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Formalization of Fuzzy Control in Possibility Theory via Rule Extraction

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ABSTRACT Possibility system has been recently recognized as a potential foundational theory for fuzzy set theory though the concept of possibility originated from the membership function of fuzzy sets. As an application of fuzzy set theory, fuzzy control has been widely used in engineering practices, where control laws are described by fuzzy if-then rules. This paper intends to formulate fuzzy control in the framework of possibility theory by regarding the fuzzification procedure of fuzzy control as extraction process of the fuzzy variable from crisp input, and the fuzzy if-then rules as extracted rules between fuzzy input/output. Typical procedures of fuzzy control, such as fuzzification, implication, aggregation, and defuzzification, are eventually formulated as a series of conditional possibilities operated by ''max-product'' operators. The reformulated fuzzy controller has a more general form, which encompasses the Mamdani fuzzy controller as a special case. Some fundamental concepts of possibility theory, such as randomness/fuzziness, possibility, and conditional possibility, are also discussed, which may be helpful for the correct understanding of possibility theory. Efforts have been made to bridge the two systems of possibility theory and fuzzy sets by the derivation of composition rule of fuzzy relations from conditional possibility. All results are derived for both normalized possibility and non-normalized possibility. This paper strengthened the role of possibility theory as a foundation for fuzzy sets, and as a complementary method to probability theory for handling information with fuzzy uncertainty.

INDEX TERMS Fuzzy control, fuzzy logic, fuzzy sets, possibility theory, rule extraction.

I. INTRODUCTION

Over the last three decades, the number and variety of applications of fuzzy sets, especially fuzzy logic, have increased significantly [1]–[5]. The applications range from consumer products such as cameras, washing machines, and microwave ovens to industrial process control, medical instrumentation, and decision-support systems. Originally, the objective of fuzzy logic control systems is to control complex processes by means of human experience [6]. However in the recent years, the emphasis on knowledge representation issues in works of fuzzy control tends to fade. More and more attention has been shifting into the fuzzy modeling and control of nonlinear systems, where fuzzy systems, as universal approximators, are usually applied to identify unknown uncertainty terms required in control design [7], [8], [37]. Continuing efforts on applications of fuzzy control exhibit great interest in the integration of fuzzy logic with other methods, such as neural-network, Kalman filter and genetic algorithm. The goal is to improve the control of complex, nonlinear dynamic plants, for example, human body [1]–[3], [9]–[11].

The first application of fuzzy set theory to the control of systems was by Mamdani and Assilian, who reported on the control of a laboratory model steam engine [6], [9], [12]. It is interesting to note that the first industrial application of fuzzy control was the control of a cement kiln in Denmark [13]. By the end of the 1980s, fuzzy control began to establish itself as a recognized control paradigm after Japanese manufacturers launched a wide range of products with fuzzy controlled parts and systems. Theoretical origin of fuzzy logic systems may attribute to the paper ''Outline of new approach to the analysis of complex systems and decision processes'' by Zadeh [14], where the compositional rule of inference is considered to be the spine of all fuzzy logic models [6]. The compositional rule of inference is a special kind of composition of fuzzy relations with one relation being unary.

In spite of the widespreading applications of fuzzy control, there are few works on logical interpretation for fuzzy control. It was claimed in [15] that most things named fuzzy inference

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can be naturally understood as logical deduction; and in [16] another logical approach to fuzzy control was proposed based on fuzzy logic programming. The link between possibility and fuzzy logic was discussed in [35] and [39], but semantically. Though the concept of possibility originated from membership function of fuzzy sets, in recent years possibility system has been recognized as a potential foundational theory for fuzzy set theory with fuzzy membership regarded as possibility likelihood [33], [36]. This work intends to formulate compositional rule of inference, especially fuzzy control, in the framework of possibility theory, where compositional rule of inference is taken as composition of conditional possibility. All results are derived for both normalized possibility and non-normalized possibility, considering that we human beings tend to present fuzzy concepts in a flexible and incomplete manner.

The rest of the paper is organized as follows. To help the understandings and applications of possibility theory, section 2 and 3 discuss some fundamental problems related to the concepts of randomness, fuzziness, and possibility. Section 4 intends to bridge the two systems of possibility theory and fuzzy sets, and derives composition rule of fuzzy relations from conditional possibility. Section 5 formalizes fuzzy control in possibility theory via fuzzy rule extraction. The reformulated fuzzy controller has a more general form, which encompasses Mamdani fuzzy controller as a special case. Section 6 gives an example of heating system to illustrate the conciseness-in-form and the ease-to-use of the reformulated fuzzy controller over the current Mamdani fuzzy controller. Section 7 concludes the paper.

II. POSSIBILITY: FOR DESCRIPTION OF FUZZINESS

Traditionally, the uncertainty of information is believed to be stochastic in nature, as such, must be dealt with methods provided by probability theory [17]–[21]. Recent decades, fuzziness, as the counterpart of randomness, has been recognized as another kind of uncertainty of information, with which fuzzy sets and possibility theory were proposed by Zadeh to cope [22]. It should be noted that the concept of possibility stems from membership degree of fuzzy sets. Nevertheless the notion and calculation rules of possibility may at least go back to the works of one economist Shackle [23], [24]. To have a clear understanding of possibility theory, we in this section start from the concepts of randomness and fuzziness, and then discuss both intuitive and axiomatic definitions of possibility.

A. FROM RANDOMNESS TO FUZZINESS

Fuzziness is a different kind of uncertainty that is easy to be mixed up with randomness. A definition of randomness presented in [25] is given below.

Definition 1: Randomness is the occurrence uncertainty of the either-or outcome of a random experiment, characterized by the lack of predictability in mutually exclusive outcomes.

