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Mittag-Leffler Stabilization of an Unstable Time Fractional Hyperbolic PDE

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ABSTRACT This paper aims to study the Mittag–Leffler stabilization of an unstable time fractional hyperbolic partial differential equation by boundary control and boundary measurement. The backstepping method, the fractional Lyapunov method, and the semigroup theory are adopted in the investigation. A novel state feedback control via the Dirichlet boundary is designed to stabilize the controlled system. Based on the output signal, we first construct an observer that can recover the state of the original system, and then, we propose an observer-based stabilizing control law, under which the closed-loop system is shown to admit a unique solution and to be Mittag–Leffler stable. Finally, a benchmark example is presented to test the proposed theory.

INDEX TERMS Mittag-Leffler stability, output feedback, boundary control, backstepping method.

I. INTRODUCTION

Mittag-Leffler stability for fractional-order system, as a generalization of classical exponential stability for integer-order system, can hardly be considered as a fundamental issue in control theory. Since the seminal paper of Li et al. in 2009 [1] where the concept “Mittag-Leffler stability” is introduced and the fractional Lyapunov method is first presented, the problems of Mittag-Leffler stability and stabilization have received a huge interest from the control community. Many interesting results about the stability/stabilization of fractional order systems have been reported in [2]–[5]. In [3], by merging the contraction mapping principle, the Lyapunov method, the graph theoretic approach, the global Mittag-Leffler stability of a coupled system of fractional-order differential equations on network with feedback controls is studied. In [4], an impulsive controller for fractional-order nonlinear systems with impulses is proposed and the Mittag-Leffler stability for the addressed model is ensured by the Lyapunov stability analysis. In [5], Mittag-Leffler stability of fractional-order Hopfield neural networks is analyzed and a set of sufficient conditions to

guarantee this stability is derived. The stabilization problem of a fractional order linear system subject to input saturation is discussed in [6]. In [7], the authors investigate the robust stabilization problem of a class of nonlinear fractional order uncertain systems and establish a sufficient condition for the robust asymptotic stability of the observer-based nonlinear fractional order uncertain systems.

However very little attention has been paid to the stabilization of fractional partial differential equations (FPDEs) that can better characterize real-world [8]–[16]. Thus, it is natural to attempt to extend such work to systems described by FPDEs. Pioneering work on boundary stabilization for time fractional diffusion-wave equation is investigated in [17], where mainly numerical simulations without developing rigorous mathematical proof are presented to illustrate the effectiveness of the boundary control. The boundary feedback stabilization for unstable time fractional reaction diffusion equations is explored in [18], where the backstepping method and Riesz basis method for the Dirichlet and Neumann boundary controls are studied and the state space is $L^2(0, 1)$. The more smooth state space $H^1(0, 1)$ is considered in [19], where two types of boundary control conditions with collocated/noncollocated boundary output are investigated. Recently, the fractional reaction diffusion system

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with mixed or Robin boundary conditions is studied in [20], where the new kernel function whose boundary condition is different from the traditional boundary condition is proposed. The diffusion coefficient of [18], [20] is a constant. The spatially-varying (non-constant) diffusion coefficient for fractional reaction diffusion can be found in [21]. The boundary control suffering from disturbance refers to [22]. A backstepping observer for semilinear subdiffusion system can be founded in [23]. However, as far as we know, the subject of stabilization of time fractional hyperbolic partial differential equation (FHPDEs) under consideration has not been addressed before. Note that FHPDEs is described by an equation which contains a first order spatial derivative and a time-fractional derivative, which has many practical applications, for instances, see [24], [25].

Motivated by the above discussions, this paper will propose a systematic approach to design the output feedback control law for FHPDEs based on the backstepping method, the semigroup theory and the Lyapunov method, which provides an elegant design method of output feedback stabilization. From a theoretical point of view, this paper provides some insights into the qualitative analysis of the backstepping method and fractional Lyapunov method based boundary control of time fractional order system.

