i,a)-\frac{1}{h+1}\sum_{j=a}^{a+h}x(i,j),$$
$$\alpha_3=\frac{1}{12}x(i,a+h)-\frac{1}{12}x(i,a)+\frac{1}{2(h+1)}\sum_{j=a}^{a+h}x(i,j),$$
$$-\frac{1}{(h+1)(h+2)}\sum_{s=a}^{a+h}\sum_{j=s}^{a+h}x(i,j).$$

Remark 3: Define the arbitrary vector and the slack matrices in lemma 5 as following:

$$\eta_1(i,j)=col\Big\{x(a+h,j),x(a,j),\frac{1}{h+1}\sum_{i=a}^{a+h}x(i,j)\Big\},$$
$$M_{11}=diag\{X,0\},\quad M_{22}=M_{33}=0,\ M_{44}=R,$$
$$M_{14}=col\{Y,0\},\quad M_{24}=M_{34}=0,$$
$$X=YR^{-1}Y^T,\quad Y=-\frac{1}{n}[R\ -R]^T,$$

Lemma 5 reduces to (8) in Lemma 1. When the slack matrices in lemma 5 are defined as following:

$$M_{11}=M_{14}M_{44}^{-1}M_{14}^T,\quad M_{22}=M_{24}M_{44}^{-1}M_{24}^T,\ M_{33}=0,$$
$$M_{14}=-\frac{1}{n}[R\quad -R\quad 0]^T,\ M_{24}=-\frac{6}{n}[R\quad -R\quad 2R]^T,$$
$$M_{44}=R,$$

Lemma 5 reduces to (10) in Lemma 1. When the following arbitrary vector and slack matrices are chosen in lemma 5:

$$\eta_1(i,j)=col\Big\{x(a+h,j),x(a,j),\frac{1}{h+1}\sum_{i=a}^{a+h}x(i,j),$$
$$\frac{1}{(h+1)(h+2)}\sum_{s=a}^{a+h}\sum_{i=s}^{a+h}x(i,j)\Big\},$$
$$M_{ii}=M_{i4}M_{44}^{-1}M_{i4}^T(i=1,2,3),$$
$$M_{14}=-\frac{1}{h}[R\quad -R\quad 0\quad 0]^T,$$
$$M_{24}=-\frac{6}{h}[R\quad R\quad -2R\quad 0]^T,$$
$$M_{34}=-\frac{60}{h}[R\quad -R\quad 6R\quad 12R]^T,\quad M_{44}=R,$$

Lemma 5 reduces to (12) in Lemma 1. Therefore, 2-D polynomials-based summation inequality contains 2-D Jensen inequality and 2-D finite-sum inequality as special cases.

To reduce the complexity of the calculation of Lemma 5, the following corollary is obtained.

Corollary 1: For $a\in\mathbb{Z}$, $h,q\in\mathbb{N}^+$, a vector function $x:\mathbb{Z}[a,a+h-1]\times\mathbb{Z}[a,a+h-1]\to\mathbb{R}^n$, for matrices $M_{i4}\in\mathbb{R}^{qn\times n}(i=1,2,3)$, a positive definite matrix $M_{44}\in\mathbb{S}^+_n$, an arbitrary vector function $\eta_1(i,j)\in R^{qn}$, the following inequalities hold:

$$\begin{aligned}-\sum_{i=a}^{a+h-1}\Delta x_1^T(i,j)M_{44}\Delta x_1(i,j)&\\ \le \eta_1^T(i,j)\Big(hM_{14}M_{44}^{-1}M_{14}^T+\frac{h}{12}M_{24}M_{44}^{-1}M_{24}^T&\\ +\frac{h}{720}M_{34}M_{44}^{-1}M_{34}^T\Big)\eta_1(i,j)+sym\Big\{\eta_1^T(i,j)M_{14}\chi_1&\\ +\eta_1^T(i,j)M_{24}\chi_2+\eta_1^T(i,j)M_{34}\chi_3\Big\},&\end{aligned}\tag{18}$$

$$-\sum_{j=a}^{a+h-1}\Delta x_2^T(i,j)M_{44}\Delta x_2(i,j)$$

$$\le \eta_2^T(i,j)\Big(hM_{14}M_{44}^{-1}M_{14}^T + \frac{h}{12}M_{24}M_{44}^{-1}M_{24}^T + \frac{h}{720}M_{34}M_{44}^{-1}M_{34}^T\Big)\eta_2(i,j) + sym\Big\{\eta_2^T(i,j)M_{14}\alpha_1 + \eta_2^T(i,j)M_{24}\alpha_2 + \eta_2^T(i,j)M_{34}\alpha_3\Big\}. \quad (19)$$

Proof: Let $M_{ii} = M_{i4}M_{44}^{-1}M_{i4}^T (i = 1, 2, 3)$ in (14) and (15). □

III. MAIN RESULTS

In the section, a novel approach of stability analysis for 2-D discrete-time system is developed. The following theorem presents a delay-variation-dependent sufficient condition for system (1) based on the above results.

Theorem 1: For given scalars d_{km}, d_{kM}, λ_{km}, λ_{kM} $(k = 1, 2)$, $d_1(i)$, $d_2(j)$ satisfy the conditions (2)-(4), $\mu_{s_1} > 0$, $\mu_{s_2} > 0$ and satisfy the conditions (5)-(6), 2-D system (1) is asymptotically stable if there exist real matrices $P_1, P_3 \in \mathbb{S}_{5n}^+$, $P_2, P_4 \in \mathbb{S}_{3n}^+$, $Q_k, R_k, S_1, S_2 \in \mathbb{S}_n^+$, $(k = 1, 2, 3, 4)$, $M_i, E_i \in \mathbb{R}^{4n\times n}$, $N_i, W_i, F_i, E_i \in \mathbb{R}^{7n\times n}$, $(i = 1, 2, 3)$, such that the following LMIs hold:

$$\begin{bmatrix} \Upsilon_1(d_1(i) = d_{1m}, \Delta d_1(i) = \lambda_{1l}) + \Upsilon_2 + \Upsilon_3 & \Theta_1 \\ * & \Omega_1 \end{bmatrix} < 0, \quad (20)$$

$$\begin{bmatrix} \Upsilon_1(d_1(i) = d_{1M}, \Delta d_1(i) = \lambda_{1l}) + \Upsilon_2 + \Upsilon_3 & \Theta_2 \\ * & \Omega_1 \end{bmatrix} < 0, \quad (21)$$

$$\begin{bmatrix} \Upsilon_4(d_2(j) = d_{2m}, \Delta d_2(j) = \lambda_{2l}) + \Upsilon_5 + \Upsilon_6 & \Theta_3 \\ * & \Omega_2 \end{bmatrix} < 0, \quad (22)$$

$$\begin{bmatrix} \Upsilon_4(d_2(j) = d_{2M}, \Delta d_2(j) = \lambda_{2l}) + \Upsilon_5 + \Upsilon_6 & \Theta_4 \\ * & \Omega_2 \end{bmatrix} < 0, \quad (23)$$

where

$$l = m, M,$$

$$\Upsilon_1(d_1(i), \Delta d_1(i)) = \Pi_1^T P_1 \Pi_1 + sym\Big\{\Pi_2^T P_1 \Pi_1\Big\} + (d_1(i) + \Delta d_1(i))\Pi_3^T P_2 \Pi_3 - d_1(i)\Pi_4^T P_2 \Pi_4,$$

$$\Upsilon_2 = e_1^T(Q_1 + \overline{\mu}_{s_1} S_1)e_1 - \frac{1}{\overline{\mu}_{s_1}} e_5^T S_1 e_5 + e_7^T(Q_2 - Q_1)e_7 - e_9^T Q_2 e_9 + (e_0 - e_1)^T(d_{1m}^2 R_1 + \overline{d}_1^2 R_2)(e_0 - e_1),$$

$$\Upsilon_3 = d_{1m} sym\{\gamma_1 M_1 \chi_1 + \gamma_1 M_2 \chi_2 + \gamma_1 M_3 \chi_3\} + \overline{d}_1 sym\{\gamma_2 N_1 \chi_4 + \gamma_2 N_2 \chi_5 + \gamma_2 N_3 \chi_6\} + \overline{d}_1 sym\{\gamma_3 W_1 \chi_7 + \gamma_3 W_2 \chi_8 + \gamma_3 W_3 \chi_9\},$$

$$\Upsilon_4(d_2(j), \Delta d_2(j)) = \Pi_5^T P_3 \Pi_5 + sym\Big\{\Pi_6^T P_3 \Pi_5\Big\} + (d_2(j) + \Delta d_2(j))\Pi_7^T P_4 \Pi_7 - d_2(j)\Pi_8^T P_4 \Pi_8,$$

