IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received May 14, 2019, accepted July 11, 2019, date of publication July 16, 2019, date of current version August 6, 2019.

Digital Object Identifier 10.1109/ACCESS.2019.2929238

Measuring the Vulnerability of Alternating Group Graphs and Split-Star Networks in Terms of Component Connectivity

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This work was supported in part by the China Postdoctoral Science Foundation under Grant 2018M631322, in part by the National Natural Science Foundation of China under Grant 11731002, in part by the 111 Project of China under Grant B16002, and in part by the Ministry of Science and Technology, Taiwan, under Grant 107-2221-E-141-001-MY3.

ABSTRACT For an integer $\ell \geqslant 2$, the ℓ-component connectivity of a graph G, denoted by $\kappa_\ell(G)$, is the minimum number of vertices whose removal from G results in a disconnected graph with at least ℓ components or a graph with fewer than ℓ vertices. This is a natural generalization of the classical connectivity of graphs defined in term of the minimum vertex-cut and a good measure of vulnerability for the graph corresponding to a network. So far, the exact values of ℓ-connectivity are known only for a few classes of networks and small ℓ's. It has been pointed out in component connectivity of the hypercubes, *International Journal of Computer Mathematics* 89 (2012) 137–145] that determining ℓ-connectivity is still unsolved for most interconnection networks such as alternating group graphs and star graphs. In this paper, by exploring the combinatorial properties and the fault-tolerance of the alternating group graphs AG_n and a variation of the star graphs called split-stars S_n^2, we study their ℓ-component connectivities. We obtain the following results: 1) $\kappa_3(AG_n) = 4n - 10$ and $\kappa_4(AG_n) = 6n - 16$ for $n \geqslant 4$, and $\kappa_5(AG_n) = 8n - 24$ for $n \geqslant 5$ and 2) $\kappa_3(S_n^2) = 4n - 8$, $\kappa_4(S_n^2) = 6n - 14$, and $\kappa_5(S_n^2) = 8n - 20$ for $n \geqslant 4$.

INDEX TERMS Alternating group graphs, component connectivity, interconnection networks, split-stars, vulnerability.

I. INTRODUCTION

An interconnection network is usually modeled as a connected graph $G(V, E)$, where the vertex set $V(= V(G))$ represents the set of processors and the edge set $E(= E(G))$ represents the set of communication channels between processors. For a subset $S \subseteq V(G)$, the graph obtained from G by removing all vertices of S is denoted by $G - S$. In particular, S is called a *vertex-cut* of G if $G - S$ is disconnected. The *connectivity* of a graph G, denoted by $\kappa(G)$, is the cardinality of a minimum vertex-cut of G, or is defined to be $|V(G)| - 1$ when G is a complete graph. For making a more thorough study on the connectivity of a graph to assess the vulnerability of its corresponding network, a concept of generalization was first introduced by Chartrand *et al.* [9]. For an integer $\ell \geqslant 2$, the *generalized* ℓ*-connectivity* of a graph G, denoted by $\kappa_\ell(G)$, is the minimum number of vertices whose removal from G results in a graph with at least ℓ components or a graph with fewer than ℓ vertices. For such a generalization, a synonym was also called the *general connectivity* [38] or ℓ*-component connectivity* [32]. Since there exist diverse definitions of generalized connectivity in the literature (e.g., see [28], [29]), hereafter we follow the use of the terminology ''ℓ-component connectivity'' (or ℓ*-connectivity* for short) to avoid confusion.

The associate editor coordinating the review of this manuscript and approving it for publication was Bora Onat.

A. PREVIOUS RESULTS OF ℓ-CONNECTIVITY

So far, the exact values of ℓ-connectivity are known only for a few classes of networks and small ℓ's. For example, ℓ-connectivity is determined on hypercube Q_n for $\ell \in [2, n+1]$ (see [32]) and $\ell \in [n + 2, 2n - 4]$ (see [51]), folded hypercube FQ_n for $\ell \in [2, n + 2]$ (see [50]), dual cube D_n for $\ell \in [2, n]$ (see [49]), hierarchical cubic network $HCN(n)$

for $\ell \in [2, n+1]$ (see [19]), complete cubic network $CCN(n)$ for $\ell \in [2, n+1]$ (see [20]), and generalized exchanged hypercube $GEH(s,t)$ for $1 \leqslant s \leqslant t$ and $\ell \in [2, s+1]$ (see [21]). Note that the number of vertices of graphs in the above classes is an exponent related to n. Also, it has been pointed out in [32] that determining ℓ-connectivity is still unsolved for most interconnection networks such as star graphs S_n and alternating group graphs AG_n. The closest results for the two classes of graph were given in [17], [18], but these are asymptotic results. Recently, Guo [26] and Guo *et al.* [27] determined the $\{3, 4\}$-connectivity of twisted cubes and locally twisted cubes, respectively. Also, Chang *et al.* [3], [4] determined the $\{3, 4\}$-connectivity of alternating group networks AN_n. Note that the two classes of AG_n and AN_n are definitely different. See also Table 3 in the final section for the details of the above component connectivities.

B. LITERATURE RELATED TO ALTERNATING GROUP GRAPH AND SPLIT-STARS

In this paper, we study ℓ-connectivity of the n-dimensional alternating group graph AG_n and the n-dimensional split-stars S_n^2 (defined later in Section II), which were introduced by Jwo *et al.* [33] and Cheng *et al.* [16], respectively, for serving as interconnection network topologies of computing systems. The two families of graphs have received much attention because they have many nice properties such as vertex-transitive, strongly hierarchical, maximally connected (i.e., the connectivity is equal to its regularity), and with a small diameter and average distance. In particular, Cheng *et al.* [14] showed that alternating group graphs and split-stars are superior to the n-cubes and star graphs under the comparison using an advanced vulnerability measure called toughness, which was defined in [22]. For the two families of graphs, many researchers were attracted to study fault tolerant routing [12], fault tolerant embedding [5], [6], [42], matching preclusion [2], [11], restricted connectivity [15], [25], [35], [36], [48] and diagnosability [10], [25], [30], [34]–[36], [41]. Moreover, alternating group graphs are also edge-transitive and possess stronger and rich properties on Hamiltonicity (e.g., it has been shown to be not only pancyclic and Hamiltonian-connected [33] but also panconnected [6], panpositionable [40] and mutually independent Hamiltonian [39]). The following structural property disclosed by Cheng *et al.* [18] is of particular interest and closely related to ℓ-component connectivity. They showed that even though linearly many faulty vertices are removed in AG_n, the rest of the graph has still a large connected component that contains almost all the surviving vertices. Therefore, this component can be used to perform original network operations without degrading most of its capability. For more further investigations on alternating group graphs and split-stars, see also [13], [46], [54].

C. APPLICATIONS OF ℓ-CONNECTIVITY AND OUR CONTRIBUTIONS

A multiprocessor system is a collection of autonomous processors linked together to enable parallel processing, where each processor has its own local memory and processors exchange data over a high-speed communication network by a technique known as "message passing". It is well known that the reliability of multiprocessor systems is an important issue for parallel computing. In particular, it must be highly fault-tolerant to ensure that the system will still function properly with a small number of processor failures. Hence, calculating the number of residual components in a faulty network will help to comprehend the vulnerability of the network. Then, further finding out the large connected components which are available in the surviving network will help to achieve fault tolerance. In general, the surviving network can be used as a functional subsystem without degrading the performance if it possesses enough big component [23]. The ℓ-connectivity is concerned with the relevance of the cardinality of a minimum vertex-cut (i.e., a set of faulty processors) and the number of residual components caused by the vertex-cut. Accordingly, finding ℓ-connectivity for certain interconnection networks is a good measure of robustness for such networks. The contribution of this work is that we obtain the ℓ-connectivity of alternating group graphs AG_n and split-stars S_n^2 for the certain cases of $\ell = 3, 4, 5$. Our main results include the following: (i) $\kappa_3(AG_n) = 4n - 10$ and $\kappa_4(AG_n) = 6n - 16$ for $n \geqslant 4$, and $\kappa_5(AG_n) = 8n - 24$ for $n \geqslant 5$; (ii) $\kappa_3(S_n^2) = 4n - 8$, $\kappa_4(S_n^2) = 6n - 14$, and $\kappa_5(S_n^2) = 8n - 20$ for $n \geqslant 4$.

The remaining part of this paper is organized as follows. Section II formally gives the definition of alternating group graphs and split-stars. In addition, we introduce some preliminary results that will be used later. Section III determines the ℓ-component connectivity of AG_n for $\ell = 3, 4, 5$. Section IV determines the ℓ-component connectivity of S_n^2 for $\ell = 3, 4, 5$. The last section contains our concluding remarks.

II. PRELIMINARIES

We first provide Table 1 that contains most of the important notations used in this paper.

For $n \geqslant 3$, let $\mathbb{Z}_n = \{1, 2, \ldots, n\}$ and $p = p_1p_2\cdots p_n$ be a permutation of elements of $\mathbb{Z}_n$, where $p_i \in \mathbb{Z}_n$ is the symbol at the position i in the permutation. Two symbols p_i and p_j are said to be a pair of *inversion* of p if $p_i < p_j$ and $i > j$. A permutation is an *even permutation* provided it has an even number of inversions. Let $\mathcal{S}_n$ (resp., $\mathcal{A}_n$) denote the set of all permutations (resp., even permutations) over $\mathbb{Z}_n$. An operation acting on a permutation that swaps symbols at positions i and j and leaves all other symbols undisturbed is denoted by g_{ij}. The composition $g_{ij}g_{k\ell}$ means that the operation is taken by swapping symbols at positions i and j,

TABLE 1. Notations.

Notations	Meaning
$\mathbb{Z}_n$	The set of integers $\{1, 2, \ldots, n\}$.
$G(V, E)$	A graph G with vertex set V and edge set E.
$V(G)$	The vertex set of a graph G.
$E(G)$	The edge set of a graph G.
$G - S$	Graph obtained from G by removing all vertices in S.
$\kappa(G)$	The connectivity of a graph G.
$\kappa_\ell(G)$	The ℓ-component connectivity of G.
$Cay(X, \Omega)$	Cayley graph on a finite group X with respect to a given generating set Ω of X.
$\mathcal{S}_n$	The set of all permutations over $\mathbb{Z}_n$.
$\mathcal{A}_n$	The set of all even permutations over $\mathbb{Z}_n$.
g_{ij}	An operation acting on a permutation that swaps symbols at positions i and j.
g_i^+	$\equiv g_{2i}g_{12}$.
g_i^-	$\equiv g_{1i}g_{12}$.
AG_n	The n-dimensional alternating group graph.
AG_n^i	The subgraph of AG_n induced by vertices with the rightmost symbol i.
S_n^2	The n-dimensional split-star.
V_n^i	The set of vertices in S_n^2 with the rightmost symbol i.
$S_n^{2:i}$	The subgraph of S_n^2 induced by V_n^i.
$S_{n,E}^2$	Subgraphs of S_n^2 induced by even permutations.
$S_{n,O}^2$	Subgraphs of S_n^2 induced by odd permutations.
$N_G(u)$	The set of neighbors of a vertex u in G.
$N_G(S)$	$\equiv \cup_{u \in S} N_G(u) \setminus S$.
$\kappa^{(h)}(G)$	The h-extra connectivity of G.

and then swapping symbols at positions k and ℓ. For $3 \leqslant i \leqslant n$, we further define two operations, g_i^+ and g_i^- on $\mathcal{A}_n$ by setting $g_i^+ = g_{2i}g_{12}$ and $g_i^- = g_{1i}g_{12}$. Accordingly, pg_i^+ (resp., pg_i^-) is the permutation obtained from p by rotating symbols at positions 1, 2 and i from left to right (resp., from right to left). Taking $\mathcal{A}_5$ as an example, if $p = 13425$, then $pg_4^+ = 21435$ and $pg_4^- = 32415$.

Recall that the *Cayley graph* $Cay(X, \Omega)$ on a finite group X with respect to a generating set Ω of X is defined to have the vertex set X and the edge set $\{(p, pg)\colon p \in X, g \in \Omega\}$. We now formally give the definition of alternating group graphs and split-stars as follows.

Definition 1 (see [33]): The *n-dimensional alternating group graph*, denoted by AG_n, is a graph consisting of the vertex set $V(AG_n) = \mathcal{A}_n$ and two vertices $p, q \in \mathcal{A}_n$ are adjacent if and only if $q \in \{pg_i^+, pg_i^-\}$ for some $i = 3, 4, \ldots, n$. That is, $AG_n = Cay(\mathcal{A}_n, \Omega)$ with $\Omega = \{g_3^+, g_3^-, g_4^+, g_4^+, \ldots, g_n^+, g_n^-\}$.

A path (resp., cycle) of length k is called a k-path (resp., k-cycle). Clearly, from the above definition, AG_3 is isomorphic to a 3-cycle. As a Cayley graph, AG_n is vertex-transitive. Also, it has been shown in [33] that AG_n contains $n!/2$ vertices, $n!(n-2)/2$ edges, and is an edge-transitive and $(2n-4)$-regular graph with diameter $\lfloor 3n/2 \rfloor - 3$. It is well known that every edge-transitive graph is maximally connected, and hence $\kappa(AG_n) = 2n - 4$. For $n \geqslant 3$ and $i \in \mathbb{Z}_n$, let AG_n^i be the subgraph of AG_n induced by vertices with the rightmost symbol i. Like most interconnection networks, AG_n can be defined recursively by a hierarchical structure. Thus, AG_n is composed of n disjoint copies of AG_n^i for $i \in \mathbb{Z}_n$, and each AG_n^i is isomorphic to AG_{n-1}. If a vertex u belongs to a subgraph AG_n^i, we simply write $u \in AG_n^i$ instead of $u \in V(AG_n^i)$. An edge joining vertices in different subgraphs is an *external edge*, and the two adjacent vertices are called *out-neighbors* to each other. By contrast, an edge joining vertices in the same subgraph is called an *internal edges*, and the two adjacent vertices are called *in-neighbors* to each other. Clearly, every vertex of AG_n has $2n - 6$ in-neighbors and two out-neighbors. For example, Fig. 1 depicts AG_3 and AG_4, where each part of shadows in AG_4 indicates a subgraph isomorphic to AG_3.

