A Dynamic Prescriptive Maintenance Model Considering System Aging and Degradation

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ABSTRACT This paper develops a dynamic maintenance strategy for a system subject to aging and degradation. The influence of degradation level and aging on system failure rate is modeled in an additive way. Based on the observed degradation level at the inspection, repair or replacement is carried out upon the system. Previous researches assume that repair will always lead to an improvement in the health condition of the system. However, in our study, repair reduces the system age but on the other hand, increases the degradation level. Considering the two-fold influence of maintenance actions, we perform reliability analysis on system reliability as a first step. The evolution of system reliability serves as a foundation for establishing the maintenance model. The optimal maintenance strategy is achieved by minimizing the long-run cost rate in terms of the repair cycle. At each inspection, the parameters of the degradation processes are updated with maximum a posteriori estimation when a new observation arrives. The effectiveness of the proposed model is illustrated through a case study of locomotive wheel-sets. The maintenance model considers the influence of degradation and aging on system failure and dynamically determines the optimal inspection time, which is more flexible than traditional stationary maintenance strategies and can provide better performance in the field.

INDEX TERMS Aging and degradation process, dynamic maintenance strategy, locomotive wheel-sets, prescriptive maintenance, sequential schedule.

I. INTRODUCTION

With the increasing integration of systems, maintenance strategies are placing more emphasis on techno-economic than technological considerations. Existing maintenance strategies include time-based maintenance, where maintenance actions are performed at failure (corrective maintenance) or based on system age (age-based maintenance), and condition-based maintenance, where maintenance decisions are provided based on the health condition of the system.

Prescriptive maintenance has gained popularity in recent years. It extends the concept of failure prediction [1], [2] by predicting maintenance measures and prescribing a course of actions based on the historical and incoming real-time data. Prescriptive maintenance strategies are updated based on the

observed/predicted degradation parameters and system state, whereas in conventional time-based maintenance, decisions only rely on historical data without considering updates. In this paper, we develop a dynamic maintenance model able to sequentially determine the optimal inspection time based on the system health condition and provide flexible maintenance advices.

Reliability modeling serves as the foundation of maintenance optimization. Traditional reliability models are constructed using failure data. In recent decades, with increased product reliability, it has become difficult to obtain failure data within a feasible time period, but sensors make degradation data available during system operation. As a result, degradation models are taking the place of failure-data-based reliability models. Degradation models can be either continuous or discrete. In a discrete degradation model, the system condition is divided into a finite number of states,

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which is usually characterized by a Markov or semi-Markov chain [3]–[7]. The disadvantage of Markov or semi-Markov models lies in the arbitrary classification of the system states and fails to fully characterize its degradation evolution.

Increasingly improved sensing technologies enable accurate monitoring of the system state, prompting the use of a continuous degradation model. Usually, the continuous degradation processes are described by stochastic process models or general path models [8]–[10]. The Lévy process is frequently used because of its mathematical properties and explicit physical interpretations, among which the Gamma process, Wiener process, and inverse Gaussian process have widely appeared in the reliability and maintenance literature. Although the Wiener process is extensively used for reliability modeling and maintenance optimization (e.g., [11]–[14]), it is a non-monotone process and fails to describe several deterioration processes in practice, such as the crack growth or wear process. On the other hand, with the property of monotonic independent increments, the Gamma process and inverse Gaussian process can overcome this disadvantage [15]–[17]. Reference [18] provided a comprehensive survey of the use of the Gamma process in reliability and maintenance strategies. In [17], the inverse Gaussian process was investigated as a degradation model and the physical mechanism was interpreted as a limiting compound Poisson process. Novel BN/DBN-based methodologies are developed for degradation modeling of components and systems [19].

In spite of the popularity of degradation models in reliability and maintenance modeling, an implicit assumption of the existing studies is that a failure occurs when the degradation level exceeds a specific threshold (soft failure). Yes this assumption is increasingly challenged since many systems fail before hitting the failure threshold in reality [20]–[24]. For locomotive wheels, usually the diameter of a wheel indicates the degradation level, where a soft failure is defined when the diameter reduces to a pre-specified threshold. However, during operation, wheels are reprofiled occasionally because of the increased roughness, even if the wheel diameter remains above a tolerance level. Various factors exert impacts upon the roughness of the wheels, including the weather conditions, surface smoothness of the track and running speed; any of these may cause sudden failure before the failure threshold is reached [25], [26]. Reduction of the wheel diameter through reprofiling also exacerbates the roughness. Another example is automobile tires. An automobile tire may fail suddenly because of a puncture before the wear reaches the failure threshold [27]. Reference [28] reported the limitation of the degradation-threshold failure model in the presence of continuous monitoring. When the system is subject to continuous monitoring, maintenance actions can always be implemented before the degradation level reaches the failure threshold, and this erroneously indicates that the system will never fail.

Motivated by the limitation of the degradation-threshold failure mechanism and the maintenance practices used for locomotive wheel-sets, we propose a maintenance strategy that considers the impact of degradation and catastrophic failure. It is well recognized that system failure is dependent on both degradation and age [28]–[30]. Reference [29] proposed a class of degradation-threshold-shock models that take advantage of degradation information and failure data. Reference [28] developed a condition-based maintenance strategy under continuous monitoring; they proposed respectively an additive and a multiplicative model to describe the relationship between system failure rate and degradation level.

In this paper, we develop a maintenance model that jointly incorporates the effect of both aging and degradation. In this model, the degradation level and system age have an additive impact on system failure rate. At inspection, replacement is carried out if the degradation level of the system hits a tolerance threshold and repair is implemented otherwise. Repair influences the system health condition in such a way that it reduces the system age to 0, but increases the degradation level. The degradation level and parameters are updated at inspections when new observation arrives. The maintenance strategy determines the optimal inspection time based on the updated degradation parameters and the degradation level after repair/replacement.

Our work differs from previous research in three aspects. First, we use a continuous stochastic process to describe system degradation, and we incorporate the joint influence of degradation and aging on system failure rate in an additive way. Second, we formulate a dynamic maintenance model as opposed to the static maintenance models in the literature; the parameters of the degradation process and the subsequent maintenance decisions are updated upon the arrival of new inspection data. Third, the effect of preventive maintenance action is twofold in our model: a reduction in system age and an increase in degradation level.

The rest of the article is organized as follows. Section II characterizes the degradation process and investigates system reliability. Section III describes the maintenance schedule and formulates a cost model as the maintenance criterion. Section IV discusses the procedure for estimating and updating parameters. A case study of locomotive wheel-sets is presented in Section V to illustrate the proposed maintenance strategy. Finally, Section VI provides the concluding remarks and future research directions.

II. DEGRADATION PROCESS AND RELIABILITY EVALUATION
A. SYSTEM DEGRADATION

Consider a system subject to a monotonic deterioration process. Without maintenance actions, the system is assumed to follow a stationary Gamma degradation process, which has been widely used in degradation modeling due to the property of monotonic independent increments. Denote $X(t)$ as the degradation level at time t, $X(t) \sim Ga(t; \alpha, \beta)$, where α is the shape parameter and β is the scale parameter. The degradation increment at any two time epochs t and l ($t > l$),,

$\Delta X(t - l) = X(t) - X(l)$, follows a Gamma distribution with the probability density function (pdf)

$$f(x; \alpha(t - l)\beta) = \frac{\beta^{\alpha(t-l)}}{\Gamma(\alpha(t - l))} x^{\alpha(t-l)-1} e^{-\beta x} I_{\{x>0\}} \quad (1)$$

where $\Gamma(\cdot)$ is the complete gamma function, and $I_{\{x>0\}}$ is the indicator function.