In this paper, we consider the stabilization of time fractional hyperbolic partial differential equation with Dirichlet boundary control:

$$\begin{cases} {}_0^C D_t^\alpha w(x, t) = w_x(x, t) + g(x)w(0, t) \\ \quad + \int_0^x f(x, y)w(y, t)dy, \quad x \in (0, 1), \quad t \geq 0, \\ w(1, t) = u(t), \quad y_o(t) = w(0, t), \quad t \geq 0, \\ w(x, 0) = w_0(x), \quad 0 \leq x \leq 1, \end{cases} \quad (1)$$

where $\alpha \in (0, 1)$, ${}_0^C D_t^\alpha w(x, t)$ is the Caputo fractional derivative of order α of w with respect to t ; $y_o = w(0, t)$ is the output (measurement), that is, the boundary pointwise signal $w(0, t)$ is measured; the initial datum $w_0 \in L^2(0, 1)$; $u \in C[0, \infty)$ is the input (control).

The objective of this paper is to design an output feedback control input u , using only the measurements y_o , such that the state of the closed-loop system (1) converges to zero, asymptotically. Throughout this paper, we assume that f and g are continuous functions on $\mathcal{F} := \{(x, y) : 0 \leq y \leq x \leq 1\}$ and $[0, 1]$, respectively.

First of all, we present an example to show that system (1) can be unstable without control.

Example 1: Take $g(x) = e^{\lambda x}$ with $\lambda > 0$, $f \equiv 0$ and take $w_0(x) = e^{\lambda x}(1 - x)$ in (1). Then, (1) becomes

$$\begin{cases} {}_0^C D_t^\alpha w(x, t) = w_x(x, t) + e^{\lambda x}w(0, t), \quad x \in (0, 1), \quad t \geq 0, \\ w(1, 0) = 0, \quad t \geq 0, \\ w(x, 0) = e^{\lambda x}(1 - x), \quad 0 \leq x \leq 1. \end{cases} \quad (2)$$

It is easy to verify that $w(x, t) = E_\alpha(\lambda t^\alpha)e^{\lambda x}(1 - x)$ solves (2). Moreover, since $\lambda > 0$, we have $\|w(\cdot, t)\|_{L^2(0,1)} = E_\alpha(\lambda t^\alpha)\|e^{\lambda x}(1 - x)\|_{L^2(0,1)} \rightarrow +\infty$ as $t \rightarrow +\infty$.

We proceed as follows. In Section 2, some concepts and facts are recalled, which will be used throughout this paper. In section 3, we consider the full state feedback stabilization by backstepping transformation. Section 4 is about the output feedback stabilization based on the observer and the state feedback control law. Finally, an example and some concluding remarks are presented.

II. PRELIMINARIES

In this section, we recall some definitions of fractional calculus and several important lemmas.

Definition 1: The Riemann-Liouville fractional integral of order $\alpha > 0$ is defined as

$${}_0 I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s) ds, \quad t > 0. \quad (3)$$

Definition 2: The left-side Caputo fractional derivatives of order $\alpha > 0$ are defined by the operators

$${}_0^C D_t^\alpha f(t) = {}_0 I_t^{n-\alpha} f^{(n)}(t), \quad (4)$$

provided they exist almost everywhere on $[0, +\infty)$, where $n = \lceil \alpha \rceil$. In particular, when $\alpha \in (0, 1)$,

$${}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - s)^{-\alpha} f'(s) ds. \quad (5)$$

Definition 3: The Mittag-Leffler function $E_\alpha(z)$ with $\alpha > 0$ is defined by the following series representation, valid in the whole complex plane:

$$E_\alpha(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + 1)}, \quad \alpha > 0, \quad z \in \mathbb{C}. \quad (6)$$

The two-parameter Mittag-Leffler function $E_{\alpha,\beta}(z)$ $\alpha, \beta > 0$ is defined by the following series representation:

$$E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)}, \quad \alpha > 0, \quad z \in \mathbb{C}. \quad (7)$$

The Laplace transform of two-parameter Mittag-Leffler function is given as

$$\mathcal{L}(t^{\beta-1} E_{\alpha,\beta}(-\lambda t^\alpha)) = \frac{s^{-\beta}}{s^\alpha + \lambda}, \quad \text{Re}(s) > |\lambda|^{\frac{1}{\alpha}}, \quad (8)$$

where $t \geq 0$, $\text{Re}(s)$ denotes the real part of s , and $\lambda \in \mathbb{R}$.

For the properties of the above Mittag-Leffler functions, one can refer to [26].