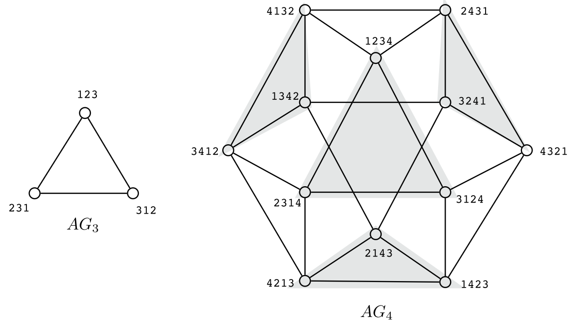

FIGURE 1. Alternating group graphs AG_3 and AG_4.

Cheng *et al.* [16] propose the Split-star networks as alternatives to the star graphs and companion graphs with the alternating group graphs.

Definition 2 (see [16]): The *n-dimensional split-star*, denoted by S_n^2, is a graph consisting of the vertex set $V(S_n^2) = \mathcal{S}_n$ and two vertices $p, q \in \mathcal{S}_n$ are adjacent if and only if $q = pg_{12}$ or $q \in \{pg_i^+, pg_i^-\}$ for some $i = 3, 4, \ldots, n$. That is, $S_n^2 = Cay(\mathcal{S}_n, \Omega)$ with $\Omega = \{g_{12}, g_3^+, g_3^-, g_4^+, g_4^+, \ldots, g_n^+, g_n^-\}$.

In the above definition, the edge generated by the operation g_{12} is called a *2-exchange edge*, and others are called *3-rotation edges*. Let V_n^i be the set of all vertices in S_n^2 with the rightmost symbol i, i.e., $V_n^i = \{p\colon p = p_1p_2\cdots p_{n-1}i, p_j \in \mathbb{Z}_n \setminus \{i\} \text{ for } 1 \leqslant j \leqslant n-1\}$. Also, let $S_n^{2:i}$ denote the subgraph of S_n^2 induced by V_n^i. Clearly, the set $\{V_n^i\colon 1 \leqslant i \leqslant n\}$ forms a partition of $V(S_n^2)$ and $S_n^{2:i}$ is isomorphic to S_{n-1}^2. It is similar to AG_n that every vertex $v \in S_n^{2:i}$ has two out-neighbors, which are joined to v by external edges. Let $S_{n,E}^2$ and $S_{n,O}^2$ be subgraphs of S_n^2 induced by the sets of even permutations and odd permutation, respectively, in which the adjacency applied to each subgraph is precisely using the edge of 3-rotation. Clearly, $S_{n,E}^2$ is the alternating group graph AG_n, and $S_{n,O}^2$ is isomorphic $S_{n,E}^2$ via a mapping $\phi(p_1p_2p_3\cdots p_n) = p_2p_1p_3\cdots p_n$ defined by 2-exchange. Accordingly, there are $n!/2$ edges between $S_{n,E}^2$ and $S_{n,O}^2$, called *matching edges*. Fig. 2 depicts S_4^2, where dashed lines indicate matching edges.

An *independent set* of a graph G is a subset $S \subseteq V(G)$ such that any two vertices of S are nonadjacent in G. For $u \in V(G)$, we define $N_G(u) = \{v \in V(G) : (u, v) \in E(G)\}$, i.e., the set of neighbors of u. Moreover, for $S \subseteq V(G)$, we define $N_G(S) = \{v \in V(G) \setminus S : \exists\, u \in S \text{ such that } (u, v) \in E(G)\}$.

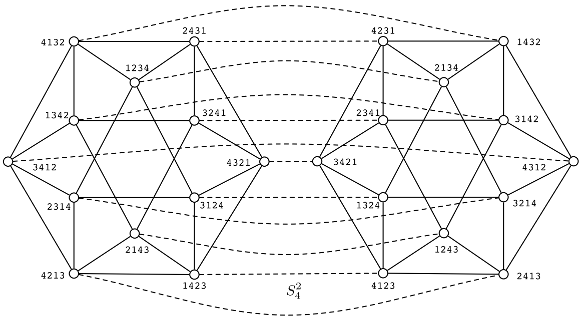

FIGURE 2. Split-star S_4^2.

When the graph G is clear from the context, the subscript in the above notations are omitted. In what follows, we present some useful properties of AG_n, which will be adopted later.

A. ALTERNATING GROUP GRAPHS AND THEIR PROPERTIES

Lemma 1 (see [30]): For AG_n with $n \geqslant 4$, the following properties hold:

(1) There are $(n-2)!$ external edges between any two distinct subgraphs AG_n^i and AG_n^j for $i, j \in \mathbb{Z}_n$ and $i \neq j$.
(2) The two out-neighbors of every vertex of AG_n are contained in different subgraphs.
(3) If u, v are two nonadjacent vertices of AG_n, then $|N(u) \cap N(v)| \leqslant 2$.

Lemma 2 (see [18]): Let F be a vertex-cut of AG_n with $|F| \leqslant 4n-11$. If $n \geqslant 5$, then one of the following conditions holds:

(1) $AG_n - F$ has two components, one of which is a singleton (i.e., a trivial component).
(2) $AG_n - F$ has two components, one of which is an edge, say (u, v). In particular, $|F| = |N(\{u, v\})| = 4n - 11$.

Also, if $n = 4$, the above description still holds except for the following two exceptions. In both cases $AG_4 - F$ has two components, one of which is a 4-cycle and the other is either a 4-cycle (if $|F| = 4$) or a 2-path (if $|F| = 5$).

For example, $F = \{1234, 2143, 3412, 4321\}$ and $F = \{1234, 2143, 3412, 4321, 2314\}$ are two exceptions of $AG_4 - F$ described in Lemma 2, respectively (see Fig. 3).

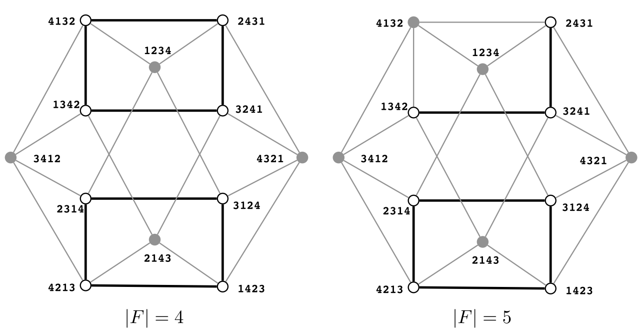

FIGURE 3. Two exception cases of $AG_4 - F$, where the set of gray vertices is a vertex-cut.

A graph is said to be *hyper-connected* [30], [36] or *tightly super-connected* [1] if each minimum vertex-cut creates exactly two components, one of which is a singleton. Since $\kappa(AG_4) = 4$, the first exception illustrates that AG_4 is not hyper-connected. Here we point out a minor flaw in the literatures (e.g., see Proposition 2.4 in [30] and Lemma 1 in [36]), which misrepresents that AG_4 is hyper-connected. As a matter of fact, AG_4 is isomorphic to the line graph of Q_3 (i.e., a 3-dimensional hypercube), and the latter is contained in a list of vertex- and edge-transitive graphs without hyper-connectivity characterized by Meng [37]. For $n \geqslant 5$, since $\kappa(AG_n) = 2n-4 < 4n-11$, by Lemma 2, AG_n is hyper-connected.

The following results are extensions of Lemma 2.

Lemma 3 (see [17]): For $n \geqslant 5$, if F is a vertex-cut of AG_n with $|F| \leqslant 6n-20$, then one of the following conditions holds:

(1) $AG_n - F$ has two components, one of which is a singleton or an edge.
(2) $AG_n - F$ has three components, two of which are singletons.

Lemma 4 (see [30]): For $n \geqslant 5$, if F is a vertex-cut of AG_n with $|F| \leqslant 6n-19$, then one of the following conditions holds:

(1) $AG_n - F$ has two components, one of which is a singleton, an edge or a 2-path.
(2) $AG_n - F$ has three components, two of which are singletons.

Lemma 5 (see [36]): For $n \geqslant 5$, if F is a vertex-cut of AG_n with $|F| \leqslant 8n-29$, then one of the following conditions holds:

(1) $AG_n - F$ has two components, one of which is a singleton, an edge, a 2-path or a 3-cycle.
(2) $AG_n - F$ has three components, two of which are singletons or a singleton and an edge.
(3) $AG_n - F$ has four components, three of which are singletons.

Lemma 6: Let S be an independent set of AG_n for $n \geqslant 4$. Then the following assertions hold.

(1) If $|S| = 3$, then $|N(S)| \geqslant 6n - 16$.
(2) If $|S| = 4$, then $|N(S)| \geqslant 8n - 24$.

Proof: Since AG_n is vertex-transitive, one may choose the identity permutation, denoted by e, as a vertex in S. Since AG_n is $(2n-4)$-regular, if $|S| = 3$ (resp., $|S| = 4$) and there exists no common neighbor between any two vertices of S, then $|N(S)| = 3(2n-4) = 6n-12 \geqslant 6n-16$ (resp., $|N(S)| = 4(2n-4) = 8n-16 \geqslant 8n-24$), as required. In what follows, we assume that $N(e) \cap N(S \setminus \{e\}) \neq \emptyset$ and let $N^+ = \{eg_i^+ : i \in \mathbb{Z}_n \setminus \{1, 2\}\}$ and $N^- = \{eg_i^- : i \in \mathbb{Z}_n \setminus \{1, 2\}\}$. Clearly, $N(e) = N^+ \cup N^-$ and every vertex in $N(e)$ has the symbol 1, 2 or n at the last position. We further define

$$N^{++} = \{(eg_i^+)g_j^+ : i, \quad j \in \mathbb{Z}_n \setminus \{1, 2\} \text{ and } i \neq j\},$$
$$N^{+-} = \{(eg_i^+)g_j^- : i, \quad j \in \mathbb{Z}_n \setminus \{1, 2\} \text{ and } i \neq j\},$$
$$N^{-+} = \{(eg_i^-)g_j^+ : i, \quad j \in \mathbb{Z}_n \setminus \{1, 2\} \text{ and } i \neq j\},$$
$$N^{--} = \{(eg_i^-)g_j^- : i, \quad j \in \mathbb{Z}_n \setminus \{1, 2\} \text{ and } i \neq j\}.$$

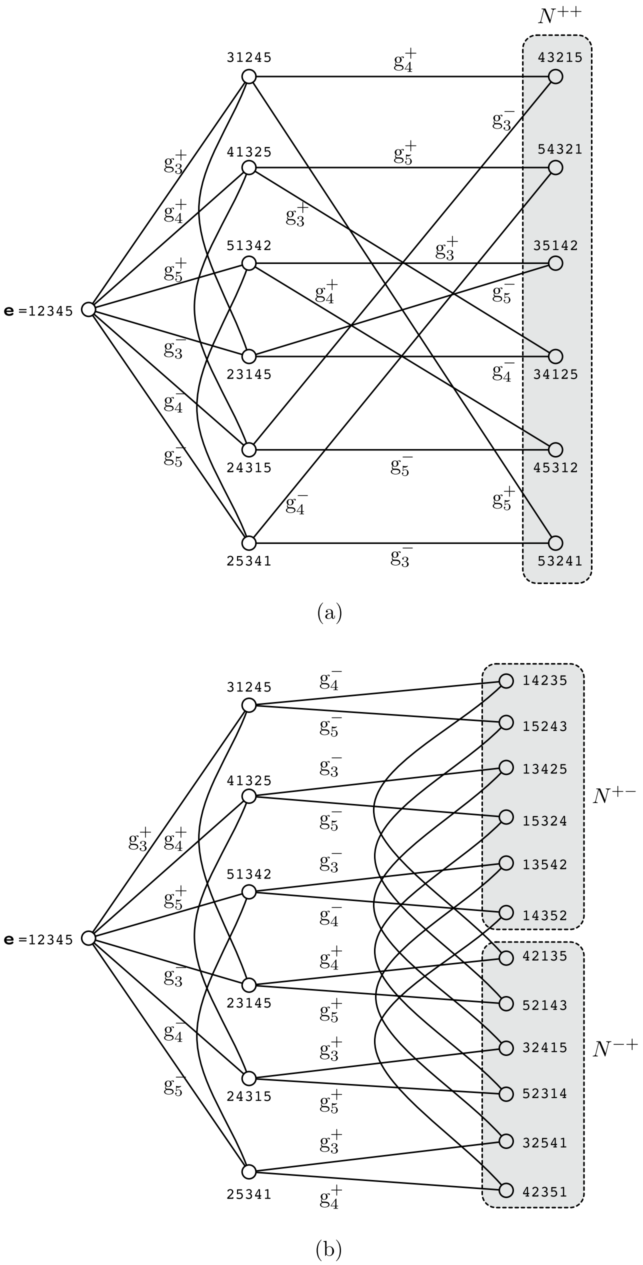

FIGURE 4. Illustration of Lemma 6, where each operation g_i^+ or g_i^- is attached to an edge between vertices (from left to right).

Since $(eg_i^+)g_j^+ = (eg_j^-)g_i^-$, the two sets N^{++} and N^{--} are identical. If $x = (eg_i^+)g_j^+ = (eg_j^-)g_i^-$, then x has the symbol j at the first position and symbol i at the second position. In this case, we have $N(e) \cap N(x) = \{eg_i^+, eg_j^-\}$, which meets the upper bound of Lemma 1(3) (see Fig. 4(a) for an illustration).

