IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received June 27, 2019, accepted July 7, 2019, date of publication July 16, 2019, date of current version August 12, 2019.

Digital Object Identifier 10.1109/ACCESS.2019.2929269

Higher-Order State and Disturbance Observer With $O(T^3)$ Errors for Linear Systems

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This work was supported in part by the Natural Science Foundation of China under Grant 61573090, and in part by the Research Funds for the Central Universities under Grant N160801001.

ABSTRACT In this paper, a new observer, named as the higher-order observer, is proposed for simultaneously estimating the system state and disturbance. When the usual assumptions on the disturbance are satisfied, the magnitudes of the state and disturbance estimate errors of the system are proved to be on the order of $O(T^3)$, which is much less than $O(T^2)$ produced by the proportional-integral-observer (PIO). The simulation results also show the effectiveness and superiority of the proposed method.

INDEX TERMS Discrete-time, higher-order observer, state and disturbance estimate, linear system.

I. INTRODUCTION

Observers, also known as estimators or filters are an integral part of control theory and engineering. The function of them is estimating immeasurable variables by using measurable input and output data. Over the years, observers have been widely used to design controllers and control systems [1]–[3], which can be divided into two branches. One is state observers, the other is disturbance observers.

A state observer provides the estimates of the internal states of a system, which is implemented based on the system model. Since the 60s of last century, many design methods of state observers have been reported, among which Luenberger observers [4], Kalman observers [5], proportional-integral-observers (PIO) [6] and unknown-input-observers (UIO) [7] are most widely used and extended. For example, in [8], Luenberger observer-based H_∞-compensators were proposed for different nonstandard cases by using the bounded real lemma; in [9], an extended Luenberger observer was designed by using a transformation into the nonlinear observer canonical form and an extended linearization; in [10], a nonlinear Luenberger observer for an extended nonlinear system resulting from a parameterized linear single-input-single-output (SISO) system was studied; in [11], the generalized Luenberger observer was applied to the problems of failure detection and failure diagnosis; in [12], an optimizing Kalman optimal observer for state affine systems was proposed by input selection; in [13], a strong tracking extended Kalman observer was designed for nonlinear discrete-time systems; in [14], a new nonlinear state estimate approach combining classical Kalman filter theory with Takagi-Sugeno modeling was proposed; in [15], a new PIO was designed and applied to a robust controller; in [16], an adaptive PIO design method with σ-modification was proposed; in [17], a proportional multi-integral observer was proposed and used to reconstruct actuator and sensor faults; in [18]–[20], observers were designed for linear systems with unknown inputs.

A disturbance observer gives the estimates of the unknown inputs or uncertainties of a system. There are two kinds of disturbance observer design methods, i.e., the transfer-function-based method and the state-space-based method. Since a transfer-function-based disturbance observer is designed for a SISO system, it cannot be applied to an multi-input-multi-output (MIMO) system directly, whereas a state-space-based method can do. Therefore, the latter method has been extensively studied until now. For example, in [21] and [22], output-based disturbance observers with reduced order were presented respectively for a class of continuous-time linear systems and discrete-time linear systems; in [23], the reduced order disturbance observer was further studied by formulating it as a functional observer design problem; in [24], a PIO was designed for single-output uncertain linear systems which could attenuate either measurement noise or modeling errors. There also exist other methods for estimating disturbances or uncertainties, such as neural networks [25], fuzzy inferring systems [26], etc.

The associate editor coordinating the review of this manuscript and approving it for publication was Lin Zhang.

As a matter of fact, the above mentioned observers can only estimate either states or external disturbances. To overcome this limitation, the extended state observer (ESO), which regards the external disturbance as an extended state, and estimates the state and disturbance simultaneously, was proposed in [27]. The ESO is actually extracted from an active disturbance rejection control scheme [28], [29]. Until now, most of ESOs were designed for continuous-time systems [30], [31]. The existing discrete-time ESOs were designed either for a lower-order linear system, such as [32], or for a particular nonlinear system, such as [33]. For a general linear system, an ESO, also named as PIO, was presented in [34] to simultaneously estimate the system state and unknown disturbance.

This paper proposes a new observer, named as higher-order observer, which can also estimate the system state and disturbance simultaneously. The proposed method is actually the extension of the PIO in [34]. It is proved in [34], the magnitudes of the state and disturbance estimate errors of the system are on the order of $O(T^2)$, where T is the sampling interval. In this paper, the higher-order observer produces the order $O(T^3)$, which is much less than $O(T^2)$ as long as T is sufficiently small.

The remainder of this paper is organized as follows: The problem description is presented in Section II. Section III shows the higher-order observer design and the theoretical results. In Section IV, the simulation results are given to demonstrate the effectiveness of the proposed observer. Finally, some conclusions and future works are presented in Section V.

Notation: The symbol $R^{m\times n}$ denotes the set of m by n real-valued matrices, R^n denotes $R^{n\times 1}$, $|x|$ denote the Euclidean norm of a vector x and $\det\{A\}$ denote the determinant of a matrix A.

