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Decomposition Method for Belief Reliability Analysis of Complex Uncertain Random Systems

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ABSTRACT Belief reliability is a new proposed reliability metric considering both aleatory and epistemic uncertainty. In belief reliability theory, system reliability analysis is a key component. Traditional system belief reliability theory for systems with random and uncertain components is based on a complex belief reliability formula, which is not understandable and efficient enough in engineering practise. In this paper, we put forward a novel system belief reliability analysis method, called decomposition method to cope with the problem. An algorithm of this method is proposed according to the properties of the cut sets of systems and the complexity of the algorithm is analyzed and compared with that of the reliability formula method. Finally, the effectiveness and efficiency of this method is further illustrated with a comparative case study.

INDEX TERMS Belief reliability, reliability analysis, decomposition method, chance theory.

NOTATIONS AND ABBREVIATIONS

Γ : universal set in uncertainty space
$\mathcal{L}$: $\sigma-$algebra over Γ
$\mathcal{M}$: uncertain measure
Λ : uncertain event
Ω : universal set in probability space
$\mathcal{A}$: $\sigma-$algebra over Ω
Pr : probability measure
ξ : uncertain random variable / system state variable
η : random variable
τ : uncertain variable
Ξ : feasible domain
G : system state mode
R_B : belief reliability
$R_B^{(P)}$: belief reliability of random components or systems
$R_B^{(U)}$: belief reliability of uncertain components or systems
x : component state mode
ϕ : structure function
m : number of random components
n : number of uncertain components
c : component number
u : state mode of random components
v : state mode of uncertain components
C : minimal cut sets (MCS)
CS : cut sets (CS)
l : the number of MCS
r : the number of CS of the decomposed system
MCS : minimal cut set
RBD : reliability block diagram
CAN : controller area network
FPGA : field-programmable gate array
MOSFET : metal-oxide-semiconductor field-effect transistor

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I. INTRODUCTION

System reliability analysis is a key component in reliability engineering. Traditional system reliability analysis is based on probability theory, i.e., the reliability of components are described as probability estimated through failure data and the system reliability is obtained through reliability block diagram (RBD) method, minimal cut/path method, structural function method, etc. [1]–[4].

In real cases, however, since the lack of failure data, we cannot even give the reliability of components, let alone the system. To give a reasonable reliability metric, the physics of failure (PoF) method is developed based on the failure mechanisms of the components [5]. In this method, failures of the components are described by physical models, whose parameters are modeled as probability distributions [6], [7]. The component reliability can be calculated by propagating the uncertainty through the model, and the system reliability can be, then, obtained based on classical reliability analysis methods mentioned above.

The limitation of the PoF-based method is that it only models aleatory uncertainty (the inherent randomness of the real world modeled by probability distributions [8]) without considering epistemic uncertainty (uncertainty caused by our lack of knowledge [9], [10]). For example, the physical model may not be perfectly accurate due to the limit knowledge of failure mechanisms and the distribution of model parameters may not be precise because of the indeterministic real working conditions. To deal with this problem, several reliability analysis methods considering epistemic uncertainty is developed based on different theories, such as evidence theory [11], [12], interval analysis theory [13], [14], fuzzy set theory [15], possibility theory [16], [17], etc. However, the reliability methods based on the first three theories may cause interval extension problems in the process of system reliability analysis because the component reliability is given as intervals [18]. As for the method based on possibility theory, it may cause counterintuitive results because the theory itself does not satisfy duality properties [18].

To overcome these shortcomings, Zeng *et al.* [19] utilize uncertainty theory (proposed by Liu in 2007 [20]) to model epistemic uncertainty and first put forward the name belief reliability. Because of the axioms of uncertainty theory, belief reliability satisfy duality property and can perfectly avoid the interval extension problems [18]. Nevertheless, this metric still cannot solve the reliability analysis problem of most real systems consisting of both aleatory and epistemic uncertainty. Specifically, the real systems are usually consist of two kinds of components, i.e., random components mainly affected by aleatory uncertainty and uncertain components mainly affected by epistemic uncertainty. Since the reliability of random and uncertain components are modeled by probability theory and uncertainty theory, respectively, the original belief reliability analysis method cannot give a reasonable value of the system reliability [21].

To solve this problem, Wen and Kang [21] and Zhang *et al.* [22] extended the concept of belief reliability and developed system reliability analysis methods using chance theory, which is regarded to be a combination of probability theory and uncertainty theory [23]. In their proposed methods, the belief reliability of simple systems which can be divided into random and uncertain subsystems can be easily calculated, while the belief reliability of other relatively complex systems can only be calculated using a complex reliability formula. Although the formula works well for some system configurations, it still has two disadvantages in practice. First, the formula is too complex for reliability engineers to understand. Second, in the calculation process, we have to figure out all possible combinations of the components states and make a lot of comparisons, so the computational complexity for using this formula seems to be too high. Therefore, in this paper, we develop a new belief reliability analysis method, called decomposition method, to improve the effectiveness and efficiency of the formula.

The basic idea of the decomposition method is to decompose the original system configuration into several sub-configurations only consisting of uncertain components according to the states of random components, and then calculate the system reliability based on the additivity property of probability theory. To program this method in computer, we develop an applicable algorithm using the system cut sets after studying their properties. By comparing the computation complexity of the proposed algorithm and the reliability formula method, we find that the new method is more understandable and more efficient in most cases. Since the reliability analysis process of simple systems, such as series and parallel systems, is very simple (shown in [22]), we only apply the decomposition method to complex systems in this paper.

