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Anti-Synchronization for Complex-Valued Bidirectional Associative Memory Neural Networks With Time-Varying Delays

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ABSTRACT This paper focuses on the anti-synchronization problem of complex-valued bidirectional associative memory (BAM) neural networks with time-varying delays. Based on a suitable Lyapunov functional, a sufficient condition for guaranteeing the anti-synchronization of the considered system is derived by using the inequality techniques. For delayed complex-valued BAM neural networks, it is the first time that the anti-synchronization control problem is addressed. So, our work not only fills the gap in this field but also complements the previous results. In the end, two numerical examples are provided to show the effectiveness of the obtained result.

INDEX TERMS Complex-valued BAM neural networks, anti-synchronization, time-varying delays.

I. INTRODUCTION

The bidirectional associative memory (BAM) neural networks model was first proposed by Kosko [1] in 1987, which contains two layers of neurons represented by F_X layer and F_Y layer. As an important neural networks model, it is widely used in image processing, signal processing and pattern recognition. Moreover, owing to the inevitable existence of time delays in practice [2]–[7], more and more scholars have shown great interests in studying delayed BAM neural networks and quite a number of fruitful dynamic achievements have been proposed [8]–[12]. For instance, global lagrange stability problem is discussed for neural-type Cohen-Grossberg BAM neural networks with mixed time-varying delays in [11]. Global exponential stability criterion is established in terms of LMIs for neutral delayed BAM neural networks with delays in leakage terms via new inequality technique in [12].

It should be noted that the above-mentioned results are based on the case of delayed real-valued neural networks. Although the real-valued dynamical systems have been applied in various fields and a lot of achievements have been made, some inherent limitations also exist at the same time. Compared with real-valued neural networks, complex-valued neural networks have different and more complex dynamic behaviors. It not only has complex-valued states, activation functions and connection weights but can deal with many problems that cannot be solved by real-valued neural networks, such as the XOR problem [13]. Therefore, it is very important to study the dynamics of complex-valued neural networks. So far, many papers related to delayed complex-valued neural networks have been published [14]–[37], in which, some achievements refer to delayed complex-valued BAM neural networks [29]–[37]. For complex-valued BAM neural networks with time-varying delays, lagrange exponential stability is investigated by combining the Lyapunov function approach with some inequalities techniques in [29]. For complex-valued BAM neutral-type neural networks with time delays, delay-independent stability criteria are established in [33]. The exponential input-to-state stability for delayed complex-valued memristor-based BAM neural networks model is considered in [34]. In [37], some

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novel sufficient conditions to guarantee the dissipativity of complex-valued BAM neural networks are obtained by using the inequality techniques, Halanay inequality, and upper right Dini derivative concepts.

On the other hand, when analyzing dynamic behavior of systems, synchronization and anti-synchronization are very hot topic [38]–[40] since they have been successfully applied in engineering applications and hardware implementations, such as image processing, information science and so on. As a result, it becomes very important and necessary to study the synchronization and anti-synchronization control problems in theoretical work. Moreover, as we know, for delayed complex-valued neural networks, some results involving these problems have been reported, such as synchronization of fractional-order delayed complex-valued neural networks [41], [42], finite-time synchronization for delayed complex-valued neural networks [43]–[46] and anti-synchronization of complex-valued memristor-based delayed neural networks [47], [48]. At the same time, for complex-valued BAM neural networks models, many researchers are interested in the synchronization problem of complex-valued BAM neural networks and have achieved some results. For example, a sufficient condition on global asymptotic periodic synchronization of complex-valued BAM neural networks is established by using novel LMI method in [49]. However, for the anti-synchronization control problem, there has been still no information. This motivates our interests.

According to the discussions above, we know that it is necessary to investigate the anti-synchronization control problem of complex-valued BAM neural networks. In this paper, we will study this issue for complex-valued BAM neural networks with time-varying delays. The main contributions of our work can be shown in the following points:

(1) Compared with the previous results, it is the first time that the anti-synchronization control problem of complex-valued BAM neural networks with time-varying delays is investigated.

(2) Via a suitable Lyapunov functional and the inequality techniques, a sufficient condition is established to ensure the anti-synchronization of the considered system.

(3) According to Hölder inequality, the right inequalities different from those in the existing references are used to derive the main result.

II. PROBLEM FORMULATION AND PRELIMINARIES

Throughout this paper, R^n and C^n denote n-dimensional Euclidean space and Unitary space, respectively. $i = \sqrt{-1}$ denotes the imaginary unit. $\mathcal{C}([-\tau, 0], R^n)$ and $\mathcal{C}([-\tau, 0], C^n)$ denote the family of continuous functions ϑ from $[-\tau, 0]$ to R^n and C^n, respectively. For $\vartheta(s) = (\vartheta_1(s), \vartheta_2(s), \ldots, \vartheta_n(s))^T \in \mathcal{C}([-\tau, 0], R^n)$, the norm $\| \vartheta \| = \sup_{-\tau \le t \le 0}\{\sum_{k=1}^{n} |\vartheta_k(s)|^{\varrho}\}^{\frac{1}{\varrho}}$ is equipped, where $\varrho > 1$ is a constant.

In this paper, we consider the following complex-valued BAM neural networks model:

$$\begin{cases} \dot{z}_k(t) = -d_{1k} z_k(t) + \sum_{l=1}^{m} a_{1lk} f_l(h_l(t)) \\ \qquad + \sum_{l=1}^{m} b_{1lk} f_l(h_l(t - \tau_{2kl}(t))) \\ \dot{h}_l(t) = -d_{2l} h_l(t) + \sum_{k=1}^{n} a_{2kl} g_k(z_k(t)) \\ \qquad + \sum_{k=1}^{n} b_{2kl} g_k(z_k(t - \tau_{1lk}(t))) \end{cases} \tag{1}$$

with the initial conditions

$$z_k(s) = \psi_{1k}(s), \quad h_l(s) = \psi_{2l}(s), \; s \in [-\tau, 0] \tag{2}$$

where $k = 1, 2, \ldots, n$ and $l = 1, 2, \ldots, m$; $z_k(t)$ and $h_l(t)$ denote the complex-valued state variables; $d_{1k} > 0$ and $d_{2l} > 0$ are constants; a_{1lk}, b_{1lk}, a_{2kl} and b_{2kl} are complex-valued connection weights; $f_l(h_l(t))$, $g_k(z_k(t))$, $f_l(h_l(t - \tau_{2kl}(t)))$ and $g_k(z_k(t - \tau_{1lk}(t)))$ denote the complex-valued activation functions; $\tau_{2kl}(t)$ and $\tau_{1lk}(t)$ are time-varying delays with $0 \le \tau_{2kl}(t) \le \tau_1$, $\dot{\tau}_{2kl}(t) \le \rho_1 \le 1$, $0 \le \tau_{1lk}(t) \le \tau_2$, $\dot{\tau}_{1lk}(t) \le \rho_2 \le 1$, τ_1, τ_2, ρ_1, ρ_2 are positive constants. Let $\tau = \max\{\tau_1, \tau_2\}$, $\psi_{1k}(s)$ and $\psi_{2l}(s) \in \mathcal{C}([-\tau, 0], C)$.

Moreover, complex-valued activation functions $f_l(h_l(t))$ and $g_k(z_k(t))$ can be separated into the real and imaginary parts as

$$f_l(h_l) = f_l^R(x_{2l}, y_{2l}) + i f_l^I(x_{2l}, y_{2l}),$$
$$g_k(z_k) = g_k^R(x_{1k}, y_{1k}) + i g_k^I(x_{1k}, y_{1k})$$

where $h_l(t) = x_{2l}(t) + i y_{2l}(t)$, $z_k(t) = x_{1k}(t) + i y_{1k}(t)$, and $h_l(t)$, $z_k(t)$, $x_{2l}(t)$, $y_{2l}(t)$, $x_{1k}(t)$, $y_{1k}(t)$ are simplified as h_l, z_k, x_{2l}, y_{2l}, x_{1k}, y_{1k}, respectively. Additionally, $f_l(-h_l) = -f_l(h_l)$, $g_k(-z_k) = -g_k(z_k)$, and satisfy the following assumption.

Assumption 1: For any x_{1k}, y_{1k}, x'_{1k}, y'_{1k}, x_{2l}, y_{2l}, x'_{2l} and $y'_{2l} \in R$, there exist positive constants λ_{1l}^{RR}, λ_{1l}^{RI}, λ_{1l}^{IR}, λ_{1l}^{II}, λ_{2k}^{RR}, λ_{2k}^{RI}, λ_{2k}^{IR} and λ_{2k}^{II} such that

$$\begin{aligned} |f_l^R(x_{2l}, y_{2l}) - f_l^R(x'_{2l}, y'_{2l})| &\le \lambda_{1l}^{RR}|x_{2l} - x'_{2l}| \\ &\quad + \lambda_{1l}^{RI}|y_{2l} - y'_{2l}| \\ |f_l^I(x_{2l}, y_{2l}) - f_l^I(x'_{2l}, y'_{2l})| &\le \lambda_{1l}^{IR}|x_{2l} - x'_{2l}| \\ &\quad + \lambda_{1l}^{II}|y_{2l} - y'_{2l}| \\ |g_k^R(x_{1k}, y_{1k}) - g_k^R(x'_{1k}, y'_{1k})| &\le \lambda_{2k}^{RR}|x_{1k} - x'_{1k}| \\ &\quad + \lambda_{2k}^{RI}|y_{1k} - y'_{1k}| \\ |g_k^I(x_{1k}, y_{1k}) - g_k^I(x'_{1k}, y'_{1k})| &\le \lambda_{2k}^{IR}|x_{1k} - x'_{1k}| \\ &\quad + \lambda_{2k}^{II}|y_{1k} - y'_{1k}|. \end{aligned} \tag{3}$$

In this paper, the corresponding response system is described as follows:

$$\begin{cases} \dot{z}_k^*(t) = -d_{1k} z_k^*(t) + \sum_{l=1}^{m} a_{1lk} f_l(h_l^*(t)) \\ \qquad\qquad + \sum_{l=1}^{m} b_{1lk} f_l(h_l^*(t-\tau_{2kl}(t))) + u_{1k}(t) \\ \dot{h}_l^*(t) = -d_{2l} h_l^*(t) + \sum_{k=1}^{n} a_{2kl} g_k(z_k^*(t)) \\ \qquad\qquad + \sum_{k=1}^{n} b_{2kl} g_k(z_k^*(t-\tau_{1lk}(t))) + u_{2l}(t) \end{cases} \tag{4}$$

with the initial conditions

$$z_k^*(s) = \theta_{1k}(s), \quad h_l^*(s) = \theta_{2l}(s), \ s \in [-\tau, 0] \tag{5}$$

where $k = 1, 2, \ldots, n$ and $l = 1, 2, \ldots, m$, $u_{1k}(t)$ and $u_{2l}(t)$ are to be designed to achieve a certain control objective. $\theta_{1k}(s)$ and $\theta_{2l}(s) \in \mathcal{C}([-\tau, 0], C)$.

Let $e_{1k}(t) = z_k^*(t) + z_k(t)$ and $e_{2l}(t) = h_l^*(t) + h_l(t)$, for $k = 1, 2, \ldots, n$ and $l = 1, 2, \ldots, m$, the anti-synchronization error system between (1) and (4) can be given by

$$\begin{cases} \dot{e}_{1k}(t) = -d_{1k} e_{1k}(t) + \sum_{l=1}^{m} a_{1lk} \mathcal{F}_{1l}(e_{2l}(t)) \\ \qquad\qquad + \sum_{l=1}^{m} b_{1lk} \mathcal{F}_{1l}(e_{2l}(t-\tau_{2kl}(t))) + u_{1k}(t) \\ \dot{e}_{2l}(t) = -d_{2l} e_{2l}(t) + \sum_{k=1}^{n} a_{2kl} \mathcal{F}_{2k}(e_{1k}(t)) \\ \qquad\qquad + \sum_{k=1}^{n} b_{2kl} \mathcal{F}_{2k}(e_{1k}(t-\tau_{1lk}(t))) + u_{2l}(t) \end{cases} \tag{6}$$

with the initial conditions

$$e_{1k}(s) = \phi_{1k}(s), \quad e_{2l}(s) = \phi_{2l}(s), \ s \in [-\tau, 0] \tag{7}$$

where $\mathcal{F}_{1l}(e_{2l}(t)) = f_l(h_l^*(t)) + f_l(h_l(t))$, $\mathcal{F}_{1l}(e_{2l}(t - \tau_{2kl}(t))) = f_l(h_l^*(t-\tau_{2kl}(t))) + f_l(h_l(t-\tau_{2kl}(t)))$, $\mathcal{F}_{2k}(e_{1k}(t)) = g_k(z_k^*(t)) + g_k(z_k(t))$, $\mathcal{F}_{2k}(e_{1k}(t - \tau_{1lk}(t))) = g_k(z_k^*(t - \tau_{1lk}(t))) + g_k(z_k(t - \tau_{1lk}(t)))$, $\phi_{1k}(s) = \psi_{1k}(s) + \theta_{1k}(s)$ and $\phi_{2l}(s) = \psi_{2l}(s) + \theta_{2l}(s)$.