Either-or here means the experiment outcomes are mutually exclusive, that is at each experiment one can have only one outcome among the sample space (a well-defined set of possible outcomes). The tossing of a six-sided dice can be a classical example of illustrating random experiment or randomness. Obviously, each time one can only get one side among all six possible sides, which is unpredictable and is either this or that (e.g., either one or two, or others). Sample space with mutually exclusive outcomes of a random experiment can be further defined as random sample space [25]. The random phenomenon, the random state or even the random experiment can be mathematically represented by a random variable. A random variable X is a variable whose value x_i is subject to variation due to randomness, which can take on a set of possible values of a random phenomenon (state) in a random sample space Ω, each with an associated probability, $p(x_i)$.

Outcomes of a random experiment are mutually exclusive and lacking in predictability. In contrast, outcomes of a fuzzy experiment are non-exclusive and lacking in clear boundary. The definition of fuzziness is given below [25].

Definition 2: Fuzziness is the cognition uncertainty of the both-and outcome of a fuzzy experiment, characterized by the lack of clear boundary between non-exclusive outcomes.

Both-and here means the experiment outcomes are not exclusive, that is at each experiment one could have multiple outcomes at the same time. The classifying of age group can be an example of illustrating fuzzy experiment or fuzziness. Suppose you are informed of a person with his picture or his age (e.g., 42 years old), and you are invited to classify this person into age group = {aged, middle, young}, which is a judgment procedure of a fuzzy phenomenon. Most likely you would regard age of 42 as middle, but at the same time you may also think this person is still young to some extent, and you definitely will not like to regard this person as aged since we elderly people always refuse to acknowledge that we are old guy. One step further, you may consider the person as aged, middle, young, with confidence 0, 1, and 0.5, respectively. The outcomes (aged, middle, young) of the classifying experiment are uncertain due to the lack of clear boundary between them, and are not exclusive. In other words, one can obtain a both-and outcome (both middle and young) that contains multiple non-exclusive outcomes, simultaneously.

Sample space with non-exclusive outcomes of a fuzzy experiment can be defined as fuzzy sample space [25]. A fuzzy variable X is a variable whose value x_i is subject to variation due to fuzziness, which can take on a set of possible values of a fuzzy phenomenon (state) in a fuzzy sample space Ψ, each with an associated possibility, $\pi(x_i)$. Under this definition, a fuzzy variable intrinsically represents the fuzzy phenomenon, the fuzzy state or even the fuzzy experiment.

B. INTUITIVE/AXIOMATIC DEFINITIONS OF POSSIBILITY

After a clear definition of fuzziness, we below introduce the concept of possibility in forms of both intuitive definition [25] and axiomatic definition [23], [25].

Definition 3: Possibility (intuitive definition) is the measure of the both-and fuzziness, which is the classification confidence of every outcome of an experiment. Possibility $\pi(x_i)$ of the outcome x_i can be numerically described by the compatibility (or similarity) between the fuzzy variable (the state being judged) X and its prospective outcome x_i, which can be formulated as

$$\pi(x_i) = \text{comp}(X, x_i), \tag{1}$$

where ''comp'' means compatibility.

In the example above of classifying age group, possibilities of the person being aged, middle, young would be 0, 1, and 0.5, respectively. That is, your confidence of classifying the person as aged, middle, young are 0, 1, and 0.5, respectively, which represents the compatibility of the state being judged (age group of the person) with prospective outcomes (aged, middle, young).

Definition 4: Possibility (axiomatic definition) can be built up upon a possibility space (Ψ, Σ, Π), which is a mathematical construct that models a real-world process (or ''experiment'') consisting of fuzzy states that are lack of clear boundary. A possibility space consists of three parts:

1) the fuzzy sample space Ψ (recall that the sample space in the axiomatic definition of probability is in fact a random sample space), which is the set of all possible non-exclusive outcomes.

2) the σ-algebra $\Sigma \subseteq 2^{\Psi}$, which is the event space consisting of a set of events $\{A_i\}$. Note each event A_i is a set containing zero or groups of outcomes which might be of more practical use whereas an outcome x_i is the result of a single execution of the experiment.

3) the possibility measure $\Pi : \Sigma \to [0, 1]$, which is a function on Σ such that it satisfies the three axioms below: [22], [32], [35]

Axiom 1. (Nonnegativity Axiom) $\pi(\phi) = 0$ for empty set ϕ.

Axiom 2. (Normality Axiom) the measure of entire fuzzy sample space is equal to one: $\pi(\Psi) = 1$.

Axiom 3. (Maxitivity Axiom), $\forall A, B \subseteq \Sigma, \pi(A \cup B) = \max\{\pi(A), \pi(B)\}$.

The widely accepted definition of possibility above was originally proposed in [22], and was reformulated in [25] following a structure parallel to that of the axiomatic definition of probability.

It is straightforward to justify the consistency between the intuitive definition 3 and the axiomatic definition 4. Under the compatibility interpretation of the intuitive definition of possibility, we can see that if a state is compatible with events A and B with confidence $\pi(A)$ and $\pi(B)$, respectively, then it would be compatible with event $A \cup B$ with confidence $\pi(A \cup B) = \max\{\pi(A), \pi(B)\}$. For example, if a person is judged as middle and young with confidence 1 and 0.5, then he will be judged as mature (the union of middle and young) with confidence 1. This holds under the logic that if we admit that a person is of middle age then we would admit of no exception he is mature. Equation $\pi(A \cup B) = \max\{\pi(A), \pi(B)\}$ in this work is named as the maximum principle of possibility (compatibility or similarity), which is different from the addition principle of probability. Because of the maximum principle of possibility, the intuitive definition 3 can serve as a foundation for the axiomatic definition 4.