II. PROBLEM DESCRIPTION

Consider a continuous-time MIMO linear system with unknown state and disturbance described by

$$\dot{x}(t) = Hx(t) + Du(t) + Ff(t) \quad (1)$$

$$y(t) = Cx(t) \quad (2)$$

where $x \in R^n$ is the system state vector, $y \in R^l$ is the system output vector, $u \in R^m$ and $f \in R^p$ are the control input vector and disturbance input vector respectively; $H \in R^{n\times n}$, $D \in R^{n\times m}$, $F \in R^{n\times p}$ and $C \in R^{l\times n}$ are constant matrices. Same as [34], it is supposed that the sampling interval is T and a zero-order-holder is adopted for Eq. (1). Denoting $x(k) = x(kT)$, $y(k) = y(kT)$, $u(k) = u(kT)$ and $d(k) = d(kT)$, then the discrete-time model can be derived as

$$x(k+1) = Ax(k) + Bu(k) + d(k) \quad (3)$$

$$y(k) = Cx(k) \quad (4)$$

where $A = \exp(HT) \in R^{n\times n}$, $B = \int_0^T \exp(H\tau)Dd\tau \in R^{n\times m}$, and $d(k) = \int_0^T e^{H\tau}Ff((k+1)T-\tau)d\tau \in R^n$.

In order to proceed further, the definitions of Equivalence and Large order are introduced:

Definition 1[35] *(Equivalence):* If there exist positive constants $M_i (i = 1, 2)$ and k_0 such that

$$|\chi(k)| \le M_1 \max_{\tau\le k} |\phi(\tau)| + M_2, \forall k \ge k_0 \quad (5)$$

$$|\phi(k)| \le M_1 \max_{\tau\le k} |\chi(\tau)| + M_2, \forall k \ge k_0 \quad (6)$$

where $\chi(k)$ and $\phi(k)$ are scalar functions or vector valued functions of discrete time k, then we refer to $\chi(k)$ and $\phi(k)$ as being equivalent and denote it as $\chi(k) \sim \phi(k)$. It follows directly that this equivalence relation is reflexive, symmetric and transitive, so that the symbol $\sim$ represents an equivalence class.

Definition 2[36] *(Large Order):* The magnitude of a variable $\upsilon(k)$ is said to be $O(T^N)$, i.e., $\upsilon(k) = O(T^N)$, if and only if there exists a constant $C_0 > 0$ such that for any sufficiently small sampling interval $T \in (0, 1)$, the following inequality holds

$$|\upsilon(k)| \le C_0T^N$$

where $N \in Z$ with Z being an integer set. Note that $O(T^N)$ can be viewed as a scalar function or a vector valued function.

Remark 1: According to Definition 1, Eq. (5) represents $\chi(k)$ does not grow faster than $\phi(k)$, and Eq. (6) represents $\phi(k)$ does not grow faster than $\chi(k)$. Then $\chi(k) \sim \phi(k)$ means that $\chi(k)$ grows at the same rate as $\phi(k)$.

Remark 2: According to Definition 2, $\upsilon(k) = O(T^N)$ means that $\upsilon(k)$ is on the order of $O(T^N)$. In fact, if $\upsilon_1(k) = O(T^N)$ and $\upsilon_2(k) = O(T^{N+1})$, then there exist two positive constants C_1 and C_2 such that $|\upsilon_1(k)| \le C_1T^N$ and $|\upsilon_2(k)| \le C_2T^{N+1}$. In this case, as long as $T \ll \frac{C_1}{C_2}$, we have $C_2T^{N+1} \ll C_1T^N$ and consequently $\upsilon_2 \ll \upsilon_1$. Moreover, the following relations also hold

$$O(T^N) + O(T^{N+1}) = O(T^N)$$

$$O(T^N) + O(1) = O(T^N)$$

$$O(T^N) \cdot O(T^{-M}) = O(T^{N-M})$$

where N and $M \in Z$ with Z being the integer set.

To design the higher-order observer, same as [34] and [36], the following assumptions are made.

Assumption 1: The disturbance $f(t)$ is smooth and bounded.

Assumption 2: The sampling interval $T \in (0, 1)$ is sufficiently small such that the disturbance do not vary too much between two consecutive sampling instant.

Remark 3: From the above assumptions and the formulation of $d(k)$, it can be shown that the magnitude of $d(k)$ is on the order of $O(T)$, i.e., $d(k) = O(T)$ (See Appendix A).

Lemma 1: If the disturbance input $f(t)$ in Eq. (1) satisfies Assumption 1 and the sampling interval T satisfies Assumption 2, then the magnitude of $\Delta^2 d(k)$ is in the small region of $O(T^3)$, i.e.,

$$\Delta^2 d(k) = O(T^3)$$

where $\Delta = 1 - z^{-1}$ is a first order difference operator with z^{-1} being the backward shift operator, and then $\Delta^2 d(k) = d(k) - 2d(k-1) + d(k-2)$.

Proof: See Appendix A.

The purpose of this paper is to design a higher-order observer such that the estimate errors of states and disturbances can be constrained in the small region of $O(T^3)$.