The remainder of this paper is organized as follows. First, we will briefly introduce some necessary knowledge about uncertainty theory and chance theory in section II. Section III gives a review on system belief reliability theory, and introduces the belief reliability analysis method of uncertain random systems and uncertain systems. In section IV, the decomposition method is introduced in detail. To put forward the algorithm, a theorem about the decomposed system is proposed and proved. The complexity analysis is also conducted in this section. Finally, we compare the proposed method with the belief reliability formula method through a numerical case study in section V.

II. PRELIMINARY

In this section, some basic concepts and results of uncertainty theory and chance theory are introduced.

A. UNCERTAINTY THEORY

Uncertainty theory is a new branch of axiomatic mathematics built on four axioms, i.e., Normality, Duality, Subadditivity and Product Axioms. Founded by Liu [20] in 2007 and refined by Liu [24] in 2010, uncertainty theory has been widely applied as a new tool for modeling subjective (especially human) uncertainties. In uncertainty theory, belief degrees of events are quantified by defining uncertain measures:

Definition 1 (Uncertain Measure [20]): Let Γ be a nonempty set, and $\mathcal{L}$ be a σ-algebra over Γ. A set function $\mathcal{M}$ is called an uncertain measure if it satisfies the following three axioms,

Axiom 2 (Normality Axiom): $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.

Axiom 3 (Duality Axiom): $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event $\Lambda \in \mathcal{L}$.

Axiom 4 (Subadditivity Axiom): For every countable sequence of events $\Lambda_1, \Lambda_2, \cdots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}$$

Uncertain measures of product events are calculated following the product axiom [25]:

Axiom 5 (Product Axiom): Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$. The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}$$

where $\mathcal{L}_k$ are σ-algebras over Γ_k, and Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \ldots$, respectively.

Definition 6 (Uncertain Variable [20]): An uncertain variable is a function τ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that $\{\tau \in \mathcal{B}\}$ is an event for any Borel set $\mathcal{B}$ of real numbers.

B. CHANCE THEORY

Chance theory is founded by Liu [23], [26] as a mixture of uncertainty theory and probability theory, to deal with problems affected by both aleatory uncertainty (randomness) and epistemic uncertainty. The basic concept in chance theory is the chance measure of an event in a chance space.

Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space, and $(\Omega, \mathcal{A}, \Pr)$ be a probability space. Then $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ is called a chance space.

Definition 7 (Chance Measure [23]): Let $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ be a chance space, and let $\Theta \in \mathcal{L} \times \mathcal{A}$ be an event. Then the chance measure of Θ is defined as

$$\mathrm{Ch}\{\Theta\} = \int_0^1 \Pr\{\omega \in \Omega | \mathcal{M}\{\gamma \in \Gamma | (\gamma, \omega) \in \Theta\} \geq x\} \, dx. \quad \text{(II.1)}$$

Theorem 8 ([23]): Let $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ be a chance space. Then

$$\mathrm{Ch}\{\Lambda \times A\} = \mathcal{M}\{\Lambda\} \times \Pr\{A\} \quad \text{(II.2)}$$

for any $\Lambda \in \mathcal{L}$ and any $A \in \mathcal{A}$. Especially, we have

$$\mathrm{Ch}\{\emptyset\} = 0, \quad \mathrm{Ch}\{\Gamma \times \Omega\} = 1. \quad \text{(II.3)}$$

Definition 9 (Uncertain Random Variable [23]): An uncertain random variable is a function ξ from a chance space $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ to the set of real numbers such that $\{\xi \in B\}$ is an event in $\mathcal{L} \times \mathcal{A}$ for any Borel set B of real numbers.

Random variables and uncertain variables are two special cases of uncertain random variables. If an uncertain random variable $\xi(\gamma, \omega)$ does not vary with γ, it degenerates to a random variable. If an uncertain random variable $\xi(\gamma, \omega)$ does not vary with ω, it degenerates to an uncertain variable.

Example: Let $\eta_1, \eta_2, \ldots, \eta_m$ be random variables and $\tau_1, \tau_2, \ldots, \tau_n$ be uncertain variables. If f is a measurable function, then

$$\xi = f(\eta_1, \eta_2, \ldots, \eta_m, \tau_1, \tau_2, \ldots, \tau_n)$$

is an uncertain random variable determined by

$$\xi(\gamma, \omega) = f(\eta_1(\omega), \ldots, \eta_m(\omega), \tau_1(\gamma), \ldots, \tau_n(\gamma))$$

for all $(\gamma, \omega) \in \Gamma \times \Omega$.

III. SYSTEM BELIEF RELIABILITY AND ANALYSIS METHOD

A. BELIEF RELIABILITY IN TERMS OF STATE MODE

Belief reliability is a newly proposed reliability metric that aims to measure the reliability of uncertain random systems affected by both aleatory and epistemic uncertainties [22]. In the framework of belief reliability theory, we are usually interested in two factors: a system state variable which describes system function or failure behaviors, and a feasible domain of the state variable representing failure criteria. Since the state variable always embodies two kinds of uncertainty, the belief reliability metric is defined based on chance theory.

Definition 10 (Belief Reliability [22]): Let a system state variable ξ be an uncertain random variable, and Ξ be the feasible domain of the product's state. Then the belief reliability is defined as the chance that the system state is within the feasible domain, i.e.,

$$R_B = \mathrm{Ch}\{\xi \in \Xi\}. \quad \text{(III.1)}$$

In this definition, the system state variable ξ can represent different physical quantities, for example, the failure time, performance margin, state mode (interpreted as the function level in [22]), etc. In this paper, we are mainly concerned about the circumstance that ξ takes the state mode, denoted as G, which describes the behavioral status of a system as it performs its specified function.