Definition 4 is also named as the normal or normalized possibility [32], [38]. Clearly, at least one of the elements of Ψ should be fully possible, i.e. $\exists x_i$, such that $\pi(x_i) = 1$. However, when such element does no longer exist, it leads to a subnormal (or non-normalized) possibility distribution. This situation may arise from incomplete data [38], or in our viewpoint is simply due to human's incomplete presentation of fuzzy concepts. For example, given a range of temperature ($t = 0°C \sim 40°C$), we can defined two fuzzy concepts ''high'' (h) and ''low'' (l), by using conditional possibility as will be discussed in the next section. It is often that, for example given temperature 20°C, both conditional possibility of high and conditional possibility of low will be lower than one, most likely being around 0.5, which means $\pi(\Psi|t) < 1$. A mapping $\Pi : \Sigma \to [0, 1]$ satisfying Axioms 1 and 3 is called a non-normalized possibility measure [32], [38].

We can definitely avoid such a situation by introducing more fuzzy concepts to make fuzzy sample space complete, that is

$$\pi(\Psi) = \max_{x_i} \pi(x_i) = 1. \tag{2}$$

For example, we can define one more fuzzy concept ''moderate'' (m) to guarantee that $\pi(\Psi|t) = \max\{\pi(\text{h}|t), \pi(\text{m}|t), \pi(\text{l}|t)\} = 1$. Nevertheless, as can be seen by an example of fuzzy control in section 6, we human beings tend to handle things in a concise and flexible way, hence in an incomplete manner. In such situations, definition 4 can be slightly adapted to incorporate non-normalized possibility by replacing Axiom 2 with Axiom 2' below [32], [38].

Axiom 2'. $0 < \pi(\Psi) \leq 1$.

This work suggests and follows hereafter the following nomination for possibility.

Normal possibility (or normalized possibility): possibility measure satisfying Axiom 1, Axiom 2 and Axiom 3.

Subnormal possibility (or non-normalized possibility): possibility measure satisfying Axiom 1 and Axiom 3.

General possibility (or possibility in short): possibility measure satisfying Axiom 1, Axiom 2' and Axiom 3.

Note that possibility includes both normalized possibility and non-normalized possibility.

III. CONDITIONAL POSSIBILITY AND POSSIBILITY UPDATE

A key issue of uncertainty theory is bout uncertain inference, which in probability theory relates to conditional probability and probability update. We in this section introduce conditional possibility and possibility update.

A. CONDITIONAL POSSIBILITY AND 2-ARY POSSIBILITY

Conditional possibility $\pi(y_j|x_i)$ has been defined by Hisdal similarly to probability theory using a Bayesian-like equation of the form [34], [35]

$$\pi(x_i y_j) = \min(\pi(y_j|x_i), \pi(x_i)), \tag{3}$$

where $\pi(x_i y_j)$ is the 2-ary joint possibility function of two dependent fuzzy variables X and Y, and $\pi(y_j|x_i)$ is the conditional possibility distribution of Y given X.

Equation (3) is the min-based notion of conditional possibility. On continuous numerical universe, this form of conditioning induces undesirable discontinuities and the maxitivity axiom is not preserved [23], [26]. Conditional possibility $\pi(y_j|x_i)$ is better defined using the product rule below

$$\pi(x_i y_j) = \pi(y_j|x_i)\pi(x_i), \tag{4}$$

where by the use of the "max" disjunctive operator, we have

$$\pi(x_i) = \frac{1}{\alpha(x_i)} \max_{y_j} \pi(x_i y_j), \tag{5}$$

$$\pi(y_j) = \frac{1}{\beta(y_j)} \max_{x_i} \pi(x_i y_j), \tag{6}$$

where

$$\alpha(x_i) = \max_{y_j} \pi(y_j|x_i) \le 1, \tag{7}$$

$$\beta(y_j) = \max_{x_i} \pi(x_i|y_j) \le 1. \tag{8}$$

De Baets *et al.* [26] provide a mathematical justification of the product-based notion of conditional possibility as in (4) in a numerical universe, as opposed to the min-based conditioning of qualitative possibility theory.

In the special case, fuzzy variables X and Y are independent, (4) can be rewritten as the "product" conjunctive operation below

$$\pi(x_i y_j) = \pi(x_i)\pi(y_j). \tag{9}$$

The preference of the product-based notion of conditional possibility over the min-based notion could be justified by intuitive argumentation. This can be illustrated by an example when two fuzzy variables X and Y are independent, and the conclusion can be extended to the general case when they are not independent. Suppose we are investigating people's physique, which is usually a joint fuzzy phenomenon of height X and shape Y. The fuzzy sample space of height X is {tall, middle, short} and shape Y = {strong, between, thin}. In symbolic representation, $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$. If we are informed of $\pi(x_i)$ and $\pi(y_j)$ as in Table 1 and 2, respectively, then what is the possibility that you would like to consider a person's physique as tall & strong? Similarly, the possibility of being middle & between? Mathematically, given $\pi(x_i)$ and $\pi(y_j)$, how do we treat $\pi(x_i y_j)$?

TABLE 1. Possibility $\pi(x_i)$.

tall	midle	short
1	0.5	0

TABLE 2. Possibility $\pi(y_j)$.

strong	between	thin
1	0.6	0

Obviously, the possibility of being tall & strong would be 1, where the applying of "min" and "product" will lead to the same results. However, the possibility of being middle & between is 0.5 with "min" operator and is 0.3 with "product" operator. We prefer to admit that if we are uncertain about two fuzzy phenomena with confidence less than one, we will be more uncertain about the joint fuzzy phenomenon with confidence less than the "min", but not with the "min" confidence of them. The results of $\pi(x_i y_j) = \pi(x_i)\pi(y_j)$ are listed in Table 3.

TABLE 3. Possibility $\pi(x_i y_j) = \pi(x_i)\pi(y_j)$.

	strong	between	thin
tall	1	0.6	0
middle	0.5	0.3	0
short	0	0	0

As a counterpart of conjunctive operation, disjunctive operator of possibility follows directly from the maximum principle of possibility and is rewritten as

$$\pi(x_i \cup x_j) = \max\{\pi(x_i), \pi(x_j)\}. \tag{10}$$

We emphasize that the "max" disjunctive operator of possibility is totally different from the "additivity" disjunctive operator of probability whereas probability/possibility follow the same conjunctive operator of "product."