Definition 4 (Mittag-Leffler Stability): The solution of (1) is said to be Mittag-Leffler stable if

$$\|w(\cdot, t)\|_{L^2(0,1)} \leq \{m(\|w(\cdot, 0)\|_{L^2(0,1)})E_\alpha(-\lambda t^\alpha)\}^b,$$

where $\alpha \in (0, 1)$, $\lambda > 0$, $b > 0$, $m(0) = 0$, $m(s) \geq 0$, and $m(s)$ is locally Lipschitz on $s \in \mathbb{R}$ with Lipschitz constant m_0 .

Remark 1: In the above definition, Mittag-Leffler stability implies the asymptotic stability. This is because it follows from [12, Chapter 1, Theorem 1.6] that $E_\alpha(-\lambda t^\alpha) = \mathcal{O}\left(\frac{1}{\lambda t^\alpha}\right)$ as $t \rightarrow \infty$ and, for some $M > 0$,

$$E_\alpha(-\lambda t^\alpha) \leq \frac{M}{1 + \lambda t^\alpha}, \quad \text{for all } t \geq 0. \quad (9)$$

The following lemma will provide huge help for the well-posedness of the closed-loop system.

Lemma 1 [27]: Let $\alpha \in (0, 1)$. Let A be a closed linear operator and be densely defined in a Banach space H . If A generates a C_0 -semigroup on H , then Cauchy problem ${}^C_0D_t^\alpha X(t) = AX(t)$, $X(0) = x$ has a unique solution $X \in C(0, \infty; H)$.

Lemma 2 [28]: Let $x(t) \in \mathbb{R}$ be a continuous and derivable function. Then, for any time instant $t \geq 0$,

$${}^C_0D_t^\alpha x^2(t) \leq 2x(t){}^C_0D_t^\alpha x(t), \quad \forall \alpha \in (0, 1). \quad (10)$$

Remark 2: Lemma 2 indicates that the quadratic function, x^2 , is a good Lyapunov candidate function, which makes the fractional Lyapunov theorem ([1]) and fractional inequality with delay ([29], [30]) more applicable to obtain the stability result. For the absolute value function, $|x|$, to be Lyapunov candidate function, we refer to [5].

Lemma 3: Let $\alpha \in (0, 1)$. Let $V(t)$ be a negative continuous function and satisfies ${}^C_0D_t^\alpha V(t) \leq -\gamma V(t)$, then $V(t) \leq V(0)E_\alpha(-\gamma t^\alpha)$, where γ is a positive constant.

Proof: The proof of this lemma is very similar to the proof in [1]. Hence, we omit it here.

III. STATE FEEDBACK STABILIZATION

In this section, we will present a state feedback control $u(t)$ which is a functional of $w(\cdot, t)$ and can stabilize the system (1).

To find the state feedback control u , we first introduce a target system which is shown to be well posed and Mittag-Leffler stable:

$$\begin{cases} {}^C_0D_t^\alpha z(x, t) = z_x(x, t), & x \in (0, 1), \quad t \geq 0, \\ z(1, t) = 0, & t \geq 0, \\ z(x, 0) = z_0(x), & 0 \leq x \leq 1. \end{cases} \quad (11)$$

Lemma 4: For any $z_0 \in L^2(0, 1)$, the target system (11) admits a unique solution $z(\cdot, t) \in C(0, \infty; L^2(0, 1))$ which is Mittag-Leffler stable in the sense that

$$\|z(\cdot, t)\|_{L^2(0,1)}^2 \leq ME_\alpha(-\mu t^\alpha)\|z_0\|_{L^2(0,1)}^2, \quad (12)$$

for some positive constants $M, \mu > 0$ that are independent of initial values z_0 .

Proof: We first show that the target system (11) admits a unique solution. To this end, we define the operator \mathcal{A} as follows:

$$\mathcal{A}\phi(x) = \phi'(x), \quad D(\mathcal{A}) = \{\phi \in H^1(0, 1) : \phi(1) = 0\}.$$

It is well known that \mathcal{A} generates C_0 -semigroup $T(t)$ given by

$$T(t)\phi(x) = \begin{cases} \phi(x+t), & x+t \leq 1, \\ 0, & x+t > 0. \end{cases}$$

It follows from Lemma 1 that the target system (11) admits a unique solution $z(\cdot, t) \in C(0, \infty; L^2(0, 1))$.