Assume the system has k different state modes $G = i, i = 0, 1, \ldots, k$. Among all the state modes, some are used to describe the functional behavior of the system to perform its specified function, some are used to describe the degraded behavior because of the degradations of different levels, and the rest describe the failure behavior when the system cannot accomplish its function. Suppose the state mode that ensure the system to be in the feasible domain is $G = j, j = s, s+1, \cdots, k$. Then, by setting $\Xi = \{s, s+1, \ldots, k\}$, we have the system belief reliability in terms of state mode:

$$R_B = \mathrm{Ch}\{G \in \{s, s+1, \ldots, k\}\}. \quad \text{(III.2)}$$

It is noted that Eq. (III.2) shows the belief reliability of multistate systems. In this paper, we tent to focus on a special case, where the system only has two state modes, namely, complete failure with $G = 0$ and perfectly function with $G = 1$. Then the system belief reliability can be described by

$$R_B = \mathrm{Ch}\{G = 1\}. \quad \text{(III.3)}$$

To calculate the belief reliability given in (III.3), Wen and Kang [21] proposed a reliability formula using the operational laws of chance theory, which will be introduced in section III-B. It is also noted that Eq. (III.3) has two degeneration cases. If the state mode is mainly affected by aleatory uncertainty, the belief reliability will be the classical probability theory-based reliability of a binary system, which can be evaluated using statistical methods [3]. If G is mainly affected by epistemic uncertainty, the belief reliability will become the uncertainty theory-based reliability of a binary system, i.e.,

$$R_B = \mathcal{M}\{G = 1\}. \tag{III.4}$$

In this case, we need to use the minimal cut theorem developed by Zeng *et al.* [27] to calculate the belief reliability. The theorem is one of the basis of the proposed decomposition method in this paper, so we will further introduce it in section III-C.

B. BELIEF RELIABILITY ANALYSIS OF UNCERTAIN RANDOM SYSTEMS

Uncertain random systems refer to the systems with both uncertain components and random components, where the two kinds of components are mainly affected by epistemic uncertainty and aleatory uncertainty respectively. To model the reliability of uncertain random systems, chance theory is inevitable to be used. Since the interpretation of state mode in belief reliability theory is a basic one which is easier and more directly to be understood, in this paper, we tend to use the Eq. (III.3) to clarify the belief reliability analysis method of uncertain random systems.

The belief reliability analysis method of uncertain random systems include two cases. First, for simple systems which can be divided into a random subsystem and an uncertain subsystem, Zhang *et al.* have provided two belief reliability formula to calculate their belief reliability [22]. Second, for the systems that cannot be separated to two subsystems, called complex systems in this paper, only a complex formula which is developed by Wen and Kang according to the operational laws of chance theory, can be used.

Before we introduce the formula, the concept of structure function should be clarified.

Definition 11 (Structure Function): Let $x_i (1 \leq i \leq n)$ denote the state mode of the ith component, where

$$x_i = \begin{cases} 1, & \text{if the } i\text{th component is working} \\ 0, & \text{if the } i\text{th component fails} \end{cases}$$

Suppose the system state mode G is a function of the component state mode vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, denoted by

$$G = \phi(\mathbf{x}) = \phi(x_1, x_2, \ldots, x_n).$$

Then, the function $\phi(\cdot)$ is called the structure function of the system.

We can see that the system structure function is a mapping from $\{0, 1\}^n$ to $\{0, 1\}$, where $G = 0$ and $G = 1$ denote the system is failed and working, respectively. Then we can give the belief reliability formula of uncertain random systems.

Theorem 12 (Wen and Kang [21]): Assume that a Boolean system has a structure function ϕ and contains random components with belief reliabilities $R^{(P)}_{B,i}, i = 1, 2, \ldots, m$ and uncertain components with belief reliabilities $R^{(U)}_{B,j}, j = 1, 2, \ldots, n$. Then the belief reliability of the system is

$$\begin{aligned} R_{B,S} &= \mathrm{Ch}\{G = 1\} \\ &= \sum_{(y_1,\ldots,y_m)\in\{0,1\}^m} \left(\prod_{i=1}^{m} \mu_i(y_i)\right) \cdot Z(y_1, y_2, \ldots, y_m), \end{aligned} \tag{III.5}$$

where

$$Z(y_1, y_2, \ldots, y_m) = \begin{cases} \sup\limits_{\phi(y_1,\ldots,y_m,z_1,\ldots,z_n)=1} \min\limits_{1\leq j\leq n} \nu_j(z_j), \\ \qquad \text{if } \sup\limits_{\phi(y_1,\ldots,y_m,z_1,\ldots,z_n)=1} \min\limits_{1\leq j\leq n} \nu_j(z_j) < 0.5, \\ 1 - \sup\limits_{\phi(y_1,\ldots,y_m,z_1,\ldots,z_n)=0} \min\limits_{1\leq j\leq n} \nu_j(z_j), \\ \qquad \text{if } \sup\limits_{\phi(y_1,\ldots,y_m,z_1,\ldots,z_n)=1} \min\limits_{1\leq j\leq n} \nu_j(z_j) \geq 0.5, \end{cases}$$

$$\mu_j(y_i) = \begin{cases} R^{(P)}_{B,i}, & \text{if } y_i = 1, \\ 1 - R^{(P)}_{B,i}, & \text{if } y_i = 0, \end{cases} \quad (i = 1, 2, \ldots, m),$$

$$\nu_j(z_j) = \begin{cases} R^{(U)}_{B,i}, & \text{if } z_j = 1, \\ 1 - R^{(U)}_{B,i}, & \text{if } z_j = 0, \end{cases} \quad (j = 1, 2, \ldots, n).$$