III. HIGHER-ORDER OBSERVER DESIGN

The higher-order observer is designed as

$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + P_0[y(k) - \hat{y}(k)] + \hat{d}(k) \quad (7)$$

$$\hat{d}(k+1) = \hat{d}(k) + L_0[y(k) - \hat{y}(k)] + \Delta\hat{d}(k) + L_1[\Delta y(k) - \Delta\hat{y}(k)] \quad (8)$$

$$\hat{y}(k) = C\hat{x}(k) \quad (9)$$

where $\hat{x}(k)$, $\hat{y}(k)$ and $\hat{d}(k)$ are respectively the estimates of $x(k)$, $y(k)$ and $d(k)$ at time k; $\Delta y(k) = y(k) - y(k-1)$, $\Delta\hat{d}(k) = \hat{d}(k) - \hat{d}(k-1)$ and $\Delta\hat{y}(k) = \hat{y}(k) - \hat{y}(k-1)$ are respectively the differences of $y(k)$, $\hat{d}(k)$ and $\hat{y}(k)$ at time k; $P_0 \in R^{n\times l}$ and $L_j \in R^{n\times l}$, $j = 0, 1$ are constant matrices which will be designed in the sequel.

Remark 4: Since when $L_1 = 0$ and $\Delta\hat{d}(k) = 0$, Eqs. (7)-(9) degrades to the PIO proposed in [34], PIO can be viewed as an special case of the higher-order observer Eqs. (7)-(9).

Theorem 1: If Assumptions 1 and 2 are satisfied, and there exist constant matrices $P_0 \in R^{n\times l}$ and $L_j \in R^{n\times l}$, $j = 0, 1$, such that for any $|z| \geq 1$,

$$\det\{[1 - z^{-1}G(z^{-1})][I_n - (A - P_0C)z^{-1}] + z^{-2}Q(z^{-1})C\} \neq 0 \quad (10)$$

where

$$G(z^{-1}) = 1 + \Delta \quad (11)$$

$$Q(z^{-1}) = L_0 + L_1\Delta \quad (12)$$

then the state observe error and disturbance observe error satisfy

$$e(k) = O(T^3)$$
$$q(k) = O(T^3)$$

where $e(k) = x(k) - \hat{x}(k)$, $q(k) = d(k) - \hat{d}(k)$.

Proof: From Eq. (3) and Eq. (7), it yields that

$$e(k+1) = Ae(k) + q(k) - P_0Ce(k) \quad (13)$$

i.e.,

$$q(k-1) = [I_n - (A - P_0C)z^{-1}]e(k) \quad (14)$$

For any disturbance $d(k)$, it can be rewritten as

$$d(k) = d(k-1) + \Delta d(k-1) + \Delta^2 d(k) = (1+\Delta)d(k-1) + \Delta^2 d(k) \quad (15)$$

From Eqs. (8), (9), (11), (12) and (15), we obtain that

$$\begin{aligned} q(k) &= d(k) - \hat{d}(k) \\ &= (1+\Delta)d(k-1) + \Delta^2 d(k) - (1+\Delta)\hat{d}(k-1) \\ &\quad - (L_0 + L_1\Delta)[y(k-1) - \hat{y}(k-1)] \\ &= (1+\Delta)q(k-1) - (L_0 + L_1\Delta)Ce(k-1) + \Delta^2 d(k) \\ &= G(z^{-1})q(k-1) - Q(z^{-1})Ce(k-1) + \Delta^2 d(k) \end{aligned} \quad (16)$$

Substituting Eq. (14) into Eq. (16), it yields

$$\{[1 - z^{-1}G(z^{-1})][I_n - (A - P_0C)z^{-1}] + z^{-2}Q(z^{-1})C\}e(k) = \Delta^2 d(k-1) \quad (17)$$

Since P_0, L_0 and L_1 can be found such that Eq. (10) is satisfied, according to the **Key Technical Lemma** (See Appendix B), it can be concluded that there exist constants $0 < C_i < \infty (i = 3, 5)$ and $0 \leq C_j < \infty (j = 4, 6)$ satisfying

$$|e(k)| \leq C_3 \max_{0<\tau<k} |\Delta^2 d(\tau - 1)| + C_4, \quad \forall k > 0$$

$$|\Delta^2 d(k-1)| \leq C_5 \max_{0<\tau<k-1} |e(\tau)| + C_6, \quad \forall k > 0$$

According to Definition 1, we have $e(k) \sim \Delta^2 d(k-1)$. From Lemma 1, we know $\Delta^2 d(k-1) = O(T^3)$, then $e(k) = O(T^3)$. Similarly, for Eq. (14), there exist $0 < C_7 < \infty$ and $0 \leq C_8 < \infty$ such that

$$|q(k-1)| \leq C_7 \max_{0<\tau<k-1} |e(\tau)| + C_8, \ \forall k > 0.$$

According to Definition 1, $q(k-1)$ does not grow faster than $e(k)$. Since $e(k) \sim \Delta^2 d(k-1)$, $q(k-1)$ does not grow faster than $\Delta^2 d(k-1)$. Therefore, $q(k-1) = O(T^3)$.