B. RELIABILITY EVALUATION

The system may experience sudden failure in addition to the continuous degradation process, where both system age and degradation level contribute to the increase of failure rate. Traditional age-based maintenance strategies fail to capture the heterogeneity of the operating systems since the failure rate is only dependent on system age. For example, for locomotive wheels operating in different environments, such as the track condition, weather, and running speed, they may suffer heterogeneous failure rates even with identical age. Hence, modeling the failure rate as a function of system age and degradation level can better describe the failure mechanism of real systems. Note that the impact of degradation lies in increasing the failure rate, while itself will not lead to system failure. Let $\lambda(t, X(t))$ denote the failure rate at time t and system degradation level $X(t)$. Conditioned on the degradation path till time t, $X_0^t = \{X(s), 0 \le s \le t\}$, system reliability is given as

$$R(t|X_0^t) = \exp\left(-\int_0^t \lambda(s, X(s)) ds\right) \quad (2)$$

Denote T_f as the time to failure. The expectation of system reliability is expressed as

$$\bar{R}(t) = P\{T_f > t\} = E\left[\exp\left(-\int_0^t \lambda(s, X(s)) ds\right)\right] \quad (3)$$

In general, numerous forms of $\lambda(t, X(t))$ can be employed as long as it can capture the influence of system age and degradation level on the failure rate. In presence of historical data and physical mechanism, the specific form of $\lambda(t, X(t))$ can be determined in detail. For simplicity, an additive model is used [28],

$$\lambda(t, X(t)) = \lambda_1(t) + \lambda_2(X(t)) \quad (4)$$

Equation (4) implies the influence of system age and degradation level on the failure rate in an additive way. In particular, $\lambda_1(t)$ describes the age-dependent failure rate, and $\lambda_2(X(t))$ depends on the degradation level.

For the failure rate $\lambda_2(X(t))$, we assume a linear function in terms of the degradation level $X(t)$, $\lambda_2(X(t)) = \gamma X(t)$, where γ is a positive constant, scaling the impact of degradation level. Given the initial degradation level x_0 and the linear formula of $\lambda_2(X(t))$, the system reliability is expressed as follows.

Proposition 1: For a system subject to an age- and degradation-dependent degradation process with an additive effect, system reliability is given as

$$\bar{R}(t; x_0) = \exp\left(-\int_0^t \lambda_1(s) + \gamma x_0 + \alpha \log(1 + \gamma s/\beta) ds\right)$$
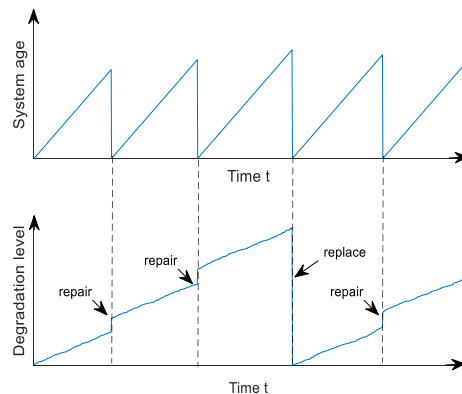

FIGURE 1. Evolution of system age and degradation level at maintenance actions.

Proof of the proposition is provided in the appendix. Note that the time to failure T_f can be regarded as the first instance of a doubly stochastic process, with a stochastic intensity $\lambda(t, X(t))$ that depends on both the degradation level and system age [29], [31].

Corollary 1: If the age-dependent failure rate is constant, $\lambda_1(t) = \lambda_0$, system reliability is then given as

$$\bar{R}(t; x_0) = \exp(-(\lambda_0 + \gamma x_0) t - \alpha((\beta/\gamma + t) \log(1 + \gamma t/\beta) - t))$$

Corollary 1 can be readily obtained from the reliability function of Proposition 1.

III. MAINTENANCE SCHEDULING

In the proposed maintenance model, the system is subject to three maintenance actions, namely, inspection, repair, and replacement. Repair is implemented preventively to prevent the system from failure or correctively to restore the system from failure. Unlike the existing imperfect repair models, in the proposed model, the influence of repair is twofold. On one hand, repair reduces the system age but on the other hand, increases the degradation level. In the present maintenance of locomotive wheels, the wheels are transported to the maintenance station for repair. During repair, they are reprofiled to restore them from anomaly. The reprofiling process diminishes the diameter of the wheels. In other words, the re-profiling process during repair increases the degradation level. In this paper the sudden failure is referred to as an anomaly during operation.

Motivated by the current practice of maintenance on locomotive wheels, repair or replacement has to be implemented at each inspection because of the high setup cost. In other words, inspection is always accompanied by repair or replacement. Inspection provides the observation of the degradation level before and after repair. Replacement is carried out when the observed degradation level hits a tolerance level. For illustrative purpose, we sketch the evolution of system age and degradation level at maintenance in Fig. 1. Note that the system age used in this paper follows the literature on imperfect maintenance [32]–[34].

A dynamic maintenance strategy is developed in this paper, wherein maintenance decisions have to be made at each inspection, including the time for the next inspection t_i, and repair or replacement upon the system. Denote X_i^- as the degradation level before the ith repair and X_i^+ the degradation level after repair. Let Y_i be the increment of degradation level at the ith repair. Y_i is assumed to be independent of the degradation level X_i^- and follows a Gaussian distribution with mean μ and variance σ^2, $Y_i \sim N\left(\mu, \sigma^2\right)$. With the above definition, we have

$$X_i^+ = X_i^- + Y_i$$
$$X_{i+1}^- = X_i^+ + \Delta X(t_i)$$

Let $\pi_i \in \{0, 1\}$ denote the maintenance action at the ith inspection, where $\pi_i = 1$ stands for replacement, and $\pi_i = 0$ indicates repair. Replacement is carried out upon the system when the degradation after repair reaches a pre-specified tolerance level, i.e.,

$$P\left(x_i^- + Y_i < \zeta\right) > \eta$$

where ζ is the threshold for replacement and η the tolerance level. Since Y_i follows a Gaussian distribution, it follows that

$$\Phi\left(\frac{\zeta - \mu - x_i^-}{\sigma}\right) > \eta$$

The maintenance action at the ith inspection can be obtained with simple algebra, given as

$$\pi_i = \begin{cases} 0, & \text{if } x_i^- \geq \zeta - \Phi^{-1}(\eta) \cdot \sigma - \mu \\ 1, & \text{otherwise} \end{cases} \tag{5}$$

Besides the maintenance actions at inspections, the decision maker will also have to determine the time for the next inspection, t_i, given the degradation level after repair or replacement, X_i^+. In addition, at each inspection where new degradation data arrive, the degradation parameters and system state are updated based on the arrival of new observations. It is difficult to obtain a stationary optimal maintenance decision, since all possible observation values have to be considered ahead of time [35]. As an approximation, we achieve the optimal maintenance decision by minimizing the expected average cost within a repair cycle.

The approximation model has two advantages. First, compared to a stationary maintenance model, the computational burden is significantly reduced. Second, the proposed one-repair-cycle optimization provides more flexibility for the optimal maintenance strategy. Although we can theoretically obtain the optimal inspection time based on the degradation process and observations, yet in practice, the actual maintenance time is influenced by the workload of the system and may deviate from the optimal maintenance time. In terms of actual maintenance time, the proposed model can be applied with a tiny modification of the repair cycle. The system should be maintained as close as possible to the optimal time to reduce the maintenance cost.