Define $e_{1k}(t) = e_{1k}^R(t) + ie_{1k}^I(t)$, $e_{2l}(t) = e_{2l}^R(t) + ie_{2l}^I(t)$, $u_{1k}(t) = u_{1k}^R(t) + iu_{1k}^I(t)$, $u_{2l}(t) = u_{2l}^R(t) + iu_{2l}^I(t)$, $a_{1lk} = a_{1lk}^R + ia_{1lk}^I$, $b_{1lk} = b_{1lk}^R + ib_{1lk}^I$, $a_{2kl} = a_{2kl}^R + ia_{2kl}^I$ and $b_{2kl} = b_{2kl}^R + ib_{2kl}^I$. Obviously, $e_{1k}^R(t) = x_{1k}(t) + x_{1k}^*(t)$, $e_{1k}^I(t) = y_{1k}(t) + y_{1k}^*(t)$, $e_{2l}^R(t) = x_{2l}(t) + x_{2l}^*(t)$ and $e_{2l}^I(t) = y_{2l}(t) + y_{2l}^*(t)$. For brevity, $e_{1k}(t)$, $e_{2l}(t)$, $e_{1k}(t - \tau_{1lk}(t))$, $e_{2l}(t - \tau_{2kl}(t))$, $e_{1k}^R(t)$, $e_{1k}^I(t)$, $e_{2l}^R(t)$, $e_{2l}^I(t)$, $e_{1k}^R(t - \tau_{1lk}(t))$, $e_{1k}^I(t - \tau_{1lk}(t))$, $e_{2l}^R(t - \tau_{2kl}(t))$, $e_{2l}^I(t - \tau_{2kl}(t))$, $u_{1k}^R(t)$, $u_{1k}^I(t)$, $u_{2l}^R(t)$, $u_{2l}^I(t)$, $x_{2l}(t - \tau_{2kl}(t))$, $y_{2l}(t - \tau_{2kl}(t))$, $x_{1k}(t - \tau_{1lk}(t))$ and $y_{1k}(t - \tau_{1lk}(t))$ are simplified as $e_{1k}, e_{2l}, e_{1k}^\tau, e_{2l}^\tau, e_{1k}^R, e_{1k}^I, e_{2l}^R, e_{2l}^I, e_{1k}^{R\tau}, e_{1k}^{I\tau}, e_{2l}^{R\tau}, e_{2l}^{I\tau}, u_{1k}^R, u_{1k}^I, u_{2l}^R, u_{2l}^I, x_{2l}^\tau, y_{2l}^\tau, x_{1k}^\tau$ and y_{1k}^τ. Then, system (6) can be separated into real and imaginary parts as

$$\begin{aligned} \dot{e}_{1k}^R &= -d_{1k} e_{1k}^R + \sum_{l=1}^{m} a_{1lk}^R \mathcal{F}_{1l}^R(e_{2l}^R, e_{2l}^I) \\ &\quad - \sum_{l=1}^{m} a_{1lk}^I \mathcal{F}_{1l}^I(e_{2l}^R, e_{2l}^I) + \sum_{l=1}^{m} b_{1lk}^R \mathcal{F}_{1l}^R(e_{2l}^{R\tau}, e_{2l}^{I\tau}) \\ &\quad - \sum_{l=1}^{m} b_{1lk}^I \mathcal{F}_{1l}^I(e_{2l}^{R\tau}, e_{2l}^{I\tau}) + u_{1k}^R \\ \dot{e}_{1k}^I &= -d_{1k} e_{1k}^I + \sum_{l=1}^{m} a_{1lk}^R \mathcal{F}_{1l}^I(e_{2l}^R, e_{2l}^I) \\ &\quad + \sum_{l=1}^{m} a_{1lk}^I \mathcal{F}_{1l}^R(e_{2l}^R, e_{2l}^I) + \sum_{l=1}^{m} b_{1lk}^R \mathcal{F}_{1l}^I(e_{2l}^{R\tau}, e_{2l}^{I\tau}) \\ &\quad + \sum_{l=1}^{m} b_{1lk}^I \mathcal{F}_{1l}^R(e_{2l}^{R\tau}, e_{2l}^{I\tau}) + u_{1k}^I \\ \dot{e}_{2l}^R &= -d_{2l} e_{2l}^R + \sum_{k=1}^{n} a_{2kl}^R \mathcal{F}_{2k}^R(e_{1k}^R, e_{1k}^I) \\ &\quad - \sum_{k=1}^{n} a_{2kl}^I \mathcal{F}_{2k}^I(e_{1k}^R, e_{1k}^I) + \sum_{k=1}^{n} b_{2kl}^R \mathcal{F}_{2k}^R(e_{1k}^{R\tau}, e_{1k}^{I\tau}) \\ &\quad - \sum_{k=1}^{n} b_{2kl}^I \mathcal{F}_{2k}^I(e_{1k}^{R\tau}, e_{1k}^{I\tau}) + u_{2l}^R \\ \dot{e}_{2l}^I &= -d_{2l} e_{2l}^I + \sum_{k=1}^{n} a_{2kl}^R \mathcal{F}_{2k}^I(e_{1k}^R, e_{1k}^I) \\ &\quad + \sum_{k=1}^{n} a_{2kl}^I \mathcal{F}_{2k}^R(e_{1k}^R, e_{1k}^I) + \sum_{k=1}^{n} b_{2kl}^R \mathcal{F}_{2k}^I(e_{1k}^{R\tau}, e_{1k}^{I\tau}) \\ &\quad + \sum_{k=1}^{n} b_{2kl}^I \mathcal{F}_{2k}^R(e_{1k}^{R\tau}, e_{1k}^{I\tau}) + u_{2l}^I \end{aligned} \tag{8}$$

where

$$\begin{aligned} \mathcal{F}_{1l}^R(e_{2l}^R, e_{2l}^I) &= f_l^R(x_{2l}^*, y_{2l}^*) + f_l^R(x_{2l}, y_{2l}), \\ \mathcal{F}_{1l}^I(e_{2l}^R, e_{2l}^I) &= f_l^I(x_{2l}^*, y_{2l}^*) + f_l^I(x_{2l}, y_{2l}), \\ \mathcal{F}_{1l}^R(e_{2l}^{R\tau}, e_{2l}^{I\tau}) &= f_l^R(x_{2l}^{*\tau}, y_{2l}^{*\tau}) + f_l(x_{2l}^\tau, y_{2l}^\tau), \\ \mathcal{F}_{1l}^I(e_{2l}^{R\tau}, e_{2l}^{I\tau}) &= f_l^I(x_{2l}^{*\tau}, y_{2l}^{*\tau}) + f_l^I(x_{2l}^\tau, y_{2l}^\tau), \\ \mathcal{F}_{2k}^R(e_{1k}^R, e_{1k}^I) &= g_k^R(x_{1k}^*, y_{1k}^*) + g_k^R(x_{1k}, y_{1k}), \\ \mathcal{F}_{2k}^I(e_{1k}^R, e_{1k}^I) &= g_k^I(x_{1k}^*, y_{1k}^*) + g_k^I(x_{1k}, y_{1k}), \\ \mathcal{F}_{2k}^R(e_{1k}^{R\tau}, e_{1k}^{I\tau}) &= g_k^R(x_{1k}^{*\tau}, y_{1k}^{*\tau}) + g_k^R(x_{1k}^\tau, y_{1k}^\tau), \\ \mathcal{F}_{2k}^I(e_{1k}^{R\tau}, e_{1k}^{I\tau}) &= g_k^I(x_{1k}^{*\tau}, y_{1k}^{*\tau}) + g_k^I(x_{1k}^\tau, y_{1k}^\tau). \end{aligned}$$

Now, we rewrite the system (8) as the following vector forms:

$$\begin{cases} \dot{e}_1^R = -D_1 e_1^R + A_1^R \mathcal{F}_1^R(e_2^R, e_2^I) - A_1^I \mathcal{F}_1^I(e_2^R, e_2^I) \\ \qquad + B_1^R \mathcal{F}_1^R(e_2^{R\tau}, e_2^{I\tau}) - B_1^I \mathcal{F}_1^I(e_2^{R\tau}, e_2^{I\tau}) + u_1^R \\ \dot{e}_1^I = -D_1 e_1^I + A_1^R \mathcal{F}_1^I(e_2^R, e_2^I) + A_1^I \mathcal{F}_1^R(e_2^R, e_2^I) \\ \qquad + B_1^R \mathcal{F}_1^I(e_2^{R\tau}, e_2^{I\tau}) + B_1^I \mathcal{F}_1^R(e_2^{R\tau}, e_2^{I\tau}) + u_1^I \\ \dot{e}_2^R = -D_2 e_2^R + A_2^R \mathcal{F}_2^R(e_1^R, e_1^I) - A_2^I \mathcal{F}_2^I(e_1^R, e_1^I) \\ \qquad + B_2^R \mathcal{F}_2^R(e_1^{R\tau}, e_1^{I\tau}) - B_2^I \mathcal{F}_2^I(e_1^{R\tau}, e_1^{I\tau}) + u_2^R \\ \dot{e}_2^I = -D_2 e_2^I + A_2^R \mathcal{F}_2^I(e_1^R, e_1^I) + A_2^I \mathcal{F}_2^R(e_1^R, e_1^I) \\ \qquad + B_2^R \mathcal{F}_2^I(e_1^{R\tau}, e_2^{I\tau}) + B_2^I \mathcal{F}_2^R(e_1^{R\tau}, e_1^{I\tau}) + u_2^I \end{cases} \tag{9}$$

where

$$\begin{aligned}
e_1^R &= (e_{11}^R, e_{12}^R, \ldots, e_{1n}^R)^T,\ e_1^I = (e_{11}^I, e_{12}^I, \ldots, e_{1n}^I)^T,\\
e_2^R &= (e_{21}^R, e_{22}^R, \ldots, e_{2m}^R)^T,\ e_2^I = (e_{21}^I, e_{22}^I, \ldots, e_{2m}^I)^T,\\
e_1^{R\tau} &= (e_{11}^{R\tau}, e_{12}^{R\tau}, \ldots, e_{1n}^{R\tau})^T,\ e_1^{I\tau} = (e_{11}^{I\tau}, e_{12}^{I\tau}, \ldots, e_{1n}^{I\tau})^T,\\
e_2^{R\tau} &= (e_{21}^{R\tau}, e_{22}^{R\tau}, \ldots, e_{2m}^{R\tau})^T,\ e_2^{I\tau} = (e_{21}^{I\tau}, e_{22}^{I\tau}, \ldots, e_{2m}^{I\tau})^T,\\
&\mathcal{F}_1^R(e_2^R, e_2^I)\\
&= (\mathcal{F}_{11}^R(e_{21}^R, e_{21}^I), \mathcal{F}_{12}^R(e_{22}^R, e_{22}^I), \ldots, \mathcal{F}_{1m}^R(e_{2m}^R, e_{2m}^I))^T,\\
&\mathcal{F}_1^I(e_2^R, e_2^I)\\
&= (\mathcal{F}_{11}^I(e_{21}^R, e_{21}^I), \mathcal{F}_{12}^I(e_{22}^R, e_{22}^I), \ldots, \mathcal{F}_{1m}^I(e_{2m}^R, e_{2m}^I))^T,\\
&\mathcal{F}_1^R(e_2^{R\tau}, e_2^{I\tau})\\
&= (\mathcal{F}_{11}^R(e_{21}^{R\tau}, e_{21}^{I\tau}), \mathcal{F}_{12}^R(e_{22}^{R\tau}, e_{22}^{I\tau}), \ldots, \mathcal{F}_{1m}^R(e_{2m}^{R\tau}, e_{2m}^{I\tau}))^T,\\
&\mathcal{F}_1^I(e_2^{R\tau}, e_2^{I\tau})\\
&= (\mathcal{F}_{11}^I(e_{21}^{R\tau}, e_{21}^{I\tau}), \mathcal{F}_{12}^I(e_{22}^{R\tau}, e_{22}^{I\tau}), \ldots, \mathcal{F}_{1m}^I(e_{2m}^{R\tau}, e_{2m}^{I\tau}))^T,\\
&\mathcal{F}_2^R(e_1^R, e_1^I)\\
&= (\mathcal{F}_{21}^R(e_{11}^R, e_{11}^I), \mathcal{F}_{22}^R(e_{12}^R, e_{12}^I), \ldots, \mathcal{F}_{2n}^R(e_{1n}^R, e_{1n}^I))^T,\\
&\mathcal{F}_2^I(e_1^R, e_1^I)\\
&= (\mathcal{F}_{21}^I(e_{11}^R, e_{11}^I), \mathcal{F}_{22}^I(e_{12}^R, e_{12}^I), \ldots, \mathcal{F}_{2n}^I(e_{1n}^R, e_{1n}^I))^T,\\
&\mathcal{F}_2^R(e_1^{R\tau}, e_1^{I\tau})\\
&= (\mathcal{F}_{21}^R(e_{11}^{R\tau}, e_{11}^{I\tau}), \mathcal{F}_{22}^R(e_{12}^{R\tau}, e_{12}^{I\tau}), \ldots, \mathcal{F}_{2n}^R(e_{1n}^{R\tau}, e_{1n}^{I\tau}))^T,\\
&\mathcal{F}_2^I(e_1^{R\tau}, e_1^{I\tau})\\
&= (\mathcal{F}_{21}^I(e_{11}^{R\tau}, e_{11}^{I\tau}), \mathcal{F}_{22}^I(e_{12}^{R\tau}, e_{12}^{I\tau}), \ldots, \mathcal{F}_{2n}^I(e_{1n}^{R\tau}, e_{1n}^{I\tau}))^T,\\
D_1 &= \mathrm{diag}(d_{11}, d_{12}, \ldots, d_{1n}),\ D_2 = \mathrm{diag}(d_{21}, d_{22}, \ldots, d_{2m}),\\
u_1^R &= (u_{11}^R, u_{12}^R, \ldots, u_{1n}^R)^T,\quad u_1^I = (u_{11}^I, u_{12}^I, \ldots, u_{1n}^I)^T,\\
u_2^R &= (u_{21}^R, u_{22}^R, \ldots, u_{2m}^R)^T,\quad u_2^I = (u_{21}^I, u_{22}^I, \ldots, u_{2m}^I)^T,\\
A_1^R &= ((a_{1lk}^R)_{m\times n})^T, A_1^I = ((a_{1lk}^I)_{m\times n})^T,\\
B_1^R &= ((b_{1lk}^R)_{m\times n})^T,\ B_1^I = ((b_{1lk}^I)_{m\times n})^T,\\
A_2^R &= ((a_{2kl}^R)_{n\times m})^T,\ A_2^I = ((a_{2kl}^I)_{n\times m})^T,\\
B_2^R &= ((b_{2kl}^R)_{n\times m})^T and B_2^I = ((b_{2kl}^I)_{n\times m})^T.
\end{aligned}$$