Since $\lim_{k\to\infty} |\Delta^2 d(k-1)| = 0$, according to Eq. (14) and Eq. (17), $e(k)$ and $q(k)$ will both equal to 0 when k tends to ∞.

IV. SIMULATION RESULTS

Consider the following system without the control input as in [34],

$$x(k+1) = Ax(k) + d(k)$$
$$y(k) = Cx(k)$$

where

$$A = \begin{bmatrix} 0.9630 & 0.0181 & 0.0187 \\ 0.1808 & 0.8195 & -0.0514 \\ -0.1116 & 0.0344 & 0.9586 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix}, \quad d(k) = Ed^*(k),$$

$$d^*(k) = \begin{bmatrix} 0.3\sin(0.1k) + 0.5\cos(0.03k) \\ 0.2\cos(0.05k) + 2 \end{bmatrix},$$

$$E = \begin{bmatrix} 0.0996 & 0.0213 \\ 0.0050 & 0.1277 \\ 0.1510 & 0.0406 \end{bmatrix}.$$

According to Eq. (17), by assigning the poles of the error system at the same points as in [34], i.e., $\{0.1, 0.2, 0.3 \pm 0.5j, 0.4\}$, the constant gain matrices P_0, L_0 and L_1 are obtained such that Eq. (10) is satisfied.

$$P_0 = \begin{bmatrix} 1.1902 & 0.0181 \\ 2.1356 & 2.0195 \\ -1.3507 & 0.0344 \end{bmatrix},$$

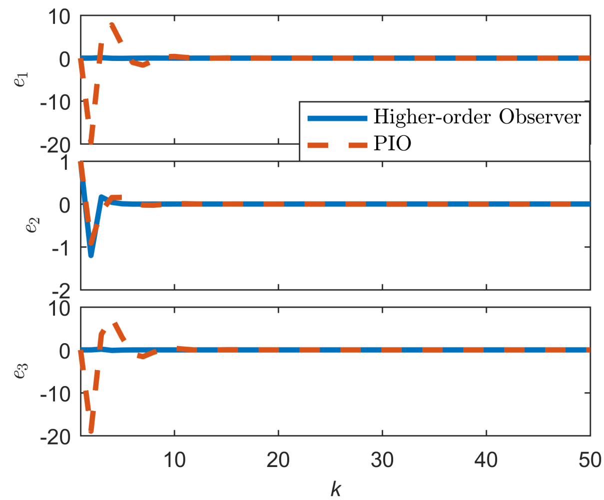

FIGURE 1. Estimate errors of the three states by using the two methods.

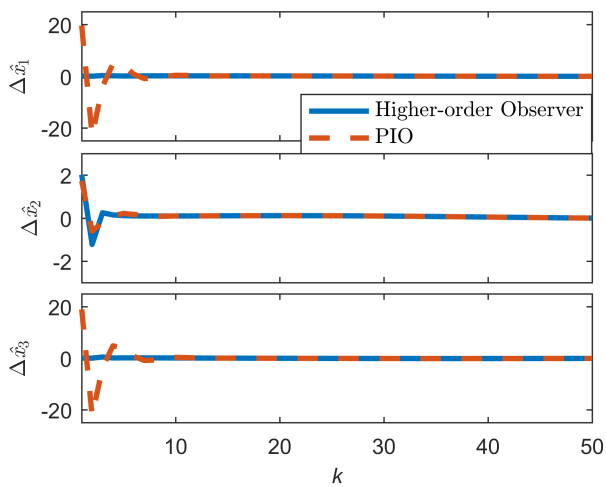

FIGURE 2. Change rates of the estimated states by using the two methods.

$$L_0 = \begin{bmatrix} 0.3700 & 0 \\ 0.3600 & 0.3600 \\ -0.3600 & 0 \end{bmatrix},$$

$$L_1 = \begin{bmatrix} 0.7000 & 0 \\ 1.2000 & 1.2000 \\ -0.8500 & 0 \end{bmatrix}.$$

For comparison, the PIO proposed in [34] is given as

$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + P_0'[y(k) - \hat{y}(k)] + E\bar{q}(k)$$
$$\bar{q}(k+1) = \bar{q}(k) + L_0'[y(k) - \hat{y}(k)]$$

where $\bar{q}$ is the estimate of the disturbance $d^*(k)$, and the gain matrices are

$$P_0' = \begin{bmatrix} 31.7392 & 19.6384 \\ 1.8918 & 1.7307 \\ 29.3767 & 19.9849 \end{bmatrix},$$

$$L_0' = \begin{bmatrix} 51.1873 & 34.5803 \\ -21.3249 & -10.6399 \end{bmatrix}.$$

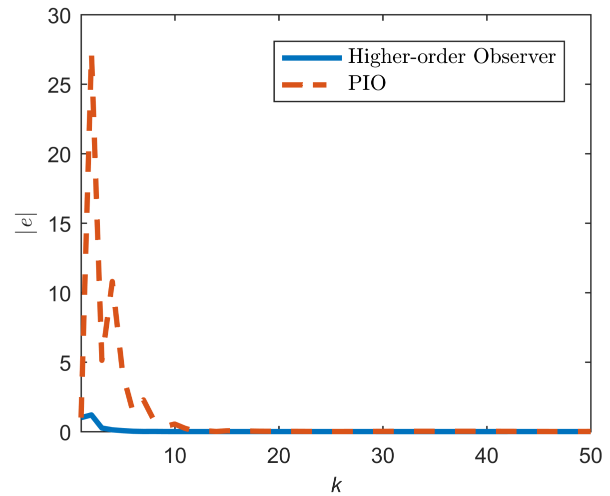

FIGURE 3. Euclidean norm of the state estimate errors by using the two methods.