At the ith inspection, given the degradation level after repair, X_i^+, the expected cost per unit time is provided as

$$
\begin{aligned}
CR&\left(t_i; x_i^+\right) \\
&= E\left[\frac{c_i + c_p \cdot 1_{\{T_f \geq t_i\}} + c_c \cdot 1_{\{T_f < t_i\}} + c_d T_d}{t_i}\right] \\
&= \frac{c_i + c_p \bar{R}(t_i; x_i^+) + c_c\left(1 - \bar{R}(t_i; x_i^+)\right) + c_d E[T_d]}{t_i}
\end{aligned} \tag{6}
$$

where c_i, c_p, and c_c denote respectively the cost of inspection, preventive repair and corrective repair, c_d is the downtime per unit time, $1_{\{\cdot\}}$ is the indicator function, t_i is the time interval between the ith and $(i+1)$th inspection, and T_d is the downtime. The expected downtime within the interval t_i can be obtained by conditioning on the degradation path $X_0^{t_i}$ and is given as

$$E[T_d] = E\left[\int_0^{t_i}\left(1 - \exp\left(-\int_0^t \lambda(s, X(s))\, ds\right)\right) dt \bigg| X_0^{t_i}\right] \tag{7}$$

A detailed derivation of the equation is provided in the appendix. Numerical approaches like Monte Carlo integration can be employed to evaluated Equation (7). The optimal inspection time t_i^* can be obtained as

$$t_i^* = \arg\min_{t_i} CR\left(t_i; x_i^+\right) \tag{8}$$

For the system degrades over time with small variance, we can approximate the expected downtime $E[T_d]$ as

$$E[T_d] \approx \int_0^{t_i}\left(1 - \bar{R}(t; x_i^+)\right) dt \tag{9}$$

and the expected average cost $CR\left(t_i; x_i^+\right)$ as

$$
\begin{aligned}
CR&\left(t_i; x_i^+\right) \\
&\approx \frac{c_i + c_c - (c_c - c_p)\bar{R}(t_i; x_i^+) + c_d \int_0^{t_i}\left(1 - \bar{R}(t; x_i^+)\right) dt}{t_i}
\end{aligned} \tag{10}
$$

For a degradation process with small variance, given the degradation level after reprofiling, x_i^+, the expected remaining useful life is denoted as $\nu = -\int_0^\infty t\, d\bar{R}(t; x_i^+)$. In Fig. 2, we plot the evolution of inspection time (repair or replacement time) to illustrate the maintenance scheduling process. The system is inspected (together with repair or replacement) at epoch s_i. After repair or replacement, the decision maker needs to decide the operating interval before the next inspection time (t_i).
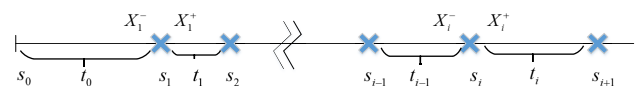

FIGURE 2. Sketch of maintenance schedule.

Corollary 2: When a degradation process with small variance satisfies $\nu > (c_i + c_c)/c_d$, the expected average cost

$CR(t_i)$ decreases with t_i, for $t_i \to 0^+$, and increases with t_i for $t_i \to \infty$.

A detailed proof of the corollary is given in the appendix. Corollary 2 indicates the existence of the optimal inspection time, t_i^*. For $\nu > (c_i + c_c)/c_d$, when t_i varies from zero to infinity, the expected cost per unit time $CR(t_i)$ shows a decreasing trend at the beginning and then an increasing time, thus implying that t_i^* exists.

Based on the previous discussion, the maintenance schedule is summarized as follows:

1) Determine initial input: initial degradation level, $x_0 = 0$, and let $i = 0$.
2) Calculate the optimal inspection time, $t_i^* = \arg\min_{t_i} CR\left(t_i; x_i^+\right)$.
3) At the ith inspection time, judge whether the degradation level hits the tolerance level, $x_i^- \geq \zeta - \Phi^{-1}(\eta) \cdot \sigma - \mu$. If so, replace the system and return to Step 1. Otherwise, repair takes place; let $i = i + 1$ and jump to Step 4.
4) Given the degradation level after repair, x_{i+1}^+, calculate the next inspection time,

$$t_{i+1}^* = \arg\min_{t_{i+1}} CR\left(t_{i+1}; x_{i+1}^+\right)$$

and go to Step 3.
5) Output the optimal inspection time, t_i^*, for $i = 0, 1, 2, \ldots$.

IV. PARAMETER UPDATE AT INSPECTION

In this paper, it is assumed that the degradation level of the system is accurately observed, with no measurement error. The scale and shape parameters are updated at each inspection, when newly arrived data are observed. Let $\theta = (\alpha, \beta)^T$ be the set of distribution parameters. Suppose that the prior distribution of θ is a bivariate Gaussian distribution, $\theta \sim N\left(\mu_\theta, \Sigma\right)$, where

$$\mu_\theta = \left(\mu_\alpha, \mu_\beta\right)^T \text{ and } \Sigma = \begin{pmatrix} \sigma_\alpha^2 & \rho\sigma_\alpha\sigma_\beta \\ \rho\sigma_\alpha\sigma_\beta & \sigma_\beta^2 \end{pmatrix}$$

Denote $p(\theta)$ as the pdf of prior distribution. It follows that

$$p(\theta) = \frac{\exp\left(-\frac{1}{2}\left(\theta - \mu_\theta\right)^T \Sigma^{-1}\left(\theta - \mu_\theta\right)\right)}{\sqrt{2\pi|\Sigma|}}$$

The prior distribution parameter, μ_θ, is obtained with historical data, by using the maximum likelihood estimation (MLE). Asymptotic methods are used to evaluate the uncertainty of the prior distribution. Under reasonably large sample sizes, the estimators of MLE can be well approximated by a multivariate Gaussian distribution. Denote $\hat{\theta}$ as the MLE estimator of θ. The asymptotic prior distribution of $\hat{\theta}$ is denoted as $\hat{\theta} \sim N\left(\theta, [I(\theta)]^{-1}\right)$, where $I(\theta)$ is the Fisher information assessed at θ. Therefore, it is reasonable to approximate Σ by $[I(\theta)]^{-1}$, $\Sigma = [I(\theta)]^{-1}$. A detailed description of the estimation procedure using the prior distribution parameters μ_θ and Σ is provided in the appendix.

Based on Bayes' theorem, the posteriori distribution of θ is expressed as

$$f(\theta|x) \propto f(x|\theta)p(\theta) \qquad (11)$$

where $f(x|\theta)$ is the sampling distribution of the dataset x. Maximum a posteriori probability (MAP) estimation is used to estimate the distribution parameters θ, giving the mode of the posteriori distribution. The MAP estimate $\tilde{\theta}$ is formulated as follows,

$$\tilde{\theta} = \arg\max_\theta f(\theta|x) = \arg\max_\theta L(\theta)p(\theta) \qquad (12)$$

where $L(\theta)$ is the likelihood function of the dataset. Since $\theta = (\alpha, \beta)^T$, the bivariate normal distribution is formulated as

$$p(\alpha, \beta) = \frac{1}{2\pi\sigma_\alpha\sigma_\beta\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2(1-\rho^2)}q(\alpha, \beta)\right)$$

where

$$q(\alpha, \beta) = \left[\frac{(\alpha - \mu_\alpha)^2}{\sigma_\alpha^2} + \frac{(\beta - \mu_\beta)^2}{\sigma_\beta^2} - \frac{2\rho(\alpha - \mu_\alpha)(\beta - \mu_\beta)}{\sigma_\alpha\sigma_\beta}\right]$$

Taking logarithm algebra leads to

$$\ln p(\alpha, \beta) = -\ln\left(2\pi\sigma_\alpha\sigma_\beta\sqrt{1-\rho^2}\right) - \frac{1}{2(1-\rho^2)}q(\alpha, \beta)$$