B. POSSIBILITY UPDATE

From (4), (6) and (8) we can derive a possibility update equation in a form parallel to Bayesian inference as

$$\pi(x_i|y_j) = \frac{\pi(x_i)\pi(y_j|x_i)}{\pi(y_j)} = \frac{\beta(y_j)\pi(x_i)\pi(y_j|x_i)}{\max_{x_k}\{\pi(x_k)\pi(y_j|x_k)\}}, \tag{11}$$

where $\pi(x_i|y_j)$ is posteriori possibility, $\pi(x_i)$ is priori possibility, $\pi(y_j|x_i)$ is possibility likelihood of x_i, and $\beta(y_j)$ can be calculated by (8). Equation (11) is slightly different from the possibility update equation in [27] and [28], which has no such scaling factor $\beta(y_j)$.

IV. THE POTENTIAL OF POSSIBILITY THEORY AS A FOUNDATION FOR FUZZY SETS

This section intends to bridge the two theories of fuzzy sets and possibility at two points: 1) to interpret membership function of fuzzy sets as likelihood function of possibility; 2) to derive composition rule of fuzzy relations from conditional possibility via extraction of intermediate fuzzy variable.

A. FUZZY SETS AS LIKELIHOOD FUNCTION OF POSSIBILITY

The concept of possibility, with a notion of membership function, originally stems from fuzzy set theory. Let F be a fuzzy set that is characterized by its membership function $\mu_F(x_i)$, which is interpreted as the compatibility of x_i with the concept labeled F. The possibility function $\pi(x_i)$ is defined to be numerically equal to the membership function $\mu_F(x_i)$ of F, that is [22]

$$\pi(x_i) \triangleq \mu_F(x_i). \tag{12}$$

On the other hand, there are ongoing efforts of interpreting membership function of fuzzy sets as likelihood function of probability (conditional probability with likelihood expansion, regarded as a function of the conditioning event) [29]–[32]. More recently, an interpretation of fuzzy membership as likelihood function of possibility was proposed [33], [36]. We here justify that it is not probability likelihood but possibility likelihood that is intrinsically equal to fuzzy membership.

Let us give an improved description to the interpretation of fuzzy membership $\mu_F(x_i)$ as the compatibility of x_i with the concept labeled F. We state that $\mu_F(x_i)$ is the compatibility of fuzzy concept variable Y with the concept labeled F given its fuzzy attribute variable X being x_i. Therefore $\mu_F(x_i)$ can be rewritten as

$$\mu_F(x_i) = \mu_{Y|X}(F|x_i) \triangleq \pi(F|x_i). \tag{13}$$

Note that the possibility likelihood interpretation $\pi(F|x_i)$ of fuzzy membership $\mu_F(x_i)$ is different from a probability likelihood interpretation, that is, $\pi(F|x_i) = \mu_F(x_i) \neq p(F|x_i)$. As we can see, it turns out that membership function of fuzzy sets is equal to conditional possibility with likelihood expansion, though possibility function was originally thought to be equal to membership function. This notion of fuzzy membership makes clear that possibility system could be a foundational theory for fuzzy set theory.

Fuzzy sets are generally assumed to be normalized [6], which means

$$\max_{x_i} \mu_F(x_i) = \max_{x_i} \pi(F|x_i) = 1. \tag{14}$$

However fuzzy membership $\mu_F(x_i)$ is usually the likelihood function of non-normalized possibility. Be aware that the normalization requirement of (14) for fuzzy sets is different from Axiom 2 of definition 4. The former is generally much easier to be achieved in practical applications as can be seen by the example of fuzzy control in section 6.

B. COMPOSITION RULE OF FUZZY RELATIONS DERIVED FROM CONDITIONAL POSSIBILITY

A fuzzy relation can be represented by a fuzzy set, and in fact by a membership function of the fuzzy set [6], [14]. Therefore, a fuzzy relation can be represented by conditional possibility. Suppose $\pi(y_j|x_i)$ and $\pi(z_k|y_j)$ represent fuzzy relations from X to Y and from Y to Z, respectively, then fuzzy relation from X to Z can be represented by $\pi(z_k|x_i)$ as

$$\pi(z_k|x_i) = \frac{1}{\eta(x_i)} \max_{y_l} \pi(z_k|y_l)\pi(y_l|x_i), \tag{15}$$

where

$$\eta(x_i) = \max_{y_l} \pi(y_l|x_i) \leq 1. \tag{16}$$

Equation (15) above can be derived by using the disjunctive/conjunctive operators of possibility. By extraction of intermediate fuzzy variable Y from fuzzy relation $\pi(z_k|x_i)$, we have

$$\begin{aligned}\max_{y_l} \pi(z_k, y_l|x_i) &= \max_{y_l} \pi(z_k|y_l, x_i)\pi(y_l|x_i)\\ &= \max_{y_l} \pi(z_k|y_l)\pi(y_l|x_i),\end{aligned} \tag{17}$$

where z_k and x_i are assumed to be conditionally independent given y_l. We also have

$$\begin{aligned}\max_{y_l} \pi(z_k, y_l|x_i) &= \max_{y_l} \pi(y_l|z_k, x_i)\pi(z_k|x_i)\\ &= \max_{y_l} \pi(y_l|x_i)\pi(z_k|x_i)\\ &= \eta(x_i)\pi(z_k|x_i),\end{aligned} \tag{18}$$

where y_l and z_k are assumed to be conditionally independent given x_i, and $\eta(x_i)$ is defined by (16).

By combining (17) and (18), we derive (15). As we can see from (15), composition of two fuzzy relations equals to the max-product operation of two conditional possibilities scaled by a factor $1/\eta(x_i)$.