Let

$$\begin{aligned}
\omega_1 &= \begin{bmatrix} e_1^R \\ e_1^I \end{bmatrix},\quad \widetilde{\mathcal{D}}_1 = \begin{bmatrix} D_1 & 0 \\ 0 & D_1 \end{bmatrix},\ \mathcal{U}_1 = \begin{bmatrix} u_1^R \\ u_1^I \end{bmatrix},\\
\overline{A}_1 &= \begin{bmatrix} A_1^R & -A_1^I \\ A_1^I & A_1^R \end{bmatrix},\quad \widetilde{\mathcal{F}}_1(\omega_2) = \begin{bmatrix} \mathcal{F}_1^R(e_2^R, e_2^I) \\ \mathcal{F}_1^I(e_2^R, e_2^I) \end{bmatrix},\\
\overline{B}_1 &= \begin{bmatrix} B_1^R & -B_1^I \\ B_1^I & B_1^R \end{bmatrix},\quad \widetilde{\mathcal{F}}_1(\omega_2^\tau) = \begin{bmatrix} \mathcal{F}_1^R(e_2^{R\tau}, e_2^{I\tau}) \\ \mathcal{F}_1^I(e_2^{R\tau}, e_2^{I\tau}) \end{bmatrix},\\
\omega_2 &= \begin{bmatrix} e_2^R \\ e_2^I \end{bmatrix},\quad \widetilde{\mathcal{D}}_2 = \begin{bmatrix} D_2 & 0 \\ 0 & D_2 \end{bmatrix},\ \mathcal{U}_2 = \begin{bmatrix} u_2^R \\ u_2^I \end{bmatrix},\\
\overline{A}_2 &= \begin{bmatrix} A_2^R & -A_2^I \\ A_2^I & A_2^R \end{bmatrix},\quad \widetilde{\mathcal{F}}_2(\omega_1) = \begin{bmatrix} \mathcal{F}_2^R(e_1^R, e_1^I) \\ \mathcal{F}_2^I(e_1^R, e_1^I) \end{bmatrix},\\
\overline{B}_2 &= \begin{bmatrix} B_2^R & -B_2^I \\ B_2^I & B_2^R \end{bmatrix},\quad \widetilde{\mathcal{F}}_2(\omega_1^\tau) = \begin{bmatrix} \mathcal{F}_2^R(e_1^{R\tau}, e_1^{I\tau}) \\ \mathcal{F}_2^I(e_1^{R\tau}, e_1^{I\tau}) \end{bmatrix},
\end{aligned}$$

system (9) can be arranged as

$$\begin{cases} \dot{\omega}_1 = -\widetilde{\mathcal{D}}_1\omega_1 + \overline{A}_1\widetilde{\mathcal{F}}_1(\omega_2) + \overline{B}_1\widetilde{\mathcal{F}}_1(\omega_2^\tau) + \mathcal{U}_1 \\ \dot{\omega}_2 = -\widetilde{\mathcal{D}}_2\omega_2 + \overline{A}_2\widetilde{\mathcal{F}}_2(\omega_1) + \overline{B}_2\widetilde{\mathcal{F}}_2(\omega_1^\tau) + \mathcal{U}_2 \end{cases} \tag{10}$$

with the initial conditions

$$\omega_1(s) = \Phi_1(s),\ \omega_2(s) = \Phi_2(s),\ s \in [-\tau, 0] \tag{11}$$

where $\Phi_1(s) = ((\phi_1^R(s))^T, (\phi_1^I(s))^T)^T$, $\phi_1^R(s) = (\phi_{11}^R(s), \phi_{12}^R(s), \ldots, \phi_{1n}^R(s))^T$, $\phi_{1k}^R(s) = \mathrm{Re}(\phi_{1k}(s))$, $\phi_1^I(s) = (\phi_{11}^I(s), \phi_{12}^I(s), \ldots, \phi_{1n}^I(s))^T$, $\phi_{1k}^I(s) = \mathrm{Im}(\phi_{1k}(s))$, $\Phi_2(s) = ((\phi_2^R(s))^T, (\phi_2^I(s))^T)^T$, $\phi_2^R(s) = (\phi_{21}^R(s), \phi_{22}^R(s), \ldots, \phi_{2m}^R(s))^T$, $\phi_{2l}^R(s) = \mathrm{Re}(\phi_{2l}(s))$, $\phi_2^I(s) = (\phi_{21}^I(s), \phi_{22}^I(s), \ldots, \phi_{2m}^I(s))^T$, $\phi_{2l}^I(s) = \mathrm{Im}(\phi_{2l}(s))$, $k = 1, 2, \ldots, n$ and $l = 1, 2, \ldots, m$. Then, let

$$\begin{aligned}
\omega &= \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix},\quad \widetilde{\mathcal{D}} = \begin{bmatrix} \widetilde{\mathcal{D}}_1 & 0 \\ 0 & \widetilde{\mathcal{D}}_2 \end{bmatrix},\ \mathcal{U} = \begin{bmatrix} \mathcal{U}_1 \\ \mathcal{U}_2 \end{bmatrix},\\
\overline{M} &= \begin{bmatrix} \overline{A}_1 & 0 \\ 0 & \overline{A}_2 \end{bmatrix},\quad \widehat{\mathcal{F}}_1(\omega) = \begin{bmatrix} \widetilde{\mathcal{F}}_1(\omega_2) \\ \widetilde{\mathcal{F}}_1(\omega_1) \end{bmatrix},\\
\overline{N} &= \begin{bmatrix} \overline{B}_1 & 0 \\ 0 & \overline{B}_2 \end{bmatrix},\quad \widehat{\mathcal{F}}_2(\omega^\tau) = \begin{bmatrix} \widetilde{\mathcal{F}}_2(\omega_2^\tau) \\ \widetilde{\mathcal{F}}_2(\omega_1^\tau) \end{bmatrix},
\end{aligned}$$

system (10) can be rewritten as follows:

$$\dot{\omega} = -\widetilde{\mathcal{D}}\omega + \overline{M}_1\widehat{\mathcal{F}}_1(\omega) + \overline{N}_1\widehat{\mathcal{F}}_2(\omega^\tau) + \mathcal{U} \tag{12}$$

with the initial condition

$$\omega(s) = \Phi(s),\quad s \in [-\tau, 0] \tag{13}$$

where $\Phi(s) = (\Phi_1^T(s), \Phi_2^T(s))^T$.

Next, we introduce the following definition and lemmas to get the desired result.

Definition 1: Systems (1) and (4) can achieve the exponential anti-synchronization if for arbitrary initial condition $\Phi(s) \in \mathcal{C}([-\tau, 0], R^{2n+2m})$ and properly designed feedback controllers, there exist constants $\alpha \geq 1$ and $\epsilon > 0$ such that

$$\left\{\sum_{k=1}^{2n} |w_{1k}(t)|^\varrho + \sum_{l=1}^{2m} |w_{2l}(t)|^\varrho\right\}^{\frac{1}{\varrho}} \leq \alpha e^{-\epsilon t} \parallel \Phi \parallel, \quad \text{for } \forall\ t \geq 0 \tag{14}$$

where ϵ is the estimated rate of exponential anti-synchronization.

Lemma 1: (Young Inequality). Let $a \geq 0$, $b \geq 0$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality hold:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{15}$$

Lemma 2: (Hölder Inequality). Let $a_i \geq 0$, $b_i \geq 0$ $(i = 1, 2, \ldots, n)$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality hold:

$$\sum_{i=1}^n a_i b_i \leq \Big(\sum_{i=1}^n a_i^p\Big)^{1/p}\Big(\sum_{i=1}^n b_i^q\Big)^{1/q}. \tag{16}$$

Especially, let $b_i = 1$ and $n = 2$, one has

$$(a_1 + a_2)^p \leq 2^{p-1}(a_1^p + a_2^p). \tag{17}$$

Moreover, it is obvious that the inequality (17) also hold when $p = 1$.

Remark 1: Based on Assumption 1 and Lemma 2, for $\mathcal{F}^R_{1l}(e^R_{2l}, e^I_{2l})$, $\mathcal{F}^I_{1l}(e^R_{2l}, e^I_{2l})$, $\mathcal{F}^R_{2k}(e^R_{1k}, e^I_{1k})$, $\mathcal{F}^I_{2k}(e^R_{1k}, e^I_{1k})$ and $p \geq 1$, the following inequalities hold:

$$\begin{aligned}
|\mathcal{F}^R_{1l}(e^R_{2l}, e^I_{2l})|^p &\leq 2^{p-1}[(\lambda^{RR}_{1l})^p|e^R_{2l}|^p + (\lambda^{RI}_{1l})^p|e^I_{2l}|^p] \\
|\mathcal{F}^I_{1l}(e^R_{2l}, e^I_{2l})|^p &\leq 2^{p-1}[(\lambda^{IR}_{1l})^p|e^R_{2l}|^p + (\lambda^{II}_{1l})^p|e^I_{2l}|^p] \\
|\mathcal{F}^R_{2k}(e^R_{1k}, e^I_{1k})|^p &\leq 2^{p-1}[(\lambda^{RR}_{2k})^p|e^R_{1k}|^p + (\lambda^{RI}_{2k})^p|e^I_{1k}|^p] \\
|\mathcal{F}^I_{2k}(e^R_{1k}, e^I_{1k})|^p &\leq 2^{p-1}[(\lambda^{IR}_{2k})^p|e^R_{1k}|^p + (\lambda^{II}_{2k})^p|e^I_{1k}|^p].
\end{aligned} \tag{18}$$

Here, in fact, since $f^R_l(x_{2l}, y_{2l})$ is odd function, $|\mathcal{F}^R_{1l}(e^R_{2l}, e^I_{2l})| = |f^R_l(x^*_{2l}, y^*_{2l}) + f^R_l(x_{2l}, y_{2l})| = |f^R_l(x^*_{2l}, y^*_{2l}) - f^R_l(-x_{2l}, -y_{2l})|$. Then, according to Assumption 1 and the inequality (17), we can obtain

$$\begin{aligned}
|\mathcal{F}^R_{1l}(e^R_{2l}, e^I_{2l})|^p &\leq (\lambda^{RR}_{1l}|e^R_{2l}| + \lambda^{RI}_{1l}|e^I_{2l}|)^p \\
&\leq 2^{p-1}[(\lambda^{RR}_{1l})^p|e^R_{2l}|^p + (\lambda^{RI}_{1l})^p|e^I_{2l}|^p].
\end{aligned} \tag{19}$$

Similarly, we can also obtain the other inequalities in (18). Moreover, $\mathcal{F}^R_{1l}(0, 0) = f^R_l(x_{2l}, y_{2l}) + f^R_l(-x_{2l}, -y_{2l}) = 0$, $\mathcal{F}^I_{1l}(0, 0) = 0$, $\mathcal{F}^R_{2k}(0, 0) = 0$ and $\mathcal{F}^I_{2k}(0, 0) = 0$.

Remark 2: It should be pointed out that the inequalities in Remark 1 of [47] are problematic. In Remark 1 of [47], $|\mathcal{F}^R_l(e^R_l(t), e^I_l(t))|^r \leq (\lambda^{RR}_l)^r|e^R_l(t)|^r + (\lambda^{RI}_l)^r|e^I_l(t)|^r$ and $|\mathcal{F}^I_l(e^R_l(t), e^I_l(t))|^r \leq (\lambda^{IR}_l)^r|e^R_l(t)|^r + (\lambda^{II}_l)^r|e^I_l(t)|^r$ ($r \geq 1$), where the mathematical notations are the same as those in [47] as a matter of convenience. In fact, $|\mathcal{F}^R_l(e^R_l(t), e^I_l(t))|^r \leq (\lambda^{RR}_l|e^R_l(t)| + \lambda^{RI}_l|e^I_l(t)|)^r$ whereas $(\lambda^{RR}_l|e^R_l(t)| + \lambda^{RI}_l|e^I_l(t)|)^r \geq (\lambda^{RR}_l)^r|e^R_{2l}(t)|^r + (\lambda^{RI}_l)^r|e^I_l(t)|^r$, so $|\mathcal{F}^R_l(e^R_l(t), e^I_l(t))|^r \leq (\lambda^{RR}_l)^r|e^R_l(t)|^r + (\lambda^{RI}_l)^r|e^I_l(t)|^r$ may not be true. In this paper, we rectify it and give the right inequalities (18) by using Hölder inequality.

III. MAIN RESULT

In this section, we will give a sufficient condition to ensure the anti-synchronization for system (1). For $k = 1, 2, \ldots, n$ and $l = 1, 2, \ldots, m$, the state feedback control laws are designed as

$$\begin{aligned}
u^R_{1k}(t) &= -\pi^R_{1k}(x_{1k}(t) + x^*_{1k}(t)) \\
u^I_{1k}(t) &= -\pi^I_{1k}(y_{1k}(t) + y^*_{1k}(t)) \\
u^R_{2l}(t) &= -\pi^R_{2l}(x_{2l}(t) + x^*_{2l}(t)) \\
u^I_{2l}(t) &= -\pi^I_{2l}(y_{2l}(t) + y^*_{2l}(t))
\end{aligned} \tag{20}$$

where π^R_{1k}, π^I_{1k}, π^R_{2l} and π^I_{2l} are the control gains to be determined.

Remark 3: So as to obtain the proposed result, we designed the control laws as shown above, which are simple and effective. Moreover, our work is carried out in theory. Hence, from the theoretical viewpoint, it seems good. In practice, it also has obvious advantages because of its simplicity. So, our designed control laws are suitable.

Theorem 1: Suppose that Assumption 1 holds, systems (1) and (4) can be exponentially anti-synchronized under control inputs (20) if there exist positive constants ξ_{1k}, ν_{1k}, ξ_{2l}, ν_{2l} and $r \geq 1$ such that

$$\begin{aligned}
&-r(d_{1k} + \pi^R_{1k}) \\
&\quad + \sum_{l=1}^{m}\Big\{\Big[\Big(\frac{\xi_{2l}}{\xi_{1k}}a^R_{2kl} + \frac{\nu_{2l}}{\xi_{1k}}a^I_{2kl} + \frac{\xi_{2l}\delta_{2l}}{\xi_{1k}(1-\rho_2)}\Big)(\lambda^{RR}_{2k})^r \\
&\quad + \Big(\frac{\xi_{2l}}{\xi_{1k}}a^I_{2kl} + \frac{\nu_{2l}}{\xi_{1k}}a^R_{2kl} + \frac{\nu_{2l}\eta_{2l}}{\xi_{1k}(1-\rho_2)}(\lambda^{IR}_{2k})^r\Big)\Big]2^{r-1} \\
&\quad + L_1(r-1)\Big\} \leq 0 \\
&-r(d_{1k} + \pi^I_{1k}) \\
&\quad + \sum_{l=1}^{m}\Big\{\Big[\Big(\frac{\xi_{2l}}{\nu_{1k}}a^R_{2kl} + \frac{\nu_{2l}}{\nu_{1k}}a^I_{2kl} + \frac{\xi_{2l}\delta_{2l}}{\nu_{1k}(1-\rho_2)}\Big)(\lambda^{RI}_{2k})^r \\
&\quad + \Big(\frac{\xi_{2l}}{\nu_{1k}}a^I_{2kl} + \frac{\nu_{2l}}{\nu_{1k}}a^R_{2kl} + \frac{\nu_{2l}\eta_{2l}}{\nu_{1k}(1-\rho_2)}\Big)(\lambda^{II}_{2k})^r\Big]2^{r-1} \\
&\quad + L_1(r-1)\Big\} \leq 0 \\
&-r(d_{2l} + \pi^R_{2l}) \\
&\quad + \sum_{k=1}^{n}\Big\{\Big[\Big(\frac{\xi_{1k}}{\xi_{2l}}a^R_{1lk} + \frac{\nu_{1k}}{\xi_{2l}}a^I_{1lk} + \frac{\xi_{1k}\delta_{1k}}{\xi_{2l}(1-\rho_1)}\Big)(\lambda^{RR}_{1l})^r \\
&\quad + \Big(\frac{\xi_{1k}}{\xi_{2l}}a^I_{1lk} + \frac{\nu_{1k}}{\xi_{2l}}a^R_{1lk} + \frac{\nu_{1k}\eta_{1k}}{\xi_{2l}(1-\rho_1)}\Big)(\lambda^{IR}_{1l})^r\Big]2^{r-1} \\
&\quad + L_2(r-1)\Big\} \leq 0 \\
&-r(d_{2l} + \pi^I_{2l}) \\
&\quad + \sum_{k=1}^{n}\Big\{\Big[\Big(\frac{\xi_{1k}}{\nu_{2l}}a^R_{1lk} + \frac{\nu_{1k}}{\nu_{2l}}a^I_{1lk} + \frac{\xi_{1k}\delta_{1k}}{\nu_{2l}(1-\rho_1)}\Big)(\lambda^{RI}_{1l})^r \\
&\quad + \Big(\frac{\xi_{1k}}{\nu_{2l}}a^I_{1lk} + \frac{\nu_{1k}}{\nu_{2l}}a^R_{1lk} + \frac{\nu_{1k}\eta_{1k}}{\nu_{2l}(1-\rho_1)}\Big)(\lambda^{II}_{1l})^r\Big]2^{r-1} \\
&\quad + L_2(r-1)\Big\} \leq 0
\end{aligned} \tag{21}$$

where $\delta_{1k} = b^R_{1lk} + (\nu_{1k}/\xi_{1k})b^I_{1lk}$, $\eta_{1k} = b^R_{1lk} + (\xi_{1k}/\nu_{1k})b^I_{1lk}$, $\delta_{2l} = b^R_{2kl} + (\nu_{2l}/\xi_{2l})b^I_{2kl}$, $\eta_{2l} = b^R_{2kl} + (\xi_{2l}/\nu_{2l})b^I_{2kl}$, $L_1 = a^R_{1lk} + a^I_{1lk} + b^R_{1lk} + b^I_{1lk}$, $L_2 = a^R_{2kl} + a^I_{2kl} + b^R_{2kl} + b^I_{2kl}$.