In practise, this reliability formula seems to be too complicated for reliability engineers to understand. In addition, it is easy to find that when computing $Z(y_1, y_2, \ldots, y_m)$, we have to figure out all possible combinations of the values of $y_i, i = 1, 2, \ldots, n$ and $z_j, j = 1, 2, \ldots, n$, and in one computation process, we also have to make a lot of comparisons to obtain the value of $\sup\limits_{f(y_1,\ldots,y_m,z_1,\ldots,z_n)=1} \min\limits_{1\leq j\leq n} \nu_j(z_j)$. In real engineering application, it may be difficult to utilize. Because of these limitations, there is a great need to put forward a simpler method for belief reliability analysis of uncertain random systems.

C. BELIEF RELIABILITY OF UNCERTAIN SYSTEMS

In this paper, we will develop a decomposition method in section IV. The main idea is to decompose the uncertain random system to several uncertain systems with associated probabilities. Therefore, the belief reliability analysis of uncertain systems is a crucial problem for the method.

Uncertain systems only consist of uncertain components, whose belief reliability is obtained using uncertainty theory. Liu first proposed a reliability index theorem using the operational laws of uncertainty theory to calculate the belief reliability of uncertain systems [28]. Later, Zeng *et al.* developed a more practical method based on the minimal cut set theorem [19]. In 2018, Zeng *et al.* extended the minimal cut set theorem to cut sets, which further simplifies the belief reliability analysis process [27]. The cut set theorem offers a possibility to calculate the decomposed uncertain systems in our method.

Definition 13 (Cut Set and Minimal Cut Set [29]): Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be a state mode vector of a system with structure function ϕ. If $\phi(\mathbf{x}) = 0$, then $CS = \{i|\xi_i = 0\}$ is called a cut set. Further, if $\forall \mathbf{u} > \mathbf{x}$, we have $\phi(\mathbf{u}) = 1$, then CS is called a minimal cut set.

Theorem 14 (Cut Set Theorem [27]): Suppose that an uncertain system has l minimal cut set $CS_1, CS_2, \ldots, CS_l$ and $(k-l)$ cut sets $CS_{l+1}, CS_{l+2}, \cdots CS_k$ that contain some minimal cut sets. Then, the system belief reliability can be calculated by

$$R_{B,S}^{(U)} = \bigwedge_{1 \le i \le k} \bigvee_{j \in CS_i} R_{B,j}^{(U)}. \tag{III.6}$$

IV. THE DECOMPOSITION METHOD

In this section, we will give the simpler and more efficient method based on decompositions of complex systems. It should be further explained that the so-called complex systems in this paper uses the concept in [22], i.e., the systems that cannot be separated into a random subsystem and an uncertain subsystem, and the proposed method is used to cope with these kinds of systems. To better demonstrate this method, we will first give some basic ideas through a small example. Then, some critical definitions and theorems about cut sets will be given as the basis of the decomposition method. An algorithm will be given for belief reliability analysis of complex uncertain random systems using computers. Finally, we will analyze the complexity of the method and compare it with the existed belief reliability formula.

A. BASIC IDEAS

In Theorem 12, we find the difficulty is actually calculating the $Z(y_1, y_2, \ldots, y_m)$ where $y_i (i = 1, 2, \ldots, m)$ are the state combination of random components. In fact, when one group of y_i is determined, there will be a corresponding system sub-configuration which only consists of uncertain components. For m random components in a system, there are totally 2^m state sets by considering their all possible combinations of success or failure, and each state mode set has a probability $p^{(i)}, i = 1, 2, \ldots, 2^m$. By evaluating the belief reliability $R_B^{(U,i)}$ of the sub-configurations, the system belief reliability can, then, be calculated based on the additivity of the probability measure, i.e.,

$$R_{B,S} = \sum_{i=1}^{2^m} p^{(i)} \cdot R_B^{(U,i)}. \tag{IV.1}$$

Here we use a small example to illustrate this thought.

Example(k-Out-of-n system): Consider a 2-out-of-4 system consisting of two random components and two uncertain components, as shown in Fig.1. The reliabilities of the four components are R_{P1}, R_{P2}, R_{U3} and R_{U4}, respectively. By figuring out all state modes combinations of random components, we will have 4 state mode sets with associated probabilities. To obtain the sub-configurations, we need to replace a random component with a path when the state mode of it is 1, while we should delete it if the state mode is 0. Then, 4 sub-configurations and their belief reliabilities can be acquired, which are shown in Table 1. Therefore, the system belief reliability of the 2-out-of-4 system is

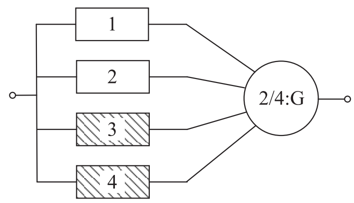
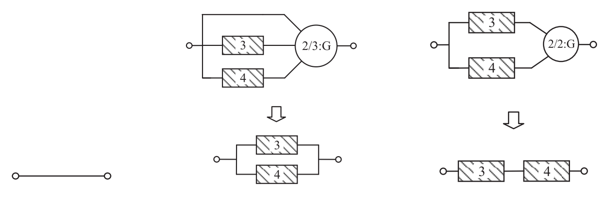


FIGURE 1. An uncertain random 2-out-of-4 system.