V. FUZZY CONTROL FORMALIZED IN POSSIBILITY THEORY VIA FEATURE EXTRACTION

In this section the classic ''Mamdani'' fuzzy controller [6] is derived in the framework of possibility theory, where compositional rule of inference is interpreted as composition of conditional possibilities.

A. A REVIEW OF FUZZY CONTROL

The widely used method of fuzzy control is characterized by its control laws described by fuzzy if-then rules. Fig. 1 depicts a generic so-called ''Mamdani'' fuzzy controller, which comprises of five parts [6]:

1) FUZZIFICATION OF THE INPUT VARIABLES

Take the inputs and determine the degree to which they belong to each of the appropriate fuzzy sets via membership functions.

2) APPLICATION OF THE FUZZY OPERATOR IN THE ANTECEDENT

After the inputs are fuzzified, we know the degree to which each part of the antecedent is satisfied for each rule. If the antecedent of a rule has more than one part, the fuzzy operator (AND or OR) is applied to obtain one number that represents the result of the rule antecedent.

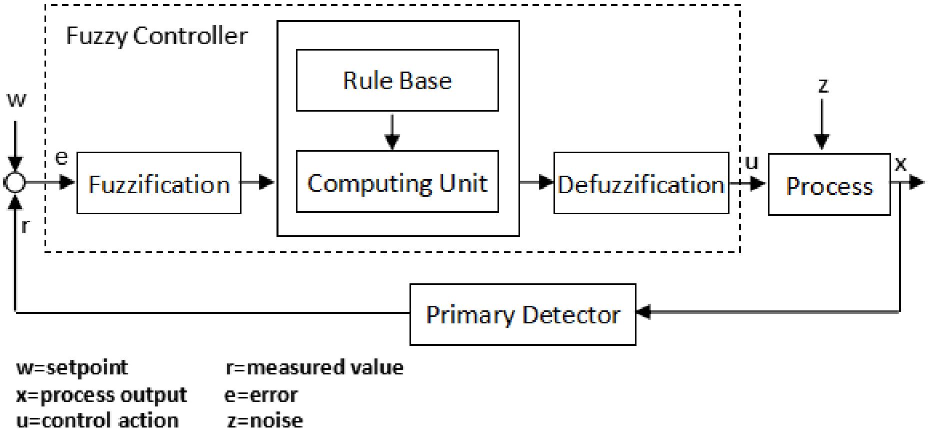

FIGURE 1. Generic "Mamdani" fuzzy controller.

3) IMPLICATION FROM THE ANTECEDENT TO THE CONSEQUENT

This step of implication is often called fuzzy if-then rules and each rule has a rule weight. By implication, the consequent is reshaped using a function associated with the antecedent. Typical implication operations include min (minimum) and prod (product). Implication is implemented for each rule with a predetermined rule weight.

4) AGGREGATION OF THE CONSEQUENTS ACROSS THE RULES

This is the process by which the fuzzy sets that represent the outputs of each rule are combined into a single fuzzy set.

5) DEFUZZIFICATION

The input for the defuzzification process is a fuzzy set (the aggregate output fuzzy set) and the output is a single number.

B. FUZZY CONTROL FORMALIZED IN POSSIBILITY THEORY VIA RULE EXTRACTION

Following the notion of Fig. 1, the end-to-end function of the fuzzy controller can be formulated as an optimum problem with objective function as the conditional possibility $\pi(u|e)$, that is

$$\text{FuzzyController}(e, u) \triangleq \max_u \pi(u|e), \tag{19}$$

where e is the crisp input of the fuzzy controller, u is the output. Equation (19) means that the objective of the fuzzy controller is to, given crisp input e, find the right output u with the maximum possibility.

By introducing a fuzzy output $\tilde{u}$ and following the process of deriving (15), possibility function $\pi(u|e)$ can be expanded into

$$\pi(u|e) = \frac{1}{\eta_1(e)} \max_{\tilde{u}} \pi(u|\tilde{u})\pi(\tilde{u}|e), \tag{20}$$

$$\eta_1(e) = \max_{\tilde{u}} \pi(\tilde{u}|e) \leq 1, \tag{21}$$

where conditional possibility $\pi(u|\tilde{u})$ is related to defuzzification, which transforms fuzzy output $\tilde{u}$ into crisp output u.

Conditional possibility $\pi(\tilde{u}|e)$ in (20) can be further expanded into

$$\pi(\tilde{u}|e) = \frac{1}{\eta_2(e)} \max_{\tilde{e}} \pi(\tilde{u}|\tilde{e})\pi(\tilde{e}|e), \tag{22}$$

$$\eta_2(e) = \max_{\tilde{e}} \pi(\tilde{e}|e) \leq 1, \tag{23}$$

which incorporates the procedures of fuzzy if-then rule $\pi(\tilde{u}|\tilde{e})$, and fuzzification $\pi(\tilde{e}|e)$. The procedure of fuzzification converses the crisp input e into a fuzzy variable $\tilde{e}$.

Combining (20) $\sim$ (23), we derive

$$\pi(u|e) = \frac{1}{\eta_1(e)\eta_2(e)} \max_{\tilde{u}} \max_{\tilde{e}} \pi(u|\tilde{u})\pi(\tilde{u}|\tilde{e})\pi(\tilde{e}|e), \tag{24}$$

$$\eta_2(e) = \max_{\tilde{e}} \pi(\tilde{e}|e) \leq 1, \tag{25}$$

$$\eta_1(e) = \frac{1}{\eta_2(e)} \max_{\tilde{u}} \max_{\tilde{e}} \pi(\tilde{u}|\tilde{e})\pi(\tilde{e}|e) \leq 1. \tag{26}$$

As we can see from (24), aggregation of the consequents $\tilde{u}$ across the rules is realized by the operator "max." And we finally have the fuzzy controller formulated in the possibility theory as

$$\begin{aligned} &\text{FuzzyController}(e, u) \\ &\quad = \frac{1}{\eta_1(e)\eta_2(e)} \max_u \max_{\tilde{u}} \max_{\tilde{e}} \pi(u|\tilde{u})\pi(\tilde{u}|\tilde{e})\pi(\tilde{e}|e) \\ &\quad = \max_u \max_{\tilde{u}} \max_{\tilde{e}} \pi(u|\tilde{u})\pi(\tilde{u}|\tilde{e})\pi(\tilde{e}|e), \end{aligned} \tag{27}$$

where $\pi(\tilde{e}|e)$ can be further expanded to consider the case of multiple antecedents. The second "=" in (27) holds because $\eta_1(e)$ and $\eta_2(e)$ are constant factors for a given e, and have no impact on deciding u.