Proof. For the detailed proof process of Theorem 1, please see Appendix.

Remark 4: In recent years, great achievements in dynamic behavior analysis for complex-valued neural networks models have developed, in which some results are related to complex-valued BAM neural networks. But there has been no information for the anti-synchronization problem of complex-valued BAM neural networks up to now. In this paper, by applying inequalities techniques, the anti-synchronization of complex-valued BAM neural networks is studied and the corresponding result is shown in Theorem 1. Hence, our work fills the gap in this respect and supplements the existing results.

Remark 5: Compared with real-valued neural networks, the states, connection weights and activation functions of complex-valued neural networks are defined in the complex domain, which can provide a simple and natural way to maintain the physical characteristics of the original problems. For instance, the XOR problem in real numbers cannot be solved with a single real-valued neuron, which can be solved by complex-valued neurons. Therefore, complex-valued neural networks models have more complex properties and wider applications than real-valued ones, and it is very necessary to study the dynamical behaviors of these models. In this paper, for complex-valued BAM neural networks with time-varying delays, the anti-synchronization problem is considered and the corresponding criterion is presented.

IV. NUMERICAL EXAMPLES

In this section, we will provide two examples to illustrate the availability of our result.

Example 1: Consider system (1) with the following parameters:

$$
\begin{aligned}
f_l(h_l) &= \frac{e^{x_{2l}} - e^{-x_{2l}}}{e^{x_{2l}} + e^{-x_{2l}}} + i\frac{e^{y_{2l}} - e^{-y_{2l}}}{e^{y_{2l}} + e^{-y_{2l}}}, (l = 1, 2),\\
g_k(z_k) &= \frac{e^{x_{1k}} - e^{-x_{1k}}}{e^{x_{1k}} + e^{-x_{1k}}} + i\frac{e^{y_{1k}} - e^{-y_{1k}}}{e^{y_{1k}} + e^{-y_{1k}}}, (k = 1, 2),\\
\tau_{2kl}(t) &= \tau_{1lk}(t) = \frac{1}{1+e^{-t}}, \quad \rho_1 = \rho_2 = 0.25,\\
&d_{11} = 2.5, \quad d_{12} = 2.5, \ d_{21} = 2.5, \ d_{22} = 2.5,\\
&a_{111} = -2+3i, \quad a_{112} = -0.5+3i, \ a_{121} = 1.2+2i,\\
&a_{122} = 0.9-0.3i, \quad b_{111} = -2-0.5i, \ b_{112} = 1.5+0.2i,\\
&b_{121} = 0.6+2i, \quad b_{122} = 1.8+2.5i, \ a_{211} = 1.6+2.5i,\\
&a_{212} = -1+2.8i, \quad a_{221} = 0.6+2i, \ a_{222} = 1.6-1.9i,\\
&b_{211} = 1.5+3i, \quad b_{212} = -2+1.9i, \ b_{221} = 1.5+2.5i,\\
&b_{222} = 2-i.
\end{aligned}
$$

It is easy to get that $\lambda_{11}^{RR} = \lambda_{11}^{II} = \lambda_{12}^{RR} = \lambda_{12}^{II} = \lambda_{21}^{RR} = \lambda_{21}^{II} = \lambda_{22}^{RR} = \lambda_{22}^{II} = 1$ and $\lambda_{11}^{IR} = \lambda_{11}^{RI} = \lambda_{12}^{IR} = \lambda_{12}^{RI} = \lambda_{21}^{IR} = \lambda_{21}^{RI} = \lambda_{22}^{IR} = \lambda_{22}^{RI} = 0$. Then, let $\pi_{11}^R = \pi_{12}^R = \pi_{11}^I = \pi_{12}^I = 10$, $\pi_{21}^R = \pi_{22}^R = \pi_{21}^I = \pi_{22}^I = 12$, $r = 2$ and $\xi_{2l} = \nu_{2l} = \xi_{1k} = \nu_{1k} = 1$, via a simple calculation, the conditions of Theorem 1 are satisfied. Hence, the drive system (1) and the corresponding response system (4) with the above parameters can achieve the exponential anti-synchronization. For simulations, the corresponding response curves are depicted in Figs. 1-4. Figs. 1 and 2 show the time responses of real and imaginary parts of variables $z_1, z_2, h_1, h_2, z_1^*, z_2^*, h_1^*, h_2^*$ for the drive-response system without external control inputs. Figs. 3 and 4 display the time responses of anti-synchronization errors e_{11}^R, e_{12}^R, e_{11}^I, e_{12}^I, e_{21}^R, e_{22}^R, e_{21}^I and e_{22}^I under external control inputs with 15 different initial conditions. Figs. 1-4 further show the effectiveness of the proposed result.

Example 2: Consider system (1) with the following parameters:

$$
\begin{aligned}
f_l(h_l) &= \frac{|x_{2l}+1| - |x_{2l}-1|}{2} + i\frac{|y_{2l}+1| - |y_{2l}-1|}{2},\\
g_k(z_k) &= \frac{x_{1k}+y_{1k}}{2} + i\frac{x_{1k}-y_{1k}}{2}, (l,\ k = 1, 2),\\
\tau_{2kl}(t) &= \tau_{1lk}(t) = \frac{1}{1+e^{-t}}, \quad \rho_1 = 0.5, \rho_2 = 0.2,\\
&d_{11} = 2, \quad d_{12} = 2, \quad d_{21} = 3, \quad d_{22} = 3,
\end{aligned}
$$

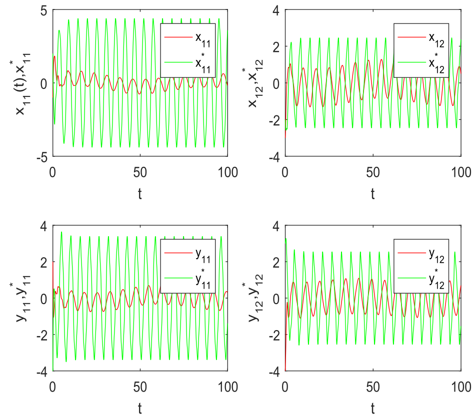

FIGURE 1. The trajectories of real and imaginary parts of variables z_1, h_1, z_1^*, h_1^* for the drive-response system without external control inputs in Example 1.

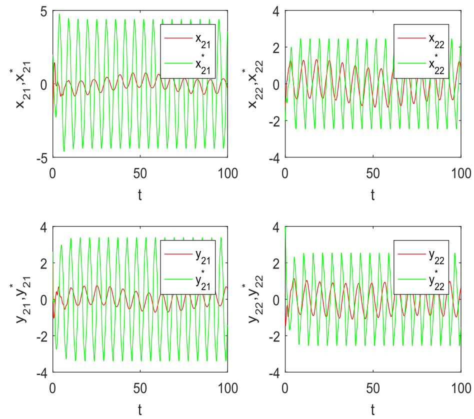

FIGURE 2. The trajectories of real and imaginary parts of variables z_2, h_2, z_2^*, h_2^* for the drive-response system without external control inputs in Example 1.

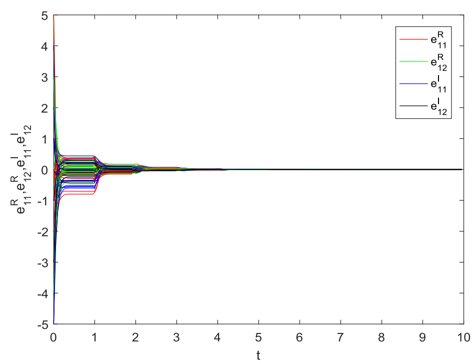

FIGURE 3. The trajectories of anti-synchronization errors e_{11}^R, e_{11}^I, e_{12}^R and e_{12}^I under external control inputs with 15 random initial conditions in Example 1.

$$
\begin{aligned}
a_{111} &= 1.1+i, \quad a_{112} = 2.5-2i, \ a_{121} = 1.2-1.5i,\\
a_{122} &= -0.5+3i, b_{111} = -1.2+2i, b_{112} = 1.44+1.5i,\\
b_{121} &= 3+2i, \quad b_{122} = -1+0.72i, \ a_{211} = -2+2.8i,\\
a_{212} &= 0.8+2i, \quad a_{221} = 1.2+i, \ a_{222} = -0.8-2i,
\end{aligned}
$$

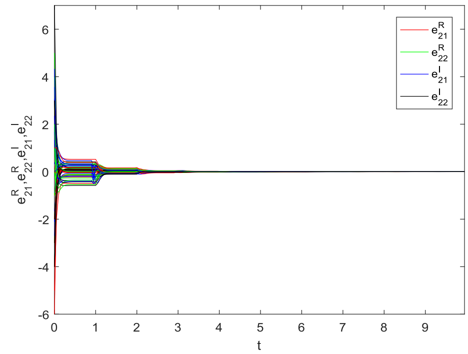

FIGURE 4. The trajectories of anti-synchronization errors e_{21}^R, e_{21}^I, e_{22}^R and e_{22}^I under external control inputs with 15 random initial conditions in Example 1.

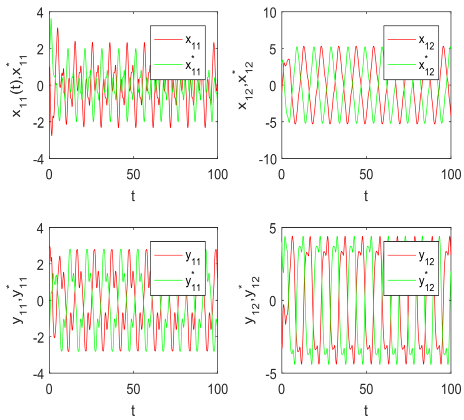

FIGURE 5. The trajectories of real and imaginary parts of variables z_1, h_1, z_1^*, h_1^* for the drive-response system without external control inputs in Example 2.

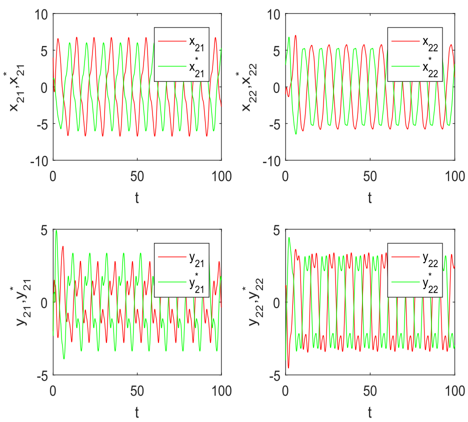

FIGURE 6. The trajectories of real and imaginary parts of variables z_2, h_2, z_2^*, h_2^* for the drive-response system without external control inputs in Example 2.

$$b_{211}=-1.5+2.1i, \quad b_{212}=1.6-0.12i, \quad b_{222}=3+1.6i,$$
$$b_{221}=1.25-0.1i.$$

It is easy to get that $\lambda_{11}^{RR}=\lambda_{11}^{II}=\lambda_{12}^{RR}=\lambda_{12}^{II}=1$, $\lambda_{11}^{IR}=\lambda_{11}^{RI}=\lambda_{12}^{IR}=\lambda_{12}^{RI}=0$ and $\lambda_{21}^{RR}=\lambda_{21}^{II}=\lambda_{22}^{RR}=\lambda_{22}^{II}=\lambda_{21}^{IR}=\lambda_{21}^{RI}=\lambda_{22}^{IR}=\lambda_{22}^{RI}=\frac{1}{2}$. Then, let $\pi_{11}^R=\pi_{12}^R=\pi_{11}^I=\pi_{12}^I=10$, $\pi_{21}^R=\pi_{22}^R=\pi_{21}^I=\pi_{22}^I=11$, $r=2$, $\xi_{2l}=\nu_{2l}=3$, and $\xi_{1k}=\nu_{1k}=4$, via a

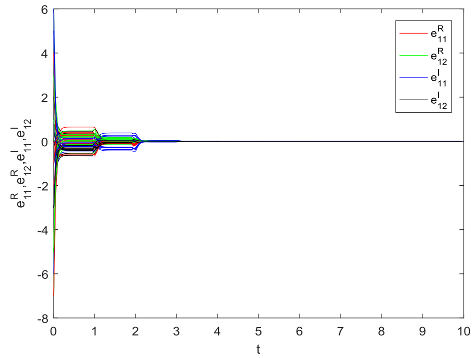

FIGURE 7. The trajectories of anti-synchronization errors e_{11}^R, e_{11}^I, e_{12}^R and e_{12}^I under external control inputs with 15 random initial conditions in Example 2.

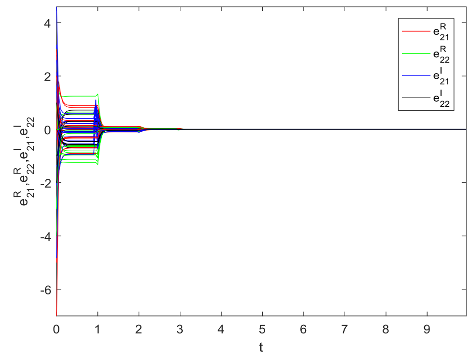

FIGURE 8. The trajectories of anti-synchronization errors e_{21}^R, e_{21}^I, e_{22}^R and e_{22}^I under external control inputs with 15 random initial conditions in Example 2.

simple calculation, the conditions of Theorem 1 are satisfied. Hence, the drive-response systems (1) and (4) with the above parameters can achieve the exponential anti-synchronization. For simulations, Figs. 5-8 depict the corresponding response curves.