TABLE 1. System belief reliability calculation process for 2-out-of-4 system using decomposition method.

i	State mode	$p^{(i)}$	Sub-fig	$R_B^{(U,i)}$
1	(1,1)	$R_{P1}R_{P2}$	Fig.2(a)	1
2	(1,0)	$R_{P1}(1-R_{P2})$	Fig.2(b)	$R_{U3} \vee R_{U4}$
3	(0,1)	$(1-R_{P1})R_{P2}$	Fig.2(b)	$R_{U3} \vee R_{U4}$
4	(0,0)	$(1-R_{P1})(1-R_{P2})$	Fig.2(c)	$R_{U3} \wedge R_{U4}$



(a) (b) (c)

FIGURE 2. RBD model for sub-configurations of the uncertain random 2-out-of-4 system.

$$\begin{aligned} R_S = &\, R_{P1}R_{P2} + R_{P1}(1-R_{P2})(R_{U3} \vee R_{U4}) \\ &+ (1-R_{P1})R_{P2}(R_{U3} \vee R_{U4}) \\ &+ (1-R_{P1})(1-R_{P2})(R_{U3} \wedge R_{U4}). \end{aligned} \tag{IV.2}$$

The result of the example is consistent with the one given by Wen and Kang [21], which shows the correctness of the ideas. However, this method cannot be easily applied in computers. Therefore, we try to use the properties of cut sets and the cut set theorem to polish the decomposition method.

B. THE CUT SETS OF DECOMPOSED SYSTEMS

The main difficulties in the above example are actually to replace the component with path or just delete the component. Since we tend to use the cut set theorem to calculate the belief reliability of uncertain systems, it is natural to directly use cut sets of the original system to get the cut sets of uncertain systems after decomposition. Therefore, we hereby give the definition of the decomposed uncertain system and show the property of its cut sets.

Definition 15 (Decomposed System): Assume a system is composed of n components with structure function ϕ, and $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is the components' state mode vector. If the state mode of the kth component is determined to be

x_{d_k}, where $1 \le k \le n$, then the system with structure function $\phi(x_1, \ldots, x_{d_k}, \ldots, x_n)$ is called the decomposed system of k.

Specially, let us consider an uncertain random system which is composed of m random components and n uncertain components with structure function $\phi(u_1, \ldots, u_m, v_1, \ldots, v_n)$, where $\mathbf{u} = (u_1, \ldots, u_m)$ and $\mathbf{v} = (v_1, \ldots, v_n)$ are the state mode vector of random and uncertain components, respectively. When the state mode vector of random components is determined to be $\mathbf{u_d} = (u_{d_1}, \ldots, u_{d_m})$, the system with structural function $\phi(u_{d_1}, \ldots, u_{d_m}, v_1, \ldots, v_n)$ can be called the decomposed uncertain system of the determined random components.

Theorem 16: Assume a system has n components with numbers $\{1, 2, \ldots, n\}$ and $CS_0 = \{c_1, c_2, \ldots, c_p\}$ is one of the minimal cut set of the system. Let x_k be the state mode of the kth component. Then we have (1)

1) If $\exists c_j \in CS_0$ and $x_{c_j} = 1$ is determined, then $CS_0 \backslash c_j$ will not be the cut set of the decomposed system of c_j anymore.
2) If $\exists c_j \in CS_0$ and $x_{c_j} = 0$ is determined, then $CS_0 \backslash c_j$ will still be the cut set (but not necessarily a minimal cut set) of the decomposed system of c_j when $CS_0 \backslash c_j \neq \varnothing$.

Proof: Without loss of generality, we assume a system only have 2 minimal cut sets, denoted as $CS_0 = \{c_1, \ldots, c_j, \ldots, c_p\}$ and $CS_1 = \{b_1, \ldots, b_q\}$, and $\forall b_i$, there is $b_i \neq c_j$, $i = 1, \ldots, q$. Suppose the state mode vector of CS_0 and CS_1 are $\mathbf{x_0} = \{x_{c_1}, \ldots, x_{c_j}, \ldots, x_{c_p}\}$ and $\mathbf{x_1} = \{x_{b_1}, \ldots, x_{b_q}\}$, respectively, so we have $\phi(\mathbf{x}) = 0$ if $\mathbf{x_0} = 0$ or $\mathbf{x_1} = 0$. Then, the structure function of the system can be written as:

$$\begin{aligned}\phi(\mathbf{x}) &= \phi_0(\mathbf{x_0}) \cdot \phi_1(\mathbf{x_1}) \\ &= \left[1 - (1 - x_{c_1}) \cdots (1 - x_{c_j}) \cdots (1 - x_{c_p})\right] \\ &\quad \cdot \left[1 - (1 - x_{b_1}) \cdots (1 - x_{b_q})\right].\end{aligned}$$

(1) When $x_{c_j} = 1$ is determined, it is easily to find that $\phi_0(\mathbf{x_0}) = 1$. Then, the structure function of the decomposed system of c_j becomes

$$\phi(\mathbf{x}) = 1 \cdot \phi_1(\mathbf{x_1}) = 1 - (1 - x_{b_1}) \cdots (1 - x_{b_q}).$$

Therefore, the set $CS_0 \backslash c_j$ will not be the cut set of the decomposed system.