Now we show that the solution of (11) is Mittag-Leffler stable. Let

$$V(t) = \int_0^1 e^x z^2(x, t) dx.$$

By Lemma 2, we have

$$\begin{aligned} {}^C_0D_t^\alpha V(t) &= \int_0^1 e^x {}^C_0D_t^\alpha z^2(x, t) dx \\ &\leq 2 \int_0^1 e^x z(x, t) {}^C_0D_t^\alpha z(x, t) dx \\ &= 2 \int_0^1 e^x z(x, t) z_x(x, t) dx \\ &= -z^2(0, t) - \int_0^1 e^x z^2(x, t) dx \leq -V(t). \end{aligned} \quad (13)$$

It then follows from Lemma 3 that $V(t) \leq V(0)E_\alpha(-t^\alpha)$, which implies that (12) holds.

Remark 3: In the above proof, we introduce the term e^x into the Lyapunov functional $V(t) = \int_0^1 e^x z^2(x, t) dx$ to get ${}^C_0D_t^\alpha V(t) \leq -V(t)$. Note that replacing e^x by 1, we only obtain ${}^C_0D_t^\alpha V(t) \leq -z^2(0, t)$ and it is difficult to get the asymptotical stability.

Remark 4: When $\alpha = 1$, the solution of (11) is given by

$$z(x, t) = \begin{cases} z_0(x+t), & 0 \leq x+t < 1, \\ 0, & x+t \geq 1. \end{cases}$$

So the solution converges to zero in finite time. However, when $\alpha \in (0, 1)$, the solution of system (11) does not converge to zero in finite time. Actually, suppose that the equilibrium $0 \in L^2(0, 1)$ of target system (11) is finite time stable. Then, for any $z(x, 0) \neq 0$, there exists a constant $T > 0$ such that for $t \geq T$,

$$\begin{cases} 0 = z(x, 0) + \frac{1}{\Gamma(\alpha)} \int_0^T (t-s)^{\alpha-1} z_x(x, s) ds, \\ z(1, t) = 0. \end{cases} \quad (14)$$

Since $z(x, 0) \neq 0$, we can choose $f \in H_0^1(0, 1)$ such that

$$\int_0^1 z(x, 0) f(x) dx \neq 0. \quad (15)$$

Multiply the first equation of (14) by f to give

$$\begin{aligned} \int_0^1 z(x, 0) f(x) dx &= -\frac{1}{\Gamma(\alpha)} \int_0^T (t-s)^{\alpha-1} \int_0^1 z_x(x, s) f(x) dx ds \\ &= \frac{1}{\Gamma(\alpha)} \int_0^T (t-s)^{\alpha-1} \int_0^1 z(x, s) f'(x) dx ds. \end{aligned}$$

Thus, for any $t > T$, we have

$$\begin{aligned} & \left| \int_0^1 z(x, 0)f(x)dx \right| \\ & \leq \frac{1}{\Gamma(\alpha)} \int_0^T (t-s)^{\alpha-1} \left| \int_0^1 z(x, s)f'(x) \right| dx ds \\ & \leq \frac{T}{\Gamma(\alpha)(t-T)^{1-\alpha}} \int_0^T \|z(\cdot, s)\|_{L^2(0,1)} \|f'(x)\|_{L^2(0,1)} ds. \end{aligned}$$

Passing to the limit as $t \rightarrow \infty$, we get

$$\int_0^1 z(x, 0)f(x)dx = 0. \tag{16}$$

This, together with (15), leads to a contradiction. The system (11) is a typical example of fractional system in the infinite dimensional space that is not finite-time stable. For non-existence of finite-time stable equilibria of the finite dimensional continuous fractional dynamic system, an interesting example can be found in [31]. We refer the reader to [32] for the finite time stable for discontinuous fractional system.

Next, by the backstepping method, we transform the system (1) into the target system (11), then we find a state feedback stabilizing control u . Introduce a transformation $w \rightarrow z$ in the form:

$$z(x, t) = w(x, t) - \int_0^x k(x, y)w(y, t)dy. \tag{17}$$

Simple computations show that

$$\begin{aligned} {}_0^C D_t^\alpha z(x, t) &= {}_0^C D_t^\alpha w(x, t) - \int_0^x k(x, y) {}_0^C D_t^\alpha w(y, t) dy \\ &= {}_0^C D_t^\alpha w(x, t) - \int_0^x k(x, y) \left(w_y(y, t) \right. \\ & \quad \left. + g(y)w(0, t) + \int_0^y f(y, \xi)w(x, \xi)d\xi \right) dy \\ &= w_x(x, t) + w(0, t) \left(g(x) - \int_0^x k(x, y)g(y)dy \right) \\ & \quad + k(x, 0)w(0, t) + \int_0^x k_y(x, y)w(y, t)dy \\ & \quad - k(x, x)w(x, t) + \int_0^x w(y, t) \left(f(x, y) \right. \\ & \quad \left. - \int_y^x k(x, \xi)f(\xi, y)d\xi \right) dy \end{aligned} \tag{18}$$