Fuzzy controller (27) is like a Mamdani fuzzy controller with the defuzzification process as "largest of the maximum" [6]. Fuzzy controller with different defuzzification can be derived straightforwardly. For example, by introducing a concept parallel to the expected value of probability, the center of gravity (COG) method is defined as [6]

$$u_{\text{COG}} = \frac{\int_u u\pi(u|e)du}{\int_u \pi(u|e)du}, \tag{28}$$

where $\pi(u|e)$ can be calculated by (24). Note that $\eta_1(e)$ and $\eta_2(e)$ have no impact on u_{COG} as well, because they will finally be reduced from numerator and denominator of (28).

C. A COMPARISON BETWEEN MAMDANI FUZZY CONTROLLER AND THE NEW REFORMULATION

The new reformulated fuzzy controller closely resembles the Mamdani fuzzy controller, which have identical basic form and five typical procedures. As can be seen from (27), typical procedures of fuzzy control, such as fuzzification, implication, aggregation and defuzzification, are eventually formulated in the framework of possibility theory as a series of conditional possibilities operated by ''max-product'' operators. Note that in this work, only max-product operation is considered for fuzzy relation, which is compatible with the disjunctive/ conjunctive operators of possibility.

If we take a closer look at these two methods, we can find three basic differences.

First, the two methods use different functions for the mapping of fuzzy output $\tilde{u}$ to crisp output u. In (24) possibility function $\pi(u|\tilde{u})$ conditioned on fuzzy output $\tilde{u}$ is used, which models the possibility of producing possible crisp output u given a certain fuzzy output $\tilde{u}$. Whereas in existing Mamdani fuzzy controller, membership function $\pi(\tilde{u}|u)$ defining fuzzy output $\tilde{u}$ is used. Though $\pi(u|\tilde{u})$ and $\pi(\tilde{u}|u)$ can be defined to be numerically equal given a certain $\tilde{u}$, we would say the former is more natural for modeling the transformation from fuzzy output $\tilde{u}$ to crisp output u.

Second, the two methods use different implication forms of fuzzy if-then rules, and different aggregation mechanisms. For Mamdani fuzzy controller, fuzzy if-then rules are generally one-to-one mappings (i.e., parallel rules [6], [35]). Though there are the cases of one-to-many mappings, they are usually reduced into one-to-one mappings [6]. Aggregation across fuzzy if-then rules is accomplished by applying the maximum operator upon the consequents $\tilde{u}$ of the rules. And every fuzzy if-then rule can be assigned a weight factor. For the proposed fuzzy controller, fuzzy if-then rules are modeled by $\pi(\tilde{u}|\tilde{e})$, which can handle more complicated many-to-many mappings. The value of $\pi(\tilde{u}|\tilde{e})$ represents the credibility (confidence level) of the fuzzy if-then rule and is equivalent to a weight factor. Aggregation across fuzzy if-then rules is accomplished by two steps as shown in (24): application of maximum operator upon fuzzy input $\tilde{e}$, and then upon the fuzzy output $\tilde{u}$. The maximum operator upon $\tilde{e}$ acts as a local optimization. When $\pi(\tilde{u}|\tilde{e})$ forms a diagonal matrix, fuzzy if-then rules reduce to one-to-one mappings, and the maximum operator over fuzzy input $\tilde{e}$ will lose its effect. Meantime, the two methods have identical implication form of fuzzy if-then rules, and the same aggregation mechanism.

Third, $\pi(u|e)$ of the new reformulated fuzzy controller has two additional scaling factors $\eta_1(e)$ and $\eta_2(e)$, which help $\pi(u|e)$ keep normalized provided $\pi(u|\tilde{u})$ is defined to be normalized conditional possibility for every $\tilde{u}$, which can be easily verified by using (20) and (21). Whereas for Mamdani fuzzy controller, $\pi(u|e)$ is expressed as membership function $\mu(u)$, which is usually a sub-normalized membership derived from some normalized memberships [6], [35]. Note again that the requirement of normalized membership (10) is different from Axiom 2 of definition 4 for normalized possibility. Nonetheless, as has been discussed previously, the final crisp output provided by (27) or (28) is not affected by the two factors.

Based on the above comparison, it is reasonable to conclude that the reformulated fuzzy controller is a more general form of fuzzy controller, which encompasses Mamdani fuzzy controller as a special case. It was derived rigorously from possibility theory, with the merit of a concise and elegant form.

VI. AN ILLUSTRATING EXAMPLE OF FUZZY CONTROL FOR ROOM HEATING SYSTEM

An example of room heating system from [6] is adapted in this section to illustrate the two fuzzy controllers. In this example, the controller outputs control action to the heating system, according to the measured temperature of the room, to maintain a comfortable room temperature. We currently have no idea of presenting examples using non-parallel if-then rules, therefore Mamdani fuzzy controller and the reformulated fuzzy controller are expected to produce the same final control action. Nevertheless, we expect that this example could illustrate the conciseness-in-form and the ease-to-use of the new formalization over the current Mamdani fuzzy controller. Besides, by this example it will be illustrated that conditional possibility function induced from membership function of fuzzy set is generally non-normalized.