1) When $x_{c_j} = 0$ is determined, we can find that $1 - x_{c_j} = 1$. If $C_0 \backslash c_j \neq \varnothing$, the structure function of the decomposed system of c_j becomes

$$\begin{aligned}\phi(\mathbf{x}) = [1 &- (1 - x_{c_1}) \cdots (1 - x_{c_j-1}) \\ &\cdot (1 - x_{c_j+1}) \cdots (1 - x_{c_p})] \\ &\cdot \left[1 - (1 - x_{b_1}) \cdots (1 - x_{b_q})\right].\end{aligned}$$

Therefore, the set $CS_0 \backslash c_j$ is still the cut set of the decomposed system, but we cannot guarantee that it is still a minimal cut set.

The theorem is proved.

C. ALGORITHM FOR THE DECOMPOSITION METHOD

In this part, we are going to put forward an algorithm for the decomposition method, which can be applied in computers. The algorithm is based on Theorem 16, and some basic assumptions are made first.

(1) The hybrid system includes m random components and n uncertain components.
(2) All the components are independent, i.e., the failure of any component will not affect other components.
(3) Each component only has two crisp state modes: failure with $G = 0$ and function with $G = 1$.

The algorithm will first decompose the system by figuring out the state mode combinations of random components and then obtain the cut sets of each decomposed uncertain systems. The cut set theorem will be used to calculate the belief reliability of decomposed uncertain systems. Finally, the belief reliability of the original system can be easily acquired using Eq. (IV.1). The algorithm is summarized in Algorithm 1.

D. COMPLEXITY ANALYSIS

In this part, the computational complexity of the proposed method is analyzed. To show the advantage of this method, a comparison of complexity is also performed between the decomposition method and the reliability formula.

In Algorithm 1, there are three loops. In the first loop, it will traverse all state mode combinations of random components, which will repeat 2^m times. Then, in the second and third loops, the algorithm will adjust the cut sets (at most l sets) according to the stat mode of each component. Finally, the belief reliability of the decomposed system is calculated in the first loop and the computation time will be at most rn. Therefore, the time complexity of the algorithm will be at most $O(2^m \cdot (ml + rn))$. We consider a worst situation, in which the number of the minimal cut sets of original system is $l = m + n$ and the number of cut sets of decomposed system is $r \le l = m + n$. Thus, the complexity of the algorithm will be less or equal to $O(2^m \cdot (m + n)^2)$.

For comparison, let us analyze the complexity of the reliability formula. The formula may only use two loops. The first loop of the method using this formula is to identify the state mode combinations of all random components (2^m times) and obtain the probability of each combination (m times of multiplication). Then, in the second loop, all the state modes of uncertain components are recognized and n times of comparisons are made. The last process is also to make n times of comparisons to get the $Z(y_1, y_2, \ldots, y_m)$. Therefore, the time complexity of the formula is $O(m \cdot 2^m \cdot (2^n \cdot n + n)) = O(mn \cdot 2^{m+n})$.

If there are more random components in the uncertain random system, i.e., $n > m$, then the computational complexity of the proposed algorithm will be less than $O(n^2 \cdot 2^m)$, which is much less than that of the formula. If there are more random components, i.e., $m > n$, then we have the computational complexity of the proposed algorithm to be less than

Algorithm 1 Decomposition Method

Input: Original RBD graph of the system and the belief reliability of each component

Output: The belief reliability of the system $R_{B,S}$

Identify all the minimal cut sets (MCSs) $C_1, C_2, \ldots, C_l$ of the original system.
Identify all the state mode vectors $X^{(i)} = (x_1^{(i)}, x_2^{(i)}, \ldots, x_m^{(i)})$ for m random components with belief reliability $R_{B,j}^{(P)}, j = 1, 2, \ldots, m$.
Set the probability of state sets to be $p^{(i)} = 1, i = 1, 2, \ldots, 2^m$.
for $i = 1$ to 2^m **do**
  Set $r = l$.
  for $j = 1$ to m **do**
    Set $q_1 = 0$ and $q_2 = 0$.
    if $s_j^{(i)} = 1$ **then**
      Set $p^{(i)} = p^{(i)} \cdot R_{B,j}^{(P)}$.
      for $k = 1$ to r **do**
        if $j \in C_k$ **then**
          Delete C_k from the group of MCSs.
          $q_1 = q_1 + 1$.
        end if
      end for
      $r = r - q_1$.
      Reorganize the MCSs as $C_1, \ldots, C_r$
    else
      Set $p^{(i)} = p^{(i)} \cdot \left(1 - R_{B,j}^{(P)}\right)$.
      for $k = 1$ to r **do**
        if $j \in C_k$ **then**
          Delete j from C_k.
          if $C_k = \varnothing$ **then**
            $q_2 = 1$, break.
          end if
        end if
      end for
    end if
  end for
  if $q_2 = 1$ **then**
    $R_B^{(U,i)} = 0$.
  else if $r = 0$ **then**
    $R_B^{(U,i)} = 1$.
  else
    $R_B^{(U,i)} = \bigvee_{1 \le k \le r} \bigwedge_{j \in C_k} R_{B,j}^{(U)}$.
  end if
end for
Calculate system belief reliability by $R_{B,S} = \sum_{i=1}^{2^m} p^{(i)} \cdot R_B^{(U,i)}$.
return $R_{B,S}$.