$$\Upsilon_5 = e_2^T(Q_2 + \overline{\mu}_{s_2} S_2)e_2 - \frac{1}{\overline{\mu}_{s_2}} e_6^T S_2 e_6 + e_8^T(Q_4 - Q_3)e_8 - e_{10}^T Q_4 e_{10} + (e_0 - e_2)^T(d_{2m}^2 R_3 + \overline{d}_2^2 R_4)(e_0 - e_2),$$

$$\Upsilon_6 = d_{2m} sym\{\beta_1 E_1 \alpha_1 + \beta_1 E_2 \alpha_2 + \beta_1 E_3 \alpha_3\} + \overline{d}_2 sym\{\beta_2 F_1 \alpha_4 + \beta_2 F_2 \alpha_5 + \beta_2 F_3 \alpha_6\} + \overline{d}_2 sym\{\beta_3 Z_1 \alpha_7 + \beta_3 Z_2 \alpha_8 + \beta_3 Z_3 \alpha_9\},$$

$$\Pi_1 = \begin{bmatrix} e_0 - e_1 \\ e_{11} \\ e_1 - e_7 \\ e_7 - e_9 \\ (d_{1m}+1)(e_1 - e_{13}) \end{bmatrix}, \quad \Pi_3 = \begin{bmatrix} e_0 \\ e_3 + e_{11} \\ (d_{1m}+1)e_{13} - e_7 \end{bmatrix},$$

$$\Pi_5 = \begin{bmatrix} e_0 - e_2 \\ e_{12} \\ e_2 - e_8 \\ e_8 - e_{10} \\ (d_{2m}+1)(e_2 - e_{14}) \end{bmatrix}, \quad \Pi_4 = \begin{bmatrix} e_1 \\ e_3 \\ (d_{1m}+1)e_{13} - e_1 \end{bmatrix},$$

$$\Pi_7 = \begin{bmatrix} e_0 \\ e_4 + e_{12} \\ (d_{2m}+1)e_{14} - e_8 \end{bmatrix}, \quad \Pi_8 = \begin{bmatrix} e_2 \\ e_4 \\ (d_{2m}+1)e_{14} - e_2 \end{bmatrix},$$

$$\Pi_2 = \begin{bmatrix} e_0 \\ e_3 + e_{11} \\ (d_{1m}+1)e_{13} - e_7 \\ \overline{d}_{1m}(i)e_{15} + \overline{d}_{1M}(i)e_{17} - e_3 - e_7 \\ (d_{1m}+1)(d_{1m}+2)e_{19} - (d_{1m}+1)e_1 \end{bmatrix},$$

$$\Pi_6 = \begin{bmatrix} e_0 \\ e_4 + e_{12} \\ (d_{2m}+1)e_{14} - e_8 \\ \overline{d}_{2m}(j)e_{16} + \overline{d}_{2M}(j)e_{18} - e_4 - e_8 \\ (d_{2m}+1)(d_{2m}+2)e_{20} - (d_{2m}+1)e_2 \end{bmatrix},$$

$$\Theta_1 = \begin{bmatrix} d_{1m}\gamma_1 M_1 & d_{1m}\gamma_1 M_2 & d_{1m}\gamma_1 M_3 & \overline{d}_1\gamma_3 W_1 & \overline{d}_1\gamma_3 W_2 & \overline{d}_1\gamma_3 W_3 \end{bmatrix},$$

$$\Theta_2 = \begin{bmatrix} d_{1m}\gamma_1 M_1 & d_{1m}\gamma_1 M_2 & d_{1m}\gamma_1 M_3 & \overline{d}_1\gamma_2 N_1 & \overline{d}_1\gamma_2 N_2 & \overline{d}_1\gamma_2 N_3 \end{bmatrix},$$

$$\Omega_1 = diag\{R_1, 12R_1, 720R_1, R_2, 12R_2, 720R_2\},$$

$$\Theta_3 = \begin{bmatrix} d_{2m}\beta_1 E_1 & d_{2m}\beta_1 E_2 & d_{2m}\beta_1 E_3 & \overline{d}_2\beta_3 Z_1 & \overline{d}_2\beta_3 Z_2 & \overline{d}_2\beta_3 Z_3 \end{bmatrix},$$

$$\Theta_4 = \begin{bmatrix} d_{2m}\beta_1 E_1 & d_{2m}\beta_1 E_2 & d_{2m}\beta_1 E_3 & \overline{d}_2\beta_3 F_1 & \overline{d}_2\beta_3 F_2 & \overline{d}_2\beta_3 F_3 \end{bmatrix},$$

$$\Omega_2 = diag\{R_3, 12R_3, 720R_3, R_4, 12R_4, 720R_4\},$$

$$\begin{aligned}
\gamma_1 &= \begin{bmatrix} e_1^T & e_7^T & e_{13}^T & e_{19}^T \end{bmatrix},\\
\gamma_2 = \gamma_3 &= \begin{bmatrix} e_7^T & e_3^T & e_9^T & e_{15}^T & e_{17}^T & e_{21}^T & e_{23}^T \end{bmatrix},\\
\beta_1 &= \begin{bmatrix} e_2^T\ e_8^T & e_{14}^T\ e_{20}^T \end{bmatrix},\\
\beta_2 = \beta_3 &= \begin{bmatrix} e_8^T & e_4^T & e_{10}^T & e_{16}^T & e_{18}^T & e_{22}^T & e_{24}^T \end{bmatrix},\\
\overline{d}_1 &= d_{1M} - d_{1m}, \quad \overline{d}_2 = d_{2M} - d_{2m},
\end{aligned}$$

$$\overline{d}_{1m}(i) = d_1(i) - d_{1m} + 1, \quad \overline{d}_{1M}(i) = d_{1M} - d_1(i) + 1,$$

$$\overline{d}_{2m}(j) = d_2(j) - d_{2m} + 1, \quad \overline{d}_{2M}(j) = d_{2M} - d_2(j) + 1.$$

Proof: Choose a Lyapunov function candidate for system (1) as

$$V = \overline{V} + \widehat{V} = \sum_{k=1}^{7} \overline{V}_k + \sum_{k=1}^{7} \widehat{V}_k, \tag{24}$$

with

$$\begin{aligned}
\overline{V}_1 &= \xi_1^T P_1 \xi_1,\\
\overline{V}_2 &= d_1(i) \xi_2^T P_2 \xi_2,\\
\overline{V}_3 &= \sum_{s=-d_{1m}}^{-1} x^T(i+s,j) Q_1 x(i+s,j),\\
\overline{V}_4 &= \sum_{s=-d_{1M}}^{-d_{1m}-1} x^T(i+s,j) Q_2 x(i+s,j),\\
\overline{V}_5 &= \sum_{s_1}^{+\infty} \mu_{s_1} \sum_{s=-s_1}^{-1} x^T(i+s,j) S_1 x(i+s,j),\\
\overline{V}_6 &= d_{1m} \sum_{l=-d_{1m}}^{-1} \sum_{s=l}^{-1} \Delta x^T(i+s,j) R_1 \Delta x(i+s,j),\\
\overline{V}_7 &= \overline{d}_1 \sum_{l=-d_{1M}}^{-d_{1m}-1} \sum_{s=l}^{-1} \Delta x^T(i+s,j) R_2 \Delta x(i+s,j),\\
\widehat{V}_1 &= \xi_3^T P_3 \xi_3,\\
\widehat{V}_2 &= d_2(j) \xi_4^T P_2 \xi_4,\\
\widehat{V}_3 &= \sum_{s=-d_{2m}}^{-1} x^T(i,j+s) Q_3 x(i,j+s),\\
\widehat{V}_4 &= \sum_{s=-d_{2M}}^{-d_{2m}-1} x^T(i,j+s) Q_4 x(i,j+s),\\
\widehat{V}_5 &= \sum_{s_2}^{+\infty} \mu_{s_2} \sum_{s=-s_2}^{-1} x^T(i,j+s) S_2 x(i,j+s),\\
\widehat{V}_6 &= d_{2m} \sum_{l=-d_{2m}}^{-1} \sum_{s=l}^{-1} \Delta x^T(i,j+s) R_3 \Delta x(i,j+s),\\
\widehat{V}_7 &= \overline{d}_2 \sum_{l=-d_{2M}}^{-d_{2m}-1} \sum_{s=l}^{-1} \Delta x^T(i,j+s) R_4 \Delta x(i,j+s),
\end{aligned}$$