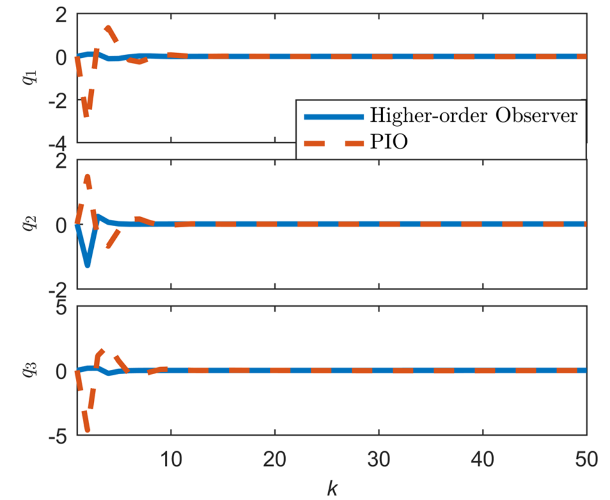

FIGURE 4. Estimate errors of the three disturbances by using the two methods.

Same as [34], the initial state

$$x(0) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T, \quad \hat{x}(0) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T,$$
$$x(-1) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T, \quad \hat{x}(-1) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T,$$

and the initial disturbance

$$d^*(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T,$$
$$\hat{d}^*(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T,$$
$$\hat{d}^*(-1) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T,$$

are chosen.

In order to evaluate the performance of the proposed observer, simulations without and with measurement noise are both conducted by using the proposed observer and the PIO in [34].

Figs. 1-8 are the results without noise. Fig. 1 shows the magnitudes of the observation errors of the three states for $0 \leq k \leq 50$. Fig. 2 shows the change rates of the three

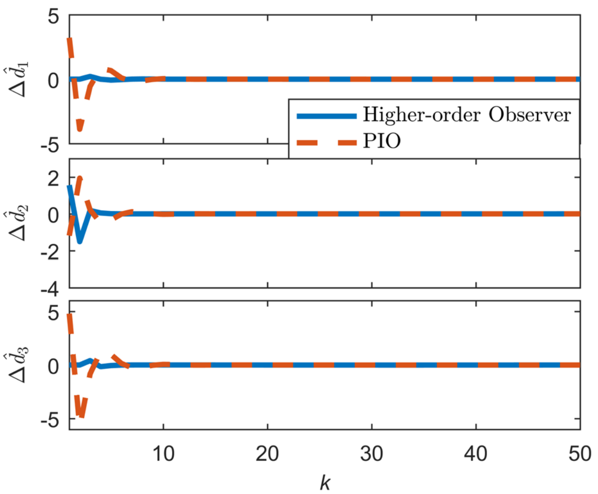

FIGURE 5. Change rates of the estimated disturbances by using the two methods.

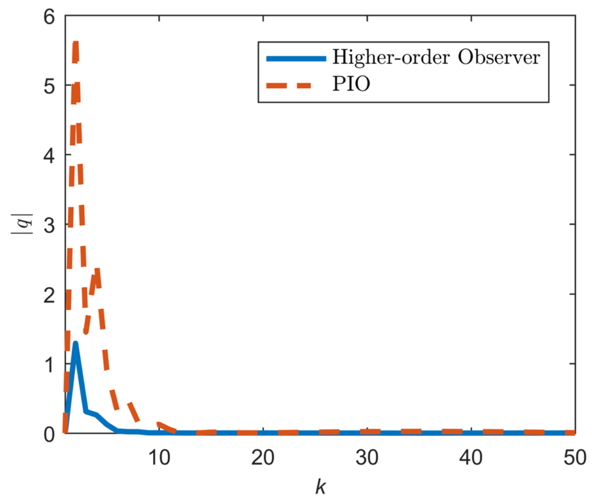

FIGURE 6. Euclidean norm of the disturbance estimate errors by using the two methods.