$O(m^2 \cdot 2^m)$. Thus, in the most extreme cases, our method is simpler as long as the number of the random components is not much more than the number of uncertain components. It also should be noted that we have analyzed the worst case of our method, while in most cases, the proposed method does not cost too much computational resource and is more understandable in applications.

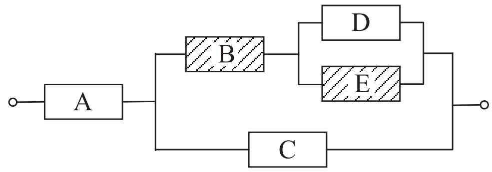


FIGURE 3. Reliability block diagram of the interested system.

TABLE 2. Belief reliability of the decomposed systems.

$S^{(i)}$	$p^{(i)}$	$R_B^{(U,i)}$
(0,×,×)	0.05	0
(1,0,0)	0.00475	0.85
(1,0,1)	0.04275	0.9
(1,1,0)	0.09025	1
(1,1,1)	0.81225	1

V. NUMERICAL CASE STUDY AND COMPARISON

In this section, the proposed decomposition method will be compared with the original belief reliability formula through a numerical case study. We focus on a system with 3 random components and 2 uncertain components, shown in Figure 3. The belief reliability of each component is given as $R_{P,A} = 0.95$, $R_{U,B} = 0.9$, $R_{P,C} = 0.95$, $R_{P,D} = 0.9$ and $R_{U,E} = 0.85$. Then, we use the decomposition method and the belief reliability formula method to calculate the belief reliability of the system respectively and compare the two method.

We first use the proposed decomposition method to calculate system belief reliability. In this method, the minimal cut sets (MCSs) of the system need to be identified first. Using the definition of MCS, we can easily obtain the MCSs of this system to be:

$$MCS_1 = \{A\}, MCS_2 = \{B, D\}, MCS_3 = \{B, C, E\}.$$

Using Algorithm 1, the state mode vectors and the belief reliability of the decomposed system of each vector can be obtained, shown in Table 2.

Therefore, the system belief reliability is

$$R_{B,S} = \sum_i p^{(i)} R_B(U, i) = 0.94501.$$

In this method, the total calculation times is obtained to be 48.

We also use the belief reliability formula shown in Theorem 12 to calculate the belief reliability of the example system. First, the structural function of the system need to be obtained, which can be calculated using the minimal cut sets identified before. Assume the state mode of the five components are y_A, z_B, y_C, y_D and z_E, respectively.

TABLE 3. System belief reliability calculation process using belief reliability formula.

(y_A, y_C, y_D)	(z_B, z_E)	f	$\mu(y_A)\cdot\mu(y_C)\cdot\mu(y_D)$	Z	Calculation times
(0,0,0)	$(\times,\times)$	0	0	0	0
(0,0,1)	$(\times,\times)$	0	0	0	0
(0,1,0)	$(\times,\times)$	0	0	0	0
(0,1,1)	$(\times,\times)$	0	0	0	0
(1,0,0)	(0,0)	0	0.00475	0.85	19
	(0,1)	0			
	(1,0)	0			
	(1,1)	1			
(1,0,1)	(0,0)	0	0.04275	0.9	19
	(0,1)	0			
	(1,0)	1			
	(1,1)	1			
(1,1,0)	(0,0)	1	0.09025	1	19
	(0,1)	1			
	(1,0)	1			
	(1,1)	1			
(1,1,1)	(0,0)	1	0.81225	1	19
	(0,1)	1			
	(1,0)	1			
	(1,1)	1			

Thus, the structural function can be written as:

$$\begin{aligned} f(y_A, y_C, y_D, z_B, z_E) &= [1-(1-y_A)]\cdot[1-(1-z_B)(1-y_D)] \\ &\quad \cdot[1-(1-z_B)(1-y_C)(1-z_E)] \\ &= y_A\cdot[y_C+(z_B\cdot(y_D+y_E))]. \end{aligned} \tag{V.1}$$

Then, all the combinations of components state should be figured out and the belief reliability is calculated based on equation (III.5). Table 3 shows the whole process. It should be noted that in one calculation process, we have to make several comparisons to obtain the value of $\sup\limits_{f(y_1,\ldots,y_m,z_1,\ldots,z_n)=1}\ \min\limits_{1\le j\le n} \nu_j(z_j,t)$. To clarify the computational cost, we also show the times of calculation in the last column of the table.

Therefore, the system belief reliability is

$$\begin{aligned} R_{B,S} &= \sum_{(y_A,y_C,y_D)\in\{0,1\}^3} \mu(y_A)\cdot\mu(y_C)\cdot\mu(y_D)\cdot Z(y_A,y_C,y_D) \\ &= 0.94501. \end{aligned}$$

We can also get the calculation times of this method to be 76.

From the calculation process of the two methods, we can find that it is much easier to obtain the system belief reliability through the decomposition method than the reliability formula. There may be two reasons:

Firstly, the proposed method only requires us to figure out the state mode combinations of random components, not all of them. In the decomposition method, since there are 3 random components, we need to consider 8 situations according to the state modes of random components. However, when using the reliability formula, all 32 state sets should be listed. This may cause a dramatic increase in computational cost.