Claim 1: For any two distinct vertices $x, y \in N^{++}$, $|N(x) \cap N(y)| \leqslant 1$. Moreover, if $z \in N(x) \cap N(y)$, then $z \in N(e)$.

Proof of Claim 1: Let $x = (eg_i^+)g_j^+$ and $y = (eg_{i'}^+)g_{j'}^+$. Consider the following situations: (i) $i = i'$ and $j \neq j'$. In this case, if there exists a common neighbor, say z, of x and y, then $z = xg_j^- = ((eg_i^+)g_j^+)g_j^- = ((eg_{i'}^+)g_{j'}^+)g_{j'}^- = yg_{j'}^-$. Thus, $z = eg_i^+ \in N^+$ (see, e.g., $x = 43215$, $y = 53241$ and $z = 31245$ in Fig. 4(a)); (ii) $i \neq i'$ and $j = j'$. In this case, if there exists a common neighbor, say z, of x and y, then $z = xg_i^- = ((eg_i^+)g_j^+)g_i^- = ((eg_{i'}^+)g_{j'}^+)g_{i'}^- = yg_{i'}^-$. Thus, $z = eg_j^- \in N^-$ (see, e.g., $x = 43215$, $y = 45312$ and $z = 24315$ in Fig. 4(a)); (iii) $i \neq i'$ and $j \neq j'$. In this case, it is clear that $N(x) \cap N(y) = \emptyset$ (see, e.g., $x = 43215$ and $y = 54321$ in Fig. 4(a)). This settles Claim 1.

On the other hand, the two sets N^{+-} and N^{-+} are not identical. Since every vertex in N(**e**) has two neighbors in $N^{+-} \cup N^{-+}$ and no two vertices of N(**e**) share a common neighbor, if $x \in N^{+-} \cup N^{-+}$, then $|N(e) \cap N(x)| = 1$. In fact, every vertex in N^{+-} has the symbol 1 at the first position, and every vertex in N^{-+} has the symbol 2 at the second position. Thus, both N^{+-} and N^{-+} are independent sets. Since the two symbols 1 and 2 are fixed in the first two positions for vertices in N^{+-} and N^{-+} respectively, every vertex in N^{+-} can be adjacent to at most one vertex of N^{-+}, and vice versa (see Fig. 4(b) for an illustration).

Claim 2: For any two distinct vertices $x, y \in N^{+-}$ or $x, y \in N^{-+}$, $|N(x) \cap N(y)| \leqslant 1$.

Proof of Claim 2: Without loss of generality, we consider $x, y \in N^{+-}$. Let $x = (eg_i^+)g_j^-$ and $y = (eg_{i'}^+)g_{j'}^-$. Consider the following situations: (i) $i = i'$ and $j \neq j'$. In this case, if there exists a common neighbor, say z, of x and y, then $z = xg_j^+ = ((eg_i^+)g_j^-)g_j^+ = ((eg_{i'}^+)g_{j'}^-)g_{j'}^+ = yg_{j'}^+$. Thus, $z = eg_i^+ \in N^+$ (see, e.g., $x = 14235$, $y = 15243$ and $z = 31245$ in Fig. 4(b)); (ii) $i \neq i'$ and $j = j'$. In this case, if there exists a common neighbor, say z, of x and y, then $z = xg_i^+ = ((eg_i^+)g_j^+)g_i^+ = ((eg_{i'}^+)g_{j'}^+)g_{i'}^+ = yg_{i'}^+$ (see, e.g., $x = 14235$, $y = 13425$ and $z = 21435$ in Fig. 4(b)); (iii) $i \neq i'$ and $j \neq j'$. In this case, it is clear that $N(x) \cap N(y) = \emptyset$ (see, e.g., $x = 14235$ and $y = 15324$ in Fig. 4(b)). This settles Claim 2.

Note that two vertices $x \in N^{+-}$ and $y \in N^{-+}$ may have two common neighbors (see, e.g., $x = 14235 \in N^{+-}$ and $y = 32415 \in N^{-+}$ in Fig. 4(b). Then $N(x) \cap N(y) = \{43215, 21435\}$).

Claim 3: If $x \in N^{+-} \cup N^{-+}$ and $y \in N^{++}$, either x and y are adjacent or $|N(x) \cap N(y)| \leqslant 1$.

Proof of Claim 3: Without loss of generality, we consider $x \in N^{+-}$. Let $x = (eg_i^+)g_j^-$ and $y = (eg_{i'}^+)g_{j'}^+$. Consider the following situations: (i) $i = i'$ and $j = j'$. In this case, we have $y = (eg_{i'}^+)g_{j'}^+ = ((eg_i^+)g_j^-)g_j^- = xg_j^-$, and thus x and y are adjacent. (ii) $i = i'$ and $j \neq j'$. In this case, if there exists a common neighbor, say z, of x and y, then $z = xg_j^+ = ((eg_i^+)g_j^-)g_j^+ = ((eg_{i'}^+)g_{j'}^+)g_{j'}^- = yg_{j'}^-$. Thus, $z = eg_i^+ \in N^+$ (see, e.g., $x = 14235$, $y = 53241$ and $z = 31245$ in Fig. 4); (iii) $i \neq i'$. In this case, it is clear that $N(x) \cap N(y) = \emptyset$. This settles Claim 3.

We are now ready to conclude the proof of the lemma. Let $v_0 = e$ and $N_{i,j} = N(v_i) \cap N(v_j)$ for any tow vertices $v_i, v_j \in S$. Consider the following conditions:

For (1), let $S = \{v_0, v_1, v_2\}$. Since $N(v_0) \cap N(S \setminus \{v_0\}) \neq \emptyset$, at least one vertex v_i for $i = 1, 2$ belongs to the sets $N^{++} \cup N^{+-} \cup N^{-+}$. If $v_1, v_2 \in N^{+-} \cup N^{-+}$, then

$|N_{0,1}| = |N_{0,2}| = 1$. Since $|N_{1,2}| \leqslant 2$ by Lemma 1(3), it implies $|N_{0,1} \cup N_{0,2} \cup N_{1,2}| \leqslant 4$. If $v_1, v_2 \in N^{++}$, then $|N_{0,1}| = |N_{0,2}| = 2$. By Claim 1, we have $N_{1,2} \subset N_{0,1} \cup N_{0,2}$. Thus, $|N_{0,1} \cup N_{0,2} \cup N_{1,2}| \leqslant 4$. If $v_1 \in N^{+-} \cup N^{-+}$ and $v_2 \in N^{++}$ (resp., $v_2 \in N^{+-} \cup N^{-+}$ and $v_1 \in N^{++}$), by Claim 3 either v_1 and v_2 are adjacent, which contradicts that S is an independent set, or $|N_{1,2}| \leqslant 1$. Since $|N_{1,2}| \leqslant 1 = |N_{0,1}|$ and $|N_{0,2}| = 2$, it follows that $|N_{0,1} \cup N_{0,2} \cup N_{1,2}| \leqslant 4$. Therefore, we have $|N(S)| = 3(2n-4) - |N_{0,1} \cup N_{0,2} \cup N_{1,2}| \geqslant 6n-16$ for all above situations. Also, it is clear that if $v_1 \notin N^{++} \cup N^{+-} \cup N^{-+}$ or $v_2 \notin N^{++} \cup N^{+-} \cup N^{-+}$, then $|N(S)| \geqslant 6n - 16$.

For (2), let $S = \{v_0, v_1, v_2, v_3\}$. Since $N(v_0) \cap N(S \setminus \{v_0\}) \neq \emptyset$, at least one vertex v_i for $i = 1, 2, 3$ belongs to the sets $N^{++} \cup N^{+-} \cup N^{-+}$. Let $I = \mathbb{Z}_3 \cup \{0\}$ and $J = |\bigcup_{i,j \in I, i \neq j} N_{i,j}|$. If $v_1, v_2, v_3 \in N^{++}$, then $|N_{0,i}| = 2$ for $i \in \mathbb{Z}_3$ and $N_{i,j} \subset N_{0,i} \cup N_{0,j}$ for $i, j \in \mathbb{Z}_3$ and $i \neq j$ (by Claim 1). Thus, $J = 6$. If $v_1, v_2 \in N^{++}$ and $v_3 \in N^{+-} \cup N^{-+}$, we have $|N_{0,1}| = |N_{0,2}| = 2$, $|N_{0,3}| = 1$, $N_{1,2} \subset N_{0,1} \cup N_{0,2}$ (by Claim 1), and $|N_{1,3}|, |N_{2,3}| \leqslant 1$ (by Claim 3). Thus, $J \leqslant 7$. If $v_1 \in N^{++}$ and $v_2, v_3 \in N^{+-}$ (resp., $v_1 \in N^{++}$ and $v_2, v_3 \in N^{-+}$), we have $|N_{0,1}| = 2$, $|N_{0,2}| = |N_{0,3}| = 1$, $|N_{2,3}| \leqslant 1$ (by Claim 2), and $|N_{1,2}|, |N_{1,3}| \leqslant 1$ (by Claim 3). Thus, $J \leqslant 7$. If $v_1 \in N^{++}$, $v_2 \in N^{+-}$ and $v_3 \in N^{-+}$, we have $|N_{0,1}| = 2$, $|N_{0,2}| = |N_{0,3}| = 1$, $|N_{2,3}| \leqslant 2$ (by Lemma 1(3)), and $|N_{1,2}|, |N_{1,3}| \leqslant 1$ (by Claim 3). Thus, $J \leqslant 8$. If $v_1, v_2, v_3 \in N^{+-}$ (resp., $v_1, v_2, v_3 \in N^{-+}$), then $|N_{0,i}| = 1$ for $i \in \mathbb{Z}_3$ and $|N_{i,j}| \leqslant 1$ for $i, j \in \mathbb{Z}_3$ and $i \neq j$ (by Claim 2). Thus, $J \leqslant 6$. If $v_1, v_2 \in N^{+-}$ and $v_3 \in N^{-+}$ (resp., $v_1, v_2 \in N^{-+}$ and $v_3 \in N^{+-}$), we have $|N_{0,i}| = 1$ for $i \in \mathbb{Z}_3$, $|N_{1,2}| \leqslant 1$ (by Claim 2), and $|N_{1,3}|, |N_{2,3}| \leqslant 2$ (by Lemma 1(3)). Thus, $J \leqslant 8$. Therefore, we have $|N(S)| = 4(2n-4) - J \geqslant 8n-24$ for all above situations. Also, if $v_i \notin N^{++} \cup N^{+-} \cup N^{-+}$ for any $i \in \mathbb{Z}_3$, we have $|N(S)| = |N(S \setminus \{v_i\})| + |N(v_i)| \geqslant (6n-16) + (2n-4) \geqslant 8n-24$. □

Form Fig. 1 it easy to check that the set $S = \{e = 1234, (eg_3^+)g_4^+ = 4321, (eg_4^+)g_3^+ = 3412\}$ (resp., $S = \{e = 1234, (eg_3^+)g_4^+ = 4321, (eg_4^+)g_3^+ = 3412, ((eg_4^+)g_3^-)g_4^+ = 2143\}$) is an independent set of AG_4 such that $N(S) = 8$. Clearly, these examples show that the bounds on the assertions of Lemma 6 are tight for $n = 4$. Indeed, based on this observation, the following properties can easily be proved by induction on n.

Remark 1: For $n \geqslant 4$, the following assertions hold:

(1) The set $S = \{e, (eg_i^+)g_j^+, (eg_j^+)g_i^+\}$ for $i, j \in \mathbb{Z}_n \setminus \{1, 2\}$ and $i \neq j$ is an independent set such that $N(S) = 6n-16$.
(2) The set $S = \{e, (eg_i^+)g_j^+, (eg_j^+)g_i^+, ((eg_j^+)g_i^-)g_j^+\}$ for $i, j \in \mathbb{Z}_n \setminus \{1, 2\}$ and $i \neq j$ is an independent set such that $N(S) = 8n - 24$.

B. SPLIT-STARS AND THEIR PROPERTIES

Lemma 7 (see [13], [15], [16]): For S_n^2 with $n \geqslant 4$, the following properties hold:

(1) S_n^2 is $(2n-3)$-regular and $\kappa(S_n^2) = 2n - 3$ for $n \geqslant 2$.
(2) The two out-neighbors of every vertex in $S_n^{2:i}$ are contained in different subgraphs and these two out-neighbors are adjacent. For any two vertices in the same subgraph $S_n^{2:i}$, their out-neighbors in other subgraphs are different. There are $2(n-2)!$ external edges between any two distinct subgraphs $S_n^{2:i}$ and $S_n^{2:j}$ for $i, j \in \mathbb{Z}_n$ and $i \neq j$.
(3) If x, y are any two vertices of S_n^2, then
$$|N(x) \cap N(y)| \leqslant \begin{cases} 1 & \text{if } d(x, y) = 1; \\ 2 & \text{if } d(x, y) = 2; \\ 0 & \text{if } d(x, y) \geqslant 3, \end{cases}$$
where $d(x, y)$ stands for the distance (i.e., the number of edges in a shortest path) between x and y in S_n^2.

Lemma 8 (see [13]): For $n \geqslant 4$, if F is a vertex-cut of S_n^2 with $|F| \leqslant 4n - 8$, then one of the following conditions holds:

(1) $S_n^2 - F$ has two components, one of which is a singleton.
(2) $S_n^2 - F$ has two components, one of which is an edge, say (u, v). If (u, v) is a 2-exchange edge, then $|F| = |N(\{u, v\})| = 4n - 8$; otherwise, $F = F_1 \cup F_2$, where $F_1 = N(\{u, v\})$, $|N(u) \cap N(v)| = 1$, and $|F_2| \leqslant 1$.
(3) $S_n^2 - F$ has three components, two of which are singletons, say u and v. Moreover, $F = N(u) \cup N(v)$ and $|N(u) \cap N(v)| = 2$, hence $|F| = 4n - 8$.