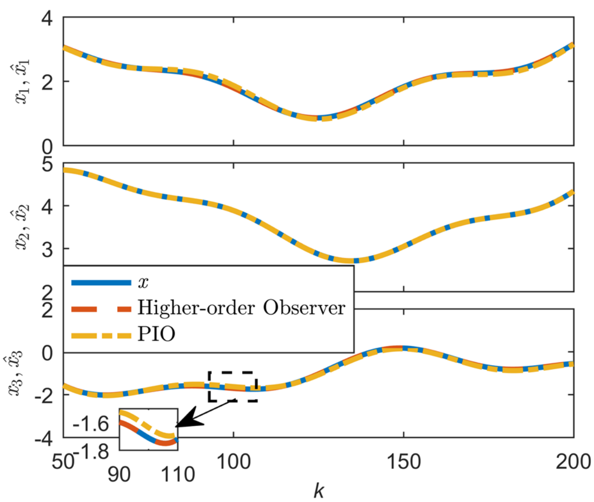

FIGURE 7. True and estimate values of the three states by using the two methods.

estimated states. Fig. 3 is the Euclidean norm of the state estimate errors. Fig. 4 illustrates the magnitude of the observation

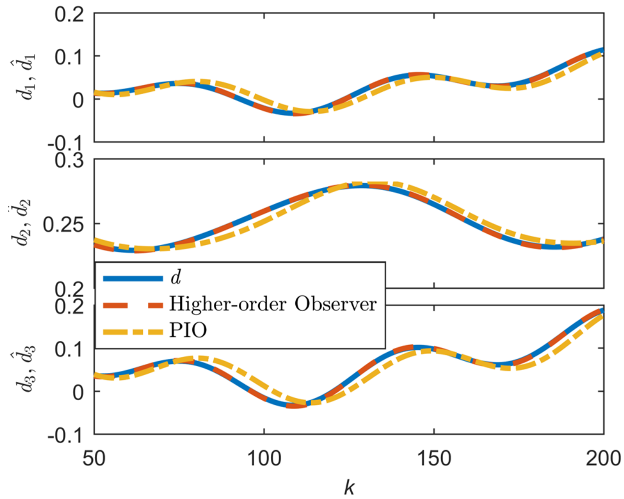

FIGURE 8. Given and estimate values of the three disturbances by using the two methods.

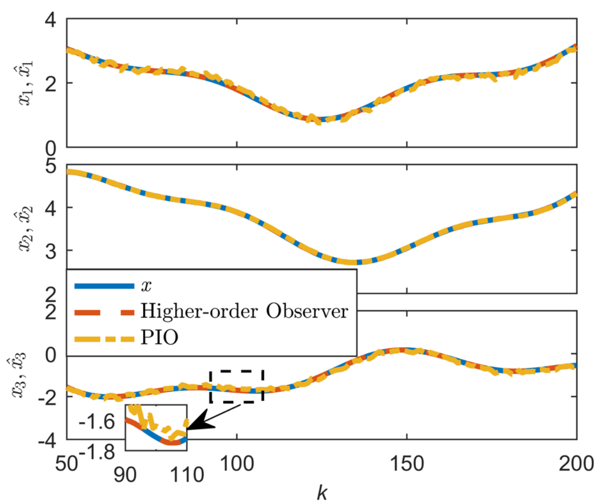

FIGURE 9. True and estimate values of the three states under measurement noise by using the two methods.

errors of the three kinds of disturbances for $0 \leq k \leq 50$. Fig. 5 shows the change rates of the three estimated disturbances. Fig. 6 is the Euclidean norm of the disturbance estimate errors. Fig. 7 shows the responses of the estimated states and the true states for $50 \leq k \leq 200$. Fig. 8 shows the responses of the estimated disturbances and the given disturbances for $50 \leq k \leq 200$. It can be seen from Figs. 1-6, the responses and the disturbance attenuation properties at transient state obtained by using the proposed method are evidently better than those obtained by using PIO in [34]. Meanwhile, it can also be seen from Figs. 2 and 5 that the change rates of the estimated states and disturbances of the higher-order observer are less than those of PIO. As can be seen from Figs. 7 and 8, the responses and the disturbance attenuation properties at steady state obtained by using the proposed method are also evidently better than those obtained by using PIO in [34].

Figs. 9 and 10 are respectively the estimated states and disturbances obtained by considering the measurement gaussian

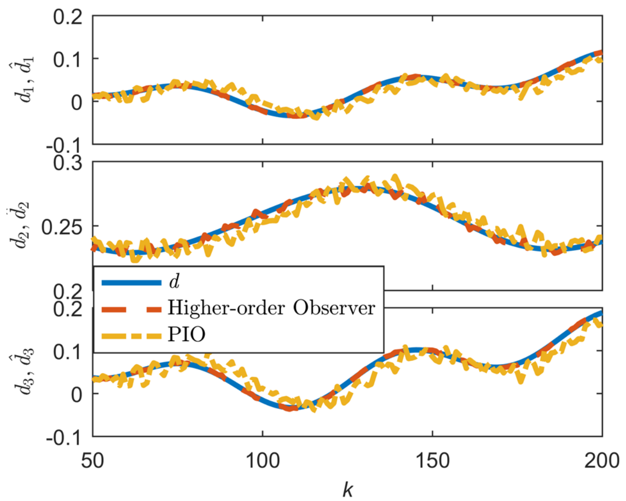

FIGURE 10. Given and estimate values of the three disturbances under measurement noise by using the two methods.

white noise with the magnitude 0.001. From Figs. 9 and 10 it can be seen that the estimated states and disturbance obtained by using the proposed higher-order observer are less affected by noise than those obtained by PIO in [34].