Suppose the dataset contains I units and J time records for each unit. Denote τ_j as the jth time record, $j = 1, 2, \ldots, J$, and x_{ij} as the degradation increment of the ith unit in time interval τ_j, $i = 1, 2, \ldots, I$. Maximizing the posteriori distribution $f(\theta|x)$ is equivalent to maximizing

$$\begin{aligned} h(\alpha, \beta) &= l(\alpha, \beta) + \ln p(\alpha, \beta) \\ &= \sum_{i=1}^{I}\sum_{j=1}^{J}(\alpha\tau_j - 1)\ln(x_{ij}) + I\sum_{j=1}^{J}\alpha\tau_j\ln(\beta) \\ &\quad - \beta\sum_{i=1}^{I}\sum_{j=1}^{J}x_{ij} - I\sum_{j=1}^{J}\ln\left(\Gamma(\alpha\tau_j)\right) \\ &\quad - \frac{1}{2(1-\rho^2)}q(\alpha, \beta) - \ln\left(2\pi\sigma_\alpha\sigma_\beta\sqrt{1-\rho^2}\right) \end{aligned} \qquad (13)$$

where $l(\alpha, \beta)$ is the log-likelihood of the MAP estimation for the dataset. By taking the derivatives of α and β, it follows that

$$\begin{aligned} \frac{\partial h}{\partial \alpha} &= I\ln(\beta)\sum_{j=1}^{J}\tau_j + \sum_{i=1}^{I}\sum_{j=1}^{J}\tau_j\ln(x_{ij}) - I\sum_{j=1}^{J}\varphi(\alpha\tau_j)\tau_j \\ &\quad - \frac{1}{2(1-\rho^2)}\left[\frac{2(\alpha - \mu_\alpha)}{\sigma_\alpha^2} - \frac{2\rho(\beta - \mu_\beta)}{\sigma_\alpha\sigma_\beta}\right] \end{aligned}$$

and

$$\frac{\partial h}{\partial \beta} = \frac{I\alpha \sum\limits_{j=1}^{J} \tau_j}{\beta} - \sum_{i=1}^{I} \sum_{j=1}^{J} x_{ij}$$
$$- \frac{1}{2(1-\rho^2)} \left[\frac{2(\beta - \mu_\beta)}{\sigma_\beta^2} - \frac{2\rho(\alpha - \mu_\alpha)}{\sigma_\alpha \sigma_\beta} \right]$$

The MAP estimates, $\tilde{\theta} = \left(\tilde{\alpha}, \tilde{\beta}\right)^T$, can be obtained as the roots of $\partial h/\partial \alpha = 0$ and $\partial h/\partial \beta = 0$, expressed as

$$\tilde{\alpha}, \tilde{\beta} \in \left\{ \alpha, \beta : \frac{\partial h}{\partial \alpha} = 0, \frac{\partial h}{\partial \beta} = 0 \right\}$$

However, since a Gaussian distribution is not a conjugate prior of a Gamma distribution, we cannot achieve a closed form of the mode of the posteriori distribution. As an alternative, numerical methods, e.g., Newton's method, can be employed to compute the MAP estimate of $\tilde{\theta}$. Let $g(\theta) = (g_1(\theta), g_2(\theta))^T$, where $g_1(\theta) = \partial h/\partial \alpha$ and $g_2(\theta) = \partial h/\partial \beta$. Under the framework of Newton's method, the iteration equation is formulated as

$$\theta_{n+1} = \theta_n - J(\theta_n)^{-1} g(\theta_n) \quad (14)$$

where $J(\theta)$ is the Jacobian matrix,

$$J(\theta) = \begin{bmatrix} \dfrac{\partial g_1(\theta)}{\partial \alpha} & \dfrac{\partial g_1(\theta)}{\partial \beta} \\ \dfrac{\partial g_2(\theta)}{\partial \alpha} & \dfrac{\partial g_2(\theta)}{\partial \beta} \end{bmatrix}$$

and

$$\frac{\partial g_1(\theta)}{\partial \alpha} = -I \sum_{j=1}^{J} \varphi_1(\alpha \tau_j) \tau_j^2 - \frac{1}{(1-\rho^2)\sigma_\alpha^2}$$

$$\frac{\partial g_1(\theta)}{\partial \beta} = \frac{\partial g_2(\theta)}{\partial \alpha} = \frac{I \sum\limits_{j=1}^{J} \tau_j}{\beta} + \frac{\rho}{(1-\rho^2)\sigma_\alpha \sigma_\beta}$$

$$\frac{\partial g_2(\theta)}{\partial \beta} = -\frac{I\alpha \sum\limits_{j=1}^{J} \tau_j}{\beta^2} - \frac{1}{(1-\rho^2)\sigma_\beta^2}$$

V. APPLICATION IN LOCOMOTIVE WHEEL-SETS

A case study on the wheel-sets of a heavy haul locomotive is employed to illustrate the proposed maintenance strategy. During operation, the locomotive wheel-sets may suffer an anomaly and are sent to the maintenance station for reprofiling. The anomaly includes increased roughness, asymmetry of wheel-sets and so on. Reprofiling can restore the wheels to the normal state but will reduce the wheel diameter, which is used as the degradation index. There are multiple indexes to measure the degradation level of wheel-sets, but the diameter is most commonly used. Therefore, we use wheel diameter as the degradation indicator and develop maintenance strategies accordingly. Fig. 3 shows the process of reprofiling.

FIGURE 3. Reprofiling process.

TABLE 1. Measurements of wheel diameters of bogie 195904.

Wheel	Diameters/mm	Running distance between two reprofiles/1000km				
		0	35.443	47.552	81.621	49.945
1	Before	1247.75	1235.28	1218.23	1193.99	1173.8
	After	1236.82	1223.42	1213.96	1180.29	1154.13
2	Before	1247.15	1235.49	1217.9	1196.15	1175.1
	After	1236.91	1223.51	1213.84	1180.33	1154.18
3	Before	1247.51	1237	1220.17	1196.72	1176.55
	After	1237.46	1224.87	1213.89	1180.34	1158.76
4	Before	1247.11	1235.12	1219.73	1196.6	1176.06
	After	1237.45	1224.84	1213.98	1180.32	1158.82
5	Before	1246.04	1234.85	1218.78	1195.71	1175.17
	After	1236.14	1225.05	1213.98	1180.23	1158.76
6	Before	1246.47	1234.35	1219.09	1196	1172.68
	After	1236.17	1224.98	1213.94	1180.27	1158.81

Two bogies, coded as 195904 and 195905, are employed to illustrate the maintenance strategy. Each bogie consists of six wheel-sets; their diameters are presented in Table 1 and Table 2. The tables show the variation of the wheel diameters before and after reprofiling in terms of the running distance. The data used for illustration were provided by a Swedish company, which were collected in a maintenance station where the wheel-sets were being inspected.

Over the life cycle of the wheels, natural wear and reprofiling contribute to the decrease of the wheel diameter [36], [37]. Natural wear occurs when the wheels are in operation and is modeled as a Gamma degradation process in this paper. Various factors contribute to natural wear, including weather, smoothness of the tracks, running speed and cracks. From Table 1 and Table II, we can find that the natural wear is actually the difference between the diameter after reprofiling and the diameter at the next inspection. For illustration, Fig. 4 presents the variation of natural wear with respect to the running distance of bogie 195904. Note that in the current example, the degradation level refers to the diameter of a wheel, and the operation time denotes the running distance. In the subsequent analysis, the data in Table 1 will be used to estimate the prior distribution and the data in Table 2 will serve for estimation of the MAP and update of the degradation parameters.