where

$$\begin{aligned}
\xi_1 &= \Bigg[x^T(i,j) \quad x^T(i-d_1(i),j) \quad \sum_{s=-d_{1m}}^{-1} x^T(i+s,j)\\
&\quad \times \sum_{s=-d_{1M}}^{-d_{1m}-1} x^T(i+s,j) \ \sum_{l=-d_{1m}}^{-1} \sum_{s=l}^{-1} x^T(i+s,j) \Bigg]^T,\\
\xi_2 &= \Bigg[x^T(i,j) \quad x^T(i-d_1(i),j) \quad \sum_{s=-d_{1m}}^{-1} x^T(i+s,j) \Bigg]^T,\\
\xi_3 &= \Bigg[x^T(i,j) \quad x^T(i,j-d_2(j)) \quad \sum_{s=-d_{2m}}^{-1} x^T(i,j+s)\\
&\quad \times \sum_{s=-d_{2M}}^{-d_{2m}-1} x^T(i,j+s) \ \sum_{l=-d_{2m}}^{-1} \sum_{s=l}^{-1} x^T(i,j+s) \Bigg]^T,\\
\xi_4 &= \Bigg[x^T(i,j) \quad x^T(i,j-d_2(j)) \quad \sum_{s=-d_{2m}}^{-1} x^T(i,j+s) \Bigg]^T.
\end{aligned}$$

Denote

$$\begin{aligned}
x_{\xi,\eta} &= x(i+\xi, j+\eta),\\
\Delta x_{-d_1(i),1} &\\
&= x(i+1-d_1(i+1), j+1) - x(i-d_1(i), j+1),\\
\Delta x_{1,-d_2(j)} &\\
&= x(i+1, j+1-d_2(j+1)) - x(i+1, j-d_2(j)),\\
\Delta V &= \Delta V(i+1, j+1)\\
&= \Delta \overline{V}(i+1, j+1) + \Delta \widehat{V}(i+1, j+1),\\
\Delta \overline{V} &= \Delta \overline{V}(i+1, j+1)\\
&= V(i+1, j+1) - V(i, j+1),\\
\Delta \widehat{V} &= \Delta \widehat{V}(i+1, j+1)\\
&= V(i+1, j+1) - V(i+1, j).
\end{aligned}$$

Then, the difference of the LKF is given as follows:

$$\Delta V = \sum_{k=1}^{7} (\Delta \overline{V}_k + \Delta \widehat{V}_k),$$

with

$$\begin{aligned}
\Delta \overline{V}_1 &= \Delta \xi_1^T P_1 \Delta \xi_1 + sym\left\{ \xi_1^T P_1 \Delta \xi_1 \right\}\\
&= \zeta^T \left(\Pi_1^T P_1 \Pi_1 + sym\left\{ \Pi_2^T P_1 \Pi_1 \right\} \right) \zeta,\\
\Delta \overline{V}_2 &= \zeta^T \left((d_1(i) + \Delta d_1(i)) \Pi_3^T P_2 \Pi_3 - d_1(i) \Pi_4^T P_2 \Pi_4 \right) \zeta,\\
\Delta \overline{V}_3 &= x_{0,1}^T Q_1 x_{0,1} - x_{-d_{1m},1}^T Q_1 x_{-d_{1m},1},\\
\Delta \overline{V}_4 &= x_{-d_{1m},1}^T Q_2 x_{-d_{1m},1} - x_{-d_{1M},1}^T Q_2 x_{-d_{1M},1},\\
\Delta \overline{V}_5 &= \overline{\mu}_{s_1} x_{0,1}^T S_1 x_{0,1}\\
&\quad - \frac{1}{\overline{\mu}_{s_1}} \left(\sum_{s_1=1}^{+\infty} \mu_{s_1} x_{-s_1,1} \right)^T S_1 \left(\sum_{s_1=1}^{+\infty} \mu_{s_1} x_{-s_1,1} \right),\\
\Delta \overline{V}_6 &= d_{1m}^2 \Delta x_{0,1}^T R_1 \Delta x_{0,1} - d_{1m} \sum_{s=-d_{1m}}^{-1} \Delta x_{s,1}^T R_1 \Delta x_{s,1},
\end{aligned}$$

$$\Delta\overline{V}_7 = \overline{d}_1^2 \Delta x_{0,1}^T R_2 \Delta x_{0,1} - \overline{d}_1 \sum_{s=-d_{1M}}^{-d_{1m}-1} \Delta x_{s,1}^T R_2 \Delta x_{s,1},$$

$$\begin{aligned}
\Delta\widehat{V}_1 &= \Delta\xi_3^T P_3 \Delta\xi_3 + sym\left\{\xi_3^T P_3 \Delta\xi_3\right\} \\
&= \zeta^T\left(\Pi_5^T P_3 \Pi_5 + sym\left\{\Pi_6^T P_3 \Pi_5\right\}\right)\zeta, \\
\Delta\widehat{V}_2 &= \zeta^T\left((d_2(j) + \Delta d_2(j))\Pi_7^T P_4 \Pi_7 - d_2(j)\Pi_8^T P_4 \Pi_8\right)\zeta, \\
\Delta\widehat{V}_3 &= x_{1,0}^T Q_3 x_{1,0} - x_{1,-d_{2m}}^T Q_3 x_{1,-d_{2m}}, \\
\Delta\widehat{V}_4 &= x_{1,-d_{2m}}^T Q_4 x_{1,-d_{2m}} - x_{1,-d_{2M}}^T Q_4 x_{1,-d_{2M}}, \\
\Delta\widehat{V}_5 &= \overline{\mu}_{s_2} x_{1,0}^T S_2 x_{1,0} \\
&\quad - \frac{1}{\overline{\mu}_{s_2}}\left(\sum_{s_2=1}^{+\infty}\mu_{s_2} x_{1,-s_2}\right)^T S_2 \left(\sum_{s_2=1}^{+\infty}\mu_{s_2} x_{1,-s_2}\right), \\
\Delta\widehat{V}_6 &= d_{2m}^2 \Delta x_{1,0}^T R_3 \Delta x_{1,0} - d_{2m}\sum_{s=-d_{2m}}^{-1} \Delta x_{1,s}^T R_3 \Delta x_{1,s}, \\
\Delta\widehat{V}_7 &= \overline{d}_2^2 \Delta x_{1,0}^T R_4 \Delta x_{1,0} - \overline{d}_2 \sum_{s=-d_{2M}}^{-d_{2m}-1} \Delta x_{1,s}^T R_4 \Delta x_{1,s},
\end{aligned}$$

where

$$\begin{aligned}
\overline{v}_1 &= \sum_{s=-d_{1m}}^{0} \frac{x_{s,1}}{d_{1m}+1}, \quad \widehat{v}_1 = \sum_{s=-d_{2m}}^{0} \frac{x_{1,s}}{d_{2m}+1}, \\
\overline{v}_2 &= \sum_{s=-d_1(i)}^{-d_{1m}} \frac{x_{s,1}}{\overline{d}_{1m}(i)}, \quad \widehat{v}_2 = \sum_{s=-d_2(j)}^{-d_{2m}} \frac{x_{1,s}}{\overline{d}_{2m}(j)}, \\
\overline{v}_3 &= \sum_{s=-d_{1M}}^{-d_1(i)} \frac{x_{s,1}}{\overline{d}_{1M}(i)}, \quad \widehat{v}_3 = \sum_{s=-d_{2M}}^{-d_2(i)} \frac{x_{1,s}}{\overline{d}_{2M}(j)}, \\
\overline{v}_4 &= \frac{1}{(d_{1m}+1)(d_{1m}+2)} \sum_{l=-d_{1m}}^{0}\sum_{s=l}^{0} x_{s,1}, \\
\widehat{v}_4 &= \frac{1}{(d_{2m}+1)(d_{2m}+2)} \sum_{l=-d_{2m}}^{0}\sum_{s=l}^{0} x_{1,s}, \\
\overline{v}_5 &= \frac{1}{\overline{d}_{1m}(i)(\overline{d}_{1m}(i)+1)} \sum_{l=-d_1(i)}^{-d_{1m}}\sum_{s=l}^{-d_{1m}} x_{s,1}, \\
\widehat{v}_5 &= \frac{1}{\overline{d}_{2m}(j)(\overline{d}_{2m}(j)+1)} \sum_{l=-d_2(j)}^{-d_{2m}}\sum_{s=l}^{-d_{2m}} x_{1,s}, \\
\overline{v}_6 &= \frac{1}{\overline{d}_{1M}(i)(\overline{d}_{1M}(i)+1)} \sum_{l=-d_{1M}}^{-d_1(i)}\sum_{s=l}^{-d_1(i)} x_{s,1}, \\
\widehat{v}_6 &= \frac{1}{\overline{d}_{2M}(j)(\overline{d}_{2M}(j)+1)} \sum_{l=-d_{2M}}^{-d_2(j)}\sum_{s=l}^{-d_2(j)} x_{1,s}, \\
e_i &= \begin{bmatrix} 0_{n\times(i-1)n} & I_{n\times n} & 0_{n\times(24-i)n} \end{bmatrix}, \\
e_0 &= \begin{bmatrix} A_1 & A_2 & A_{1d} & A_{2d} & A_3 & A_4 & 0_{n\times 18n} \end{bmatrix}, \\
\zeta &= col\Big\{x_{0,1}, x_{1,0}, x_{-d_1(i),1}, x_{1,-d_2(j)}, \sum_{s_1=1}^{+\infty}\mu_{s_1} x_{-s_1,1}, \\
&\quad \sum_{s_2=1}^{+\infty}\mu_{s_2} x_{1,-s_2}, x_{-d_{1m},1}, x_{1,-d_{2m}} x_{-d_{1M},1}, x_{1,-d_{2M},1}, \\
&\quad \Delta x_{-d_1(i),1}, \Delta x_{1,-d_2(j)}, \overline{v}_1, \widehat{v}_1, \overline{v}_2, \widehat{v}_2, \overline{v}_3, \widehat{v}_3, \\
&\quad \overline{v}_4, \widehat{v}_4, \overline{v}_5, \widehat{v}_5, \overline{v}_6, \widehat{v}_6\Big\}.
\end{aligned}$$