Lemma 9 (see [34]): For $n \geqslant 5$, if F is a vertex-cut of S_n^2 with $|F| \leqslant 6n - 17$, then one of the following conditions holds:

(1) $S_n^2 - F$ has two components, one of which is a singleton, an edge or a 2-path.
(2) $S_n^2 - F$ has three components, two of which are singletons.

Lemma 10 (see [34]): For $n \geqslant 5$, if F is a vertex-cut of S_n^2 with $|F| \leqslant 8n - 25$, then one of the following conditions holds:

(1) $S_n^2 - F$ has two components, one of which is a singleton, an edge, a 2-path or a 3-cycle.
(2) $S_n^2 - F$ has three components, two of which are singletons or a singleton and an edge.
(3) $S_n^2 - F$ has four components, three of which are singletons.

Lemma 11: Let S be an independent set of S_n^2 for $n \geqslant 4$. Then the following assertions hold.

(1) If $|S| = 2$, then $|N(S)| \geqslant 4n - 8$.
(2) If $|S| = 3$, then $|N(S)| \geqslant 6n - 14$.
(3) If $|S| = 4$, then $|N(S)| \geqslant 8n - 20$.

Proof: Recall that S_n^2 contains two copies of AG_n, namely $S_{n,E}^2$ and $S_{n,O}^2$. For notational convenience, we simply write $N_{S_n^2}(U)$, $N_{S_{n,E}^2}(U)$ and $N_{S_{n,O}^2}(U)$ as $N(U)$, $N_E(U)$ and $N_O(U)$ for any subset of vertices $U \subset V(S_n^2)$, respectively. Consider the following conditions:

For (1), let $S = \{v_1, v_2\}$. By Lemma 7(3), v_1 and v_2 has at most two common neighbors, $|N(S)| = |N(v_1)| + |N(v_2)| - |N(v_1) \cap N(v_2)| \geqslant 2(2n-3) - 2 = 4n - 8$.

For (2), let $S = \{v_1, v_2, v_3\}$. We consider the following cases.

Case 2.1: Three vertices v_1, v_2, v_3 are contained in a common subgraph. Without loss of generality, assume $v_1, v_2, v_3 \in S^2_{n,E}$. Since $S^2_{n,E}$ is isomorphic to AG_n, by Lemma 6(1), $|N_E(S)| \geqslant 6n - 16$. Since each vertex of $\{v_1, v_2, v_3\}$ is joined a neighbor by a matching edge, we have $|N(S)| = |N_E(S)| + |N_O(S)| \geqslant (6n - 16) + 3 \geqslant 6n - 14$.

Case 2.2: Three vertices v_1, v_2, v_3 are distributed in two distinct subgraphs. Without loss of generality, assume $v_1, v_2 \in S^2_{n,E}$ and $v_3 \in S^2_{n,O}$. Since both $S^2_{n,E}$ and $S^2_{n,O}$ are isomorphic to AG_n, by Lemma 1(3), $|N_E(\{v_1, v_2\})| \geqslant 2(2n-4) - 2 = 4n - 10$ and $|N_E(v_3)| = 2n - 4$. Thus, $|N(S)| \geqslant |N_E(\{v_1, v_2\})| + |N_O(v_3)| \geqslant (4n-10) + (2n-4) = 6n - 14$.

For (3), let $S = \{v_1, v_2, v_3, v_4\}$. We consider the following cases.

Case 3.1: Four vertices v_1, v_2, v_3, v_4 are contained in a common subgraph. Without loss of generality, assume $v_1, v_2, v_3, v_4 \in S^2_{n,E}$. Since $S^2_{n,E}$ is isomorphic to AG_n, by Lemma 6(2), $|N_E(S)| \geqslant 8n - 24$. Since each vertex of $\{v_1, v_2, v_3\}$ is joined a neighbor by a matching edge, we have $|N(S)| = |N_E(S)| + |N_O(S) \geqslant (8n - 24) + 4 = 8n - 20$.

Case 3.2: Four vertices v_1, v_2, v_3, v_4 are distributed equally in two distinct subgraphs. Without loss of generality, assume $v_1, v_2 \in S^2_{n,E}$ and $v_3, v_4 \in S^2_{n,O}$. By Lemma 1(3), $|N_E(\{v_1, v_2\})| = |N_O(\{v_3, v_4\})| \geqslant 2(2n-4) - 2 = 4n - 10$. Thus, $|N(S)| \geqslant |N_E(\{v_1, v_2\})| + |N_O(\{v_3, v_4\})| \geqslant 8n - 20$.

Case 3.3: Four vertices v_1, v_2, v_3, v_4 are distributed nonequally in two distinct subgraphs. Without loss of generality, assume $v_1, v_2, v_3 \in S^2_{n,E}$ and $v_4 \in S^2_{n,O}$. Since both $S^2_{n,E}$ and $S^2_{n,O}$ are isomorphic to AG_n, by Lemma 6(1), $|N_E(\{v_1, v_2, v_3\})| \geqslant 6n - 16$ and $|N_O(v_4)| = 2n - 4$. Thus, $|N(S)| \geqslant |N_E(\{v_1, v_2, v_3\})| + |N_O(\{v_4\})| \geqslant 8n - 20$. □

III. THE ℓ-COMPONENT CONNECTIVITY OF AG_N

Lemma 12: For $n \geqslant 4$, $\kappa_3(AG_n) = 4n - 10$.

Proof: By Lemma 2, if F is a vertex-cut with $|F| \leqslant 4n - 11$, then $AG_n - F$ has exact two components. Thus, $\kappa_3(AG_n) \geqslant 4n - 10$. We now prove $\kappa_3(AG_n) \leqslant 4n - 10$ as follows. For $n \geqslant 4$, since AG_n is pancyclic, let (w, x, y, z, w) be a 4-cycle. Also, let $F = N(\{w, y\})$. By Lemma 1(3), we have $N(w) \cap N(y) = \{x, z\}$. Since every vertex of AG_n has $2n-4$ neighbors and w and y share exactly two common neighbors, we have $|F| = 2(2n-4) - 2 = 4n - 10$. Clearly, the removal of F from AG_n results in a surviving graph with a large connected component and two singletons w and y. This attains the upper bound. □

Suppose that S is an independent set with the maximum cardinality in AG_4 and let $F = V(AG_4) \setminus S$. Obviously, $|S| = 4$ (e.g., $S = \{1234, 2143, 3412, 4321\}$) and F is a vertex-cut of AG_4. Thus, $\kappa_4(AG_4) \leqslant 8$. From the maximality of S, if we choose a vertex $u \in S$, the remaining three vertices of S are determined involuntary. Since AG_4 is vertex-transitive, F is the unique vertex-cut of size 8 (up to isomorphism) in AG_4 such that $AG_4 - F$ has four components. Thus, there is no vertex-cut F with $|F| \leqslant 7$ such that $AG_4 - F$

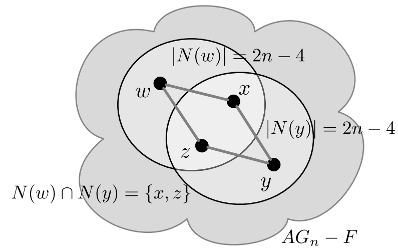

FIGURE 5. An illustration of Lemma 12, where a shape of cloud indicates the large component of $AG_n - F$.

contains four components. This shows that $\kappa_4(AG_4) \geqslant 8$. As a result, we have the following lemma.

Lemma 13: $\kappa_4(AG_4) = 8$.

We denote by $c(G)$ the number of components in a graph G. Hereafter, we suppose that F is a vertex-cut of AG_n and, for convenience, vertices in F (resp., not in F) are called *faulty vertices* (resp., *fault-free vertices*). For each $i \in \mathbb{Z}_n$, let $F_i = F \cap V(AG^i_n)$, $G_i = AG^i_n - F_i$, $f_i = |F_i|$, and $c(G_i)$ be the number of components of G_i. Also, let $I = \{i \in \mathbb{Z}_n :$ G_i is disconnected$\}$ and $J = \mathbb{Z}_n \setminus I$. In addition, we adopt the following notations:

$$F_I = \bigcup_{i \in I} F_i,\ F_J = \bigcup_{j \in J} F_j,\ AG^I_n = \bigcup_{i \in I} AG^i_n, \text{ and}$$
$$AG^J_n = \bigcup_{j \in J} AG^j_n.$$

Lemma 14: $\kappa_4(AG_5) \geqslant 14$.

Proof: Let F be a vertex-cut of AG_5 with $|F| \leqslant 13$. Since each subgraph AG^i_5 is isomorphic to AG_4, we have $\kappa(AG^i_5) = 4$. If $|I| \geqslant 4$, then $|F| \geqslant 4|I| \geqslant 16$, a contradiction. Thus, $|I| \leqslant 3$. By the definition of J, G_j is connected for $j \in J$. If $I = \emptyset$, then $J = \mathbb{Z}_5$. By Lemma 1(1), there are $(5-2)! = 6$ independent edges between AG^i_5 and AG^j_5 for $i, j \in J$ with $i \neq j$. Since $|F| \leqslant 13 < 3 \times (5-2)!$, every G_i is connected to at least two subgraphs G_j and G_k for $j, k \in J \setminus \{i\}$ when $I = \emptyset$. This further implies that $AG_5 - F$ is connected, a contradiction. So, $1 \leqslant |I| \leqslant 3$. Let H be the union of components of $AG_5 - F$ such that all vertices of H are contained in $\bigcup_{i \in I} V(G_i)$. We claim that $AG^J_5 - F_J$ is connected and $c(H) \leqslant 2$. Thus, counting together with the component that contains $AG^J_5 - F_J$ as a subgraph, $AG_5 - F$ contains $c(H) + 1 \leqslant 3$ components and the result follows. We now prove our claim by the following three cases:

Case 1: $|I| = 1$. Without loss of generality, assume $I = \{1\}$. In this case, G_1 is disconnected and $f_1 \geqslant \kappa(AG^1_5) = 4$. By Lemma 1(1), since $|F_J| = |F| - f_1 \leqslant 13 - 4 = 9 < 2 \times (5-2)!$, every G_i for $i \in J$ is connected to at least two subgraphs G_j and G_k for $j, k \in J \setminus \{i\}$. This further implies that $AG^J_5 - F_J$ is connected. By the definition of H, we have $V(H) \subseteq V(G_1)$ and H is not connected to $AG^J_5 - F_J$. Since by Lemma 1(2) every vertex of H has exactly two faulty out-neighbors in F_J, $2|V(H)| \leqslant |F_J| \leqslant 9$, which implies $|V(H)| \leqslant 4$. If $|V(H)| = 4$, then $|F| - f_1 = |F_J| \geqslant 2|V(H)| = 8$. It follows that $f_1 \leqslant |F| - 8 \leqslant 13 - 8 = 5 = 4 \times 4 - 11$. By Lemma 2, G_1 has two components, and thus $c(H) \leqslant c(G_1) = 2$. If $|V(H)| = 3$, then $c(H) \leqslant 2$. Otherwise, H contains three singletons (i.e., an independent set of three vertices), and by Lemma 6(1), $|F| \geqslant |N_{AG_5}(V(H))| \geqslant 6 \times 5 - 16 = 14$, a contradiction. Also, if $|V(H)| \leqslant 2$, it is clear that $c(H) \leqslant |V(H)| \leqslant 2$.

Case 2: $|I| = 2$. Without loss of generality, assume $I = \{1, 2\}$. Then, both G_1 and G_2 are disconnected graphs and $f_1, f_2 \geqslant 4$. By Lemma 1(1), since $|F_J| = |F| - f_1 - f_2 \leqslant 13 - 8 = 5 < (5-2)!$, $AG_5^J - F_J$ is connected. There are two subcases as follows:

Case 2.1: $f_1, f_2 \in \{4, 5\}$. For $i \in \{1, 2\}$, since $f_i \leqslant 4 \times 4 - 11$, by Lemma 2, there are four situations as follows: (i) G_i contains a singleton and a larger component that is connected to $AG_5^J - F_J$; (ii) G_i contains an edge and a larger component that is connected to $AG_5^J - F_J$; (iii) G_i contains two disjoint 4-cycles; and (iv) G_i contains a 4-cycle and a 2-path (See Fig. 6). By Lemma 1(2), every vertex of $V(G_i)$ has exactly two out-neighbors. In the latter two situations, since $|F_J| + f_j \leqslant 5 + 5 = 10 < 2|V(G_i)|$ where $j \in I \setminus \{i\}$, it implies that at least one component of G_i must be connected to $AG_5^J - F_J$. Thus, H contains at most one component of G_i for $i = 1, 2$. This shows that $c(H) \leqslant 2$.

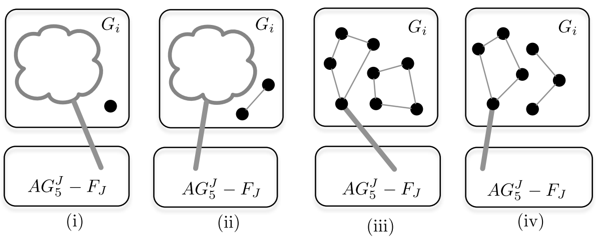

FIGURE 6. An illustration of Case 2.1 in Lemma 14, where a shape of cloud indicates the large component: (i) and (ii) occur when $f_j \leqslant 5$, (iii) occurs when $f_j = 4$, and (iv) occurs when $f_j = 5$.