A. MAINTENANCE UNDER FIXED DEGRADATION PARAMETERS

Parameters of the Gamma process can be estimated from the degradation data using maximum likelihood

TABLE 2. Measurements of wheel diameters of bogie 195905.

| Wheel	Diameters/mm	Running distance between two reprofiles/1000km					
		0	48.01	73.169	50.918	65.359	31.214
1	Before	1250.64	1236.33	1227.92	1201.76	1190.36	1177.73
	After	1244.74	1231.99	1206.81	1194.38	1180.32	1171.14
2	Before	1251	1235	1227.64	1201.17	1189.32	1177.6
	After	1244.97	1231.98	1206.94	1194.34	1180.36	1171.31
3	Before	1250.14	1237.22	1230.28	1203.23	1192.15	1177.91
	After	1244.68	1232.05	1206.98	1195.33	1180.25	1171.21
4	Before	1251.33	1236.78	1230.27	1203.55	1191.85	1178.09
	After	1244.72	1231.95	1207.02	1195.37	1180.3	1171.26
5	Before	1250.03	1235.82	1228.61	1202.25	1190.2	1177.34
	After	1243.95	1231.98	1207.03	1194.35	1180.35	1171.22
6	Before	1250.96	1234.63	1228.26	1202.25	1188.76	1177.31
	After	1244	1232.12	1207.03	1194.26	1180.27	1171.28
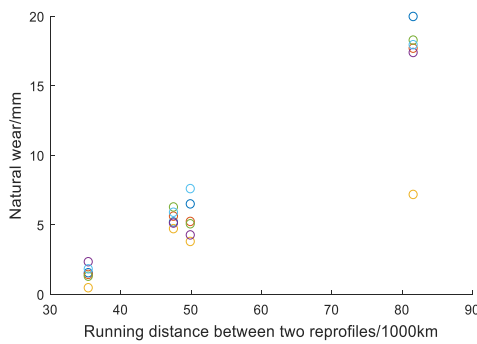
FIGURE 4. Plot of natural wear.

estimation (MLE). From the data in Table 1, the scale parameter and the shape parameter are estimated as $\hat{\alpha} = 0.0592$, $\hat{\beta} = 0.4419$, and the associated covariance matrix is obtained as

$$\hat{\Sigma} = [I(\theta)]^{-1} = \begin{bmatrix} 2.6 & 20 \\ 20 & 172 \end{bmatrix} \times 10^{-4}$$

For simplicity, the age-dependent failure rate is assumed as constant, $\lambda_1(t) = \lambda_0$, where $\lambda_0 = 0.0005$. The scaling parameter on the degradation level is set as $\gamma = 0.001$. With the abovementioned parameters, we present the variations in system reliability in Fig. 5.
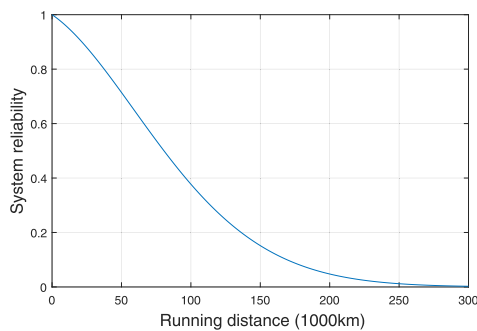

FIGURE 5. Variations of system reliability.

For a perfect wheel, the initial diameter is 1250mm, and replacement is implemented on the wheel when the diameter tails off to below 1150mm. In this case, the threshold for

replacement is set as $\zeta = 100$mm. The tolerance level is given as $\eta = 0.95$ for the purposes of illustration. However, in Table 1, we can find that the initial diameter does not exactly match 1250mm; it averages at 1247mm. The difference between the ideal and the measured diameter is due to an error of production and measurement. Thus, the initial degradation level is set as $x_0 = 3$. It is assumed that the degradation increment caused by reprofiling follows a Gaussian distribution, with the parameters estimated as $\hat{\mu} = 11.87$ and $\hat{\sigma} = 4.667$. From Equation (5), we get the threshold for replacement, and the decision at inspection is given as

$$\pi_i = \begin{cases} 0, & \text{if } x_i^- \geq 80.45 \\ 1, & \text{otherwise} \end{cases}$$

While the wheel is sent to the maintenance station for reprofiling, one has to determine the next inspection time in addition to the maintenance actions. According to Equation (6), Fig. 6 presents the variation of the cost rate in terms of the inspection time t_i; in the figure, the maintenance-related cost is set as $c_i = 10$, $c_p = 70$, $c_c = 100$, and $c_d = 10$. The optimal inspection time is obtained at $t_i^* = 49.8$, and the minimal cost rate is given as $CR^* = 3.05$. The result implies that the optimal running distance is 49,800km given the wheel diameter at 1247mm. In current practice, the optimal inspection time is advised by the manufacturers, but this strategy fails to capture the heterogeneity of the operating environment and the initial diameter of wheel-sets. In current maintenance practice, it is suggested by the manufacturer that the average running distance between reprofiling should be 40,000km regardless of the heterogeneity in wheel conditions. However, our model suggests that maintenance engineers should dynamically inspect the wheel-sets based upon the present health condition, rather than a fixed value.

Since the optimal inspection time is dependent on the current wheel conditions, we are interested to investigate the influence of the degradation levels after reprofiling on the optimal inspection time t_i^* and the associated cost rate CR^*. We sketch the variation of the optimal inspection time and cost rate with respect to the degradation level x_i^+ in Fig. 7. The figure presents an increasing trend of the optimal cost rate in terms of the degradation level after reprofiling. This is
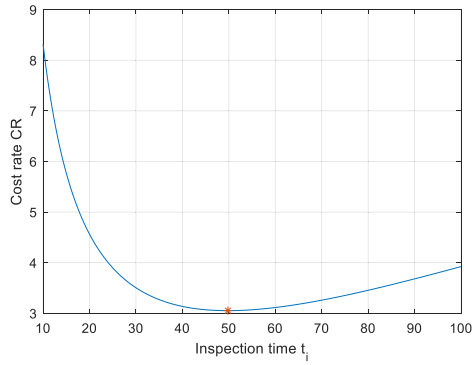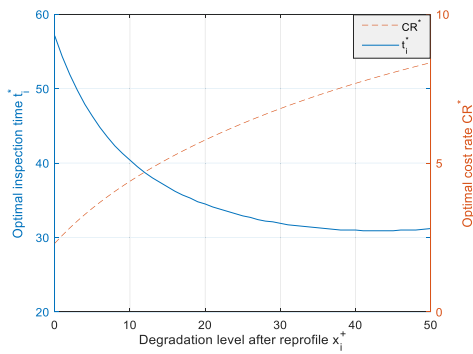
FIGURE 6. Cost rate *vs* inspection time.

FIGURE 7. t_i^* and *CR* vs* degradation level after reprofiling.

due to the fact that an increased x_i^+ makes the system more prone to failure.
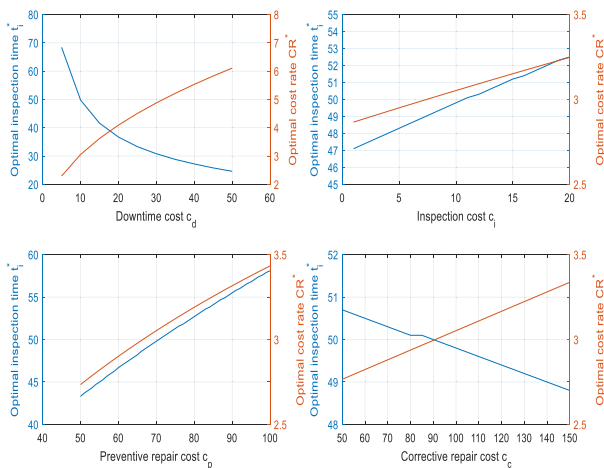

FIGURE 8. Sensitivity analysis on c_d, c_i, c_p, c_c.