Then, it can be shown that:

$$\begin{aligned}
\Delta\overline{V} &= \zeta^T(\Upsilon_1(d_1(i), \Delta d_1(i)) + \Upsilon_2)\zeta \\
&\quad - d_{1m}\sum_{s=-d_{1m}}^{-1} \Delta x_{s,1}^T R_1 \Delta x_{s,1} \\
&\quad - \overline{d}_1 \sum_{s=-d_{1M}}^{-d_{1m}-1} \Delta x_{s,1}^T R_2 \Delta x_{s,1}, \qquad (25) \\
\Delta\widehat{V} &= \zeta^T(\Upsilon_4(d_2(j), \Delta d_2(j)) + \Upsilon_5)\zeta \\
&\quad - d_{2m}\sum_{s=-d_{2m}}^{-1} \Delta x_{1,s}^T R_3 \Delta x_{1,s} \\
&\quad - \overline{d}_2 \sum_{s=-d_{2M}}^{-d_{2m}-1} \Delta x_{1,s}^T R_4 \Delta x_{1,s}. \qquad (26)
\end{aligned}$$

For the summation terms in (25), applying the summation inequality (18) in Corollary 1, and the adaptive vector is selected as follows:

$$\eta_1(i,j) = \begin{cases} \gamma_1^T \zeta(s,j), & \forall s \in [i-d_{1m}, i-1], \\ \gamma_2^T \zeta(s,j), & \forall s \in [i-d_1(i), i-d_{1m}-1], \\ \gamma_3^T \zeta(s,j), & \forall s \in [i-d_{1M}, i-d_1(i)-1]. \end{cases}$$

It is shown as:

$$\begin{aligned}
&-d_{1m}\sum_{s=-d_{1m}}^{-1} \Delta x_{s,1}^T R_1 \Delta x_{s,1} \\
&\le \zeta^T\Big(d_{1m}^2\gamma_1\Big(M_1 R_1^{-1} M_1^T + \frac{1}{12} M_2 R_1^{-1} M_2^T \\
&\quad + \frac{1}{720} M_3 R_1^{-1} M_3^T\Big)\gamma_1^T + d_{1m} sym\{\gamma_1 M_1 \chi_1 \\
&\quad + \gamma_1 M_2 \chi_2 + \gamma_1 M_3 \chi_3\}\Big)\zeta, \\
&\quad -\overline{d}_1 \sum_{s=-dM}^{-d_{1m}-1} \Delta x_{s,1}^T R_2 \Delta x_{s,1} \\
&= -\overline{d}_1 \sum_{s=-d_1(i)}^{-d_{1m}-1} \Delta x_{s,1}^T R_2 \Delta x_{s,1} - \overline{d}_1 \sum_{s=-d_{1M}}^{-d_1(i)-1} \Delta x_{s,1}^T R_2 \Delta x_{s,1}, \\
&\quad -\overline{d}_1 \sum_{s=-d_1(i)}^{-d_{1m}-1} \Delta x_{s,1}^T R_2 \Delta x_{s,1} \\
&\le \zeta^T\Big(\overline{d}_1(d_1(i) - d_{1m})\gamma_2\Big(N_1 R_2^{-1} N_1^T + \frac{1}{12} N_2 R_2^{-1} N_2^T \\
&\quad + \frac{1}{720} N_3 R_2^{-1} N_3^T\Big)\gamma_2^T + \overline{d}_1 sym\{\gamma_2 N_1 \chi_4 \\
&\quad + \gamma_2 N_2 \chi_5 + \gamma_2 N_3 \chi_6\}\Big)\zeta,
\end{aligned}$$

$$
\begin{aligned}
&-\bar{d}_1 \sum_{s=-d_{1M}}^{-d_1(i)-1} \Delta x_{s,1}^T R_2 \Delta x_{s,1} \\
&\le \zeta^T \Big(\bar{d}_1(d_{1M}-d_1(i))\gamma_3 \Big(W_1 R_2^{-1} W_1^T + \frac{1}{12} W_2 R_2^{-1} W_2^T \\
&\quad + \frac{1}{720} W_3 R_2^{-1} W_3^T\Big)\gamma_3^T + \bar{d}_1 sym\{\gamma_3 W_1 \chi_7 \\
&\quad + \gamma_3 W_2 \chi_8 + \gamma_3 W_3 \chi_9\}\Big)\zeta,
\end{aligned}
$$

where

$$
\begin{aligned}
&\chi_1 = e_1 - e_7, \quad \chi_4 = e_7 - e_3, \ \chi_7 = e_3 - e_9, \\
&\chi_2 = \frac{1}{2}e_1 + \frac{1}{2}e_7 - e_{13}, \quad \chi_3 = \frac{1}{12}e_1 - \frac{1}{12}e_7 + \frac{1}{2}e_{13} - e_{19}, \\
&\chi_5 = \frac{1}{2}e_7 + \frac{1}{2}e_3 - e_{15}, \quad \chi_6 = \frac{1}{12}e_7 - \frac{1}{12}e_3 + \frac{1}{2}e_{15} - e_{21}, \\
&\chi_8 = \frac{1}{2}e_3 + \frac{1}{2}e_9 - e_{17}, \quad \chi_9 = \frac{1}{12}e_3 - \frac{1}{12}e_9 + \frac{1}{2}e_{17} - e_{23}.
\end{aligned}
$$

When $d_1(i) = d_{1m}$,

$$
\begin{aligned}
&\Delta \bar{V}_{d_1(i)=d_{1m}} \\
&= \sum_{k=1}^{7} \bar{V}_k \\
&\le \zeta^T \Upsilon_1(d_1(i) = d_{1m}, \Delta d_1(i) = \lambda_{1l})\zeta + \zeta^T \Upsilon_2 \zeta \\
&\quad + d_{1m}\zeta^T sym\{\gamma_1 M_1 \chi_1 + \gamma_1 M_2 \chi_2 + \gamma_1 M_3 \chi_3\}\zeta \\
&\quad + \bar{d}_1 \zeta^T sym\{\gamma_2 N_1 \chi_4 + \gamma_2 N_2 \chi_5 + \gamma_2 N_3 \chi_6\}\zeta \\
&\quad + \bar{d}_1 \zeta^T sym\{\gamma_3 W_1 \chi_7 + \gamma_3 W_2 \chi_8 + \gamma_3 W_3 \chi_9\}\zeta \\
&\quad + \zeta^T d_{1m}\gamma_1 \Big(M_1 R_1^{-1} M_1^T + \frac{1}{12} M_2 R_1^{-1} M_2^T \\
&\quad + \frac{1}{720} M_3 R_1^{-1} M_3^T\Big) d_{1m}\gamma_1^T + \zeta^T \bar{d}_1 \gamma_3 \Big(W_1 R_2^{-1} W_1^T \\
&\quad + \frac{1}{12} W_2 R_2^{-1} W_2^T + \frac{1}{720} W_3 R_2^{-1} W_3^T\Big)\bar{d}_1 \gamma_3^T. \qquad (27)
\end{aligned}
$$