V. CONCLUSION

In this paper, the anti-synchronization control problem of delayed complex-valued BAM neural networks has been investigated. By using the inequality techniques, a sufficient condition has been obtained to ensure the anti-synchronization of the considered system. The effectiveness of our result has been verified by two numerical examples. Moreover, the proposed result is the first one for delayed complex-valued BAM neural networks, so our work fills the gap in this field and complements the previous results. In the future, we will focus on the anti-synchronization issues of more complex-valued systems and endeavour to obtain more refined achievements.

APPENDIX

Firstly, for the inequalities (21), we choose a sufficient small constant $\epsilon > 0$ such that

$$r(\epsilon - d_{1k} - \pi_{1k}^R)$$
$$+\sum_{l=1}^{m}\Big\{\Big[\Big(\frac{\xi_{2l}}{\xi_{1k}}a_{2kl}^R+\frac{\nu_{2l}}{\xi_{1k}}a_{2kl}^I+\frac{\xi_{2l}e^{r\epsilon\tau_2}}{\xi_{1k}(1-\rho_2)}\delta_{2l}\Big)(\lambda_{2k}^{RR})^r$$

$$
\begin{aligned}
&+\Big(\frac{\xi_{2l}}{\xi_{1k}}a^I_{2kl}+\frac{\nu_{2l}}{\xi_{1k}}a^R_{2kl}+\frac{\nu_{2l}e^{r\epsilon\tau_2}}{\xi_{1k}(1-\rho_2)}\eta_{2l}\Big)(\lambda^{IR}_{2k})^r\Big]2^{r-1}\\
&+L_1(r-1)\Big\}\le 0\\
&r(\epsilon-d_{1k}-\pi^I_{1k})\\
&+\sum_{l=1}^{m}\Big\{\Big[\Big(\frac{\xi_{2l}}{\nu_{1k}}a^R_{2kl}+\frac{\nu_{2l}}{\nu_{1k}}a^I_{2kl}+\frac{\xi_{2l}e^{r\epsilon\tau_2}}{\nu_{1k}(1-\rho_2)}\delta_{2l}\Big)(\lambda^{RI}_{2k})^r\\
&+\Big(\frac{\xi_{2l}}{\nu_{1k}}a^I_{2kl}+\frac{\nu_{2l}}{\nu_{1k}}a^R_{2kl}+\frac{\nu_{2l}e^{r\epsilon\tau_2}}{\nu_{1k}(1-\rho_2)}\eta_{2l}\Big)(\lambda^{II}_{2k})^r\Big]2^{r-1}\\
&+L_1(r-1)\Big\}\le 0\\
&r(\epsilon-d_{2l}-\pi^R_{2l})\\
&+\sum_{k=1}^{n}\Big\{\Big[\Big(\frac{\xi_{1k}}{\xi_{2l}}a^R_{1lk}+\frac{\nu_{1k}}{\xi_{2l}}a^I_{1lk}+\frac{\xi_{1k}e^{r\epsilon\tau_1}}{\xi_{2l}(1-\rho_1)}\delta_{1k}\Big)(\lambda^{RR}_{1l})^r\\
&+\Big(\frac{\xi_{1k}}{\xi_{2l}}a^I_{1lk}+\frac{\nu_{1k}}{\xi_{2l}}a^R_{1lk}+\frac{\nu_{1k}e^{r\epsilon\tau_1}}{\xi_{2l}(1-\rho_1)}\eta_{1k}\Big)(\lambda^{IR}_{1l})^r\Big]2^{r-1}\\
&+L_2(r-1)\Big\}\le 0\\
&r(\epsilon-d_{2l}-\pi^I_{2l})\\
&+\sum_{k=1}^{n}\Big\{\Big[\Big(\frac{\xi_{1k}}{\nu_{2l}}a^R_{1lk}+\frac{\nu_{1k}}{\nu_{2l}}a^I_{1lk}+\frac{\xi_{1k}e^{r\epsilon\tau_1}}{\nu_{2l}(1-\rho_1)}\delta_{1k}\Big)(\lambda^{RI}_{1l})^r\\
&+\Big(\frac{\xi_{1k}}{\nu_{2l}}a^I_{1lk}+\frac{\nu_{1k}}{\nu_{2l}}a^R_{1lk}+\frac{\nu_{1k}e^{r\epsilon\tau_1}}{\nu_{2l}(1-\rho_1)}\eta_{1k}\Big)(\lambda^{II}_{1l})^r\Big]2^{r-1}\\
&+L_2(r-1)\Big\}\le 0.
\end{aligned}\tag{22}
$$

Next, we construct a Lyapunov functional as follows:

$$
\begin{aligned}
V(w(t),t)=&\sum_{k=1}^{2n}\zeta_{1k}\Big\{|w_{1k}(t)|^re^{r\epsilon t}+\frac{1}{1-\rho_1}\sum_{l=1}^{2m}\beta_{1k}\\
&\times\int_{t-\tau_{2kl}(t)}^{t}|\mathcal{F}_{1l}(w_{2l}(s))|^re^{r\epsilon(s+\tau_{2kl}(s))}ds\Big\}\\
&+\sum_{l=1}^{2m}\zeta_{2l}\Big\{|w_{2l}(t)|^re^{r\epsilon t}+\frac{1}{1-\rho_2}\sum_{k=1}^{2n}\beta_{2l}\\
&\times\int_{t-\tau_{1lk}(t)}^{t}|\mathcal{F}_{2k}(w_{1k}(s))|^re^{r\epsilon(s+\tau_{1lk}(s))}ds\Big\}
\end{aligned}\tag{23}
$$

where

$$
w_{1k}(t)=\begin{cases}e^R_{1k}(t), & k=1,2,\ldots,n\\ e^I_{1k}(t), & k=n+1,\ n+2,\ldots,2n\end{cases}
$$

$$
\zeta_{1k}=\begin{cases}\xi_{1k}, & k=1,2,\ldots,n\\ \nu_{1k}, & k=n+1,\ n+2,\ldots,2n\end{cases}
$$

$$
w_{2l}(t)=\begin{cases}e^R_{2l}(t), & l=1,2,\ldots,m\\ e^I_{2l}(t), & l=m+1,\ m+2,\ldots,2m\end{cases}
$$

$$
\zeta_{2l}=\begin{cases}\xi_{2l}, & l=1,2,\ldots,m\\ \nu_{2l}, & l=m+1,\ m+2,\ldots,2m\end{cases}
$$

$$
\beta_{1k}=\begin{cases}\delta_{1k}, & k=1,2,\ldots,n\\ \eta_{1k}, & k=n+1,\ n+2,\ldots,2n\end{cases}
$$

$$
\beta_{2l}=\begin{cases}\delta_{2l}, & l=1,2,\ldots,m\\ \eta_{2l}, & l=m+1,\ m+2,\ldots,2m\end{cases}
$$

$$
\mathcal{F}_{1l}(w_{2l}(t))=\begin{cases}\mathcal{F}^R_{1l}(e^R_{2l}(t),e^I_{2l}(t)), & l=1,2,\ldots,m\\ \mathcal{F}^I_{1l}(e^R_{2l}(t),e^I_{2l}(t)), & l=m+1,\ldots,2m\end{cases}
$$

$$
\mathcal{F}_{2k}(w_{1k}(t))=\begin{cases}\mathcal{F}^R_{2k}(e^R_{1k}(t),e^I_{1k}(t)), & k=1,2,\ldots,n\\ \mathcal{F}^I_{2k}(e^R_{1k}(t),e^I_{1k}(t)), & k=n+1,\ldots,2n.\end{cases}
$$

Then,

$$
\begin{aligned}
V(w(t),t)=&\sum_{k=1}^{n}\xi_{1k}|e^R_{1k}(t)|^re^{r\epsilon t}+\frac{1}{1-\rho_1}\sum_{k=1}^{n}\sum_{l=1}^{m}\xi_{1k}\delta_{1k}\\
&\times\int_{t-\tau_{2kl}(t)}^{t}|\mathcal{F}^R_{1l}(e^R_{2l}(s),e^I_{2l}(s))|^re^{r\epsilon(s+\tau_{2kl}(s))}ds\\
&+\sum_{k=1}^{n}\nu_{1k}|e^I_{1k}(t)|^re^{r\epsilon t}+\frac{1}{1-\rho_1}\sum_{k=1}^{n}\sum_{l=1}^{m}\nu_{1k}\eta_{1k}\\
&\times\int_{t-\tau_{2kl}(t)}^{t}|\mathcal{F}^I_{1l}(e^R_{2l}(s),e^I_{2l}(s))|^re^{r\epsilon(s+\tau_{2kl}(s))}ds\\
&+\sum_{l=1}^{m}\xi_{2l}|e^R_{2l}(t)|^re^{r\epsilon t}+\frac{1}{1-\rho_2}\sum_{l=1}^{m}\sum_{k=1}^{n}\xi_{2l}\delta_{2l}\\
&\times\int_{t-\tau_{1lk}(t)}^{t}|\mathcal{F}^R_{2k}(e^R_{1k}(s),e^I_{1k}(s))|^re^{r\epsilon(s+\tau_{1lk}(s))}ds\\
&+\sum_{l=1}^{m}\nu_{2l}|e^I_{2l}(t)|^re^{r\epsilon t}+\frac{1}{1-\rho_2}\sum_{l=1}^{m}\sum_{k=1}^{n}\nu_{2l}\eta_{2l}\\
&\times\int_{t-\tau_{1lk}(t)}^{t}|\mathcal{F}^I_{2k}(e^R_{1k}(s),e^I_{1k}(s))|^re^{r\epsilon(s+\tau_{1lk}(s))}ds.
\end{aligned}\tag{24}
$$

By taking the upper Dini-derivative of $V(w(t),t)$ along the solution trajectories of system (8), we have