Sensitivity analysis is conducted to investigate the effect of cost parameters on the optimal inspection time, t_i^*, and the associated cost rate, CR^*. Fig. 8 presents the results. As can be observed, CR^* always increases with cost. This is intuitive, since a high cost item lead to a higher cost rate. In addition, t_i^* shows a decreasing trend with the downtime cost, c_d, and the corrective repair cost, c_c, and an increasing trend
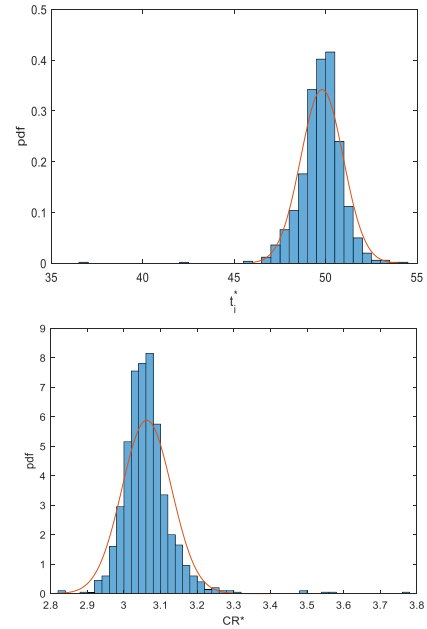

FIGURE 9. Histogram of t_i^* and *CR**, and associated fitted normal pdf curves.

with the preventive repair cost, c_p and the inspection cost, c_i. The system is less likely to fail under higher downtime and corrective repair costs, as this leads to a conservative strategy, i.e., a shorter inspection interval. In contrast, for a high c_i and c_p, maintenance actions are postponed to reduce the frequency of inspection and the likelihood of preventive repair.

Next, we investigate the influence of parameter estimation uncertainty on the optimal maintenance decision. The maximum likelihood estimator asymptotically follows a multivariate normal distribution, $\hat{\boldsymbol{\theta}} \sim N\left(\boldsymbol{\theta}, [I(\boldsymbol{\theta})]^{-1}\right)$, as $\hat{\boldsymbol{\theta}}$ will converge to $\boldsymbol{\theta}$ with increased sample size. Therefore, we employ $N\left(\hat{\boldsymbol{\theta}}, \left[I(\hat{\boldsymbol{\theta}})\right]^{-1}\right)$ to approximate the parameter distribution under large sample sizes. Samples that characterize the uncertainty of MLE can be obtained by drawing from the distribution $N\left(\hat{\boldsymbol{\theta}}, \left[I(\hat{\boldsymbol{\theta}})\right]^{-1}\right)$. Using this distribution generates 1000 samples, with an optimal inspection time t_i^* and an associated expected cost rate CR^* for each sample. Fig. 9 plots the histogram of t_i^* and CR^* and the associated fitted normal pdf curves. The mean of t_i^* is achieved as 49.76, and the standard deviation is 1.3907. For CR^*, the mean and standard deviation are obtained as 3.06 and 0.0679. The small standard deviation of t_i^* and CR^* indicates the effectiveness of MLE.

B. MAINTENANCE WITH UPDATED PARAMETERS
At each inspection where new observations arrive, the distribution parameter $\boldsymbol{\theta}$ will be updated. The prior distribution of $\boldsymbol{\theta}$ is obtained using the MLE. Maximum a posteriori estimation is employed to update the parameters with the arrival of new observations. Data from Table 2 serve as the new arrivals and are used for updating. It should be noted that

TABLE 3. Update of parameters at inspections.

Wheel	Estimate	Index of inspection				
		1	2	3	4	5
1	α	0.0616	0.057	0.0586	0.0564	0.0578
	β	0.4511	0.4394	0.4555	0.4538	0.4675
2	α	0.061	0.0571	0.0591	0.0587	0.0602
	β	0.441	0.434	0.4511	0.4591	0.4732
3	α	0.0618	0.0453	0.0461	0.0429	0.0446
	β	0.4557	0.3708	0.3839	0.3759	0.3917
4	α	0.0618	0.0444	0.0448	0.0427	0.0442
	β	0.4535	0.3634	0.3746	0.3726	0.3876
5	α	0.0617	0.0546	0.0561	0.0544	0.0563
	β	0.4526	0.4267	0.4425	0.4432	0.4590
6	α	0.0613	0.0559	0.0573	0.0577	0.0594
	β	0.4452	0.4299	0.4455	0.4569	0.4718

TABLE 4. Iteration of Newton's method for the sixth wheel-set.

Estimate	Iteration	Index of inspection				
		1	2	3	4	5
α	0	0.0592	0.0592	0.0592	0.0592	0.0592
α	1	0.0613	0.0559	0.0573	0.0576	0.0593
α	2	0.0613	0.0559	0.0573	0.0577	0.0594
α	3	0.0613	0.0559	0.0573	0.0577	0.0594
β	0	0.4419	0.4419	0.4419	0.4419	0.4419
β	1	0.4451	0.4297	0.4452	0.456	0.4707
β	2	0.4452	0.4299	0.4455	0.4569	0.4718
β	3	0.4452	0.4299	0.4455	0.4569	0.4718

we update the six wheel-sets of Table 2 separately; i.e., the distribution parameters of each wheel-set are estimated only by the prior information and the associated observations. With the MAP procedure presented in Section IV, we can compute the updated parameters, as shown in Table 3. Unlike the estimates using MLE, the estimates using MAP vary for different wheel-sets, thus describing the heterogeneity of the wheel-sets. In addition, variations of θ at each inspection indicate the flexibility and power of incorporating the new information.

Since the Gaussian distribution is not a conjugate prior, a closed-form solution of Equation (12) cannot be achieved. Therefore, we use Newton's method to compute the estimates from MAP. The initial guess of Newton's method is identical to the prior distribution, $\theta_0 = (0.0592, 0.4419)$. Following the iteration procedure in Equation (14), we can obtain the updated parameters given the observed data at each inspection. Taking the sixth wheel-set for example, Table 4 presents the variation of the parameters at each iteration. It can be observed that the estimates converge quickly, mostly in two iterations, indicating that the updated estimates are closed to the initial guess. At each inspection, given the current degradation level and the updated parameters, the optimal subsequent inspection time can be achieved by minimizing the average cost rate in Equation (6).

C. COMPARISON WITH CONTINUOUS MONITORING

This section presents a case where the degradation level of the system can be continuously monitored. Compared with the discrete inspection in Section III, under continuous

monitoring, repair or replacement is carried out immediately if the system is found to have failed. Therefore, no downtime is considered in this scenario. In addition, the inspection cost is suppressed since the system is under continuous monitoring. The maintenance strategy under continuous monitoring works as follows: given the degradation level after repair x_i^+, the system is repaired preventively at time T_i or correctively at failure T_f. The corresponding cost rate is given as

$$CR\left(T_i; x_i^+\right) = \frac{c_p + (c_c - c_p)\left(1 - \bar{R}(T_i; x_i^+)\right)}{E\left[\min\{T_i, T_f\}\right]} \quad (15)$$

where the expected cycle length $E\left[\min\{T_i, T_f\}\right]$ is given as

$$E\left[\min\{T_i, T_f\}\right] = E\left[\int_0^{T_i}\left(1 - F(t|X_0^{T_i})\right) dt | X_0^{T_i}\right]$$

Upon repair, the system age is reduced to 0, while the degradation level increases. In other words, after repair or replacement, the system age is reduced to 0. Therefore, in the subsequent repair cycle, the system age will always start from 0. In Equation (15), the reliability function performs differently in terms of the repair or replacement (the degradation level x_i^+ differs).