When $d_1(i) = d_{1M}$,

$$
\begin{aligned}
&\Delta \bar{V}_{d_1(i)=d_{1M}} \\
&= \sum_{k=1}^{7} \bar{V}_k \\
&\le \zeta^T \Upsilon_1(d_1(i) = d_{1M}, \Delta d_1(i) = \lambda_{1l})\zeta + \zeta^T \Upsilon_2 \zeta \\
&\quad + d_{1m}\zeta^T sym\{\gamma_1 M_1 \chi_1 + \gamma_1 M_2 \chi_2 + \gamma_1 M_3 \chi_3\}\zeta \\
&\quad + \bar{d}_1 \zeta^T sym\{\gamma_2 N_1 \chi_4 + \gamma_2 N_2 \chi_5 + \gamma_2 N_3 \chi_6\}\zeta \\
&\quad + \bar{d}_1 \zeta^T sym\{\gamma_3 W_1 \chi_7 + \gamma_3 W_2 \chi_8 + \gamma_3 W_3 \chi_9\}\zeta \\
&\quad + \zeta^T d_{1m}\gamma_1 \Big(M_1 R_1^{-1} M_1^T + \frac{1}{12} M_2 R_1^{-1} M_2^T \\
&\quad + \frac{1}{720} M_3 R_1^{-1} M_3^T\Big) d_{1m}\gamma_1^T + \zeta^T \bar{d}_1 \gamma_2 \Big(N_1 R_2^{-1} N_1^T \\
&\quad + \frac{1}{12} N_2 R_2^{-1} N_2^T + \frac{1}{720} N_3 R_2^{-1} N_3^T\Big)\bar{d}_1 \gamma_2^T. \qquad (28)
\end{aligned}
$$

For the summation terms in (26), applying the summation inequality (19) in Corollary 1, and the adaptive vector is selected as follows:

$$
\eta_2(i,j) = \begin{cases} \beta_1^T \zeta(i,s), & \forall s \in [j-d_{2m}, j-1], \\ \beta_2^T \zeta(i,s), & \forall s \in [j-d_2(j), j-d_{2m}-1], \\ \beta_3^T \zeta(i,s), & \forall s \in [j-d_{2M}, j-d_2(j)-1]. \end{cases}
$$

It is shown as:

$$
\begin{aligned}
&-d_{2m} \sum_{s=-d_{2m}}^{-1} \Delta x_{1,s}^T R_3 \Delta x_{1,s} \\
&\quad \le \zeta^T \Big(d_{2m}^2 \beta_1 \Big(E_1 R_3^{-1} E_1^T + \frac{1}{12} E_2 R_3^{-1} E_2^T \\
&\qquad + \frac{1}{720} E_3 R_3^{-1} E_3^T\Big)\beta_1^T + d_{2m} sym\{\beta_1 E_1 \alpha_1 \\
&\qquad + \beta_1 E_2 \alpha_2 + \beta_1 E_3 \alpha_3\}\Big)\zeta, \\
&-\bar{d}_2 \sum_{s=-d_{2M}}^{-d_{2m}-1} \Delta x_{1,s}^T R_4 \Delta x_{1,s} \\
&\quad = -\bar{d}_2 \sum_{s=-d_2(j)}^{-d_{2m}-1} \Delta x_{1,s}^T R_4 \Delta x_{1,s} \\
&\qquad - \bar{d}_2 \sum_{s=-d_{2M}}^{-d_2(j)-1} \Delta x_{1,s}^T R_4 \Delta x_{1,s}, \\
&-\bar{d}_2 \sum_{s=-d_2(j)}^{-d_{2m}-1} \Delta x_{1,s}^T R_4 \Delta x_{1,s} \\
&\quad \le \zeta^T \Big(\bar{d}_2(d_2(j) - d_{2m})\beta_2 \Big(F_1 R_4^{-1} F_1^T + \frac{1}{12} F_2 R_4^{-1} F_2^T \\
&\qquad + \frac{1}{720} F_3 R_4^{-1} F_3^T\Big)\beta_2^T + \bar{d}_2 sym\{\beta_2 F_1 \alpha_4 \\
&\qquad + \beta_2 F_2 \alpha_5 + \beta_2 F_3 \alpha_6\}\Big)\zeta, \\
&-\bar{d}_2 \sum_{s=-d_{2M}}^{-d_2(j)-1} \Delta x_{1,s}^T R_4 \Delta x_{1,s} \\
&\quad \le \zeta^T \Big(\bar{d}_2(d_{2M} - d_2(j))\beta_3 \Big(Z_1 R_4^{-1} Z_1^T + \frac{1}{12} Z_2 R_4^{-1} Z_2^T \\
&\qquad + \frac{1}{720} Z_3 R_4^{-1} Z_3^T\Big)\beta_3^T + \bar{d}_2 sym\{\beta_3 Z_1 \alpha_7 \\
&\qquad + \beta_3 Z_2 \alpha_8 + \beta_3 Z_3 \alpha_9\}\Big)\zeta,
\end{aligned}
$$

where

$$
\begin{aligned}
&\alpha_1 = e_2 - e_8, \quad \alpha_4 = e_8 - e_4, \ \alpha_7 = e_4 - e_{10}, \\
&\alpha_2 = \frac{1}{2}e_2 + \frac{1}{2}e_8 - e_{14}, \quad \alpha_3 = \frac{1}{12}e_2 - \frac{1}{12}e_8 + \frac{1}{2}e_{14} - e_{20}, \\
&\alpha_5 = \frac{1}{2}e_8 + \frac{1}{2}e_4 - e_{16}, \quad \alpha_6 = \frac{1}{12}e_8 - \frac{1}{12}e_4 + \frac{1}{2}e_{16} - e_{22}, \\
&\alpha_8 = \frac{1}{2}e_4 + \frac{1}{2}e_{10} - e_{18}, \quad \alpha_9 = \frac{1}{12}e_4 - \frac{1}{12}e_{10} + \frac{1}{2}e_{18} - e_{24}.
\end{aligned}
$$

When $d_2(j) = d_{2m}$,

$$\begin{aligned}
&\Delta \widehat{V}_{(d_2(j)=d_{2m})} \\
&= \sum_{k=1}^{7} \widehat{V}_k \\
&\le \zeta^T \Upsilon_4(d_2(j) = d_{2m}, \Delta d_2(j) = \lambda_{2s})\zeta + \zeta^T \Upsilon_5 \zeta \\
&\quad + d_{2m}\zeta^T sym\{\beta_1 E_1 \alpha_1 + \beta_1 E_2 \alpha_2 + \beta_1 E_3 \alpha_3\}\zeta \\
&\quad + \overline{d}_2 \zeta^T sym\{\beta_2 F_1 \alpha_4 + \beta_2 F_2 \alpha_5 + \beta_2 F_3 \alpha_6\}\zeta \\
&\quad + \overline{d}_2 \zeta^T sym\{\beta_3 Z_1 \alpha_7 + \beta_3 Z_2 \alpha_8 + \beta_3 Z_3 \alpha_9\}\zeta \\
&\quad + \zeta^T d_{2m}\beta_1\Big(E_1 R_3^{-1} E_1^T + \frac{1}{12} E_2 R_3^{-1} E_2^T \\
&\quad + \frac{1}{720} E_3 R_3^{-1} E_3^T\Big) d_{2m}\beta_1^T + \zeta^T \overline{d}_2 \beta_3 \Big(Z_1 R_4^{-1} Z_1^T \\
&\quad + \frac{1}{12} Z_2 R_4^{-1} Z_2^T + \frac{1}{720} Z_3 R_4^{-1} Z_3^T\Big)\overline{d}_2 \beta_3^T. \qquad (29)
\end{aligned}$$