$$
\begin{aligned}
&D^+V(w(t),t)\\
&=e^{r\epsilon t}\sum_{k=1}^{n}r\xi_{1k}\Big\{\epsilon|e^R_{1k}|^r+|e^R_{1k}|^{r-1}\mathrm{sgn}(e^R_{1k})\Big[-(d_{1k}\\
&\quad+\pi^R_{1k})e^R_{1k}+\sum_{l=1}^{m}a^R_{1lk}\mathcal{F}^R_{1l}(e^R_{2l},e^I_{2l})-\sum_{l=1}^{m}a^I_{1lk}\mathcal{F}^I_{1l}(e^R_{2l},e^I_{2l})\\
&\quad+\sum_{l=1}^{m}b^R_{1lk}\mathcal{F}^R_{1l}(e^{R\tau}_{2l},e^{I\tau}_{2l})-\sum_{l=1}^{m}b^I_{1lk}\mathcal{F}^I_{1l}(e^{R\tau}_{2l},e^{I\tau}_{2l})\Big]\Big\}\\
&\quad+e^{r\epsilon t}\sum_{k=1}^{n}r\nu_{1k}\Big\{\epsilon|e^I_{1k}|^r+|e^I_{1k}|^{r-1}\mathrm{sgn}(e^I_{1k})\Big[-(d_{1k}\\
&\quad+\pi^I_{1k})e^I_{1k}+\sum_{l=1}^{m}a^R_{1lk}\mathcal{F}^I_{1l}(e^R_{2l},e^I_{2l})+\sum_{l=1}^{m}a^I_{1lk}\mathcal{F}^R_{1l}(e^R_{2l},e^I_{2l})\\
&\quad+\sum_{l=1}^{m}b^R_{1lk}\mathcal{F}^I_{1l}(e^{R\tau}_{2l},e^{I\tau}_{2l})+\sum_{l=1}^{m}b^I_{1lk}\mathcal{F}^R_{1l}(e^{R\tau}_{2l},e^{I\tau}_{2l})\Big]\Big\}\\
&\quad+\frac{1}{1-\rho_1}\sum_{k=1}^{n}\sum_{l=1}^{m}e^{r\epsilon t}\xi_{1k}\delta_{1k}\Big\{|\mathcal{F}^R_{1l}(e^R_{2l},e^I_{2l})|^re^{r\epsilon\tau_{2kl}(t)}\\
&\quad-(1-\dot{\tau}_{2kl}(t))|\mathcal{F}^R_{1l}(e^{R\tau}_{2l},e^{I\tau}_{2l})|^r\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{1-\rho_1}\sum_{k=1}^{n}\sum_{l=1}^{m}e^{r\epsilon t}\nu_{1k}\eta_{1k}\Big\{|\mathcal{F}_{1l}^{I}(e_{2l}^{R},e_{2l}^{I})|^{r}e^{r\epsilon\tau_{2kl}(t)}\\
&-(1-\dot{\tau}_{2kl}(t))|\mathcal{F}_{1l}^{I}(e_{2l}^{R\tau},e_{2l}^{I\tau})|^{r}\Big\}\\
&+e^{r\epsilon t}\sum_{l=1}^{m}r\xi_{2l}\Big\{\epsilon|e_{2l}^{R}|^{r}+|e_{2l}^{R}|^{r-1}\mathrm{sgn}(e_{2l}^{R})\Big[-(d_{2l}\\
&+\pi_{2l}^{R})e_{2l}^{R}+\sum_{k=1}^{n}a_{2kl}^{R}\mathcal{F}_{2k}^{R}(e_{1k}^{R},e_{1k}^{I})-\sum_{k=1}^{n}a_{2kl}^{I}\mathcal{F}_{2k}^{I}(e_{1k}^{R},e_{1k}^{I})\\
&+\sum_{k=1}^{n}b_{2kl}^{R}\mathcal{F}_{2k}^{R}(e_{1k}^{R\tau},e_{1k}^{I\tau})-\sum_{k=1}^{n}b_{2kl}^{I}\mathcal{F}_{2k}^{I}(e_{1k}^{R\tau},e_{1k}^{I\tau})\Big]\Big\}\\
&+e^{r\epsilon t}\sum_{l=1}^{m}r\nu_{2l}\Big\{\epsilon|e_{2l}^{I}|^{r}+|e_{2l}^{I}|^{r-1}\mathrm{sgn}(e_{2l}^{I})\Big[-(d_{2l}\\
&+\pi_{2l}^{I})e_{2l}^{I}+\sum_{k=1}^{n}a_{2kl}^{R}\mathcal{F}_{2k}^{I}(e_{1k}^{R},e_{1k}^{I})+\sum_{k=1}^{n}a_{2kl}^{I}\mathcal{F}_{2k}^{R}(e_{1k}^{R},e_{1k}^{I})\\
&+\sum_{k=1}^{n}b_{2kl}^{R}\mathcal{F}_{2k}^{I}(e_{1k}^{R\tau},e_{1k}^{I\tau})+\sum_{k=1}^{n}b_{2kl}^{I}\mathcal{F}_{2k}^{R}(e_{1k}^{R\tau},e_{1k}^{I\tau})\Big]\Big\}\\
&+\frac{1}{1-\rho_2}\sum_{l=1}^{m}\sum_{k=1}^{n}e^{r\epsilon t}\xi_{2l}\delta_{2l}\Big\{|\mathcal{F}_{2k}^{R}(e_{1k}^{R},e_{1k}^{I})|^{r}e^{r\epsilon\tau_{1lk}(t)}\\
&-(1-\dot{\tau}_{1lk}(t))|\mathcal{F}_{2k}^{R}(e_{1k}^{R\tau},e_{1k}^{I\tau})|^{r}\Big\}\\
&+\frac{1}{1-\rho_2}\sum_{l=1}^{m}\sum_{k=1}^{n}e^{r\epsilon t}\nu_{2l}\eta_{2l}\Big\{|\mathcal{F}_{2k}^{I}(e_{1k}^{R},e_{1k}^{I})|^{r}e^{r\epsilon\tau_{1lk}(t)}\\
&-(1-\dot{\tau}_{1lk}(t))|\mathcal{F}_{2k}^{I}(e_{1k}^{R\tau},e_{1k}^{I\tau})|^{r}\Big\}\\
\le\ & e^{r\epsilon t}\sum_{k=1}^{n}\Big\{r\xi_{1k}(\epsilon-d_{1k}-\pi_{1k}^{R})|e_{1k}^{R}|^{r}\\
&+\sum_{l=1}^{m}a_{1lk}^{R}r\xi_{1k}|\mathcal{F}_{1l}^{R}(e_{2l}^{R},e_{2l}^{I})||e_{1k}^{R}|^{r-1}\\
&+\sum_{l=1}^{m}a_{1lk}^{I}r\xi_{1k}|\mathcal{F}_{1l}^{I}(e_{2l}^{R},e_{2l}^{I})||e_{1k}^{R}|^{r-1}\\
&+\sum_{l=1}^{m}b_{1lk}^{R}r\xi_{1k}|\mathcal{F}_{1l}^{R}(e_{2l}^{R\tau},e_{2l}^{I\tau})||e_{1k}^{R}|^{r-1}\\
&+\sum_{l=1}^{m}b_{1lk}^{I}r\xi_{1k}|\mathcal{F}_{1l}^{I}(e_{2l}^{R\tau},e_{2l}^{I\tau})||e_{1k}^{R}|^{r-1}\\
&+\frac{e^{r\epsilon\tau_1}}{1-\rho_1}\sum_{l=1}^{m}\xi_{1k}\delta_{1k}|\mathcal{F}_{1l}^{R}(e_{2l}^{R},e_{2l}^{I})|^{r}\\
&-\sum_{l=1}^{m}\xi_{1k}\delta_{1k}|\mathcal{F}_{1l}^{R}(e_{2l}^{R\tau},e_{2l}^{I\tau})|^{r}\Big\}\\
&+e^{r\epsilon t}\sum_{k=1}^{n}\Big\{r\nu_{1k}(\epsilon-d_{1k}-\pi_{1k}^{I})|e_{1k}^{I}|^{r}\\
&+\sum_{l=1}^{m}a_{1lk}^{R}r\nu_{1k}|\mathcal{F}_{1l}^{I}(e_{2l}^{R},e_{2l}^{I})||e_{1k}^{I}|^{r-1}\\
&+\sum_{l=1}^{m}a_{1lk}^{I}r\nu_{1k}|\mathcal{F}_{1l}^{R}(e_{2l}^{R},e_{2l}^{I})||e_{1k}^{I}|^{r-1}\\
&+\sum_{l=1}^{m}b_{1lk}^{R}r\nu_{1k}|\mathcal{F}_{1l}^{I}(e_{2l}^{R\tau},e_{2l}^{I\tau})||e_{1k}^{I}|^{r-1}\\
&+\sum_{l=1}^{m}b_{1lk}^{I}r\nu_{1k}|\mathcal{F}_{1l}^{R}(e_{2l}^{R\tau},e_{2l}^{I\tau})||e_{1k}^{I}|^{r-1}\\
&+\frac{e^{r\epsilon\tau_1}}{1-\rho_1}\sum_{l=1}^{m}\nu_{1k}\eta_{1k}|\mathcal{F}_{1l}^{I}(e_{2l}^{R},e_{2l}^{I})|^{r}\\
&-\sum_{l=1}^{m}\nu_{1k}\eta_{1k}|\mathcal{F}_{1l}^{I}(e_{2l}^{R\tau},e_{2l}^{I\tau})|^{r}\Big\}\\
&+e^{r\epsilon t}\sum_{l=1}^{m}\Big\{r\xi_{2l}(\epsilon-d_{2l}-\pi_{2l}^{R})|e_{2l}^{R}|^{r}\\
&+\sum_{k=1}^{n}a_{2kl}^{R}r\xi_{2l}|\mathcal{F}_{2k}^{R}(e_{1k}^{R},e_{1k}^{I})||e_{2l}^{R}|^{r-1}\\
&+\sum_{k=1}^{n}a_{2kl}^{I}r\xi_{2l}|\mathcal{F}_{2k}^{I}(e_{1k}^{R},e_{1k}^{I})||e_{2l}^{R}|^{r-1}\\
&+\sum_{k=1}^{n}b_{2kl}^{R}r\xi_{2l}|\mathcal{F}_{2k}^{R}(e_{1k}^{R\tau},e_{1k}^{I\tau})||e_{2l}^{R}|^{r-1}\\
&+\sum_{k=1}^{n}b_{2kl}^{I}r\xi_{2l}|\mathcal{F}_{2k}^{I}(e_{1k}^{R\tau},e_{1k}^{I\tau})||e_{2l}^{R}|^{r-1}\\
&+\frac{e^{r\epsilon\tau_2}}{1-\rho_2}\sum_{k=1}^{n}\xi_{2l}\delta_{2l}|\mathcal{F}_{2k}^{R}(e_{1k}^{R},e_{1k}^{I})|^{r}\\
&-\sum_{k=1}^{n}\xi_{2l}\delta_{2l}|\mathcal{F}_{2k}^{R}(e_{1k}^{R\tau},e_{1k}^{I\tau})|^{r}\Big\}\\
&+e^{r\epsilon t}\sum_{l=1}^{m}\Big\{r\nu_{2l}(\epsilon-d_{2l}-\pi_{2l}^{I})|e_{2l}^{I}|^{r}\\
&+\sum_{k=1}^{n}a_{2kl}^{R}r\nu_{2l}|\mathcal{F}_{2k}^{I}(e_{1k}^{R},e_{1k}^{I})||e_{2l}^{I}|^{r-1}\\
&+\sum_{k=1}^{n}a_{2kl}^{I}r\nu_{2l}|\mathcal{F}_{2k}^{R}(e_{1k}^{R},e_{1k}^{I})||e_{2l}^{I}|^{r-1}\\
&+\sum_{k=1}^{n}b_{2kl}^{R}r\nu_{2l}|\mathcal{F}_{2k}^{I}(e_{1k}^{R\tau},e_{1k}^{I\tau})||e_{2l}^{I}|^{r-1}\\
&+\sum_{k=1}^{n}b_{2kl}^{I}r\nu_{2l}|\mathcal{F}_{2k}^{R}(e_{1k}^{R\tau},e_{1k}^{I\tau})||e_{2l}^{I}|^{r-1}\\
&+\frac{e^{r\epsilon\tau_2}}{1-\rho_2}\sum_{k=1}^{n}\nu_{2l}\eta_{2l}|\mathcal{F}_{2k}^{I}(e_{1k}^{R},e_{1k}^{I})|^{r}\\
&-\sum_{k=1}^{n}\nu_{2l}\eta_{2l}|\mathcal{F}_{2k}^{I}(e_{1k}^{R\tau},e_{1k}^{I\tau})|^{r}\Big\}. \qquad (25)
\end{aligned}
$$

By using Lemma 1, one has

$$
\begin{aligned}
&D^{+}V(w(t),t)\\
&\leq e^{r\epsilon t}\sum_{k=1}^{n}\Big\{r\xi_{1k}(\epsilon-d_{1k}-\pi_{1k}^{R})|e_{1k}^{R}|^{r}\\
&\quad+\sum_{l=1}^{m}a_{1lk}^{R}r\xi_{1k}\Big[\frac{1}{r}|\mathcal{F}_{1l}^{R}(e_{2l}^{R},e_{2l}^{I})|^{r}+\frac{r-1}{r}(|e_{1k}^{R}|^{r-1})^{\frac{r}{r-1}}\Big]\\
&\quad+\sum_{l=1}^{m}a_{1lk}^{I}r\xi_{1k}\Big[\frac{1}{r}|\mathcal{F}_{1l}^{I}(e_{2l}^{R},e_{2l}^{I})|^{r}+\frac{r-1}{r}(|e_{1k}^{R}|^{r-1})^{\frac{r}{r-1}}\Big]\\
&\quad+\sum_{l=1}^{m}b_{1lk}^{R}r\xi_{1k}\Big[\frac{1}{r}|\mathcal{F}_{1l}^{R}(e_{2l}^{R\tau},e_{2l}^{I\tau})|^{r}+\frac{r-1}{r}(|e_{1k}^{R}|^{r-1})^{\frac{r}{r-1}}\Big]\\
&\quad+\sum_{l=1}^{m}b_{1lk}^{I}r\xi_{1k}\Big[\frac{1}{r}|\mathcal{F}_{1l}^{I}(e_{2l}^{R\tau},e_{2l}^{I\tau})|^{r}+\frac{r-1}{r}(|e_{1k}^{R}|^{r-1})^{\frac{r}{r-1}}\Big]\\
&\quad+\frac{e^{r\epsilon\tau_{1}}}{1-\rho_{1}}\sum_{l=1}^{m}\xi_{1k}\delta_{1k}|\mathcal{F}_{1l}^{R}(e_{2l}^{R},e_{2l}^{I})|^{r}\\
&\quad-\sum_{l=1}^{m}\xi_{1k}\delta_{1k}|\mathcal{F}_{1l}^{R}(e_{2l}^{R\tau},e_{2l}^{I\tau})|^{r}\Big\}\\
&\quad+e^{r\epsilon t}\sum_{k=1}^{n}\Big\{r\nu_{1k}(\epsilon-d_{1k}-\pi_{1k}^{I})|e_{1k}^{I}|^{r}\\
&\quad+\sum_{l=1}^{m}a_{1lk}^{R}r\nu_{1k}\Big[\frac{1}{r}|\mathcal{F}_{1l}^{I}(e_{2l}^{R},e_{2l}^{I})|^{r}+\frac{r-1}{r}(|e_{1k}^{I}|^{r-1})^{\frac{r}{r-1}}\Big]\\
&\quad+\sum_{l=1}^{m}a_{1lk}^{I}r\nu_{1k}\Big[\frac{1}{r}|\mathcal{F}_{1l}^{R}(e_{2l}^{R},e_{2l}^{I})|^{r}+\frac{r-1}{r}(|e_{1k}^{I}|^{r-1})^{\frac{r}{r-1}}\Big]\\
&\quad+\sum_{l=1}^{m}b_{1lk}^{R}r\nu_{1k}\Big[\frac{1}{r}|\mathcal{F}_{1l}^{I}(e_{2l}^{R\tau},e_{2l}^{I\tau})|^{r}+\frac{r-1}{r}(|e_{1k}^{I}|^{r-1})^{\frac{r}{r-1}}\Big]\\
&\quad+\sum_{l=1}^{m}b_{1lk}^{I}r\nu_{1k}\Big[\frac{1}{r}|\mathcal{F}_{1l}^{R}(e_{2l}^{R\tau},e_{2l}^{I\tau})|^{r}+\frac{r-1}{r}(|e_{1k}^{I}|^{r-1})^{\frac{r}{r-1}}\Big]\\
&\quad+\frac{e^{r\epsilon\tau_{1}}}{1-\rho_{1}}\sum_{l=1}^{m}\nu_{1k}\eta_{1k}|\mathcal{F}_{1l}^{I}(e_{2l}^{R},e_{2l}^{I})|^{r}\\
&\quad-\sum_{l=1}^{m}\nu_{1k}\eta_{1k}|\mathcal{F}_{1l}^{I}(e_{2l}^{R\tau},e_{2l}^{I\tau})|^{r}\Big\}\\
&\quad+e^{r\epsilon t}\sum_{l=1}^{m}\Big\{r\xi_{2l}(\epsilon-d_{2l}-\pi_{2l}^{R})|e_{2l}^{R}|^{r}\\
&\quad+\sum_{k=1}^{n}a_{2kl}^{R}r\xi_{2l}\Big[\frac{1}{r}|\mathcal{F}_{2k}^{R}(e_{1k}^{R},e_{1k}^{I})|^{r}+\frac{r-1}{r}(|e_{2l}^{R}|^{r-1})^{\frac{r}{r-1}}\Big]\\
&\quad+\sum_{k=1}^{n}a_{2kl}^{I}r\xi_{2l}\Big[\frac{1}{r}|\mathcal{F}_{2k}^{I}(e_{1k}^{R},e_{1k}^{I})|^{r}+\frac{r-1}{r}(|e_{2l}^{R}|^{r-1})^{\frac{r}{r-1}}\Big]\\
&\quad+\sum_{k=1}^{n}b_{2kl}^{R}r\xi_{2l}\Big[\frac{1}{r}|\mathcal{F}_{2k}^{R}(e_{1k}^{R\tau},e_{1k}^{R\tau})|^{r}+\frac{r-1}{r}(|e_{2l}^{R}|^{r-1})^{\frac{r}{r-1}}\Big]\\
&\quad+\sum_{k=1}^{n}b_{2kl}^{I}r\xi_{2l}\Big[\frac{1}{r}|\mathcal{F}_{2k}^{I}(e_{1k}^{R\tau},e_{1k}^{R\tau})|^{r}+\frac{r-1}{r}(|e_{2l}^{R}|^{r-1})^{\frac{r}{r-1}}\Big]\\
&\quad+\frac{e^{r\epsilon\tau_{2}}}{1-\rho_{2}}\sum_{k=1}^{n}\xi_{2l}\delta_{2l}|\mathcal{F}_{2k}^{R}(e_{1k}^{R},e_{1k}^{I})|^{r}\\
&\quad-\sum_{k=1}^{n}\xi_{2l}\delta_{2l}|\mathcal{F}_{2k}^{R}(e_{1k}^{R\tau},e_{1k}^{I\tau})|^{r}\Big\}\\
&\quad+e^{r\epsilon t}\sum_{l=1}^{m}\Big\{r\nu_{2l}(\epsilon-d_{2l}-\pi_{2l}^{I})|e_{2l}^{I}|^{r}\\
&\quad+\sum_{k=1}^{n}a_{2kl}^{R}r\nu_{2l}\Big[\frac{1}{r}|\mathcal{F}_{2k}^{I}(e_{1k}^{R},e_{1k}^{I})|^{r}+\frac{r-1}{r}(|e_{2l}^{I}|^{r-1})^{\frac{r}{r-1}}\Big]\\
&\quad+\sum_{k=1}^{n}a_{2kl}^{I}r\nu_{2l}\Big[\frac{1}{r}|\mathcal{F}_{2k}^{R}(e_{1k}^{R},e_{1k}^{I})|^{r}+\frac{r-1}{r}(|e_{2l}^{I}|^{r-1})^{\frac{r}{r-1}}\Big]\\
&\quad+\sum_{k=1}^{n}b_{2kl}^{R}r\nu_{2l}\Big[\frac{1}{r}|\mathcal{F}_{2k}^{I}(e_{1k}^{R\tau},e_{1k}^{I\tau})|^{r}+\frac{r-1}{r}(|e_{2l}^{I}|^{r-1})^{\frac{r}{r-1}}\Big]\\
&\quad+\sum_{k=1}^{n}b_{2kl}^{I}r\nu_{2l}\Big[\frac{1}{r}|\mathcal{F}_{2k}^{R}(e_{1k}^{R\tau},e_{1k}^{I\tau})|^{r}+\frac{r-1}{r}(|e_{2l}^{I}|^{r-1})^{\frac{r}{r-1}}\Big]\\
&\quad+\frac{e^{r\epsilon\tau_{2}}}{1-\rho_{2}}\sum_{k=1}^{n}\nu_{2l}\eta_{2l}|\mathcal{F}_{2k}^{I}(e_{1k}^{R},e_{1k}^{I})|^{r}\\
&\quad-\sum_{k=1}^{n}\nu_{2l}\eta_{2l}|\mathcal{F}_{2k}^{I}(e_{1k}^{R\tau},e_{1k}^{I\tau})|^{r}\Big\}\\
&=e^{r\epsilon t}\sum_{k=1}^{n}\Big\{r\xi_{1k}(\epsilon-d_{1k}-\pi_{1k}^{R})|e_{1k}^{R}|^{r}\\
&\quad+r\nu_{1k}(\epsilon-d_{1k}-\pi_{1k}^{I})|e_{1k}^{I}|^{r}\\
&\quad+\sum_{l=1}^{m}\Big[a_{1lk}^{R}\xi_{1k}+a_{1lk}^{I}\nu_{1k}+\frac{e^{r\epsilon\tau_{1}}}{1-\rho_{1}}\xi_{1k}\delta_{1k}\Big]|\mathcal{F}_{1l}^{R}(e_{2l}^{R},e_{2l}^{I})|^{r}\\
&\quad+\sum_{l=1}^{m}\Big[a_{1lk}^{I}\xi_{1k}+a_{1lk}^{R}\nu_{1k}+\frac{e^{r\epsilon\tau_{1}}}{1-\rho_{1}}\nu_{1k}\eta_{1k}\Big]|\mathcal{F}_{1l}^{I}(e_{2l}^{R},e_{2l}^{I})|^{r}\\
&\quad+\sum_{l=1}^{m}\xi_{1k}\Big[a_{1lk}^{R}+a_{1lk}^{I}+b_{1lk}^{R}+b_{1lk}^{I}\Big](r-1)|e_{1k}^{R}|^{r}\\
&\quad+\sum_{l=1}^{m}\nu_{1k}\Big[a_{1lk}^{R}+a_{1lk}^{I}+b_{1lk}^{R}+b_{1lk}^{I}\Big](r-1)|e_{1k}^{I}|^{r}\\
&\quad+\sum_{l=1}^{m}\Big[b_{1lk}^{R}\xi_{1k}+b_{1lk}^{I}\nu_{1k}\Big]|\mathcal{F}_{1l}^{R}(e_{2l}^{R\tau},e_{2l}^{I\tau})|^{r}\\
&\quad-\sum_{l=1}^{m}\xi_{1k}\delta_{1k}|\mathcal{F}_{1l}^{R}(e_{2l}^{R\tau},e_{2l}^{I\tau})|^{r}\\
&\quad+\sum_{l=1}^{m}\Big[b_{1lk}^{I}\xi_{1k}+b_{1lk}^{R}\nu_{1k}\Big]|\mathcal{F}_{1l}^{I}(e_{2l}^{R\tau},e_{2l}^{I\tau})|^{r}\\
&\quad-\sum_{l=1}^{m}\nu_{1k}\eta_{1k}|\mathcal{F}_{1l}^{I}(e_{2l}^{R\tau},e_{2l}^{I\tau})|^{r}\Big\}\\
&\quad+e^{r\epsilon t}\sum_{l=1}^{m}\Big\{r\xi_{2l}(\epsilon-d_{2l}-\pi_{2l}^{R})|e_{2l}^{R}|^{r}\\
&\quad+r\nu_{2l}(\epsilon-d_{2l}-\pi_{2l}^{I})|e_{2l}^{I}|^{r}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{k=1}^{n}\Big[a^R_{2kl}\xi_{2l}+a^I_{2lk}\nu_{2l}+\frac{e^{r\epsilon\tau_2}}{1-\rho_2}\xi_{2l}\delta_{2l}\Big]|\mathcal{F}^R_{2k}(e^R_{1k},e^I_{1k})|^r\\
&+\sum_{k=1}^{n}\Big[a^I_{2kl}\xi_{2l}+a^R_{2lk}\nu_{2l}+\frac{e^{r\epsilon\tau_2}}{1-\rho_2}\nu_{2l}\eta_{2l}\Big]|\mathcal{F}^I_{2k}(e^R_{1k},e^I_{1k})|^r\\
&+\sum_{k=1}^{n}\xi_{2l}\Big[a^R_{2kl}+a^I_{2kl}+b^R_{2kl}+b^I_{2kl}\Big](r-1)|e^R_{2l}|^r\\
&+\sum_{k=1}^{n}\nu_{2l}\Big[a^R_{2kl}+a^I_{2kl}+b^R_{2kl}+b^I_{2kl}\Big](r-1)|e^I_{2l}|^r\\
&+\sum_{k=1}^{n}\Big[b^R_{2kl}\xi_{2l}+b^I_{2kl}\nu_{2l}\Big]|\mathcal{F}^R_{2k}(e^{R\tau}_{1k},e^{I\tau}_{1k})|^r\\
&-\sum_{k=1}^{n}\xi_{2l}\delta_{2l}|\mathcal{F}^R_{2k}(e^{R\tau}_{1k},e^{I\tau}_{1k})|^r\\
&+\sum_{k=1}^{n}\Big[b^I_{2kl}\xi_{2l}+b^R_{2kl}\nu_{2l}\Big]|\mathcal{F}^I_{2k}(e^{R\tau}_{1k},e^{I\tau}_{1k})|^r\\
&-\sum_{k=1}^{n}\nu_{2l}\eta_{2l}|\mathcal{F}^I_{2k}(e^{R\tau}_{1k},e^{I\tau}_{1k})|^r\Big\}. \qquad (26)
\end{aligned}
$$