V. CONCLUSION

In this paper, for a class of linear systems with unmeasurable state and disturbance inputs, a new higher-order observer simultaneously estimating the system state and disturbance is proposed. It can be viewed as the extension of the PIO reported in the literature. It is proved without adding any other assumptions that the magnitudes of the state and disturbance estimate errors of the system are on the order of $O(T^3)$, which is much less than $O(T^2)$ produced by the PIO. Simulations are conducted and the results show the effectiveness and superior of the proposed method.

The higher-order observer is designed without considering the issues such as unknown parameters and unmodeled dynamics, which are always suffered by practical systems [37]. Those issues will be further considered in the future research. Moreover, since practical systems are always nonlinear, designing higher-order observers for nonlinear systems will be our future direction.

APPENDIX A

Proof: From Eq. (3), the disturbance $d(k)$ is

$$d(k) = \int_0^T e^{H\tau} Ff((k+1)T-\tau)d\tau \tag{18}$$

As the function f is smooth and bound, the function $f((k+1)T-\tau)$ can be expanded as following through Taylor's series at kT.

$$\begin{aligned} f(kT+T-\tau) &= f(kT) + \frac{df(t)}{dt}\mid_{t=kT}(T-\tau) \\ &+ \frac{1}{2!}\frac{d^2f(t)}{dt^2}\mid_{t=kT}(T-\tau)^2 \\ &+ \frac{1}{3!}\frac{d^3f(t)}{dt^3}\mid_{t=kT}(T-\tau)^3 \\ &+\cdots+\frac{1}{n!}\frac{d^nf(t)}{dt^n}\mid_{t=\xi}(T-\tau)^n \end{aligned} \tag{19}$$

where $\xi \in (kT, kT+T)$.

Assuming that

$$V_t^m = \frac{1}{m!}\frac{d^mf(t)}{dt^m}$$

where $m = 1, 2, 3, \cdots, n-1$, and V_t^m is also smooth and bound. Then for any m, there exist constant $C_m > 0$ such that

$$\mid V_t^m \mid \leq C_m.$$

For simplicity, k is used to instead of kT. Then Eq. (19) is rewritten as following

$$\begin{aligned} f(k+1-\tau) &= f(k) + V_k^1(T-\tau) + V_k^2(T-\tau)^2 \\ &+ V_k^3(T-\tau)^3 + \cdots + V_\xi^n(T-\tau)^n \end{aligned}$$

where $V_k^m = V_t^m \mid_{t=k}$.

Similar as in [36], we assume that $\tau = 0$ and $\tau = 2T$ respectively, we have

$$\begin{aligned} f(k+1) &= f(k) + V_k^1T + V_k^2T^2 + V_k^3T^3 + \cdots \\ &+ V_\xi^nT^n \end{aligned} \tag{20}$$

$$\begin{aligned} f(k-1) &= f(k) - V_k^1T + V_k^2T^2 - V_k^3T^3 + \cdots \\ &+ (-1)^nV_\eta^nT^n \end{aligned} \tag{21}$$

where $\eta \in (k-1, k)$.

Subtracting Eq. (21) from Eq. (20), we obtain

$$f(k+1) - 2f(k) + f(k-1) = V_k^2T^2 + V_k^4T^4 + \cdots + V_\delta^\rho T^\rho$$

where $\delta \in (\eta, \xi)$, $\rho \leq n$ is even number.

Similarly, for the function V_t^1, there is

$$\begin{aligned} V_{k+1}^1 &= V_k^1 + V_k^2T + V_k^3T^2 + V_k^4T^3 + \cdots \\ &+ V_\xi^{n-1}T^{n-2} \end{aligned} \tag{22}$$

$$\begin{aligned} V_{k-1}^1 &= V_k^1 - V_k^2T + V_k^3T^2 - V_k^4T^3 + \cdots \\ &+ (-1)^{n-2}V_\xi^{n-1}T^{n-2} \end{aligned} \tag{23}$$

and then

$$V_{k+1}^1 - 2V_k^1 + V_{k-1}^1 = V_k^3T^2 + V_k^5T^4 + \cdots + V_\delta^{\rho-1}T^{\rho-2}$$

As we have known

$$\mathcal{C} = \int_0^T e^{H\tau}Fd\tau = FT + \frac{1}{2!}HFT^2 + \frac{1}{3!}H^2FT^3 + \frac{1}{4!}H^3FT^4 + \cdots$$

is constant matrix.