When $d_2(j) = d_{2M}$,

$$\begin{aligned}
&\Delta \widehat{V}_{(d_2(j)=d_{2M})} \\
&= \sum_{k=1}^{7} \widehat{V}_k \\
&\le \zeta^T \Upsilon_4(d_2(j) = d_{2M}, \Delta d_2(j) = \lambda_{2s})\zeta + \zeta^T \Upsilon_5 \zeta \\
&\quad + d_{2m}\zeta^T sym\{\beta_1 E_1 \alpha_1 + \beta_1 E_2 \alpha_2 + \beta_1 E_3 \alpha_3\}\zeta \\
&\quad + \overline{d}_2 \zeta^T sym\{\beta_2 F_1 \alpha_4 + \beta_2 F_2 \alpha_5 + \beta_2 F_3 \alpha_6\}\zeta \\
&\quad + \overline{d}_2 \zeta^T sym\{\beta_3 Z_1 \alpha_7 + \beta_3 Z_2 \alpha_8 + \beta_3 Z_3 \alpha_9\}\zeta \\
&\quad + \zeta^T d_{2m}\beta_1\Big(E_1 R_3^{-1} E_1^T + \frac{1}{12} E_2 R_3^{-1} E_2^T \\
&\quad + \frac{1}{720} E_3 R_3^{-1} E_3^T\Big) d_{2m}\beta_1^T + \zeta^T \overline{d}_2 \beta_2 \Big(F_1 R_4^{-1} F_1^T \\
&\quad + \frac{1}{12} F_2 R_4^{-1} F_2^T + \frac{1}{720} F_3 R_4^{-1} F_3^T\Big)\overline{d}_2 \beta_2^T. \qquad (30)
\end{aligned}$$

According to Schur's complement, negativity conditions of inequalities (27)-(30) are equivalent to inequalities (20)-(23), which implies $\Delta V(i+1, j+1) = \Delta \overline{V}(i+1, j+1) + \Delta \widehat{V}(i+1, j+1) < 0$ for all nonzero ζ. The inequality means

$$\overline{V}(i+1, j+1) + \widehat{V}(i+1, j+1) < \overline{V}(i, j+1) + \widehat{V}(i+1, j). \qquad (31)$$

According to inequality (31) and the boundary conditions (7), for any integer $k > \max\{r_1, r_2\}$, it will be obtained that

$$\begin{aligned}
&\sum_{i+j=k+1} V(i,j) = \sum_{i+j=k+1} \big(\overline{V}(i,j) + \widehat{V}(i,j)\big) \\
&= \overline{V}(k,1) + \widehat{V}(k,1) + \overline{V}(k-1,2) + \widehat{V}(k-1,2) \\
&\quad + \cdots + \overline{V}(1,k) + \widehat{V}(1,k) \\
&< \overline{V}(k-1,1) + \widehat{V}(k,0) + \overline{V}(k-2,2) + \widehat{V}(k-1,1) \\
&\quad + \cdots + \overline{V}(0,k) + \widehat{V}(1,k-1) \\
&= \overline{V}(k-1,1) + \widehat{V}(k-1,1) + \overline{V}(k-2,2) + \widehat{V}(k-2,2) \\
&\quad + \cdots + \overline{V}(1,k-1) + \widehat{V}(1,k-1) + \overline{V}(k,0) + \widehat{V}(0,k) \\
&= \sum_{i+j=k} V(i,j). \qquad (32)
\end{aligned}$$

Denote a separation set $D_k = \{(i,j) : i + j = k\}$, $d = \max\{d_{1M}, d_{2M}\}$. Inequality (32) implies that the energy stored at all points in $D_{k+1} \bigcup \cdots \bigcup D_{k-d+1}$ is less than the energy stored at all points in $D_k \bigcup \cdots \bigcup D_{k-d}$ [13]. Thus, it's obtained that $\lim_{i+j\to\infty} V(i,j) = 0$, which implies $\lim_{i+j=\infty} \|x(i,j)\| = 0$. By Definition 1, the system (1) is asymptotically stable. □

Remark 4: The LKF proposed in this paper is quite different from the previous literature of 2-D systems. To activate the advantage of the 2-D polynomials-based summation inequalities, $\overline{V}_1$ and $\widehat{V}_1$ are proposed which contain more summation terms. Due to the delay-product type terms are introduced in $\overline{V}_2$ and $\widehat{V}_2$, the differences of $\overline{V}_2$ and $\widehat{V}_2$ contain more delay changing information, which make the stability criteria is delay-variation-dependent. The state vector $x(i-d_1(i), j)$ and $x(i, j-d_2(j))$ in the augmented LKF are developed according to similar terms for continuous-time systems [20]. Simulation examples will illustrate the effectiveness of the proposed LKF.

If distributed time delays are not considered, then system (1) reduces to the following model:

$$\begin{aligned}
x(i+1, j+1) &= A_1 x(i, j+1) + A_2 x(i+1, j) \\
&\quad + A_{1d} x(i - d_1(i), j+1) \\
&\quad + A_{2d} x(i+1, j - d_2(j)). \qquad (33)
\end{aligned}$$

Since there is no stability criterion about 2-D discrete-time systems with mixed time delays, in order to make an effective comparison, the following corollary is derived.

Corollary 2: For given scalars d_{km}, d_{kM}, λ_{km}, λ_{kM} $(k = 1, 2)$, $d_1(i)$, $d_2(j)$ satisfy the conditions (2)-(4), the 2-D system (33) is asymptotically stable if there exist real matrices $P_1, P_3 \in \mathbb{S}_{5n}^+$, $P_2, P_4 \in \mathbb{S}_{3n}^+$, Q_k, R_k, $(k = 1, 2, 3, 4)$, $M_i, E_i \in \mathbb{R}^{4n\times n}$, $N_i, W_i, F_i, E_i \in \mathbb{R}^{7n\times n}$, $(i = 1, 2, 3)$, such that the following LMIs hold:

$$\begin{bmatrix} \Gamma_1(d_1(i) = d_{1m}, \Delta d_1(i) = \lambda_{1l}) + \Gamma_2 + \Gamma_3 & \Phi_1 \\ * & \Lambda_1 \end{bmatrix} < 0, \qquad (34)$$

$$\begin{bmatrix} \Gamma_1(d_1(i) = d_{1M}, \Delta d_1(i) = \lambda_{1l}) + \Gamma_2 + \Gamma_3 & \Phi_1 \\ * & \Lambda_1 \end{bmatrix} < 0, \qquad (35)$$

$$\begin{bmatrix} \Gamma_4(d_2(j) = d_{2m}, \Delta d_2(j) = \lambda_{2l}) + \Gamma_5 + \Gamma_6 & \Phi_3 \\ * & \Lambda_2 \end{bmatrix} < 0, \qquad (36)$$

$$\begin{bmatrix} \Gamma_4(d_2(j) = d_{2M}, \Delta d_2(j) = \lambda_{2l}) + \Gamma_5 + \Gamma_6 & \Phi_4 \\ * & \Lambda_2 \end{bmatrix} < 0, \qquad (37)$$

where

$$\begin{aligned}
l &= m, M, \\
\Gamma_1(d_1(i)) &= \Xi_1^T P_1 \Xi_1 + sym\{\Xi_2^T P_1 \Xi_2\} + (d_1(i) \\
&\quad + \Delta d_1(i))\Xi_3^T P_2 \Xi_3 - d_1(i)\Xi_4^T P_2 \Xi_4, \\
\Gamma_2 &= \widetilde{e}_1^T Q_1 \widetilde{e}_1 + \widetilde{e}_5^T (Q_2 - Q_1)\widetilde{e}_5 - \widetilde{e}_7^T Q_2 \widetilde{e}_7 \\
&\quad + (\widetilde{e}_0 - \widetilde{e}_1)^T \big(d_{1m}^2 R_1 + \overline{d}_1^2 R_2\big)(\widetilde{e}_0 - \widetilde{e}_1),
\end{aligned}$$

$$\Gamma_3 = d_{1m} sym\{\widetilde{\gamma}_1 M_1 \widetilde{\chi}_1 + \widetilde{\gamma}_1 M_2 \widetilde{\chi}_2 + \widetilde{\gamma}_1 M_3 \widetilde{\chi}_3\} + \overline{d}_1 sym\{\widetilde{\gamma}_2 N_1 \widetilde{\chi}_4 + \widetilde{\gamma}_2 N_2 \widetilde{\chi}_5 + \widetilde{\gamma}_2 N_3 \widetilde{\chi}_6\} + \overline{d}_1 sym\{\widetilde{\gamma}_3 W_1 \widetilde{\chi}_7 + \widetilde{\gamma}_3 W_2 \widetilde{\chi}_8 + \widetilde{\gamma}_3 W_3 \widetilde{\chi}_9\},$$

$$\Gamma_4(d_2(j)) = \Xi_5^T P_3 \Xi_5 + sym\{\Xi_6^T P_3 \Xi_6\} + (d_2(j) + \Delta d_2(j))\Xi_7^T P_2 \Xi_7 - d_2(j)\Xi_8^T P_2 \Xi_8,$$