and

$$\begin{aligned} z_x(x, t) &= w_x(x, t) - k(x, x)w(x, t) \\ & \quad - \int_0^x k_x(x, y)w(y, t)dy. \end{aligned} \tag{19}$$

Substituting (18) and (19) into (11), it follows that the kernel functions $k(x, y)$ must satisfy

$$\begin{cases} k_x(x, y) + k_y(x, y) = \int_y^x k(x, \xi)f(\xi, y)d\xi - f(x, y), \\ k(x, 0) = \int_0^x k(x, y)g(y)dy - g(x). \end{cases} \tag{20}$$

By [33], (20) has unique solution $k \in C^1(\mathcal{F})$. Moreover, from [33], the inverse transformation of (17) has the form:

$$w(x, t) = z(x, t) + \int_0^x l(x, y)z(y, t)dy, \tag{21}$$

where $l(x, y)$ satisfies

$$\begin{cases} l_x(x, y) + l_y(x, y) = - \int_y^x f(x, \xi)l(\xi, y)d\xi - f(x, y), \\ l(x, 0) = -g(x). \end{cases} \tag{22}$$

It is easy to verify that under the transformation (21), the target system (11) can be transformed into the original system (1).

We design a full state feedback control as follows:

$$u(t) = \int_0^1 k(1, y)w(y, t)dy, \tag{23}$$

under which the system (1) becomes

$$\begin{cases} {}_0^C D_t^\alpha w(x, t) = w_x(x, t) + g(x)w(0, t) \\ \quad + \int_0^x f(x, y)w(y, t)dy, \\ w(1, t) = \int_0^1 k(1, y)w(y, t)dy, \quad t \geq 0, \\ w(x, 0) = w_0(x), \quad 0 \leq x \leq 1, \end{cases} \tag{24}$$

which is equivalent to the target system (11). Thus, from Lemma 4, we obtain Theorem 1.

Theorem 1: For any initial value $w_0 \in L^2(0, 1)$, under the state feedback control (23), the closed-loop system (24) admits a unique solution $w(\cdot, t) \in C(0, \infty; L^2(0, 1))$ which is Mittag-Leffler stable in the sense of

$$\|w(\cdot, t)\|_{L^2(0,1)}^2 \leq ME_\alpha(-\mu t^\alpha) \|w_0\|_{L^2(0,1)}^2, \tag{25}$$

for some positive constants $M, \mu > 0$ that are independent of initial values w_0 .

Remark 5: By Remark 4 and the equivalence between the system (24) and the system (11), when $\alpha = 1$, the system (24) is finite time stable, but it is not finite time stable when $\alpha \in (0, 1)$.

IV. OUTPUT FEEDBACK STABILIZATION

From the view of practice, the full state information may be not available directly due to a cost of measurement spatial and temporal dynamics information or a lack of suitable physical sensors capturing the dynamics state information. To remove this limitation, it is natural to apply the idea of output feedback, that is, the control output measurement signal should be as little as possible.

In this section, we will present an output feedback control u where we use measurable information $w(0, t)$ only and this control u can stabilize the system (1).

To find the output feedback control, we need to recover the state of (1). Since it is difficult to design an observer for (1) in

the direct way, we first introduce an invertible transformation $w \rightarrow v$ in the form:

$$v(x, t) = w(x, t) - \int_0^x q(x, y)w(y, t)dy, \quad (26)$$

where the kernel functions $q(x, y)$ satisfies

$$\begin{cases} q_x(x, y) + q_y(x, y) = \int_y^x q(x, \xi)f(\xi, y)d\xi - f(x, y), \\ q(1, y) = 0. \end{cases} \quad (27)$$