Substituting Eq. (19) into Eq. (18), and taking $n = 2$, then we get

$$\begin{aligned} d(k) &= \int_0^T e^{H\tau}F[f(k) + V_k^1(T-\tau) + V_\xi^2(T-\tau)^2]d\tau \\ &= \int_0^T e^{H\tau}Fd\tau f(k) + \int_0^T e^{H\tau}F(T-\tau)d\tau V_k^1 \\ &+ \int_0^T e^{H\tau}F(T-\tau)^2d\tau V_\xi^2 \end{aligned}$$

where

$$\int_0^T e^{H\tau} F d\tau f(k) = \mathcal{C} f(k)$$
$$\int_0^T e^{H\tau} F(T-\tau) d\tau V_k^1 = \frac{1}{2}\mathcal{C} V_k^1 T + \mathcal{R}$$

where

$$\mathcal{R} = (\frac{1}{12} HFV_k^1 + \frac{1}{8} H^2 F V_k^1 T + \cdots) T^3$$

According to Definition 2, there is

$$\mathcal{R} = O(T^3)$$

So

$$\int_0^T e^{H\tau} F(T-\tau) d\tau V_k^1 = \frac{1}{2}\mathcal{C} V_k^1 T + O(T^3).$$

In the same way, there is

$$\int_0^T e^{H\tau} F(T-\tau)^2 d\tau V_\xi^2 = O(T^3).$$

Therefore, we have

$$d(k) = \mathcal{C} f(k) + \frac{1}{2}\mathcal{C} V_k^1 T + O(T^3) \tag{24}$$

According to Definition 2 and substituting $\mathcal{C}$ into the above equality, the $\mathcal{C} f(k)$ and $\frac{1}{2}\mathcal{C} V_k^1 T$ satisfy

$$\mathcal{C} f(k) = O(T)$$
$$\frac{1}{2}\mathcal{C} V_k^1 T = O(T^2),$$

then

$$d(k) = O(T).$$

And

$$d(k-1) = \mathcal{C} f(k-1) + \frac{1}{2}\mathcal{C} V_{k-1}^1 T + O(T^3) \tag{25}$$

From Eq.(23) and Eq. (24), we have

$$d(k) - d(k-1) = \mathcal{C}(f(k) - f(k-1)) + \frac{1}{2}\mathcal{C}(V_k^1 - V_{k-1}^1)T + O(T^3)$$

Combining to Eq. (20) and Eq. (22), there are

$$\mathcal{C}(f(k) - f(k-1)) = O(T^2),$$
$$\mathcal{C}(V_k^1 - V_{k-1}^1)T = O(T^3),$$

then

$$d(k) - d(k-1) = O(T^2)$$

or

$$\Delta d(k) = O(T^2).$$

Similarly, there are

$$\begin{aligned} d(k) - 2d(k-1) + d(k-2) &= \mathcal{C}(f(k) - 2f(k-1) + f(k-2)) \\ &\quad + \frac{1}{2}\mathcal{C}(V_k^1 - 2V_{k-1}^1 + V_{k-2}^1)T + O(T^3) \end{aligned}$$

and

$$\mathcal{C}(f(k) - 2f(k-1) + f(k-2)) = O(T^3)$$
$$\frac{1}{2}\mathcal{C}(V_k^1 - 2V_{k-1}^1 + V_{k-2}^1)T = O(T^4).$$

Then

$$d(k) - 2d(k-1) + d(k-2) = O(T^3)$$

or

$$\Delta^2 d(k) = O(T^3).$$

APPENDIX B

Key Technical Lemma [38]*:* For the following system

$$A(z^{-1})y(k) = \begin{bmatrix} z^{-d_{11}} B_{11}(z^{-1}) & \cdots & z^{-d_{1r}} B_{1r}(z^{-1}) \\ \vdots & \ddots & \vdots \\ z^{-d_{m1}} B_{m1}(z^{-1}) & \cdots & z^{-d_{mr}} B_{mr}(z^{-1}) \end{bmatrix} u(k)$$

where $A(z^{-1})$ and $B_{ij}(z^{-1})(i = 1, \cdots, m; j = 1, \cdots, r)$ denote scalar polynomials in the backward shift operator z^{-1}, and the factors $z^{-d_{ij}}$ represent pure time delays. With $r = m$, the system subjects to

$$\det \begin{bmatrix} z^{d_{11}-d_1} B_{11}(z) & \cdots & z^{d_{1m}-d_1} B_{1m}(z) \\ \vdots & \ddots & \vdots \\ z^{d_{m1}-d_m} B_{m1}(z) & \cdots & z^{d_{mm}-d_m} B_{mm}(z) \end{bmatrix} \neq 0$$

for $| z | \leq 1$ where

$$d_i = \min_{1 \leq j \leq m} d_{ij} \quad i = 1, \cdots, m.$$

If

$$\max_{\substack{0 \leq k \leq T_0 \\ 1 \leq i \leq m}} | y_i(k + d_i) | = m_2,$$

then there exit positive constants m_3 and m_4 which are independent of T_0 with $0 \leq m_3 < \infty$, $0 < m_4 < \infty$ such that

$$| u_i(k) | < m_2 m_4 + m_3 \quad 0 \leq k \leq T_0, \; i = 1, \cdots, m.$$

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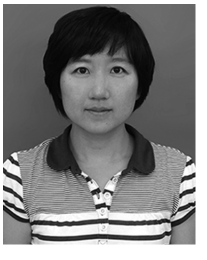

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