Case 2.2: $f_1 \geqslant 6$ (resp., $f_2 \geqslant 6$). Then $|F_J| = |F| - f_1 - f_2 \leqslant 13 - 6 - 4 = 3$. By Lemma 1(2), if a vertex $u \in F_J$ have two fault-free out-neighbors, say u_1 and u_2, in H, then $u_1 \in V(G_1)$ and $u_2 \in V(G_2)$ (or vice versa). In this case, the vertex u must be the form with a permutation $12\cdots k$ where $k \in J$. Clearly, $u_1 = 2k\cdots 1$ and $u_2 = k1\cdots 2$. So u_1 and u_2 are adjacent in H. Since $|F_J| \leqslant 3$, H contains at most three components, say H_i for $i = 1, 2, 3$ if they exist (See Fig. 7). Now, we show that $c(H) \leqslant 2$ by contradiction. Suppose that there exists a vertex $v_i \in V(H_i)$ for every $i \in \{1, 2, 3\}$. Since H_i and H_j are not connected in H for any $i, j \in \{1, 2, 3\}$ with $i \neq j$, $\{v_1, v_2, v_3\}$ is an independent set of AG_5. Clearly, $N_{AG_5}(V(H_i))$ is a vertex-cut of AG_5 for each $i \in \{1, 2, 3\}$. Since AG_5 is hyper-connected, $|N_{AG_5}(V(H_i))| \geqslant \kappa(AG_5) = |N_{AG_5}(v_i)|$. By Lemma 6(1), $|F| \geqslant |N_{AG_5}(V(H_1) \cup V(H_2) \cup V(H_3))| \geqslant |N_{AG_5}(\{v_1, v_2, v_3\})| \geqslant 6 \times 5 - 16 = 14$, a contradiction.

Case 3: $|I| = 3$. Without loss of generality, assume $I = \{1, 2, 3\}$. Since $|F| \leqslant 13$ and $f_i \geqslant 4$ for $i \in I$, it implies $|F_J| = |F| - f_1 - f_2 - f_3 \leqslant 13 - 3 \times 4 = 1$. By Lemma 1(1), $AG_5^J - F_J$ is connected. Also, we have $f_i \leqslant |F| - f_j - f_k \leqslant 13 - 4 - 4 = 5$

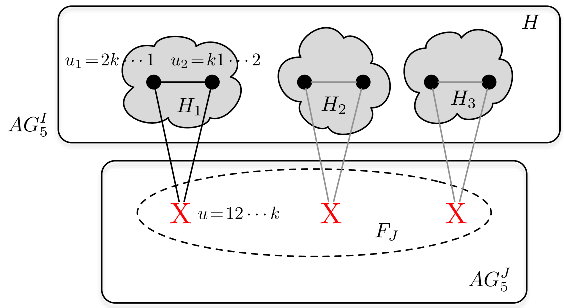

FIGURE 7. An illustration of Case 2.2 in Lemma 14.

for each $i \in I$ where $j, k \in I \setminus \{i\}$ with $j \neq k$. Since $f_i \in \{4, 5\}$, through an argument similar to Case 2.1, we can show that H contains at most one component of G_i, say H_i if it exists, for $i = 1, 2, 3$. If any two H_i and H_j are connected in H for $i, j \in I$, then $c(H) \leqslant 2$. Otherwise, through an argument similar to Case 2.2 by considering an independent set $\{v_1, v_2, v_3\}$ where $v_i \in V(H_i)$, we can show that at least one component H_i for $i \in I$ does not exist. Thus, $c(H) \leqslant 2$. □

Lemma 15: For $n \geqslant 4$, $\kappa_4(AG_n) = 6n - 16$.

Proof: If $n = 4$, the result is proved in Lemma 13. For $n \geqslant 5$, the upper bound $\kappa_4(AG_n) \leqslant 6n - 16$ can be acquired from Remark 1(1) by considering the removal of $N(\{v_0, v_1, v_2\})$, where $\{v_0, v_1, v_2\}$ is an independent set of AG_n and $|N(\{v_0, v_1, v_2\})| = 6n - 16$. Thus, the resulting graph has four components, three of which are singletons. Lemma 14 proves the lower bound $\kappa_4(AG_n) \geqslant 6n - 16$ for $n = 5$, and we now consider $n \geqslant 6$ as follows.

Let F be any vertex-cut of AG_n such that $|F| \leqslant 6n - 17$. Lemma 4 shows that the removal of a vertex-cut with no more than $6n - 19$ vertices in AG_n results in a disconnected graph with at most three components. To complete the proof, we need to show that the same result holds when $6n - 18 \leqslant |F| \leqslant 6n - 17$. Recall $I = \{i \in \mathbb{Z}_n : G_i \text{ is disconnected}\}$ and $J = \mathbb{Z}_n \setminus I$. By definition, G_j is connected for all $j \in J$. Since $|F| \leqslant 6n - 17 < (n-2)!$ when $n \geqslant 6$, $AG_n^J - F_J$ remains connected for arbitrary J. Since AG_n^i is isomorphic to AG_{n-1}, we have $\kappa(AG_n^i) = 2n - 6$. If $|I| \geqslant 4$, then $|F| \geqslant |I| \times (2n - 6) \geqslant 8n - 24 > 6n - 17$, a contradiction. Also, if $I = \emptyset$, then $AG_n - F$ is connected, a contradiction. Thus, $1 \leqslant |I| \leqslant 3$. Let H be the union of components of $AG_n - F$ such that all vertices of H are contained in $\bigcup_{i \in I} V(G_i)$. In the following, we will show that $c(H) \leqslant 2$. Thus, counting together with the component that contains $AG_n^J - F_J$ as a subgraph, $AG_n - F$ contains $c(H) + 1 \leqslant 3$ components. We consider the following three cases:

Case 1: $|I| = 1$. Without loss of generality, assume $I = \{1\}$. In this case, $V(H) \subseteq V(G_1)$. We analyze the number of faulty vertices of F_J as follows. For $|F_J| \leqslant 7$, since every vertex of H has exactly two faulty out-neighbors in F_J by Lemma 1(2), $2|V(H)| \leqslant |F_J| \leqslant 7$, which implies $|V(H)| \leqslant 3$. If $|V(H_1)| = 3$, then $c(H) \leqslant 2$. Otherwise, H_1 contains three singletons (i.e., an independent set of three vertices), and by Lemma 6(1), $|F| \geqslant |N_{AG_n}(V(H))| \geqslant 6n - 16$, a contradiction. Also, if $|V(H)| \leqslant 2$, it is clear that $c(H) \leqslant |V(H)| \leqslant 2$. On the other hand, we consider $|F_J| \geqslant 8$. Since F_1 is a vertex-cut of AG_n^1 and $f_1 = |F| - |F_J| \leqslant (6n-17) - 8 = 6(n-1) - 19$, by Lemma 4, G_1 contains at most three components in which the largest component is connected to $AG_n^J - F_J$. Thus, $c(G_1) \leqslant 3$ and $c(H) = c(G_1) - 1 \leqslant 2$.

Case 2: $|I| = 2$. Without loss of generality, assume $I = \{1, 2\}$. If $f_1 \geqslant 4n-14$ or $f_2 \geqslant 4n-14$, then $|F_J| = |F| - f_1 - f_2 \leqslant (6n-17) - (4n-14) - (2n-6) = 3$. By Lemma 1(2), every vertex of H has at least one faulty out-neighbor in F_J. Thus, $c(H) \leqslant |V(H)| \leqslant |F_J| \leqslant 3$. If $c(H) = 3$, then each component is a singleton. By Lemma 6(1), $|F| \geqslant N(H) \geqslant 6n - 16$, a contradiction. Thus $c(H) \leqslant 2$. We now consider $f_1, f_2 \leqslant 4n-15 = 4(n-1) - 11$. For $i \in \{1, 2\}$, by Lemma 2, G_i contains two components, one is either a singleton or an edge, and the other is a larger component connecting to $AG_n^J - F_J$. Thus, $c(G_i) = 2$ for $i = 1, 2$ and $c(H) \leqslant c(G_1) + c(G_2) - 2 = 2$.

Case 3: $|I| = 3$. Without loss of generality, assume $I = \{1, 2, 3\}$. Since $|F| \leqslant 6n-17$ and $f_i \geqslant 2n-6$ for $i \in I$, it implies $f_i \leqslant |F| - f_j - f_k \leqslant (6n-17) - 2(2n-6) = 2n-5$ where $j, k \in I \setminus \{i\}$ with $j \neq k$. Since $f_i \leqslant 2n-5 < 4(n-1) - 11$ for $n \geqslant 6$, by Lemma 2, for each $i \in I$, G_i contains two components, one is a singleton, say v_i, and the other is a larger component connecting to $AG_n^J - F_J$. If $\{v_1, v_2, v_3\}$ is an independent set of AG_n, by Lemma 6(1), $|F| \geqslant N(\{v_1, v_2, v_3\}) \geqslant 6n-16$, a contradiction. Thus, at least two vertices of v_1, v_2 and v_3 are connected, which implies $c(H) \leqslant 2$. □

Lemma 16: $\kappa_5(AG_5) \geqslant 16$.

Proof: Let F be a vertex-cut of AG_5 with $|F| \leqslant 15$. Since each subgraph AG_5^i is isomorphic to AG_4, we have $\kappa(AG_5^i) = 4$. If $|I| \geqslant 4$, then $|F| \geqslant 4|I| \geqslant 16$, a contradiction. Thus, $|I| \leqslant 3$. By the definition of J, G_j is connected for $j \in J$. If $I = \emptyset$, then $J = \mathbb{Z}_5$. Through an argument similar to Lemma 14, we have $AG_5 - F$ is connected, a contradiction. So, $1 \leqslant |I| \leqslant 3$. Let H be the union of components of $AG_5 - F$ such that all vertices of H are contained in $\bigcup_{i \in I} V(G_i)$. We claim that $AG_5^J - F_J$ is connected and $c(H) \leqslant 3$. Thus, counting together with the component that contains $AG_5^J - F_J$ as a subgraph, $AG_5 - F$ contains $c(H) + 1 \leqslant 4$ components and the result follows. We now prove our claim by the following three cases:

Case 1: $|I| = 1$. Without loss of generality, assume $I = \{1\}$. In this case, G_1 is disconnected and $f_1 \geqslant \kappa(AG_5^1) = 4$. By Lemma 1(1), since $|F_J| = |F| - f_1 \leqslant 15 - 4 = 11 < 2 \times (5-2)!$, every G_i for $i \in J$ is connected to at least two subgraphs G_j and G_k for $j, k \in J \setminus \{i\}$. This further implies that $AG_5^J - F_J$ is connected. By the definition of H, we have $V(H) \subseteq V(G_1)$ and H is not connected to $AG_5^J - F_J$. Since by Lemma 1(2) every vertex of H has exactly two faulty out-neighbors in F_J, $2|V(H)| \leqslant |F_J| \leqslant 11$, which implies $|V(H)| \leqslant 5$. If $|V(H)| = 5$, then $|F| - f_1 = |F_J| \geqslant 2|V(H)| = 10$. It follows that $f_1 \leqslant |F| - 10 \leqslant 15 - 10 = 5 = 4 \times 4 - 11$. By Lemma 2, G_1 has two components, and thus $c(H) \leqslant c(G_1) = 2$. If $|V(H)| = 4$, then $c(H) \leqslant 3$. Otherwise, H contains four singletons (i.e., an independent set of four vertices), and by Lemma 6(1), $|F| \geqslant |N_{AG_5}(V(H))| \geqslant 8 \times 5 - 24 = 16$, a contradiction. Also, if $|V(H)| \leqslant 3$, it is clear that $c(H) \leqslant |V(H)| \leqslant 3$.

Case 2: $|I| = 2$. Without loss of generality, assume $I = \{1, 2\}$ and $f_1 \geqslant f_2$. Then, both G_1 and G_2 are disconnected graphs and $f_1 \geqslant f_2 \geqslant 4$. By Lemma 1(1), since $|F_J| = |F| - f_1 - f_2 \leqslant 15 - 8 = 7 < 3(5-2)!$, $AG_5^J - F_J$ is connected. There are three subcases as follows:

Case 2.1: $f_1, f_2 \in \{4, 5\}$. Through an argument similar to Case 2.1 in Lemma 14, we know the result holds.

Case 2.2: $f_1 \geqslant 6$ and $4 \leqslant f_2 \leqslant 5$. Then $|F_J| = |F| - f_1 - f_2 \leqslant 15 - 6 - 4 = 5$. Since $|F_J| \leqslant 5$, by the similar proof of case 2.2 of Lemma 13, we have $|V(H)| \leqslant 5$. If $|V(H)| = 5$, then $f_1 = 6$ and $f_2 = 4$. We claim $c(H) = 2 \leqslant 3$. For $i \in \{1, 2\}$, let $H_i \subseteq H$ be the set of components such that all vertices of H_i are contained in G_i. By Lemma 13 and $f_1 = 6 < \kappa_4(AG_4) = 8$, G_1 has at most three components and $c(H_1) \leqslant 2$. By Lemma 2, G_2 has two components, one of which is a singleton or a four cycle and $c(H_2) = 1$. It implies that $c(H) \leqslant 3$ (See Fig. 8 for two situations). If $|V(H)| = 4$, then $c(H) \leqslant 3$. Otherwise, H contains four singletons (i.e., an independent set of four vertices), and by Lemma 6(2), $|F| \geqslant |N_{AG_5}(V(H))| \geqslant 8 \times 5 - 24 = 16$, a contradiction. Also, if $|V(H)| \leqslant 3$, it is clear that $c(H) \leqslant |V(H)| \leqslant 3$.

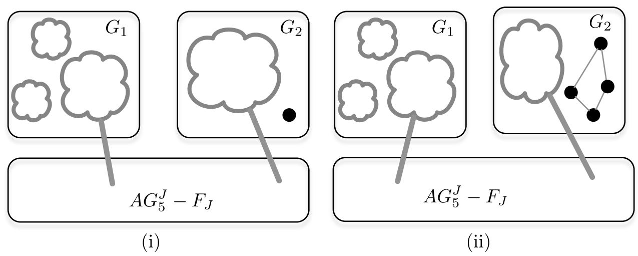

FIGURE 8. An illustration of Case 2.2 in Lemma 16, where a shape of cloud indicates a component: (i) corresponds to $|V(H_2)| = 1$ and (ii) corresponds to $|V(H_2)| = 4$.