The existence of solution to the kernel equation (27) can be found in [35], moreover, $q \in C^1([0, 1] \times [0, 1])$. Now we show the invertibility of transformation(26) by the functional analysis theory. Actually, let \mathbb{P} be a Volterra operator given by

$$\mathbb{P}h(x) = - \int_0^x q(x, y)h(y)dy.$$

It is not difficult to verify that \mathbb{P} is a bounded operator from $L^2(0, 1)$ to $C[0, 1]$. Next, we will show that $(I + \mathbb{P})$ is invertible and its inverse $(I + \mathbb{P})^{-1}$ is bounded. By Fredholm's alternative [36, Chapter 4, p.107] and bounded inverse theorem [36, Chapter 2, p.49], it suffices to prove that

$$(I + \mathbb{P})h(x) = h(x) - \int_0^x q(x, y)h(y)dy = 0 \quad (28)$$

has only the zero solution on $L^2(0, 1)$. Indeed, if there exists a function $h(x) \neq 0$ satisfying (28), by the continuity of $(\mathbb{P}h)(x)$ on $[0, 1]$, we then know that h is a continuous function. Taking $x = 0$ in (28), we get $h(0) = 0$. We assume without loss of generality that $h(x)$ does not vanish identically in any interval of the form $[0, \delta]$ with $\delta > 0$. Thus, the function $h_m(\delta) = \max\{|h(x)| : x \in [0, \delta]\}$ is nonzero and $\lim_{\delta \rightarrow 0} h_m(\delta) = 0$. Moreover, for any given $\delta > 0$, there exists a point $x_\delta \in [0, \delta]$ such that $|h(x_\delta)| = h_m(\delta)$. Denote $M = \max\{|q(x, y)| : x, y \in [0, 1]\}$. It follows from (28) that

$$\begin{aligned} h_m(\delta) &= |h(x_\delta)| \leq \int_0^{x_\delta} |q(x_\delta, y)||h(y)|dy \\ &\leq Mh_m(x_\delta)x_\delta \leq Mh_m(\delta)\delta. \end{aligned} \quad (29)$$

This, together with $h_m(\delta) > 0$, implies $1 \leq M\delta$ for every $\delta > 0$. This yields a contradiction. Thus, (28) has only the zero solution, the existence of $(I + \mathbb{P})^{-1}$ follows from Fredholm's alternative, and the boundedness of $(I + \mathbb{P})^{-1}$ follows from bounded inverse theorem. Therefore, the inverse transformation of (26) is given by

$$w(x, t) = (I + \mathbb{P})^{-1}v(x, t). \quad (30)$$

Taking Caputo's fractional derivative for (26), using the first equation of (1) and partial integral show that

$$\begin{aligned} {}_0^C D_t^\alpha v(x, t) &= {}_0^C D_t^\alpha w(x, t) - \int_0^x q(x, y) {}_0^C D_t^\alpha w(y, t)dy \\ &= {}_0^C D_t^\alpha w(x, t) - \int_0^x q(x, y) \left(w_y(y, t) \right. \\ &\quad \left. + g(y)w(0, t) + \int_0^y f(y, \xi)w(x, \xi)d\xi \right) dy \end{aligned}$$

$$\begin{aligned} &= w_x(x, t) + w(0, t) \left(g(x) - \int_0^x q(x, y)g(y)dy \right) \\ &\quad + q(x, 0)w(0, t) + \int_0^x q_y(x, y)w(y, t)dy \\ &\quad - q(x, x)w(x, t) + \int_0^x w(y, t) \left(f(x, y) \right. \\ &\quad \left. - \int_y^x q(x, \xi)f(\xi, y)d\xi \right) dy \end{aligned} \quad (31)$$

and

$$v_x(x, t) = w_x(x, t) - q(x, x)w(x, t) - \int_0^x q_x(x, y)w(y, t)dy. \quad (32)$$

It follows from the first equation of (27), (31) and (32) that

$$\begin{aligned} &{}_0^C D_t^\alpha v(x, t) - v_x(x, t) \\ &= \left(q(x, 0) + g(x) - \int_0^x q(x, y)g(y)dy \right) w(0, t) \\ &\quad + \int_0^x \left(q_x(x, y) + q_y(x, y) + f(x, y) \right. \\ &\quad \left. - \int_y^x q(x, \xi)f(\xi, y)d\xi \right) w(y, t)dy \\ &= \left(q(x, 0) + g(x) - \int_0^x q(x, y)g(y)dy \right) v(0, t). \end{aligned} \quad (33)$$