For a degradation process with small variance, $E\left[\min\{T_i, T_f\}\right]$ can be approximated as $E\left[\min\{T_i, T_f\}\right] \approx \int_0^{T_i} \bar{R}(t; x_i^+)dt$. [38] states that the optimal maintenance strategy for the above problem is a control limit strategy, with the optimal repair time expressed as

$$T_i^* = \inf\left\{t \geq 0, \ \lambda\left(t, X(t); x_i^+\right) \geq CR^* / \left(c_c/c_p - 1\right)\right\}$$

Since the optimal expected one-cycle cost rate is dependent on the associated repair time, T_i^*, an iterative algorithm is proposed for computation purposes [38], [39]. Although the iterative algorithm works well for a deterministic or discrete degradation process, it cannot be applied for a continuous degradation process, as the hazard rate function $\lambda\left(t, X(t); x_i^+\right)$ is randomized by $X(t)$. Therefore, we resort to a one-directional search algorithm to obtain the optimal repair time, T_i^*. In fact, Equation (15) can be rewritten as

$$CR\left(T_i; x_i^+\right) = c_p \frac{1 + (K - 1)\left(1 - \bar{R}(T_i; x_i^+)\right)}{E\left[\min\{T_i, T_f\}\right]}$$

where K is the cost ratio, $K = c_c/c_p$. The optimal repair time is dependent on K, instead of c_c or c_p separately, while the optimal expected cost rate, CR^*, is related to both c_c and c_p.

Under the present cost setting, $c_p = 70$ and $c_c = 100$, the optimal repair time, T_i^*, approaches to infinity, implying that the system is repaired only at failure, and the optimal maintenance decision is reduced to a block-based maintenance strategy. The repair cost is close to the replacement cost, and this diminishes the effect of preventive repair. To investigate the influence of cost parameters, in Fig. 10, we show the variation of the optimal cost rate and the associated repair time with respect to c_c and c_p under
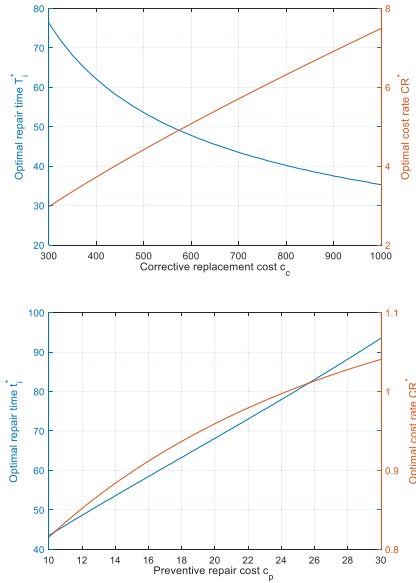
FIGURE 10. Influence of c_c and c_p under continuous monitoring.

continuous monitoring. The variation trend of T_i^* and CR^* is similar to that found in discrete inspection.

VI. CONCLUSION

This paper presents a dynamic maintenance strategy for systems subject to aging and degradation. During operation, the system goes through an aging and degradation process, which affects the failure rate of the system in an additive way. System reliability is analyzed as a first step, followed by a maintenance model evaluated based on the expected cost rate within a repair cycle. A dynamic maintenance strategy is proposed, where the degradation parameters are updated and the optimal subsequent inspection time is provided at each maintenance action. The proposed maintenance model has two advantages compared with the existant time-based preventive maintenance models. First, it incorporates the effects of both the aging and degradation. Second, it permits decision makers to dynamically determine the optimal inspection time and the associated maintenance actions based on the observed degradation level and the operating history, thereby adapting to heterogeneous operating conditions. Application of the model to a case study of locomotive wheel-sets shows its effectiveness.

Future research can take two directions. First, it is assumed in this study that the degradation level of the system can be perfectly inspected, but in reality, observations are usually contaminated by noise. In future research, filtering approaches, such as the Kalman filter and its variants, and particle filters, can be employed to estimate the degradation level and the parameters of interest. Second, the optimal maintenance strategy is achieved based on the criterion of the average cost rate. However, if the cost parameters are subject to high uncertainty, i.e., cost items cannot be accurately evaluated, criteria that are independent of the cost parameters,

such as availability, reliability, and remaining useful lifetime, could be applied.

APPENDIX
A. PROOF OF PROPOSITION 1

Consider the degradation-dependent failure rate $\lambda_2(X(t))$. Based on the scaling property of the Gamma process, it follows that $\lambda_2(X(t)) = \gamma X(t) \sim Ga(t; \alpha, \beta/\gamma)$ The Gamma process belongs to the class of Lévy processes, where the Lévy measure of $\lambda_2(X(t))$ is given as

$$v(dy) = y^{-1}\alpha \exp(-\beta y/\gamma)dy$$

The theorem (Corollary 3.3) of [31] indicates that a reliability function with an increasing Lévy failure rate process can be transformed to one with a deterministic failure rate,

$$h(t) = \int_0^\infty \left[1 - \exp(-tx)\right]v(dx)$$

Then it follows that,

$$
\begin{aligned}
&E\left[\exp\left(-\int_0^t \lambda_2(X(s))\,ds\right)\right] \\
&= \exp\left(-\gamma x_0 t - \int_0^t ds \int_0^\infty \alpha \exp(-\beta y/\gamma)y^{-1}\right. \\
&\qquad \left. \cdot (1 - \exp(-(t-s)y)\,dy)\right) \\
&= \exp\left(-\gamma x_0 t - \int_0^t \alpha \log\left(\frac{\beta/\gamma + t - s}{\beta/\gamma}\right)ds\right) \\
&= \exp\left(-\gamma x_0 t - \int_0^t \alpha \log\left(\frac{\beta/\gamma + s}{\beta/\gamma}\right)ds\right)
\end{aligned}
$$

Combined with the additive model of Equation (4), system reliability is provided as

$$
\begin{aligned}
\bar{R}(t) &= E\left[\exp\left(-\int_0^t \lambda_1(s) + \lambda_2(X(s))\,ds\right)\right] \\
&= \exp\left(-\int_0^t \lambda_1(s)\,ds\right) \cdot E\left[\exp\left(-\int_0^t \lambda_2(X(s))\,ds\right)\right] \\
&= \exp\left(-\int_0^t \lambda_1(s) + \gamma x_0 t + \alpha \log\left(\frac{\beta + \gamma s}{\beta}\right)ds\right)
\end{aligned}
$$

which completes the proof.

B. DERIVATION OF EQUATION (7)

Let T_s be the smaller of the inspection time and the time to failure, $T_s = \min\{t_i, T_f\}$. It follows that

$$
\begin{aligned}
E[T_s] &= t_i\left(1 - E\left[F(t_i|X_0^{t_i})|X_0^{t_i}\right]\right) \\
&\quad + E\left[\int_0^{t_i} t\,dF(t|X_0^{t_i})|X_0^{t_i}\right] \\
&= E\left[\int_0^{t_i}\left(1 - F(t|X_0^{t_i})\right)dt|X_0^{t_i}\right]
\end{aligned}
$$

Since downtime only occurs when the system fails before inspection/repair, we have

$$T_d = \max\{t_i - T_f, 0\}$$

By definition, T_d can be rewritten as $T_d = t_i - T_s$. Its expectation is readily obtained as

$$
\begin{aligned}
E\left[T_d\right] &= t_i F(t_i) - \int_0^{t_i} t\, dF(t) \\
&= E\left[\int_0^{t_i} F(t|X_0^{t_i}) dt \,|\, X_0^{t_i}\right] \\
&= E\left[\int_0^{t_i}\left(1 - \exp\left(-\int_0^t \lambda\left(s, X(s)\right) ds\right)\right) dt \,|\, X_0^{t_i}\right]
\end{aligned}
$$

C. PROOF OF COROLLARY 2

Denote the expected cost in a repair cycle as

$$
\begin{aligned}
CT(t_i; x_i^+) = c_i &+ c_p + \left(c_c - c_p\right)\left(1 - \bar{R}(t_i; x_i^+)\right) \\
&+ c_d \int_0^{t_i}\left(1 - \bar{R}(t_i; x_i^+)\right)
\end{aligned}
$$