Case 2.3: $f_1, f_2 \geqslant 6$. Then $|F_J| = |F| - f_1 - f_2 \leqslant 15 - 6 - 6 = 3$. This implies that $c(H) \leqslant 3$.

Case 3: $|I| = 3$. Without loss of generality, assume $I = \{1, 2, 3\}$ and $f_1 \geqslant f_2 \geqslant f_3$. Since $|F| \leqslant 15$ and $f_i \geqslant 4$ for $i \in I$, it implies $|F_J| = |F| - f_1 - f_2 - f_3 \leqslant 15 - 3 \times 4 = 3$. By Lemma 1(1), $AG_5^J - F_J$ is connected. Also, we have $f_i \leqslant |F| - f_j - f_k \leqslant 15 - 4 - 4 = 7$ for each $i \in I$ where $j, k \in I \setminus \{i\}$ with $j \neq k$. There is at most one $i \in I$ such that $f_i \geqslant 6$. Otherwise, $|F| \geqslant f_1 + f_2 + f_3 \geqslant 16 > 15$, a contradiction. We consider the following cases.

Case 3.1: $4 \leqslant f_3 \leqslant f_2 \leqslant f_1 \leqslant 5$. For $i \in \{1, 2, 3\}$, by Lemma 12 and $f_i \leqslant 5 < \kappa_3(AG_4) = 6$, G_i has two components and $c(H_i) = 1$. It implies that $c(H) \leqslant 3$.

Case 3.2: $6 \leqslant f_1 \leqslant 7$ and $4 \leqslant f_3 \leqslant f_2 \leqslant 5$. For $i \in \{2, 3\}$, by Lemma 12 and $f_i \leqslant 5 < \kappa_3(AG_4) = 6$, G_i has two components and $c(H_i) = 1$. By Lemma 13 and

$f_1 \leqslant 7 < \kappa_4(AG_4) = 8$, G_1 has at most three components and $c(H_1) \leqslant 2$. Thus, $c(H) \leqslant 4$. We claim $c(H) \leqslant 3$. Suppose not and let H_i for $i = 1, 2, 3, 4$ be components of H. Let $v_i \in V(H_i)$ for $i \in \{1, 2, 3, 4\}$. Since H_i and H_j are not connected in H for any $i, j \in \{1, 2, 3, 4\}$ with $i \neq j$, $\{v_1, v_2, v_3, v_4\}$ is an independent set of AG_5. Clearly, $N_{AG_5}(V(H_i))$ is a vertex-cut of AG_5 for each $i \in \{1, 2, 3, 4\}$. Since AG_5 is hyper-connected, $|N_{AG_5}(V(H_i))| \geqslant \kappa(AG_5) = |N_{AG_5}(v_i)|$. By Lemma 6(2), $|F| \geqslant |N_{AG_5}(V(H_1) \cup V(H_2) \cup V(H_3) \cup V(H_4))| \geqslant |N_{AG_5}(\{v_1, v_2, v_3, v_4\})| \geqslant 8 \times 5 - 24 = 16$, a contradiction. □

Lemma 17: For $n \geqslant 5$, $\kappa_5(AG_n) = 8n - 24$.

Proof: For $n \geqslant 5$, the upper bound $\kappa_5(AG_n) \leqslant 8n - 24$ can be acquired from Remark 1(2) by considering the removal of $N(\{v_0, v_1, v_2, v_3\})$, where $\{v_0, v_1, v_2, v_3\}$ is an independent set of AG_n and $|N(\{v_0, v_1, v_2, v_3\})| = 8n - 24$. Thus, the resulting graph has five components, four of which are singletons. Lemma 16 proves the lower bound $\kappa_5(AG_n) \geqslant 8n - 24$ for $n = 5$, and we now consider $n \geqslant 6$ as follows.

Let F be any vertex-cut of AG_n such that $|F| \leqslant 8n - 25$. Lemma 5 shows that the removal of a vertex-cut with no more than $8n - 29$ vertices in AG_n results in a disconnected graph with at most four components. To complete the proof, we need to show that the same result holds when $8n - 28 \leqslant |F| \leqslant 8n - 25$. Recall $I = \{i \in \mathbb{Z}_n : G_i \text{ is disconnected}\}$ and $J = \mathbb{Z}_n \setminus I$. By definition, G_j is connected for all $j \in J$. Since $|F| \leqslant 8n - 25 < (n-2)!$ when $n \geqslant 6$, $AG_n^J - F_J$ remains connected for arbitrary J. Since AG_n^i is isomorphic to AG_{n-1}, we have $\kappa(AG_n^i) = 2n - 6$. If $|I| \geqslant 4$, then $|F| \geqslant |I| \times (2n - 6) \geqslant 8n - 24 > 8n - 25$, a contradiction. Also, if $I = \emptyset$, then $AG_n - F$ is connected, a contradiction. Thus, $1 \leqslant |I| \leqslant 3$. Let H be the union of components of $AG_n - F$ such that all vertices of H are contained in $\bigcup_{i \in I} V(G_i)$. In the following, we will show that $c(H) \leqslant 3$. Thus, counting together with the component that contains $AG_n^J - F_J$ as a subgraph, $AG_n - F$ contains $c(H) + 1 \leqslant 4$ components. We consider the following three cases:

Case 1: $|I| = 1$. Without loss of generality, assume $I = \{1\}$. In this case, $V(H) \subseteq V(G_1)$. We analyze the number of faulty vertices of F_J as follows.

Case 1.1: $|F_J| \leqslant 11$. Since every vertex of H has exactly two faulty out-neighbors in F_J by Lemma 1(2), $2|V(H)| \leqslant |F_J| \leqslant 11$, which implies $|V(H)| \leqslant 5$. If $|V(H)| = 5$, then $c(H) \leqslant 3$. Otherwise, H contains five singletons or three singletons and an edge. If $V(H) = \{v_1, v_2, v_3, v_4, v_5\} = H' \cup \{v_5\}$, where $H' = \{v_1, v_2, v_3, v_4\}$, by Lemma 6(2), $|N_{AG_n}(V(H))| = |N_{AG_n}(H')| + |N_{AG_n}(v_5)| - |N_{AG_n}(H') \cap N_{AG_n}(v_5)| \geqslant (8n - 24) + (2n - 4) - 2(4 \times 1) = 10n - 36 > 8n - 25$ for $n \geqslant 6$, a contradiction. Now we assume $V(H) = \{v_1, v_2, v_3, u, w\} = H' \cup \{u, w\}$, where $H' = \{v_1, v_2, v_3, \}$ and (u, w) is an edge. Then, by Lemma 6(1), $|N_{AG_n}(V(H))| = |N_{AG_n}(H')| + |N_{AG_n}(\{u, w\})| - |N_{AG_n}(H') \cap N_{AG_n}(\{u, w\})| \geqslant (6n - 16) + 2(2n - 4) - 2(3 \times 2) = 10n - 36 > 8n - 25$ for $n \geqslant 6$, a contradiction. If $|V(H)| = 4$, then $c(H) \leqslant 3$. Otherwise, H contains four singletons (i.e., an independent set of four vertices), and by Lemma 6(2), $|F| \geqslant |N_{AG_n}(V(H))| \geqslant 8n - 24$, a contradiction. Also, if $|V(H)| \leqslant 3$, it is clear that $c(H) \leqslant |V(H)| \leqslant 3$.

Case 1.2: $|F_J| \geqslant 12$. Since F_1 is a vertex-cut of AG_n^1 and $f_1 = |F| - |F_J| \leqslant (8n - 25) - 12 = 8(n-1) - 29$, by Lemma 5, G_1 contains at most four components in which the largest component is connected to $AG_n^J - F_J$. Thus, $c(G_1) \leqslant 4$ and $c(H) = c(G_1) - 1 \leqslant 3$.

Case 2: $|I| = 2$. Without loss of generality, assume $I = \{1, 2\}$ and $f_1 \geqslant f_2$. Since $|F| \leqslant 8n - 25$ and $f_i \geqslant 2n - 6$ for $i \in I$, it implies $f_i \leqslant |F| - f_j \leqslant 6n - 19$ where $j \in I \setminus \{i\}$ with $j \neq i$. We consider the following subcases:

Case 2.1: $2n - 6 \leqslant f_2 \leqslant f_1 \leqslant 4n - 15 = 4(n-1) - 11$. For $i \in \{1, 2\}$, by Lemma 2, G_i contains two components, one is either a singleton or an edge, and the other is a larger component connecting to $AG_n^J - F_J$. Thus, $c(G_i) = 2$ for $i = 1, 2$ and $c(H) \leqslant c(G_1) + c(G_2) - 2 = 2$.

Case 2.2: $2n - 6 \leqslant f_2 \leqslant 4n - 15$ and $4n - 14 \leqslant f_1 \leqslant 6n - 19$. Since $f_2 \leqslant 4n - 15 = 4(n-1) - 11$, by Lemma 2, G_2 contains two components, one is either a singleton or an edge, and the other is a larger component connecting to $AG_n^J - F_J$. Thus $c(G_2) = 2$. If $4n - 14 \leqslant f_1 \leqslant 6n - 23$, by Lemma 15, $f_1 < 6(n-1) - 16 = \kappa_4(AG_{n-1})$, and thus G_1 contains at most three components and the largest component is connected to $AG_n^J - F_J$. Thus, $c(G_1) \leqslant 3$ and $c(H) \leqslant c(G_1) + c(G_2) - 2 \leqslant 3$. If $6n - 22 \leqslant f_1 \leqslant 6n - 19$, then $|F_J| = |F| - f_1 - f_2 \leqslant (8n - 25) - (6n - 22) - (2n - 6) = 3$. By Lemma 1(2), every vertex of H has at least one faulty out-neighbor in F_J. Thus, $c(H) \leqslant |V(H)| \leqslant |F_J| \leqslant 3$.

Case 2.3: $4n - 14 \leqslant f_2 \leqslant f_1 \leqslant 6n - 19$. In this case, $|F_J| = |F| - f_i - f_2 \leqslant (8n - 25) - 2(4n - 14) = 3$. By Lemma 1(2), every vertex of H has at least one faulty out-neighbor in F_J. Thus, $c(H) \leqslant |V(H)| \leqslant |F_J| \leqslant 3$.

Case 3: $|I| = 3$. Without loss of generality, assume $I = \{1, 2, 3\}$ and $f_1 \geqslant f_2 \geqslant f_3$. Since $|F| \leqslant 8n - 25$ and $f_i \geqslant 2n - 6$ for $i \in I$, it implies $f_i \leqslant |F| - f_j - f_k \leqslant (8n - 25) - 2(2n - 6) = 4n - 13$, where $j, k \in I \setminus \{i\}$ with $j \neq k$. We consider the following subcases:

Case 3.1: $f_i \leqslant 4n - 16 < 4(n-1) - 11$ for each $i \in I$. By Lemma 2, G_i contains two components, one is a singleton, and the other is a larger component connecting to $AG_n^J - F_J$, and thus $c(G_i) = 2$. So $c(H) \leqslant c(G_1) + c(G_2) + c(G_3) - 3 = 3 \times 2 - 3 = 3$.

Case 3.2: $f_3 \leqslant f_2 \leqslant 4n - 16 < f_1 \leqslant 4n - 13$. In this case, each of G_i for $i = 2, 3$ contains two components, one is a singleton, say v_i, and the other is a larger component connecting to $AG_n^J - F_J$. Thus $c(G_2) = c(G_3) = 2$. Since $f_1 \leqslant 4n - 13 \leqslant 6n - 25 = 6(n-1) - 19$ for $n \geqslant 6$, by Lemma 4, G_1 contains either two components, or three components and two of which are singletons, say v_1 and v_1' (see Fig. 9 for two situations). Since the largest component of G_1 is connected to $AG_n^J - F_J$, if $c(G_1) = 2$, then $c(H) \leqslant c(G_1) + c(G_2) + c(G_3) - 3 = 3 \times 2 - 3 = 3$. On the other hand, if $\{v_1, v_1', v_2, v_3\}$ is an independent set of AG_n, by Lemma 6(2), $|F| \geqslant N(\{v_1, v_1', v_2, v_3\}) \geqslant 8n - 24$, a contradiction. Thus, there exists at least one of edges (v_1, v_2),

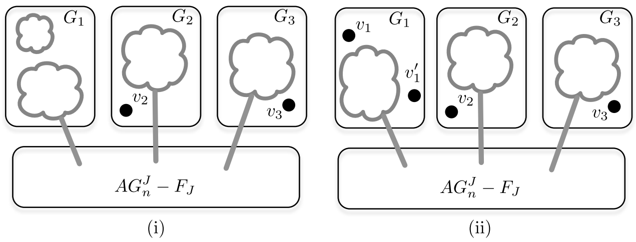

FIGURE 9. An illustration of Case 3.2 in Lemma 17, where a shape of cloud indicates a component: (i) corresponds to $c(G_1) = 2$ and (ii) corresponds to $c(G_1) = 3$.

(v_1, v_3), (v_1', v_2), (v_1', v_3) and (v_2, v_3) in AG_n, which implies $c(H) \leqslant 3$.

Case 3.3: $f_3 \leqslant 4n - 14 \leqslant f_2 \leqslant f_1 \leqslant 4n - 13$. Clearly, $f_3 \leqslant |F| - f_1 - f_2 \leqslant (8n - 25) - 2(4n - 14) = 3 < 2n - 6$ for $n \geqslant 6$, a contradiction.