It follows from the second equation of (27) that we have $v(1, t) = w(1, t) = u(t)$. Hence, under (26), the system (1) is transformed into the following system:

$$\begin{cases} {}_0^C D_t^\alpha v(x, t) = v_x(x, t) + G(x)v(0, t), \\ v(1, t) = u(t), \quad t \geq 0, \\ v(x, 0) = v_0(x), \quad 0 \leq x \leq 1, \end{cases} \quad (34)$$

where $G(x)$ is given by

$$G(x) = q(x, 0) + g(x) - \int_0^x q(x, y)g(y)dy, \quad (35)$$

and the initial state $v_0(x)$ is given by

$$v_0(x) = w_0(x) - \int_0^x q(x, y)w_0(y)dy. \quad (36)$$

Now, we design a state observer for (34) as follows:

$$\begin{cases} {}_0^C D_t^\alpha \widehat{v}(x, t) = \widehat{v}_x(x, t) + G(x)v(0, t), \\ \widehat{v}(1, t) = u(t), \quad t \geq 0, \\ \widehat{v}(x, 0) = \widehat{v}_0(x), \quad 0 \leq x \leq 1, \end{cases} \quad (37)$$

and the observed state $\widehat{w}(x, t)$ of $w(x, t)$ is given by

$$\widehat{w}(x, t) = [(I + \mathbb{P})^{-1}\widehat{v}](x, t). \quad (38)$$

Let

$$\widetilde{v}(x, t) = \widehat{v}(x, t) - v(x, t) \quad (39)$$

be an error variable. Then, we can see that $\widetilde{v}(x, t)$ satisfies

$$\begin{cases} {}_0^C D_t^\alpha \widetilde{v}(x, t) = \widetilde{v}_x(x, t), \\ \widetilde{v}(1, t) = 0, \quad t \geq 0, \\ \widetilde{v}(x, 0) = \widehat{v}_0(x) - v_0(x), \quad 0 \leq x \leq 1. \end{cases} \quad (40)$$

By Lemma 4, we know that the system (40) admits a unique solution that is Mittag-Leffler stable. Since the state feedback control (23) stabilize the system (1), and $\widehat{w}(x, t)$ given by (39) is an estimate of $w(x, t)$, an observer-based feedback should be designed naturally as:

$$\begin{aligned}
 u(t) &= \int_0^1 k(1, y)\widehat{w}(y, t)dy \\
 &= \int_0^1 k(1, y)[(I + \mathbb{P})^{-1}\widehat{v}](y, t)dy, \quad (41)
 \end{aligned}$$

under which the closed-loop system can be obtained as

$$\begin{cases}
 {}^C_0 D_t^\alpha w(x, t) = w_x(x, t) + g(x)w(0, t) \\
 \quad + \int_0^x f(x, y)w(y, t)dy, \\
 w(1, t) = \int_0^1 k(1, y)[(I + \mathbb{P})^{-1}\widehat{v}](y, t)dy, \\
 {}^C_0 D_t^\alpha \widehat{v}(x, t) = \widehat{v}_x(x, t) + G(x)v(0, t), \\
 \widehat{v}(1, t) = \int_0^1 k(1, y)[(I + \mathbb{P})^{-1}\widehat{v}](y, t)dy, \\
 \widehat{v}(x, 0) = \widehat{v}_0(x), \quad 0 \leq x \leq 1, \\
 w(x, 0) = w_0(x), \quad 0 \leq x \leq 1.
 \end{cases} \quad (42)$$

We consider system (42) in the energy Hilbert state space defined by

$$\mathbb{H} = L^2(0, 1) \times L^2(0, 1), \quad (43)$$

with the inner product induced norm given by

$$\|(\varphi, \phi)^\top\|^2 = \int_0^1 e^x \varphi^2(x)dx + \kappa \int_0^1 e^x \phi^2(x)dx,$$

where κ is a positive constant satisfying

$$\kappa > e \int_0^1 k^2(1, y)dy \|(I + \mathbb{P})^{-1}\|^2. \quad (44)$$

Theorem 2: With the output feedback control (41), for any initial value $(w_0, \widehat{v}_0) \in \mathbb{H}$, the closed-loop system (42) admits a unique solution $(w(\cdot, t), \widehat{v}(\cdot, t)) \in C(0, \infty; \mathbb{H})$ which is Mittag-Leffler stable in the sense that

$$\|(w(\cdot, t), \widehat{v}(\cdot, t))\|_{\mathbb{H}}^2 \leq ME_\alpha(-\mu t^\alpha) \|(w_0, \widehat{v}_0)\|_{\mathbb{H}}^2, \quad (45)$$

for some positive constants $M, \mu > 0$ that are independent of initial value $(w_0, \widehat{v}_0) \in \mathbb{H}$.