$$\Gamma_5 = \widetilde{e}_2^T Q_3 \widetilde{e}_2 + \widetilde{e}_6^T (Q_4 - Q_3)\widetilde{e}_6 - \widetilde{e}_8^T Q_4 \widetilde{e}_8 + (\widetilde{e}_0 - \widetilde{e}_2)^T (d_{1m}^2 R_3 + (d_{1M} - d_{1m})^2 R_4)(\widetilde{e}_0 - \widetilde{e}_2),$$

$$\Gamma_6 = d_{1m} sym\{\widetilde{\beta}_1 E_1 \widetilde{\alpha}_1 + \widetilde{\beta}_1 E_2 \widetilde{\alpha}_2 + \widetilde{\beta}_1 E_3 \widetilde{\alpha}_3\} + \overline{d}_1 sym\{\widetilde{\beta}_2 F_1 \widetilde{\alpha}_4 + \widetilde{\beta}_2 F_2 \widetilde{\alpha}_5 + \widetilde{\beta}_2 F_3 \widetilde{\alpha}_6\} + \overline{d}_1 sym\{\widetilde{\beta}_3 Z_1 \widetilde{\alpha}_7 + \widetilde{\beta}_3 Z_2 \widetilde{\alpha}_8 + \widetilde{\beta}_3 Z_3 \widetilde{\alpha}_9\},$$

$$\Xi_1 = \begin{bmatrix} \widetilde{e}_0 - \widetilde{e}_1 \\ \widetilde{e}_9 \\ \widetilde{e}_1 - \widetilde{e}_5 \\ \widetilde{e}_5 - \widetilde{e}_7 \\ (d_{1m}+1)(\widetilde{e}_1 - \widetilde{e}_{12}) \end{bmatrix},$$

$$\Xi_3 = \begin{bmatrix} e_0 \\ e_3 + e_9 \\ (d_{1m}+1)e_9 - e_5 \end{bmatrix},$$

$$\Xi_5 = \begin{bmatrix} \widetilde{e}_0 - \widetilde{e}_2 \\ \widetilde{e}_{10} \\ \widetilde{e}_2 - \widetilde{e}_6 \\ \widetilde{e}_6 - \widetilde{e}_8 \\ (d_{2m}+1)(\widetilde{e}_2 - \widetilde{e}_{12}) \end{bmatrix},$$

$$\Xi_7 = \begin{bmatrix} e_0 \\ e_4 + e_{10} \\ (d_{2m}+1)e_{12} - e_6 \end{bmatrix},$$

$$\Xi_4 = \begin{bmatrix} e_1 \\ e_3 \\ (d_{1m}+1)e_{11} - e_1 \end{bmatrix},$$

$$\Xi_8 = \begin{bmatrix} e_2 \\ e_4 \\ (d_{2m}+1)e_{12} - e_2 \end{bmatrix},$$

$$\Xi_2 = \begin{bmatrix} \widetilde{e}_0 \\ \widetilde{e}_3 + \widetilde{e}_9 \\ (d_{1m}+1)\widetilde{e}_{11} - \widetilde{e}_5 \\ \overline{d}_{1m}(i)\widetilde{e}_{13} + \overline{d}_{1M}(i)\widetilde{e}_{15} - \widetilde{e}_3 - \widetilde{e}_5 \\ (d_{1m}+1)(d_{1m}+2)\widetilde{e}_{17} - (d_{1m}+1)\widetilde{e}_1 \end{bmatrix},$$

$$\Xi_6 = \begin{bmatrix} \widetilde{e}_0 \\ \widetilde{e}_4 + \widetilde{e}_{10} \\ (d_{2m}+1)\widetilde{e}_{12} - \widetilde{e}_6 \\ \overline{d}_{2m}(j)\widetilde{e}_{14} + \overline{d}_{2M}(j)\widetilde{e}_{16} - \widetilde{e}_4 - \widetilde{e}_6 \\ (d_{2m}+1)(d_{2m}+2)\widetilde{e}_{18} - (d_{2m}+1)\widetilde{e}_2 \end{bmatrix},$$

$$\widetilde{\gamma}_1 = \begin{bmatrix} \widetilde{e}_1^T & \widetilde{e}_5^T & \widetilde{e}_{11}^T & \widetilde{e}_{17}^T, \end{bmatrix},$$

$$\widetilde{\gamma}_2 = \widetilde{\gamma}_3 = \begin{bmatrix} \widetilde{e}_5^T & \widetilde{e}_3^T & \widetilde{e}_7^T & \widetilde{e}_{13}^T & \widetilde{e}_{15}^T & \widetilde{e}_{19}^T & \widetilde{e}_{21}^T \end{bmatrix},$$

$$\widetilde{\beta}_1 = \begin{bmatrix} \widetilde{e}_2^T & \widetilde{e}_6^T & \widetilde{e}_{12}^T & \widetilde{e}_{18}^T \end{bmatrix},$$

$$\widetilde{\beta}_2 = \widetilde{\beta}_3 = \begin{bmatrix} \widetilde{e}_6^T & \widetilde{e}_4^T & \widetilde{e}_8^T & \widetilde{e}_{14}^T & \widetilde{e}_{16}^T & \widetilde{e}_{20}^T & \widetilde{e}_{22}^T \end{bmatrix},$$

$$\widetilde{\alpha}_1 = \widetilde{e}_2 - \widetilde{e}_6, \widetilde{\alpha}_2 = \frac{1}{2}\widetilde{e}_2 + \frac{1}{2}\widetilde{e}_6 - \widetilde{e}_{12},$$

$$\widetilde{\alpha}_3 = \frac{1}{12}\widetilde{e}_2 - \frac{1}{12}\widetilde{e}_6 + \frac{1}{2}\widetilde{e}_{12} - \widetilde{e}_{18},$$

$$\widetilde{\alpha}_4 = \widetilde{e}_6 - \widetilde{e}_4, \quad \widetilde{\alpha}_5 = \frac{1}{2}\widetilde{e}_6 + \frac{1}{2}\widetilde{e}_4 - \widetilde{e}_{14},$$

$$\widetilde{\alpha}_6 = \frac{1}{12}\widetilde{e}_6 - \frac{1}{12}\widetilde{e}_4 + \frac{1}{2}\widetilde{e}_{14} - \widetilde{e}_{20},$$

$$\widetilde{\alpha}_7 = \widetilde{e}_4 - \widetilde{e}_8, \widetilde{\alpha}_8 = \frac{1}{2}\widetilde{e}_4 + \frac{1}{2}\widetilde{e}_8 - \widetilde{e}_{16},$$

$$\widetilde{\alpha}_9 = \frac{1}{12}\widetilde{e}_4 - \frac{1}{12}\widetilde{e}_8 + \frac{1}{2}\widetilde{e}_6 - \widetilde{e}_{22},$$

$$\widetilde{\chi}_1 = \widetilde{e}_1 - \widetilde{e}_5, \widetilde{\chi}_2 = \frac{1}{2}\widetilde{e}_1 + \frac{1}{2}\widetilde{e}_5 - \widetilde{e}_{11},$$

$$\widetilde{\chi}_3 = \frac{1}{12}\widetilde{e}_1 - \frac{1}{12}\widetilde{e}_5 + \frac{1}{2}\widetilde{e}_{11} - \widetilde{e}_{17},$$

$$\widetilde{\chi}_4 = \widetilde{e}_5 - \widetilde{e}_3, \widetilde{\chi}_5 = \frac{1}{2}\widetilde{e}_5 + \frac{1}{2}\widetilde{e}_3 - \widetilde{e}_{13},$$

$$\widetilde{\chi}_6 = \frac{1}{12}\widetilde{e}_5 - \frac{1}{12}\widetilde{e}_3 + \frac{1}{2}\widetilde{e}_{13} - \widetilde{e}_{19},$$

$$\widetilde{\chi}_7 = \widetilde{e}_3 - \widetilde{e}_7, \widetilde{\chi}_8 = \frac{1}{2}\widetilde{e}_3 + \frac{1}{2}\widetilde{e}_7 - \widetilde{e}_{15},$$

$$\widetilde{\chi}_9 = \frac{1}{12}\widetilde{e}_3 - \frac{1}{12}\widetilde{e}_7 + \frac{1}{2}\widetilde{e}_5 - \widetilde{e}_{21},$$

$$\widetilde{e}_i = \begin{bmatrix} 0_{n\times(i-1)n} & I_{n\times n} & 0_{n\times(22-i)n} \end{bmatrix}, \quad i = 1, 2, \cdots, 22,$$

$$\widetilde{e}_0 = \begin{bmatrix} A_1 & A_2 & A_{1d} & A_{2d} & 0_{n\times 18n} \end{bmatrix}.$$

Proof: The proof method of Corollary 2 is similar to that of Theorem 1. It is omitted here. □

IV. NUMERICAL EXAMPLE

In this section, two examples are given to show the effectiveness of the proposed methods. Since the existing stability criteria are not about 2-D discrete-time systems with mixed time delays, Example 1 ignoring distributed delays is given to make an effective comparison. In order to prove the effectiveness of the proposed methods for more complex 2-D systems with mixed delays, Example 2 is given.