Based on Assumption 1, $\delta_{1k}=b^R_{1lk}+(\nu_{1k}/\xi_{1k})b^I_{1lk}$, $\eta_{1k}=b^R_{1lk}+(\xi_{1k}/\nu_{1k})b^I_{1lk}$, $\delta_{2l}=b^R_{2kl}+(\nu_{2l}/\xi_{2l})b^I_{2kl}$ and $\eta_{2l}=b^R_{2kl}+(\xi_{2l}/\nu_{2l})b^I_{2kl}$, we can further get that

$$
\begin{aligned}
&D^+V(w(t),t)\\
&\le e^{r\epsilon t}\sum_{k=1}^{n}\Big\{r\xi_{1k}(\epsilon-d_{1k}-\pi^R_{1k})|e^R_{1k}|^r\\
&\quad+r\nu_{1k}(\epsilon-d_{1k}-\pi^I_{1k})|e^I_{1k}|^r\\
&\quad+\sum_{l=1}^{m}2^{r-1}\Big[a^R_{1lk}\xi_{1k}+\frac{e^{r\epsilon\tau_1}}{1-\rho_1}\xi_{1k}\delta_{1k}+a^I_{1lk}\nu_{1k}\Big]\\
&\quad\times\Big[(\lambda^{RR}_{1l})^r|e^R_{2l}|^r+(\lambda^{RI}_{1l})^r|e^I_{2l}|^r\Big]\\
&\quad+\sum_{l=1}^{m}2^{r-1}\Big[a^I_{1lk}\xi_{1k}+\frac{e^{r\epsilon\tau_1}}{1-\rho_1}\nu_{1k}\eta_{1k}+a^R_{1lk}\nu_{1k}\Big]\\
&\quad\times\Big[(\lambda^{IR}_{1l})^r|e^R_{2l}|^r+(\lambda^{II}_{1l})^r|e^I_{2l}|^r\Big]\\
&\quad+\sum_{l=1}^{m}\xi_{1k}\Big[a^R_{1lk}+a^I_{1lk}+b^R_{1lk}+b^I_{1lk}\Big](r-1)|e^R_{1k}|^r\\
&\quad+\sum_{l=1}^{m}\nu_{1k}\Big[a^R_{1lk}+a^I_{1lk}+b^R_{1lk}+b^I_{1lk}\Big](r-1)|e^I_{1k}|^r\Big\}\\
&\quad+e^{r\epsilon t}\sum_{l=1}^{m}\Big\{r\xi_{2l}(\epsilon-d_{2l}-\pi^R_{2l})|e^R_{2l}|^r\\
&\quad+r\nu_{2l}(\epsilon-d_{2l}-\pi^I_{2l})|e^I_{2l}|^r\\
&\quad+\sum_{k=1}^{n}2^{r-1}\Big[a^R_{2kl}\xi_{2l}+\frac{e^{r\epsilon\tau_2}}{1-\rho_2}\xi_{2l}\delta_{2l}+a^I_{2kl}\nu_{2l}\Big]\\
&\quad\times\Big[(\lambda^{RR}_{2k})^r|e^R_{1k}|^r+(\lambda^{RI}_{2k})^r|e^I_{1k}|^r\Big]\\
&\quad+\sum_{k=1}^{n}2^{r-1}\Big[a^I_{2kl}\xi_{2l}+\frac{e^{r\epsilon\tau_2}}{1-\rho_2}\nu_{2l}\eta_{2l}+a^R_{2kl}\nu_{2l}\Big]\\
&\quad\times\Big[(\lambda^{IR}_{2k})^r|e^R_{1k}|^r+(\lambda^{II}_{2k})^r|e^I_{1k}|^r\Big]\\
&\quad+\sum_{k=1}^{n}\xi_{2l}\Big[a^R_{2kl}+a^I_{2kl}+b^R_{2kl}+b^I_{2kl}\Big](r-1)|e^R_{2l}|^r\\
&\quad+\sum_{k=1}^{n}\nu_{2l}\Big[a^R_{2kl}+a^I_{2kl}+b^R_{2kl}+b^I_{2kl}\Big](r-1)|e^I_{2l}|^r\Big\}\\
&=e^{r\epsilon t}\sum_{k=1}^{n}\Big\{r\xi_{1k}(\epsilon-d_{1k}-\pi^R_{1k})\\
&\quad+\sum_{l=1}^{m}\Big[\Big((a^R_{2kl}\xi_{2l}+a^I_{2kl}\nu_{2l}+\frac{e^{r\epsilon\tau_2}}{1-\rho_2}\xi_{2l}\delta_{2l})(\lambda^{RR}_{2k})^r\\
&\quad+(a^I_{2kl}\xi_{2l}+a^R_{2kl}+\frac{e^{r\epsilon\tau_2}}{1-\rho_2}\nu_{2l}\eta_{2l})(\lambda^{IR}_{2k})^r\Big)2^{r-1}\\
&\quad+\xi_{1k}(a^R_{1lk}+a^I_{1lk}+b^R_{1lk}+b^I_{1lk})(r-1)\Big]\Big\}|e^R_{1k}|^r\\
&\quad+e^{r\epsilon t}\sum_{k=1}^{n}\Big\{r\nu_{1k}(\epsilon-d_{1k}-\pi^I_{1k})\\
&\quad+\sum_{l=1}^{m}\Big[\Big((a^R_{2kl}\xi_{2l}+a^I_{2kl}\nu_{2l}+\frac{e^{r\epsilon\tau_2}}{1-\rho_2}\xi_{2l}\delta_{2l})(\lambda^{RI}_{2k})^r\\
&\quad+(a^I_{2kl}\xi_{2l}+a^R_{2kl}\nu_{2l}+\frac{e^{r\epsilon\tau_2}}{1-\rho_2}\nu_{2l}\eta_{2l})(\lambda^{II}_{2k})^r\Big)2^{r-1}\\
&\quad+\nu_{1k}(a^R_{1lk}+a^I_{1lk}+b^R_{1lk}+b^I_{1lk})(r-1)\Big]\Big\}|e^I_{1k}|^r\\
&\quad+e^{r\epsilon t}\sum_{l=1}^{m}\Big\{r\xi_{2l}(\epsilon-d_{2l}-\pi^R_{2l})\\
&\quad+\sum_{k=1}^{n}\Big[\Big((a^R_{1lk}\xi_{1k}+a^I_{1lk}\nu_{1k}+\frac{e^{r\epsilon\tau_1}}{1-\rho_1}\xi_{1k}\delta_{1k})(\lambda^{RR}_{1l})^r\\
&\quad+(a^I_{1lk}\xi_{1k}+a^R_{1lk}\nu_{1k}+\frac{e^{r\epsilon\tau_1}}{1-\rho_1}\nu_{1k}\eta_{1k})(\lambda^{IR}_{1l})^r\Big)2^{r-1}\\
&\quad+\xi_{2l}(a^R_{2kl}+a^I_{2kl}+b^R_{2kl}+b^I_{2kl})(r-1)\Big]\Big\}|e^R_{2l}|^r\\
&\quad+e^{r\epsilon t}\sum_{l=1}^{m}\Big\{r\nu_{2l}(\epsilon-d_{2l}-\pi^I_{2l})\\
&\quad+\sum_{k=1}^{n}\Big[\Big((a^R_{1lk}\xi_{1k}+a^I_{1lk}\nu_{1k}+\frac{e^{r\epsilon\tau_1}}{1-\rho_1}\xi_{1k}\delta_{1k})(\lambda^{RI}_{1l})^r\\
&\quad+(a^I_{1lk}\xi_{1k}+a^R_{1lk}\nu_{1k}+\frac{e^{r\epsilon\tau_1}}{1-\rho_1}\nu_{1k}\eta_{1k})(\lambda^{II}_{1l})^r\Big)2^{r-1}\\
&\quad+\nu_{2l}(a^R_{2kl}+a^I_{2kl}+b^R_{2kl}+b^I_{2kl})(r-1)\Big]\Big\}|e^I_{2l}|^r. \qquad (27)
\end{aligned}
$$