Case 3.4: $4n - 14 \leqslant f_3 \leqslant f_2 \leqslant f_1 \leqslant 4n - 3$. Clearly, $f_1 + f_2 + f_3 \geqslant 3(4n - 14) > 8n - 25 \geqslant |F|$ when $n \geqslant 6$, a contradiction. □

Theorem 1: $\kappa_3(AG_n) = 4n - 10$ and $\kappa_4(AG_n) = 6n - 16$ for $n \geqslant 4$, and $\kappa_5(AG_n) = 8n - 24$ for $n \geqslant 5$.

Proof: The result directly follows from Lemmas 12, 15 and 17. □

IV. THE ℓ-COMPONENT CONNECTIVITY OF S_N^2

Lemma 18: For $n \geqslant 4$, $\kappa_3(S_n^2) = 4n - 8$.

Proof: By Lemma 8, if F is a vertex-cut with $|F| \leqslant 4n - 9$, then $AG_n - F$ has exact two components. Thus, $\kappa_3(S_n^2) \geqslant 4n - 8$. The upper bound $\kappa_3(S_n^2) \leqslant 4n - 8$ can be proved using an argument similar to Lemma 12 by considering that every vertex of S_n^2 has $2n - 3$ neighbors. □

Lemma 19: $\kappa_4(S_4^2) \geqslant 10$ and $\kappa_5(S_4^2) \geqslant 12$.

Proof: Using the notations established earlier, S_4^2 contains two copies of AG_4, say $S_{4,E}^2$ and $S_{4,O}^2$, respectively. Let F be any vertex-cut of S_4^2. Let $F_O = F \cap V(S_{4,O}^2)$ and $F_E = F \cap V(S_{4,E}^2)$. Let $H = H_O \cup H_E$ be the union of small components of $S_n^2 - F$, where H_O and H_E are the set of components such that their vertices are contained in $S_{n,O}^2$ and $S_{n,E}^2$, respectively.

We first prove $\kappa_4(S_4^2) \geqslant 10$ by showing that if $|F| \leqslant 9$, then $c(H) \leqslant 3$. Note that there are $\frac{4!}{2} = 12 > |F|$ matching edges between $S_{4,O}^2$ and $S_{4,E}^2$. If both $S_{4,O}^2 - F_O$ and $S_{4,E}^2 - F_E$ are connected, then so is $S_4^2 - F$, a contradiction. Next, we consider only one of $S_{4,O}^2 - F_O$ and $S_{4,E}^2 - F_E$ is connected. Without loss of generality, assume $S_{4,O}^2 - F_O$ is connected. Then $4 = \kappa(AG_4) \leqslant |F_E| \leqslant 9$. By Lemma 13, if $4 \leqslant |F_E| \leqslant 7 < 8 = \kappa_4(AG_4)$, then $S_{4,E}^2 - F_E$ has at most three components, and thus $c(H_E) \leqslant 2$. Since $\frac{4!}{2} = 12 > |F|$, the largest component of $S_{4,E}^2 - F_E$ is connected to $S_{4,O}^2 - F_O$, and it leads to $c(H) = c(H_E) \leqslant 2$. Also, if $8 \leqslant |F_E| \leqslant 9$, then $|F_O| \leqslant 1$. Since there are $\frac{4!}{2} = 12$ matching edges between $S_{4,O}^2$ and $S_{4,E}^2$, every component of size at least 2 in $S_{4,E}^2 - F_E$ is part of the component in $S_4^2 - F$ containing $S_{n,O}^2 - F_O$, and at most one vertex in $S_{4,E}^2 - F_E$ is not part of this component containing $S_{4,O}^2 - F_O$. Thus, $|V(H_E)| \leqslant 1$ and $c(H) \leqslant |V(H_E)| \leqslant 1$. We now consider both $S_{4,O}^2 - F_O$ and $S_{4,E}^2 - F_E$ are disconnected. Without loss of generality, assume $|F_O| \geqslant |F_E| \geqslant 4$. Since $|F| \leqslant 9$, it implies $4 \leqslant |F_E| \leqslant |F_O| \leqslant 5$. By Lemma 2, each of $S_{4,O}^2 - F_O$ and $S_{4,E}^2 - F_E$ has two components. Thus, $c(H_O) = c(H_E) = 1$. Since the largest component of $S_{4,E}^2 - F_E$ is connected to $S_{4,O}^2 - F_O$, it leads to $c(H) \leqslant c(H_O) + c(H_E) = 2$.

Next, we prove $\kappa_5(S_4^2) \geqslant 12$ by showing that if $|F| \leqslant 11$, then $c(H) \leqslant 4$. Note that there are $\frac{4!}{2} = 12 > |F|$ matching edges between $S_{4,O}^2$ and $S_{4,E}^2$. If both $S_{4,O}^2 - F_O$ and $S_{4,E}^2 - F_E$ are connected, then so is $S_4^2 - F$, a contradiction. Next, we consider only one of $S_{4,O}^2 - F_O$ and $S_{4,E}^2 - F_E$ is connected. Without loss of generality, assume $S_{4,O}^2 - F_O$ is connected. Then $4 = \kappa(AG_4) \leqslant |F_E| \leqslant 11$. If $4 \leqslant |F_E| \leqslant 7 < 8 = \kappa_4(AG_4)$, we can show that $c(H) \leqslant 2$ through a similar discussion as above. So we assume $8 \leqslant |F_E| \leqslant 11$, and this implies $|F_O| \leqslant 3$. Since there are $\frac{4!}{2} = 12$ matching edges between $S_{4,O}^2$ and $S_{4,E}^2$, every component of size at least 4 in $S_{4,E}^2 - F_E$ is part of the component in $S_4^2 - F$ containing $S_{n,O}^2 - F_O$, and at most three vertex in $S_{4,E}^2 - F_E$ is not part of this component containing $S_{4,O}^2 - F_O$. Thus, $|V(H_E)| \leqslant 3$ and $c(H) \leqslant |V(H_E)| \leqslant 3$. We now consider both $S_{4,O}^2 - F_O$ and $S_{4,E}^2 - F_E$ are disconnected. Without loss of generality, assume $|F_O| \geqslant |F_E| \geqslant 4$. Since $|F| \leqslant 11$, it implies $4 \leqslant |F_E| \leqslant |F_O| \leqslant 7$ and at most one $i \in \{E, O\}$ such that $|F_i| \geqslant 6$. If $4 \leqslant |F_E| \leqslant |F_O| \leqslant 5$, we can show that $c(H) \leqslant 2$ through a similar discussion as above. Finally, we consider $6 \leqslant |F_O| \leqslant 7$ and $4 \leqslant |F_E| \leqslant 5$. By Lemma 13, $6 \leqslant |F_O| \leqslant 7 < 8 = \kappa_4(AG_4)$ implies that $S_{4,O}^2 - F_O$ has at most three components and $c(H_O) \leqslant 2$. Also, by Lemma 2, $4 \leqslant |F_E| \leqslant 5$ implies that $S_{4,E}^2 - F_E$ has two components and $c(H_E) = 1$. Since the largest component of $S_{4,E}^2 - F_E$ is connected to the largest component of $S_{4,O}^2 - F_O$, we have $c(H) \leqslant c(H_E) + c(H_O) \leqslant 3$. □

Lemma 20: For $n \geqslant 4$, $\kappa_4(S_n^2) = 6n - 14$.

Proof: For $n \geqslant 4$, the upper bound $\kappa_4(S_n^2) \leqslant 6n - 14$ can be acquired from Lemma 11(2) by considering the removal of $N_{S_n^2}(\{v_1, v_2, v_3\})$ where $\{v_1, v_2, v_3\}$ is an independent set of S_n^2, and thus the resulting graph has four components, three of which are singletons. By Lemma 19, we know $\kappa_4(S_4^2) \geqslant 10 = 6 \times 4 - 14$. So we prove the lower bound $\kappa_4(S_n^2) \geqslant 6n - 14$ for $n \geqslant 5$ as follows. Recall that S_n^2 contains two copies of AG_n, say $S_{n,E}^2$ and $S_{n,O}^2$, respectively. Let F be any vertex-cut of S_n^2 such that $|F| \leqslant 6n - 15$. Lemma 9 shows that the removal of a vertex-cut with no more than $6n - 17$ vertices in S_n^2 results in a disconnected graph with at most three components. To complete the proof, we need to show that the same result holds when $6n - 16 \leqslant |F| \leqslant 6n - 15$.

Let $F_O = F \cap V(S_{n,O}^2)$ and $F_E = F \cap V(S_{n,E}^2)$. Let $H = H_O \cup H_E$ be the union of small components of $S_n^2 - F$, where H_O and H_E are the set of components such that their vertices are contained in $S_{n,O}^2$ and $S_{n,E}^2$, respectively. Without loss of generality, assume $|F_O| \geqslant |F_E|$. Since $2(4n-11) > 6n-15$ for $n \geqslant 5$, we consider the following two cases.

Case 1: $|F_E| \leqslant |F_O| \leqslant 4n-12$. By Lemma 2, $S^2_{n,O} - F_O$ (resp., $S^2_{n,E} - F_E$) either is connected or has two components, one of which is a singleton. Let B_O (resp., B_E) be the largest component of $S^2_{n,O} - F_O$ (resp., $S^2_{n,E} - F_E$). Since $\frac{n!}{2} - (6n-15) - 2 > 0$ for $n \geqslant 5$, B_O and B_E belong to the same component in $S^2_n - F$. Note that F is a vertex-cut of S^2_n, the singletons in $S^2_{n,O} - F_O$ and $S^2_{n,E} - F_E$ can remain singleton or for two of them to form an edge in $S^2_n - F$. Thus, $S^2_n - F$ has at most three components, i.e. $c(H) \leqslant 2$. The result holds.

Case 2: $4n-11 \leqslant |F_O| \leqslant 6n-15$. It implies that $|F_E| \leqslant (6n-15)-(4n-11) \leqslant 2n-4$. Note that $S^2_{n,E}$ is isomorphic to AG_n and $2n-4 \leqslant 4n-12$ for $n \geqslant 5$, by Lemma 2, so $S^2_{n,E} - F_E$ either is connected or has two components, one of which is a singleton. Thus $V(H_E) \leqslant 1$ and $c(H_E) \leqslant 1$. If $S^2_{n,O} - F_O$ is connected, note that $\frac{n!}{2} - (6n-15) - 1 > 0$ for $n \geqslant 5$, then $S^2_n - F$ has two components, one of which is a singleton. The result holds in this case. In the following, we assume that $S^2_{n,O} - F_O$ is disconnected, and consider the following cases:

Case 2.1: $6n-18 \leqslant |F_O| \leqslant 6n-15$. It implies $|F_E| \leqslant (6n-15)-(6n-18) = 3$, and thus $S^2_{n,E} - F_E$ is connected. Note that there are $\frac{n!}{2}$ matching edges between $S^2_{n,O}$ and $S^2_{n,E}$. Since $|F_E| \leqslant 3$, every component of size at least 4 in $S^2_{n,O} - F_O$ is part of the component in $S^2_n - F$ containing $S^2_{n,E} - F_E$, and at most three vertices in $S^2_{n,O} - F_O$ are not part of this component containing $S^2_{n,E} - F_E$. Thus, $|V(H_O)| \leqslant 3$ and $|V(H)| = |V(H_O)| + |V(H_E)| \leqslant 4$. If $|V(H)| = 4$, then $c(H) \leqslant 2$. Otherwise, H contains four singletons or two singletons and an edge. If H contains four singletons, by Lemma 11(3), $|N_{S^2_n}(H)| \geqslant 8n-20 > 6n-15$ for $n \geqslant 5$, a contradiction. Now we assume that $V(H) = \{v_1, v_2, u, w\} = H' \cup \{u, w\}$, where $H' = \{v_1, v_2\}$ and (u, w) is an edge. Then, by Lemma 11(1) and Lemma 7(3), $|N_{S^2_n}(V(H))| = |N_{S^2_n}(H')| + |N_{S^2_n}(\{u, v\})| - |N_{S^2_n}(H') \cap N_{S^2_n}(\{u, v\})| \geqslant (4n-8) + 2(2n-3) - 2 \times 3 = 8n-20 > 6n-15$ for $n \geqslant 5$, a contradiction. If $|V(H)| = 3$, then $c(H) \leqslant 2$. Otherwise, H contains three singletons, and by Lemma 11(2), $|F| \geqslant |N_{S^2_n}(V(H))| \geqslant 6n-14$, a contradiction. Also, if $|V(H)| \leqslant 2$, it is clear that $c(H) \leqslant |V(H)| \leqslant 2$.

Case 2.2: $4n-11 \leqslant |F_O| \leqslant 6n-19$. It implies $|F_E| \leqslant (6n-15)-(4n-11) = 2n-4$, and thus $S^2_{n,E} - F_E$ is connected. By Lemma 4, $S^2_{n,O} - F_O$ either has two components, one of which is a singleton, an edge or a 2-path, or has three components, two of which are singletons (See Fig. 10). Let C be the largest component of $S^2_{n,O} - F_O$. Since $\frac{n!}{2} - (6n-15) - 3 > 0$ for $n \geqslant 5$, C is part of the component in $S^2_n - F$ containing $S^2_{n,E} - F_E$. Thus, $|V(H_O)| \leqslant 3$ and $|V(H)| = |V(H_O)| + |V(H_E)| \leqslant 4$. Then, through a similar argument in the above case, we can show that $c(H) \leqslant 2$. □

Lemma 21: For $n \geqslant 4$, $\kappa_5(S^2_n) = 8n-20$.