Obviously, $CT(t_i; x_i^+) > 0$ for $t_i \in (0, \infty)$. By taking the derivative of the expected average cost in Equation (10) with respect to t_i, we have

$$
CR'\left(t_i; x_i^+\right) = \frac{t_i CT'(t_i; x_i^+) - CT(t_i; x_i^+)}{t_i^2}
$$

When t_i approaches zero from the right, $t_i \to 0^+$, we clearly have $\lim_{t_i \to 0} CR'\left(t_i; x_i^+\right) < 0$. When t_i approaches infinity, it follows

$$
\begin{aligned}
&\lim_{t_i \to \infty} t_i CT'(t_i) - CT(t_i) \\
&= \lim_{t_i \to \infty} c_d\left(t_i\left(1 - \bar{R}(t_i; x_i^+)\right) - \int_0^{t_i}\left(1 - \bar{R}(t_i; x_i^+)\right)\right) - c_i - c_c \\
&= c_d v - c_i - c_c
\end{aligned}
$$

If $v > (c_i + c_c)/c_d$, we have $\lim_{t_i \to \infty} CR'\left(t_i; x_i^+\right) > 0$, which completes the proof.

D. ESTIMATION OF THE PRIOR DISTRIBUTION

Denote N as the number of units and M the number of time records of the historical data. Denote $\bar{\tau}_j$ as the jth time record, $j = 1, 2 \ldots, M$, and $\bar{x}_{ij}$ as the degradation increment of the ith unit in time interval $\bar{\tau}_j$, $i = 1, 2, \ldots, N$. The likelihood function is expressed as

$$
\begin{aligned}
L(\alpha, \beta) &= \prod_{i=1}^{N}\prod_{j=1}^{M} f(\bar{x}_{ij}|\alpha, \beta) \\
&= \prod_{i=1}^{N}\prod_{j=1}^{M} \frac{\beta^{\alpha\bar{\tau}_j}}{\Gamma(\alpha\bar{\tau}_j)} \bar{x}_{ij}^{\alpha\bar{\tau}_j - 1} e^{-\beta\bar{x}_{ij}}
\end{aligned}
$$

and the log-likelihood function is given as

$$
\begin{aligned}
l(\alpha, \beta) = N\sum_{j=1}^{M}\alpha\bar{\tau}_j \ln(\beta) &+ \sum_{i=1}^{N}\sum_{j=1}^{M}(\alpha\bar{\tau}_j - 1)\ln(\bar{x}_{ij}) \\
&- \beta\sum_{i=1}^{N}\sum_{j=1}^{M}\bar{x}_{ij} - N\sum_{j=1}^{M}\ln\left(\Gamma(\alpha t_j)\right) \quad \text{(A1)}
\end{aligned}
$$

Let the derivative of β be 0,

$$
\frac{\partial l}{\partial \beta} = \frac{N\alpha\sum_{j=1}^{M}\bar{\tau}_j}{\beta} - \sum_{i=1}^{N}\sum_{j=1}^{M}\bar{x}_{ij} = 0
$$

It follows that

$$
\hat{\beta} = \frac{N\alpha\sum_{j=1}^{M}\bar{\tau}_j}{\sum_{i=1}^{N}\sum_{j=1}^{M}\bar{x}_{ij}} \quad \text{(A2)}
$$

Taking the derivative of α leads to

$$
\frac{\partial l}{\partial \alpha} = N\ln(\beta)\sum_{j=1}^{M}\bar{\tau}_j + \sum_{i=1}^{N}\sum_{j=1}^{M}\bar{\tau}_j \ln(\bar{x}_{ij}) - N\sum_{j=1}^{M}\varphi(\alpha\bar{\tau}_j)\bar{\tau}_j \quad \text{(A3)}
$$

where $\varphi(\cdot)$ is the digamma function,

$$
\varphi(z) = \frac{\Gamma'(z)}{\Gamma(z)}
$$

Substituting Equation (A2) into Equation (A3), and letting the derivative be zero leads to

$$
\begin{aligned}
&N\sum_{j=1}^{M}\varphi(\alpha\bar{\tau}_j)\bar{\tau}_j - N\ln(\alpha)\sum_{j=1}^{M}\bar{\tau}_j \\
&= N\left(\ln\left(N\sum_{j=1}^{M}\bar{\tau}_j\right)\right)\sum_{j=1}^{M}\bar{\tau}_j - N\ln\left(\sum_{i=1}^{N}\sum_{j=1}^{M}\bar{x}_{ij}\right)\sum_{j=1}^{M}\bar{\tau}_j \\
&\quad + \sum_{i=1}^{N}\sum_{j=1}^{M}\bar{\tau}_j \ln(\bar{x}_{ij}) \quad \text{(A4)}
\end{aligned}
$$

Equation (A4) can be solved via numerical methods, e.g., Newton's method. Based on the estimated $\hat{\alpha}$, the estimate of β can be readily obtained from Equation (A2).

The Fisher information matrix is obtained by taking the second derivation of α and β. It follows that

$$
I(\boldsymbol{\theta}) = \begin{bmatrix} N\sum_{j=1}^{M}\varphi_1(\alpha\bar{\tau}_j)\bar{\tau}_j^2 & -\dfrac{N\sum_{j=1}^{M}\bar{\tau}_j}{\beta} \\[2em] -\dfrac{N\sum_{j=1}^{M}\bar{\tau}_j}{\beta} & \dfrac{N\alpha\sum_{j=1}^{M}\bar{\tau}_j}{\beta^2} \end{bmatrix}
$$

where $\varphi_1(\cdot)$ is the trigamma function,

$$
\varphi_1(z) = \frac{d\varphi(z)}{dz}
$$

The covariance matrix $\hat{\boldsymbol{\Sigma}}$ can be readily obtained as $\hat{\boldsymbol{\Sigma}} = [I(\boldsymbol{\theta})]^{-1}$.

NOTATION

$X(t)$	Degradation level at time t
$Ga(t; \alpha, \beta)$	Gamma process with shape parameter α and scale parameter β
$\Delta X(t - l)$	Degradation increment between time t and l
$\lambda(t, X(t))$	System failure rate
$\bar{R}(t)$	Expected system reliability at time t
$\lambda_1(t)$	Age-related failure rate
$\lambda_2(X(t))$	Degradation-dependent failure rate
T_f	Time to failure
X_i^-	Degradation level before the ith repair
X_i^+	Degradation level after the ith repair
Y_i	Degradation increment at the ith repair
$\pi \in \{0, 1\}$	Maintenance decision: replacement ($\pi_i = 0$) or repair ($\pi_i = 1$)
ζ	Threshold for replacement
η	Tolerance level of degradation after repair
t_i	Time interval between the ith and $(i + 1)$th inspection
CR	Long-run cost rate
c_i, c_p, c_c	Cost of inspection, preventive repair and corrective replacement
c_d	Downtime cost per unit time
T_d	Downtime
$\boldsymbol{\theta} = (\alpha, \beta)^T$	Set of degradation parameters
$p(\boldsymbol{\theta})$	pdf of prior distribution
$f(\boldsymbol{\theta}\|\boldsymbol{x})$	Posteriori distribution of $\boldsymbol{\theta}$ given the dataset $\boldsymbol{x}$
$I(\boldsymbol{\theta})$	Fisher information evaluated at $\boldsymbol{\theta}$
$\hat{\boldsymbol{\theta}}$	MLE estimate of $\boldsymbol{\theta}$
$\tilde{\boldsymbol{\theta}}$	MAP estimate of $\boldsymbol{\theta}$
$L(\boldsymbol{\theta})$	Likelihood function

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