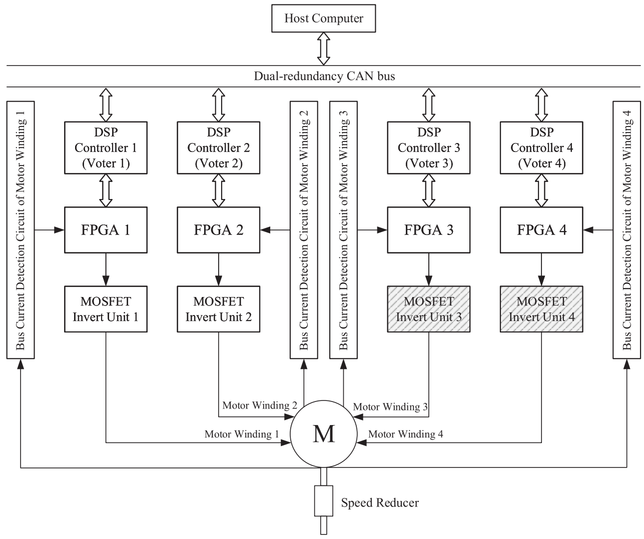


FIGURE 4. Structure of the quad redundant servo system.

Secondly, the calculation process of the decomposition method is more accessible and more understandable. We only need to know the minimal cut sets of the system in the proposed method and do not need to face the complex formula. It is easier for engineers in real applications.

VI. APPLICATION IN REAL SYSTEMS

The proposed method is applied to the belief reliability analysis of a quad redundant servo system [30] with two new MOSFET inverse unit (uncertain components). The structure of the servo system is shown in Fig. 4.

Besides a CAN bus and a motor, the most essential part of the servo system is the voting module. It consists of 3 units, i.e., central control unit, power drive unit and sensor unit, each of which has 4 redundancies. A group of the three units forms one channel of the voting module. The central control unit is composed of a DSP controller and an FPGA, which are used to control speed and displacement. The core device of the power drive unit is a MOSFET inverter, and this unit will generate control current to drive the motor. The sensor unit is responsible for collecting the bus current of the motor. In this system, two MOSFET inverters are new products which embrace sever epistemic uncertainty, thus we model them as uncertain components, while others are regarded as random components.

It is designed that if there are more than two out of four channels are working, the motor speed will be normal. If there is only one channel left, the servo system cannot work normally, but safely.According to the function of the system, the reliability block diagram (RBD) model of this system can be established, as shown in Fig. 5. In the model, $A1 \sim A4$ represent the DSP controllers, $B1 \sim B4$ denote FPGAs, $C1 \sim C4$ are the MOSFET inverter ($C1$ and $C2$ are random components while the other two are uncertain components), $D1 \sim D4$ represent the detection circuits, and M is the motor. The belief reliability of these components are

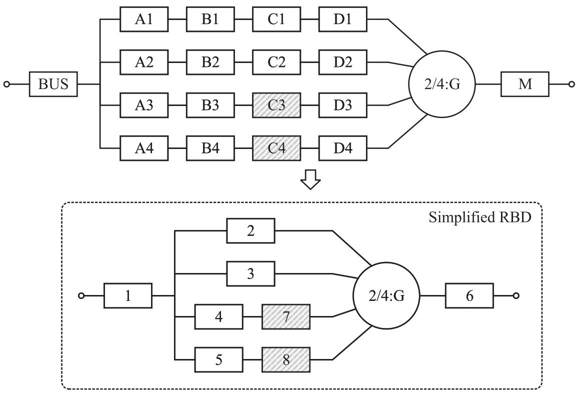


FIGURE 5. RBD model of the servo system.

given as $R_{B,BUS}^{(P)} = 0.9999$, $R_{B,Ai}^{(P)} = 0.999$ $(i = 1 \sim 4)$, $R_{B,Bi}^{(P)} = 0.995$ $(i = 1 \sim 4)$, $R_{B,C1}^{(P)} = R_{B,C2}^{(P)} = 0.998$, $R_{B,C3}^{(P)} = R_{B,C4}^{(P)} = 0.98$, $R_{B,Di}^{(P)} = 0.995$ $(i = 1 \sim 4)$.

Obviously, the whole system is a complex uncertain random system where the uncertain components and random components cannot be separated. In this case, the reliability formula method cannot work out. To use the decomposition method, we can first combine some simple subsystem according to Zhang *et al.* ([22]). The simplified RBD is also shown in Fig. 5 with 6 random components and 2 uncertain components, whose belief reliabilities are $R_{B,1}^{(P)} = 0.9999$, $R_{B,2}^{(P)} = R_{B,3}^{(P)} = 0.9871$, $R_{B,4}^{(P)} = R_{B,5}^{(P)} = 0.9890$, $R_{B,6}^{(P)} = 0.9995$, $R_{B,7}^{(U)} = R_{B,8}^{(U)} = 0.98$. Using the proposed method, we can calculate the system belief reliability to be

$$R_{B,S} = 0.9989.$$

VII. CONCLUSION

This paper presents a new method, called the decomposition method, for belief reliability analysis of complex uncertain random systems. In this method, the main idea is to decompose the original system according to the state mode vectors of random components and then calculate the system belief reliability based on the probability of each vector and the corresponding belief reliability of the decomposed uncertain system. By figuring out the properties of cut sets for decomposed system, an algorithm for belief reliability analysis of complex uncertain random systems is first summarized in this paper. The complexity analysis shows that this algorithm is more efficient in most cases compared with the belief reliability formula method. Moreover, a numerical case study is performed to compare the proposed method with belief reliability formula method. It is seen that this method is more understandable for reliability engineers and can be applied more easily in practice.

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