Proof: For $n \geqslant 4$, the upper bound $\kappa_5(S^2_n) \leqslant 8n-20$ can be acquired from Lemma 11 by considering the removal

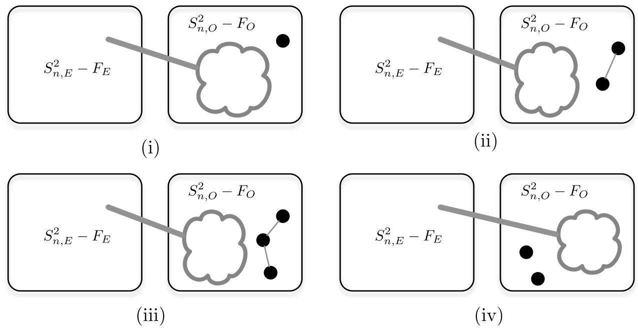

FIGURE 10. An illustration of Case 2.2 in Lemma 20, where a shape of cloud indicates the large component of $S^2_{n,O} - F_O$.

of $N_{S^2_n}(\{v_1, v_2, v_3, v_4\})$ where $\{v_1, v_2, v_3, v_4\}$ is an independent set of S^2_n, and thus the resulting graph has five components, four of which are singletons. By Lemma 19, we know $\kappa_5(S^2_4) \geqslant 12 = 8 \times 4 - 20$. So we prove the lower bound $\kappa_5(S^2_n) \geqslant 8n-20$ for $n \geqslant 5$ as follows. Let F be any vertex-cut of S^2_n such that $|F| \leqslant 8n-21$. Lemma 10 shows that the removal of a vertex-cut with no more than $8n-25$ vertices in S^2_n results in a disconnected graph with at most four components. To complete the proof, we need to show that the same result holds when $8n-24 \leqslant |F| \leqslant 8n-19$.

Let $F_O = F \cap V(S^2_{n,O})$ and $F_E = F \cap V(S^2_{n,E})$. Let $H = H_O \cup H_E$ be the union of small components of $S^2_n - F$, where H_O and H_E are the set of components such that their vertices are contained in $S^2_{n,O}$ and $S^2_{n,E}$, respectively. Without loss of generality, assume $|F_O| \geqslant |F_E|$. Since $2(6n-19) > 8n-21$ for $n \geqslant 5$, we consider the following cases.

Case 1: $|F_E| \leqslant |F_O| \leqslant 4n-12$. By Lemma 2, $S^2_{n,O} - F_O$ (resp., $S^2_{n,E} - F_E$) either is connected or has two components, one of which is a singleton. Since $\frac{n!}{2} - (8n-21) - 2 > 0$ for $n \geqslant 5$, a proof similar to Case 1 in Lemma 20 can show that $c(H) \leqslant 2$.

Case 2: $4n-11 \leqslant |F_E| \leqslant |F_O| \leqslant 6n-20$. By Lemma 3, $S^2_{n,O} - F_O$ (resp., $S^2_{n,E} - F_E$) has at most three components, and $|V(H_O)| \leqslant 2$ (resp., $|V(H_E)| \leqslant 2$). Thus, $|V(H)| \leqslant 4$. Since $\frac{n!}{2} - (8n-21) - 4 > 0$ for $n \geqslant 5$, the largest component of $S^2_{n,O} - F_O$ is connected to the largest component of $S^2_{n,E} - F_E$. If $|V(H)| = 4$, then $c(H) \leqslant 3$. Otherwise, by Lemma 11(3), $|N_{S^2_n}(H)| \geqslant 8n-20 > 8n-21$ for $n \geqslant 5$, a contradiction. Also, if $|V(H)| \leqslant 3$, it is clear that $c(H) \leqslant |V(H)| \leqslant 3$.

Case 3: $6n-19 \leqslant |F_O| \leqslant 8n-21$. In this case, $|F_E| \leqslant 8n-21-(6n-19) = 2n-2 \leqslant 4n-12$. By Lemma 2, $S^2_{n,E} - F_E$ has at most two components and $|V(H_E)| \leqslant 1$. Thus $c(H_E) \leqslant 1$. If $S^2_{n,O} - F_O$ is connected, note that $\frac{n!}{2} - (8n-21) - 1 > 0$ for $n \geqslant 5$, then $S^2_n - F$ has two components, one of which is a singleton. The result holds in this case. In the following, we assume that $S^2_{n,O} - F_O$ is disconnected, and consider the following cases:

Case 3.1: $8n-24 \leqslant |F_O| \leqslant 8n-21$. It implies $|F_E| \leqslant (8n-21)-(8n-24) = 3$, and thus $S^2_{n,E} - F_E$ is connected. Then a proof similar to Case 2.1 in Lemma 20 can show that $|V(H)| \leqslant 4$. If $|V(H_1)| = 4$, then $c(H) \leqslant 3$.

TABLE 2. The comparison of $\kappa^{(\ell-2)}(AG_n)$ and $\kappa_\ell(AG_n)$ (resp., $\kappa^{(\ell-2)}(S_n^2)$ and $\kappa_\ell(S_n^2)$) for $\ell = 3, 4, 5$.

Graph classes	h-extra connectivity	Ref.	ℓ-component connectivity	Ref.
AG_n	$\kappa^{(1)}(AG_n) = 4n - 11$ for $n \geqslant 5$ $\kappa^{(2)}(AG_n) = 6n - 19$ for $n \geqslant 5$ $\kappa^{(3)}(AG_n) = 8n - 28$ for $n \geqslant 5$	[36]	$\kappa_3(AG_n) = 4n - 10$ for $n \geqslant 4$ $\kappa_4(AG_n) = 6n - 16$ for $n \geqslant 4$ $\kappa_5(AG_n) = 8n - 24$ for $n \geqslant 5$	this paper
S_n^2	$\kappa^{(1)}(S_n^2) = 4n - 9$ for $n \geqslant 4$ $\kappa^{(2)}(S_n^2) = 6n - 16$ for $n \geqslant 4$ $\kappa^{(3)}(S_n^2) = 8n - 24$ for $n \geqslant 4$	[35]	$\kappa_3(S_n^2) = 4n - 8$ for $n \geqslant 4$ $\kappa_4(S_n^2) = 6n - 14$ for $n \geqslant 4$ $\kappa_5(S_n^2) = 8n - 20$ for $n \geqslant 4$	this paper

TABLE 3. The comparison of h-extra connectivity and ℓ-component connectivity for some networks.

Graph classes	h-extra connectivity	Ref.	ℓ-component connectivity	Ref.
Hypercubes Q_n	$\kappa^{(1)}(Q_n) = 2n - 2$ for $n \geqslant 3$ $\kappa^{(2)}(Q_n) = 3n - 5$ for $n \geqslant 4$	[44]	$\kappa_\ell(Q_n) = (\ell - 1)n - \frac{\ell(\ell-1)}{2} + 1$ for $n \geqslant 2$ and $\ell \in [2, n+1]$	[32]
	$\kappa^{(h)}(Q_n) = (h+1)n - 2h - \binom{h}{2}$ for $n \geqslant 4$ and $h \in [0, n-4]$	[45]	$\kappa_\ell(Q_n) = -\frac{(\ell-1)^2}{2} + (2n - \frac{5}{2})(\ell - 1) - n^2 + 2n + 1$ for $n \geqslant 6$ and $\ell \in [n+2, 2n-4]$	[51]
	$\kappa^{(h)}(Q_n) = \frac{n(n-1)}{2}$ for $n \geqslant 4$ and $h \in [n-3, n]$	[45]		
Folded Hypercubes FQ_n	$\kappa^{(1)}(FQ_n) = 2n$ for $n \geqslant 4$	[44]	$\kappa_\ell(FQ_n) = (\ell - 1)(n+1) - \frac{\ell(\ell-1)}{2} + 1$ for $n \geqslant 8$ and $\ell \in [2, n]$	[50]
	$\kappa^{(2)}(FQ_n)) = 3n - 2$ for $n \geqslant 8$	[55]		
	$\kappa^{(3)}(FQ_n)) = 4n - 5$ for $n \geqslant 6$	[8]		
	$\kappa^{(h)}(FQ_{n+1}) = f_{n+2}(h)$ for $n \geqslant 6$ and $h \in [0, n-2]$	[47]		
	$\kappa^{(h)}(FQ_{n+1}) = f_{n+2}(n+2)$ for $n \geqslant 6$ and $h \in [n-1, n+2]$	[47]		
Dual Cubes D_n	$\kappa^{(1)}(D_n) = 2n$ for $n \geqslant 3$ $\kappa^{(2)}(D_n) = 3n - 2$ for $n \geqslant 3$	[53]	$\kappa_\ell(D_n) = (\ell - 1)n - \frac{\ell(\ell-1)}{2} + 1$ for $n \geqslant 2$ and $\ell \in [2, n]$	[49]
Alternating Group Networks AN_n	$\kappa^{(1)}(AN_n) = 2n - 5$ for $n \geqslant 4$ $\kappa^{(2)}(AN_n) = 3n - 9$ for $n \geqslant 4$	[52]	$\kappa_3(AN_n) = 2n - 3$ for $n \geqslant 4$ $\kappa_4(AN_n) = 3n - 6$ for $n \geqslant 4$	[4] [3]
Twisted Cubes TQ_n	$\kappa^{(1)}(TQ_n) = \kappa^{(1)}(LTQ_n) = 2n - 2$ for $n \geqslant 3$	[43]	$\kappa_3(TQ_n) = \kappa_3(LTQ_n) = 2n - 2$ for $n \geqslant 3$	[26]
Locally Twisted Cubes LTQ_n	$\kappa^{(2)}(TQ_n) = \kappa^{(2)}(LTQ_n) = 3n - 5$ for $n \geqslant 5$	[7]	$\kappa_4(TQ_n) = \kappa_4(LTQ_n) = 3n - 4$ for $n \geqslant 4$	[27]

Remark: $f_n(h) = (h+1)n - \frac{h(h+3)}{2}$

Otherwise, H_1 contains four singletons, and by Lemma 11, $|F| \geqslant |N_{S_n^2}(V(H))| \geqslant 8n - 20$, a contradiction. Also, if $|V(H)| \leqslant 3$, it is clear that $c(H) \leqslant |V(H)| \leqslant 3$.

Case 3.2: $6n - 19 \leqslant |F_O| \leqslant 8n - 25$. By Lemma 17, $\kappa_5(AG_n) = 8n - 24$. Since $6n - 19 \leqslant |F_O| \leqslant 8n - 25 < 8n - 24$, $S_{n,O}^2 - F_O$ has at most four components and $c(H_O) \leqslant 3$. As before, the largest component of $S_{n,O}^2 - F_O$ is connected to the largest component of $S_{n,E}^2 - F_E$. It implies that $c(H) \leqslant c(H_O) + c(H_E) \leqslant 4$. □

Theorem 2: $\kappa_3(S_n^2) = 4n - 8$, $\kappa_4(S_n^2) = 6n - 14$, and $\kappa_5(S_n^2) = 8n - 20$ for $n \geqslant 4$.

Proof: The result directly follows from Lemmas 18, 20 and 21. □

V. CONCLUDING REMARKS

In this paper, we study the ℓ-component connectivity of alternating group graphs and split-stars. For alternating group graphs, we obtain the results: $\kappa_3(AG_n) = 4n - 10$ and $\kappa_4(AG_n) = 6n - 16$ for $n \geqslant 4$, and $\kappa_5(AG_n) = 8n - 24$ for $n \geqslant 5$. For split-stars, we obtain the results: $\kappa_3(S_n^2) = 4n - 8$ for $n \geqslant 4$, and $\kappa_4(S_n^2) = 6n - 14$ and $\kappa_5(S_n^2) = 8n - 20$ for $n \geqslant 5$. So far the problem of determining $\kappa_\ell(AG_n)$ and $\kappa_\ell(S_n^2)$ for $\ell \geqslant 6$ are still open.

Fàbrega and Fiol [24] introduced another evaluation of the reliability for interconnection networks. Given a graph G and a nonnegative integer h, the *h-extra connectivity* of G, denoted by $\kappa^{(h)}(G)$, is the cardinality of a minimum vertex-cut S of G, if it exists, such that each component of $G - S$ has at least $h + 1$ vertices. In fact, the extra connectivity plays an important indicator of a network's ability for diagnosis and fault tolerance [25], [31], [35], [36]. Currently, the known results of h-extra connectivity for alternating group graphs and split-stars were proposed in [36] and [35], respectively. Table 2 compares the two types of connectivities for alternating group graphs and split-stars. From this table, it seems that $\kappa^{(\ell-2)}(G)$ and $\kappa_\ell(G)$ have strongly close relationship for a network G. Based on the result $\kappa^{(\ell-2)}(G) < \kappa_\ell(G)$ for $G \in \{AG_n, S_n^2\}$ and $\ell \in \{3, 4, 5\}$, we know that finding $\kappa_\ell(G)$ needs more analyses than that of $\kappa^{(\ell-2)}(G)$. An interesting question is that does the relation always hold for larger ℓ?

As a matter of fact, so far the relationship between the two types of connectivities is not clear. To provide more comparisons between extra connectivity and component connectivity for other network topologies, we list the currently known results in Table 3. From this table, we already checked the following: for hypercubes, we have $\kappa^{(\ell-2)}(Q_n) = \kappa_\ell(Q_n)$

for $n \geqslant 4$ and $\ell \in [2, n-2]$; for folded hypercubes, we have $\kappa^{(\ell-2)}(FQ_n) = \kappa_\ell(FQ_n)$ for $n \geqslant 8$ and $\ell \in [2, n]$; for dual cubes, we have $\kappa^{(\ell-2)}(D_n) > \kappa_\ell(D_n)$ for $n \geqslant 3$ and $\ell \in \{3, 4\}$; for alternating group networks, we have $\kappa^{(\ell-2)}(AN_n) < \kappa_\ell(AN_n)$ for $n \geqslant 4$ and $\ell \in \{3, 4\}$. As a remark that the greater of the two types of connectivities is not absolutely certain, but is determined by the topology of the network. However, it is valuable to delve further into the details of this direction.

ACKNOWLEDGMENT

This work was accomplished while the corresponding author Jou-Ming Chang visiting the Department of Mathematics, Beijing Jiaotong University.

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