A. DESIGN PARAMETERS OF FUZZY CONTROLLER

For both fuzzy controllers, design parameters include fuzzification membership $\pi(\tilde{e}|e)$, fuzzy if-then rules in form of $\pi(\tilde{u}|\tilde{e})$, and conditional possibility $\pi(\tilde{u}|u)$ for Mamdani fuzzy controller and $\pi(u|\tilde{u})$ for the proposed fuzzy controller.

Input signal e of the fuzzy controller is room temperature (0°C $\sim$ 40°C), which can be transformed into fuzzy input $\tilde{e}$ as ''very low'' (vl), ''low'' (l), ''comfortable'' (c), and ''high'' (h). The fuzzification procedure is realized by using membership function $\mu_{\tilde{e}}(e) = \pi(\tilde{e}|e)$ as shown in Fig. 2. As we can see, for every $\tilde{e}$ ($\forall\tilde{e}$), $\mu_{\tilde{e}}(e)$ satisfies (14), whereas for a certain e ($\exists e$), $\pi(\tilde{e}|e)$ does not satisfy Axiom 2 of definition 4. Therefore, conditional possibility function $\pi(\tilde{e}|e)$ induced from membership function $\mu_{\tilde{e}}(e)$ as defined in Fig. 2 is non-normalized, which reflects that we human beings tend to handle fuzzy concepts in a concise, flexible and incomplete manner. We will later in this section give definitions for membership $\pi(\tilde{e}|e)$ in form of normalized

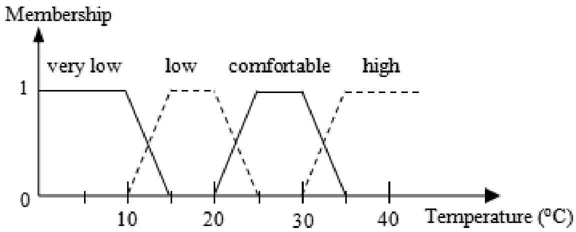

FIGURE 2. Fuzzification membership $\mu_{\tilde{e}}(e)\backslash\pi(\tilde{e}|e)$.

conditional possibility, which are complete yet non-common modelings of fuzzy concepts.

The set of fuzzy if-then rules in form of $\pi(\tilde{u}|\tilde{e})$ is defined in Table 4, which forms a diagonal matrix. Fuzzy output $\tilde{u}$ has three possible values as "big power" (b), "medium power" (m), and "small power" (s). Note that Table 4 has two columns for $\tilde{u}$ = small, which is used purposely to make fuzzy if-then rules "parallel" [6].

TABLE 4. Fuzzy if-then rules in form of $\pi(\tilde{u}|\tilde{e})$.

	big	medium	small	small
very low	1	0	0	0
low	0	1	0	0
comfortable	0	0	1	0
high	0	0	0	1

The transformation of fuzzy output $\tilde{u}$ to crisp output u is realized by $\pi(\tilde{u}|u)$ for Mamdani fuzzy controller or by $\pi(u|\tilde{u})$ for the proposed fuzzy controller, which are defined to be numerically equal as in Fig. 3. Crisp output u represents the signal amplitude (e.g. in voltage) of the control action.

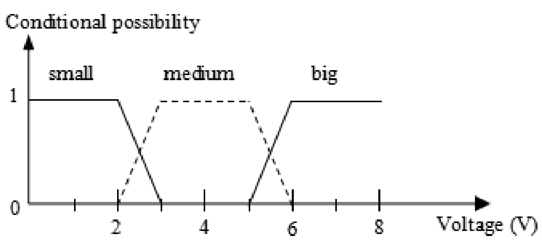

FIGURE 3. Conditional possibility $\pi(\tilde{u}|u)\backslash\pi(u|\tilde{u})$.

B. CALCULATION PROCESSES OF FUZZY CONTROLLER

Suppose the room temperature is 8°C and 14°C, respectively, let us see what are the outputs of the two fuzzy controllers. Note that $\pi(\tilde{e}|e)$ is normalized possibility for room temperature e = 8°C, but non-normalized for room temperature e = 14°C.

If input signal e = 8°C, then from Fig. 2 we know $\pi(\tilde{e} = \text{vl}|e) = 1$ and $\pi(\tilde{e} = \text{l}|e) = \pi(\tilde{e} = \text{c}|e) = \pi(\tilde{e} = \text{h}|e) = 0$. Then, only the rule "if $\tilde{e}$ = vl then $\tilde{u}$ = b" will be fired. According to (22), we have

$$\begin{aligned}\pi(\tilde{u} = b|e) &= \frac{1}{\eta_2(e)} \max_{\tilde{e}} \pi(\tilde{u} = \text{b}|\tilde{e})\pi(\tilde{e}|e) \\ &= \pi(\tilde{u} = \text{b}|\tilde{e} = \text{vl})\pi(\tilde{e} = \text{vl}|e) \\ &= 1,\end{aligned}$$

where

$$\eta_2(e) = \max_{\tilde{e}} \pi(\tilde{e}|e) = 1,$$

which means Mamdani fuzzy controller and the reformulated fuzzy controller have the same fuzzy output "big" with equal possibility of one. And by (20), we have

$$\begin{aligned}\pi(u|e) &= \frac{1}{\eta_1(e)} \max_{\tilde{u}} \pi(u|\tilde{u})\pi(\tilde{u}|e) \\ &= \pi(u|\tilde{u} = \text{b})\end{aligned}$$

where

$$\eta_1(e) = \max_{\tilde{u}} \pi(\tilde{u}|e) = 1,$$

which means the two fuzzy controllers will produce the same crisp output with equal possibility of $\pi(u|\tilde{u} = \text{b})$.