Example 1: A thermal processes in chemical reactors, heat exchangers and pipe furnaces can be expressed in a partial differential equation with time delays, which can be modeled in the 2-D FM model [11]. Consider the 2-D discrete-time system (33) with the following parameters:

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ 0.25 & 0.65 \end{bmatrix}, \quad A_{1d} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{2d} = \begin{bmatrix} 0 & 0 \\ 0 & -0.12 \end{bmatrix}. \tag{38}$$

In this example, in order to compare with the references, distributed delays are not taken into account. Take $d_1(i) = 6 + 5\sin(\frac{\pi i}{2})$, $d_2(j) = 18 + 17\sin(\frac{\pi j}{2})$, $-5 \le \Delta d_1(i) \le 5, -17 \le \Delta d_2(j) \le 17$. *The state dimension is* $n = 2$. *The simulation result is shown in Fig. 1 and Fig. 2 with the following boundary conditions.*

$$x(i,j) = \begin{cases} \begin{bmatrix} \frac{1}{5(i+1)} & \frac{1}{3(i+1)} \end{bmatrix}^T, & 0 \le i \le 20, j = 0, \\ 0, & i > 20, j = 0, \end{cases} \tag{39}$$

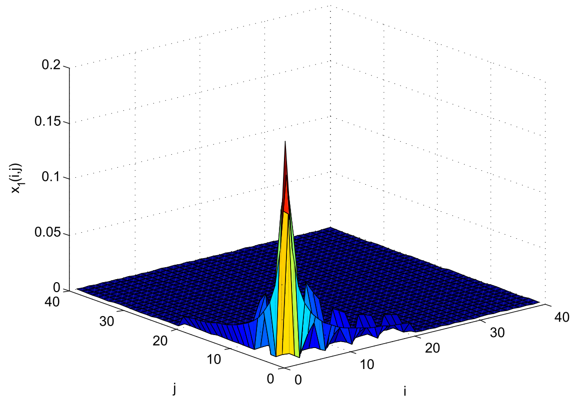

FIGURE 1. State $x_1(i,j)$ trajectory of the system (33).

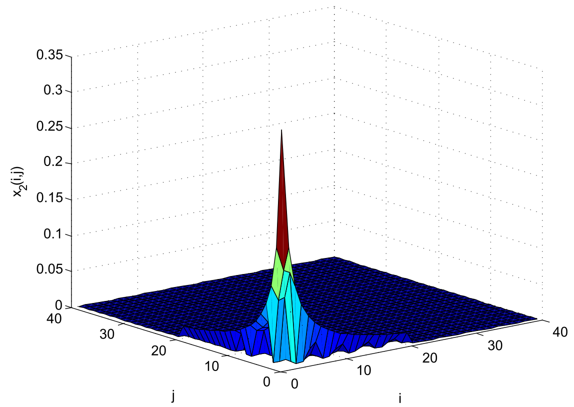

FIGURE 2. State $x_2(i,j)$ trajectory of the system (33).

$$x(i,j)=\begin{cases}\left[\frac{1}{5(j+1)}\ \ \frac{1}{3(j+1)}\right]^T, & 0\le j\le 20, i=0,\\ 0, & j>20, i=0.\end{cases} \quad (40)$$

It is seen clearly that state responses converge to origin, which means the system (33) with matrices (38) is asymptotically stable. Table 1 lists the maximum delay bounds of $d_2(j)$ obtained by the Corollary 2 and the results in the literature. Obviously, the result in this paper is better than previous methods which based on the 2-D Jensen inequalities and 2-D finite-sum inequalities.

TABLE 1. Allowable time-delay upper bounds d_{2M}.

	d_{1m}	d_{1M}	d_{2m}	d_{2M}
[19]	1	11	1	20
[29]	1	11	1	20
[33]	1	11	1	28
Corollary 2	1	11	1	36

Example 2: the 2-D system (1) with the following parameters is studied:

$$A_1=\begin{bmatrix}0.1 & 0\\ 1 & 0.2\end{bmatrix},\quad A_2=\begin{bmatrix}0.4 & 0\\ 0.2 & 0.1\end{bmatrix},$$

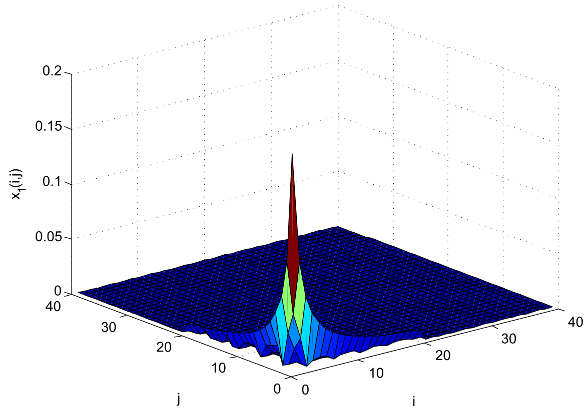

FIGURE 3. State $x_1(i,j)$ trajectory of the system (1).

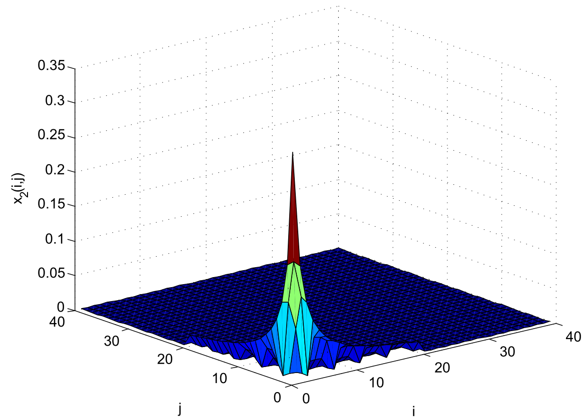

FIGURE 4. State $x_2(i,j)$ trajectory of the system (1).

$$A_{1d}=\begin{bmatrix}0.02 & 0\\ 0 & 0.01\end{bmatrix},\quad A_{2d}=\begin{bmatrix}0.01 & 0\\ 0 & 0.01\end{bmatrix},$$
$$A_3=\begin{bmatrix}0.1 & 0\\ 0 & 0.1\end{bmatrix},\quad A_4=\begin{bmatrix}0 & 0\\ 0 & -0.1\end{bmatrix}. \quad (41)$$

The time-varying delays satisfy $d_1(i)=11+10\sin(\frac{\pi i}{2})$, $d_2(j)=21+20\sin(\frac{\pi j}{2})$, $-10\le\Delta d_1(i)\le 10$, $-20\le\Delta d_2(j)\le 20$. $\mu_{s_1}=2^{-(s_1+1)}$, $\mu_{s_2}=2^{-(s_2+1)}$. It is easy to get that $d_{1m}=1$, $d_{2m}=2$, $d_{1M}=21$, $d_{2M}=41$, $\overline{\mu}_{s_1}=\overline{\mu}_{s_2}=1/2$.

Simulation results are shown in Fig. 3 and Fig. 4 with the boundary conditions (39) and (40). In the initial stage, the state curves have notable variations. This effect will gradually reduced when the system states asymptotically tend to zero. Thus, the stability of the given systems can be verified by the method proposed in this paper.

V. CONCLUSION

In this paper, the problem of stability analysis for the 2-D discrete-time systems with mixed delays has been studied. New 2-D polynomials-based summation inequalities have been proposed. It has been discussed that the inequalities can be transformed into 2-D Jensen inequalities and 2-D finite-sum inequalities by specially designing slack matrices and arbitrary vectors. The novel LKF which contains more crossing information has been constructed. Sufficient conditions on asymptotical stability in terms of linear matrix inequalities have been obtained. Finally, two examples have been presented to illustrate the availability of the proposed results.

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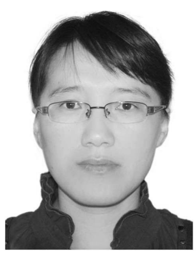

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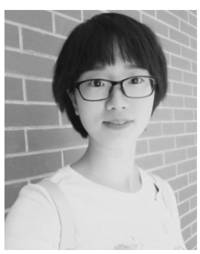

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