Then, according to the inequalities (22), we have

$$D^+V(w(t),t)\le 0. \qquad (28)$$

Hence,

$$V(w(t),t)\le V(w(0),0),\ \ t\ge 0. \qquad (29)$$

Moreover,

$$
\begin{aligned}
&V(w(0),0)\\
&\le \Big\{ \max_{1\le k\le 2n} \zeta_{1k} + \frac{\tau_1 e^{r\epsilon\tau_1}}{1-\rho_1}\lambda_1^r \sum_{l=1}^{2m} \zeta_{1l} \max_{1\le l\le 2m} \beta_{1l}\\
&\quad + \max_{1\le l\le 2m} \zeta_{2l} + \frac{\tau_2 e^{r\epsilon\tau_2}}{1-\rho_2}\lambda_2^r \sum_{k=1}^{2n} \zeta_{2k} \max_{1\le k\le 2n} \beta_{2k} \Big\} \|\Phi\|^r\\
&= \Pi \|\Phi\|^r\\
&V(w(t),t)\\
&\ge \sum_{k=1}^{2n} \zeta_{1k} |w_{1k}(t)|^r e^{r\epsilon t} + \sum_{l=1}^{2m} \zeta_{2l} |w_{2l}(t)|^r e^{r\epsilon t}\\
&\ge \Big\{ \min_{1\le k\le 2n} \zeta_{1k} \Big\} e^{r\epsilon t} \sum_{k=1}^{2n} |w_{1k}(t)|^r\\
&\quad + \Big\{ \min_{1\le l\le 2m} \zeta_{2l} \Big\} e^{r\epsilon t} \sum_{l=1}^{2m} |w_{2l}(t)|^r\\
&\ge \gamma e^{r\epsilon t} \Big\{ \sum_{k=1}^{2n} |w_k(t)|^r + \sum_{l=1}^{2m} |w_l(t)|^r \Big\} \qquad (30)
\end{aligned}
$$

where $\lambda_1 = \max\{\lambda_{1l}^{RR}, \lambda_{1l}^{RI}, \lambda_{1l}^{IR}, \lambda_{1l}^{II}\}$, $\lambda_2 = \max\{\lambda_{2k}^{RR}, \lambda_{2k}^{RI}, \lambda_{2k}^{IR}, \lambda_{2k}^{II}\}$, $\gamma = \min\{\min_{1\le k\le 2n} \zeta_{1k}, \min_{1\le l\le 2m} \zeta_{2l}\}$ and $\Pi = \max_{1\le k\le 2n} \zeta_{1k} + \frac{\tau_1 e^{r\epsilon\tau_1}}{1-\rho_1}\lambda_1^r \sum_{l=1}^{2m} \zeta_{1l} \max_{1\le l\le 2m} \beta_{1l} + \max_{1\le l\le 2m} \zeta_{2l} + \frac{\tau_2 e^{r\epsilon\tau_2}}{1-\rho_2}\lambda_2^r \sum_{k=1}^{2n} \zeta_{2k} \max_{1\le k\le 2n} \beta_{2k}$.

From (29)-(30), we have

$$\gamma e^{r\epsilon t} \Big\{ \sum_{k=1}^{2n} |w_k(t)|^r + \sum_{l=1}^{2m} |w_l(t)|^r \Big\} \le \Pi \|\Phi\|^r \qquad (31)$$

that is,

$$\Big\{ \sum_{k=1}^{2n} |w_k(t)|^r + \sum_{l=1}^{2m} |w_l(t)|^r \Big\}^{\frac{1}{r}} \le \alpha e^{-\epsilon t} \parallel \Phi \parallel \qquad (32)$$

where

$$\alpha = \Big(\frac{\Pi}{\gamma}\Big)^{\frac{1}{r}} \ge 1. \qquad (33)$$

Thus, by Definition 1, systems (1) and (4) can achieve the exponential anti-synchronization. This completes the proof.

REFERENCES

[1] B. Kosko, "Adaptive bidirectional associative memories," *Appl. Opt.*, vol. 26, no. 23, pp. 4947–4960, 1987.

[2] J. Chen, T. Zhang, Z. Zhang, C. Lin, and B. Chen, "Stability and output feedback control for singular Markovian jump delayed systems," *Math. Control Rel. Fields*, vol. 8, pp. 475–490, Jun. 2018.

[3] S. Jiao, H. Shen, Y. Wei, X. Huang, and Z. Wang, "Further results on dissipativity and stability analysis of Markov jump generalized neural networks with time-varying interval delays," *Appl. Math. Comput.*, vol. 336, pp. 338–350, Nov. 2018.

[4] Y. Wang, Z. Pan, Y. Li, and W. Zhang, "H_∞ control for nonlinear stochastic Markov systems with time-delay and multiplicative noise," *J. Syst. Sci. Complex.*, vol. 30, no. 6, pp. 1293–1315, Dec. 2017.

[5] Z. Wang, Y. Xie, J. Lu, and Y. Li, "Stability and bifurcation of a delayed generalized fractional-order prey–predator model with interspecific competition," *Appl. Math. Comput.*, vol. 347, pp. 360–369, Apr. 2019.

[6] X. Wang, Z. Wang, X. Huang, and Y. Li, "Dynamic analysis of a delayed fractional-order sir model with saturated incidence and treatment functions," *Int. J. Bifurcation Chaos*, vol. 28, no. 14, Dec. 2018, Art. no. 1850180.

[7] M. Gao, L. Sheng, and W. Zhang, "Stochastic H_2/H_∞ control of nonlinear systems with time-delay and state-dependent noise," *Appl. Math. Comput.*, vol. 266, pp. 429–440, Sep. 2015.

[8] K. Gopalsamy, "Leakage delays in BAM," *J. Math. Anal. Appl.*, vol. 325, no. 2, pp. 1117–1132, Jan. 2007.

[9] L. Li, Y. Yang, and G. Lin, "The stabilization of BAM neural networks with time-varying delays in the leakage terms via sampled-data control," *Neural Comput. Appl.*, vol. 27, pp. 447–457, Feb. 2016.

[10] S. Lakshmanan, J. H. Park, T. H. Lee, H. Y. Jung, and R. Rakkiyappan, "Stability criteria for BAM neural networks with leakage delays and probabilistic time-varying delays," *Appl. Math. Comput.*, vol. 219, no. 17, pp. 9408–9423, May 2013.

[11] J. Jian and B. Wang, "Global Lagrange stability for neutral-type Cohen–Grossberg BAM neural networks with mixed time-varying delays," *Math. Comput. Simul.*, vol. 116, pp. 1–25, Oct. 2015.

[12] W. Peng, Q. Wu, and Z. Zhang, "LMI-based global exponential stability of equilibrium point for neutral delayed BAM neural networks with delays in leakage terms via new inequality technique," *Neurocomputing*, vol. 199, pp. 103–113, Jul. 2016.

[13] T. Nitta, "Solving the XOR problem and the detection of symmetry using a single complex-valued neuron," *Neural Netw.*, vol. 16, no. 8, pp. 1101–1105, Oct. 2003.

[14] B. Zhou and Q. Song, "Boundedness and complete stability of complex-valued neural networks with time delay," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 24, no. 8, pp. 1227–1238, Aug. 2013.

[15] Z. Zhang, C. Lin, and B. Chen, "Global stability criterion for delayed complex-valued recurrent neural networks," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 25, no. 9, pp. 1704–1708, Sep. 2014.

[16] R. Rakkiyappan, J. Cao, and G. Velmurugan, "Existence and uniform stability analysis of fractional-order complex-valued neural networks with time delays," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 26, no. 1, pp. 84–97, Jan. 2015.

[17] X. X. Liu and T. T. Chen, "Global exponential stability for complex-valued recurrent neural networks with asynchronous time delays," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 27, no. 3, pp. 593–606, Mar. 2016.

[18] Z. Wang, X. Wang, Y. Li, and X. Huang, "Stability and Hopf bifurcation of fractional-order complex-valued single neuron model with time delay," *Int. J. Bifurcation Chaos*, vol. 27, no. 13, Dec. 2017, Art. no. 1750209.

[19] L. Li, Z. Wang, Y. Li, H. Shen, and J. Lu, "Hopf bifurcation analysis of a complex-valued neural network model with discrete and distributed delays," *Appl. Math. Comput.*, vol. 330, pp. 152–169, Aug. 2018.

[20] J. Shen and S. Chen, "Stability and L_∞-gain analysis for a class of nonlinear positive systems with mixed delays," *Int. J. Robust Nonlinear Control*, vol. 27, no. 1, pp. 39–49, Jan. 2017.

[21] J. Shen and W. X. Zheng, "Stability analysis of linear delay systems with cone invariance," *Automatica*, vol. 53, pp. 30–36, Mar. 2015.

[22] Z. Zhang, X. Liu, D. Zhou, C. Lin, J. Chen, and H. Wang, "Finite-time stabilizability and instabilizability for complex-valued memristive neural networks with time delays," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 48, no. 12, pp. 2371–2382, Dec. 2018.

[23] J. Jian and P. Wan, "Global exponential convergence of fuzzy complex-valued neural networks with time-varying delays and impulsive effects," *Fuzzy Sets Syst.*, vol. 338, pp. 23–39, May 2018.

[24] Z. Zhang, X. Liu, J. Chen, R. Guo, and S. Zhou, "Further stability analysis for delayed complex-valued recurrent neural networks," *Neurocomputing*, vol. 251, pp. 81–89, Aug. 2017.

[25] X. Xu, Q. Xu, Y. Peng, J. Zhang, and Y. Xu, "Stochastic exponential robust stability of delayed complex-valued neural networks with Markova jumping parameters," *IEEE Access*, vol. 6, pp. 839–849, 2018.

[26] R. Guo, Z. Zhang, and M. Gao, "State estimation for complex-valued memristive neural networks with time-varying delays," *Adv. Difference Equ.*, vol. 118, p. 118, Apr. 2018.

[27] X. Li, R. Rakkiyappan, and G. Velmurugan, "Dissipativity analysis of memristor-based complex-valued neural networks with time-varying delays," *Inf. Sci.*, vol. 294, pp. 645–665, Feb. 2015.

[28] X. Chen, Z. Zhao, Q. Song, and J. Hu, "Multistability of complex-valued neural networks with time-varying delays," *Appl. Math. Comput.*, vol. 294, pp. 18–35, Feb. 2017.

[29] Z. Zhang, R. Guo, X. Liu, and C. Lin, "Lagrange exponential stability of complex-valued BAM neural networks with time-varying delays," *IEEE Trans. Syst., Man, Cybern., Syst.*, to be published. doi: 10.1109/TSMC.2018.2840091.

[30] Z. Wang and L. Huang, "Global stability analysis for delayed complex-valued BAM neural networks," *Neurocomputing*, vol. 173, pp. 2083–2089, Jan. 2016.

[31] R. Guo, Z. Zhang, X. Liu, and C. Lin, "Existence, uniqueness, and exponential stability analysis for complex-valued memristor-based BAM neural networks with time delays," *Appl. Math. Comput.*, vol. 311, pp. 100–117, Oct. 2017.

[32] Z. Zhang, X. Liu, R. Guo, and C. Lin, "Finite-time stability for delayed complex-valued BAM neural networks," *Neural Process. Lett.*, vol. 48, no. 1, pp. 179–193, Aug. 2018.

[33] D. Xu and M. Tan, "Delay-independent stability criteria for complex-valued BAM neutral-type neural networks with time delays," *Nonlinear Dyn.*, vol. 89, no. 2, pp. 819–832, Jul. 2017.

[34] R. Guo, Z. Zhang, X. Liu, C. Lin, H. Wang, and J. Chen, "Exponential input-to-state stability for complex-valued memristor-based BAM neural networks with multiple time-varying delays," *Neurocomputing*, vol. 275, pp. 2041–2054, Jan. 2018.

[35] R. Guo, Z. Zhang, C. Lin, Y. Chu, and Y. Li, "Finite time state estimation of complex-valued BAM neutral-type neural networks with time-varying delays," *Int. J. Control Autom. Syst.*, vol. 17, no. 3, pp. 801–809, Mar. 2019.

[36] K. Subramanian and P. Muthukumar, "Existence, uniqueness, and global asymptotic stability analysis for delayed complex-valued Cohen–Grossberg BAM neural networks," *Neural Comput. Appl.*, vol. 29, no. 9, pp. 565–584, May 2018.

[37] C. Rajivganthi, F. A. Rihan, and S. Lakshmanan, "Dissipativity analysis of complex-valued BAM neural networks with time delay," *Neural Comput. Appl.*, vol. 31, no. 1, pp. 127–137, Jan. 2019.

[38] M. Xing, H. Shen, and Z. Wang, "H_∞ synchronization of semi-Markovian jump neural networks with randomly occurring time-varying delays," *Complexity*, vol. 2018, Sep. 2018, Art. no. 8094292.

[39] J. Wang, T. Ru, J. Xia, Y. Wei, and Z. Wang, "Finite-time synchronization for complex dynamic networks with semi-Markov switching topologies: An H_∞ event-triggered control scheme," *Appl. Math. Comput.*, vol. 356, pp. 235–251, Sep. 2019.

[40] J. Wang, X. Hu, Y. Wei, and Z. Wang, "Sampled-data synchronization of semi-Markov jump complex dynamical networks subject to generalized dissipativity property," *Appl. Math. Comput.*, vol. 346, pp. 853–864, Apr. 2019.

[41] H. Bao, J. H. Park, and J. Cao, "Synchronization of fractional-order complex-valued neural networks with time delay," *Neural Netw.*, vol. 81, pp. 16–28, Sep. 2016.

[42] W. Zhang, H. Zhang, J. Cao, F. E. Alsaadi, and D. Chen, "Synchronization in uncertain fractional-order memristive complex-valued neural networks with multiple time delays," *Neural Netw.*, vol. 110, pp. 186–198, Feb. 2019.

[43] J. Shen and J. Cao, "Finite-time synchronization of coupled neural networks via discontinuous controllers," *Cognit. Neurodyn.*, vol. 5, no. 4, pp. 373–385, 2011.

[44] C. Zhou, W. Zhang, X. Yang, C. Xu, and J. Feng, "Finite-time synchronization of complex-valued neural networks with mixed delays and uncertain perturbations," *Neural Process. Lett.*, vol. 46, no. 1, pp. 271–291, Aug. 2017.

[45] Z. Zhang, A. Li, and S. Yu, "Finite-time synchronization for delayed complex-valued neural networks via integrating inequality method," *Neurocomputing*, vol. 318, pp. 248–260, Nov. 2018.

[46] Z. Zhang, X. Liu, C. Lin, and B. Chen, "Finite-time synchronization for complex-valued recurrent neural networks with time delays," *Complexity*, vol. 2018, Dec. 2018, Art. no. 8456737.

[47] D. Liu, S. Zhu, and K. Sun, "Global anti-synchronization of complex-valued memristive neural networks with time delays," *IEEE Trans. Cybern.*, vol. 49, no. 5, pp. 1735–1747, May 2019.

[48] D. Liu, S. Zhu, and K. Sun, "Anti-synchronization of complex-valued memristor-based delayed neural networks," *Neural Netw.*, vol. 105, pp. 1–13, Sep. 2018.

[49] Z. Zhang, A. Li, and L. Yang, "Global asymptotic periodic synchronization for delayed complex-valued BAM neural networks via vector-valued inequality techniques," *Neural Process. Lett.*, vol. 48, no. 2, pp. 1019–1041, Oct. 2018.

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