By using the COG method of (28) and $\pi(u|\tilde{u})$ defined in Fig. 3, we figure out

$$u_{\text{COG}} = 6.7(V)$$

If input signal e = 14°C, then from Fig. 2 we know $\pi(\tilde{e} = \text{vl}|e) = 0.2$, $\pi(\tilde{e} = \text{l}|e) = 0.8$, and $\pi(\tilde{e} = \text{c}|e) = \pi(\tilde{e} = \text{h}|e) = 0$. Then, rule "if $\tilde{e}$ = vl then $\tilde{u}$ = b" and rule "if $\tilde{e}$ = l then $\tilde{u}$ = m" will be fired. According to (22), we have

$$\begin{aligned}\pi(\tilde{u} = b|e) &= \frac{1}{\eta_2(e)} \max_{\tilde{e}} \pi(\tilde{u} = \text{b}|\tilde{e})\pi(\tilde{e}|e) \\ &= 0.25 \\ \pi(\tilde{u} = m|e) &= \frac{1}{\eta_2(e)} \max_{\tilde{e}} \pi(\tilde{u} = \text{m}|\tilde{e})\pi(\tilde{e}|e) \\ &= 1\end{aligned}$$

where

$$\eta_2(e) = \max_{\tilde{e}} \pi(\tilde{e}|e) = 0.8,$$

which are the results for the reformulated fuzzy controller. For the Mamdani fuzzy controller, $\pi(\tilde{u} = b|e) = 0.2$ and $\pi(\tilde{u} = m|e) = 0.8$. And by (20), we have

$$\begin{aligned}\pi(u|e) &= \frac{1}{\eta_1(e)} \max_{\tilde{u}} \pi(u|\tilde{u})\pi(\tilde{u}|e) \\ &= \max_{\tilde{u}}\{0.25\pi(u|\tilde{u} = \text{b}), \pi(u|\tilde{u} = \text{m})\}\end{aligned}$$

where

$$\eta_1(e) = \max_{\tilde{u}} \pi(\tilde{u}|e) = 1,$$

which are the results for the reformulated fuzzy controller. For the Mamdani fuzzy controller,

$$\pi(u|e) = \max_{\tilde{u}}\{0.2\pi(u|\tilde{u} = \text{b}), 0.8\pi(u|\tilde{u} = \text{m})\}.$$

By using the COG method of (28) and $\pi(u|\tilde{u})$ defined in Fig. 3, we figure out

$$u_{\text{COG}} = 4.3(V)$$

which is the result for both fuzzy controllers.

Now, we get the final results. Given room temperature is 8°C and 14°C, both fuzzy controllers will output control action 6.7 (V) and 4.3 (V), respectively, to the heating system. However, the two methods do produce different $\pi(u|e)$.

C. COMPLETE MODELING OF FUZZIFICATION MEMBERSHIP

By complete modeling, we mean the fuzzification membership is in form of normalized conditional possibility. As we discussed earlier in this section, the conditional possibility induced from the fuzzification membership $\pi(\tilde{e}|e)$ is non-normalized. To be a normalized possibility and make fuzzy sample space complete, we can redefine $\pi(\tilde{e}|e)$ either by adjusting the shape of membership function or by introducing additional values for fuzzy inputs $\tilde{e}$. Fig. 4 defines membership $\pi(\tilde{e}|e)$ in different shapes from those of Fig. 2. As we can see, for every e $(\forall e)$, $\pi(\tilde{e}|e)$ satisfies Axiom 2 of definition 4 and is a normalized possibility. However, membership as defined in Fig. 2 is more flexible and more common than that of Fig. 4. Membership $\pi(\tilde{e}|e)$ defined in Fig. 5 is for additional values of fuzzy inputs $\tilde{e}$: "rather low" (rl), "little low" (ll), and "little high" (lh). Fig. 2 and Fig. 5 together make sure that membership $\pi(\tilde{e}|e)$ is a normalized possibility.

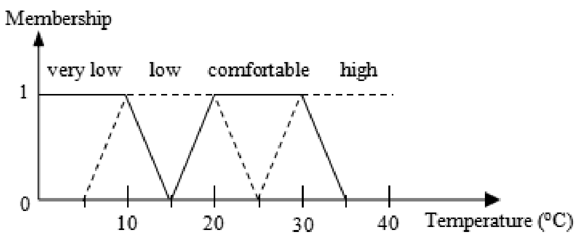

FIGURE 4. Membership $\pi(\tilde{e}|e)$ as normalized possibility.

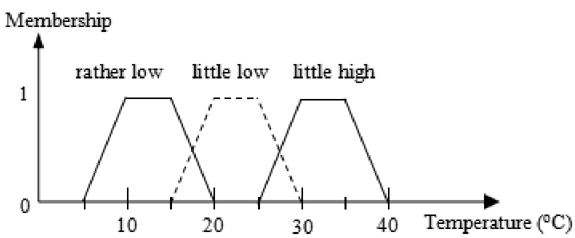

FIGURE 5. Membership $\pi(\tilde{e}|e)$ for additional fuzzy inputs.

Be aware that when the defined memberships are all normalized possibilities, $\eta_1(e)$ and $\eta_2(e)$ in (24) will all be one.

VII. CONCLUSION

This work strengthened the role of possibility theory as a foundation for fuzzy sets, and as a complementary method to probability theory for handling information with fuzzy uncertainty. The two theories of fuzzy sets and possibility are connected at two points: 1) membership function of fuzzy sets can be interpreted as likelihood function of possibility; 2) composition of fuzzy relations equals to composition of conditional possibilities. Specifically, the classic Mamdani fuzzy controller is derived rigorously in the framework of possibility theory. As future works, it is necessary to explore the application of the reformulated fuzzy controller to cases where many-to-many rules mapping is desired. Besides, it would be beneficial to explore the integration of fuzzy knowledge representation (in form of non-parallel if-then rules) with neural learning, in the framework of neural-symbolic computing [40]–[42].

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