Proof: Noticing that under the transformation (26), the closed-loop system (42) is equivalent to

$$\begin{cases}
 {}^C_0 D_t^\alpha v(x, t) = v_x(x, t) + G(x)v(0, t), \\
 v(1, t) = \int_0^1 k(1, y)[(I + \mathbb{P})^{-1}\widehat{v}](y, t)dy, \\
 {}^C_0 D_t^\alpha \widehat{v}(x, t) = \widehat{v}_x(x, t) + G(x)v(0, t), \\
 \widehat{v}(1, t) = \int_0^1 k(1, y)[(I + \mathbb{P})^{-1}\widehat{v}](y, t)dy, \\
 v(x, 0) = v_0(x) = w_0(x) - \int_0^x q(x, y)w_0(y)dy, \\
 \widehat{v}(x, 0) = \widehat{v}_0(x), \quad 0 \leq x \leq 1,
 \end{cases} \quad (46)$$

where $G(x)$ is given by (35). Using the error variable $\widetilde{v}(x, t)$ defined in (39), we can write the equivalent system of (46) as follows:

$$\begin{cases}
 {}^C_0 D_t^\alpha v(x, t) = v_x(x, t) + G(x)v(0, t), \\
 v(1, t) = \int_0^1 k(1, y)[(I + \mathbb{P})^{-1}v(y, t) \\
 \quad + (I + \mathbb{P})^{-1}\widetilde{v}(y, t)]dy, \\
 {}^C_0 D_t^\alpha \widetilde{v}(x, t) = \widetilde{v}_x(x, t), \\
 \widetilde{v}(1, t) = 0, \\
 v(x, 0) = v_0(x) = w_0(x) - \int_0^x q(x, y)w_0(y)dy, \\
 \widetilde{v}(x, 0) = \widehat{v}_0(x) - v_0(x), \quad 0 \leq x \leq 1.
 \end{cases} \quad (47)$$

Under the transformation (30), system (47) is the equivalent system of the following

$$\begin{cases}
 {}^C_0 D_t^\alpha w(x, t) = w_x(x, t) + g(x)w(0, t) \\
 \quad + \int_0^x f(x, y)w(y, t)dy, \\
 w(1, t) = \int_0^1 k(1, y)[w(y, t) \\
 \quad + [(I + \mathbb{P})^{-1}\widetilde{v}](y, t)]dy, \\
 {}^C_0 D_t^\alpha \widetilde{v}(x, t) = \widetilde{v}_x(x, t), \\
 \widetilde{v}(1, t) = 0, \\
 w(x, 0) = w_0(x), \quad 0 \leq x \leq 1, \\
 \widetilde{v}(x, 0) = \widehat{v}_0(x) - w_0(x) + \int_0^x q(x, y)w_0(y)dy.
 \end{cases} \quad (48)$$

Further, under the transformation (17), system (48) is equivalent to the following system:

$$\begin{cases}
 {}^C_0 D_t^\alpha z(x, t) = z_x(x, t), \\
 z(1, t) = \int_0^1 k(1, y)[(I + \mathbb{P})^{-1}\widetilde{v}](y, t)dy, \\
 {}^C_0 D_t^\alpha \widetilde{v}(x, t) = \widetilde{v}_x(x, t), \\
 \widetilde{v}(1, t) = 0, \\
 z(x, 0) = z_0(x) = w_0(x) - \int_0^x k(x, y)w_0(y)dy, \\
 \widetilde{v}(x, 0) = \widehat{v}_0(x) - w_0(x) + \int_0^x q(x, y)w_0(y)dy.
 \end{cases} \quad (49)$$

Thus, it suffices to prove the existence and Mittag-Leffler stable of solution of (49). For this purpose, we define the operator $\mathbb{A} : D(\mathbb{A}) \subset \mathbb{H} \rightarrow \mathbb{H}$ as follows:

$$\begin{cases}
 l\mathbb{A}(\varphi, \phi)^\top = (\varphi', \phi')^\top, \quad \forall (\varphi, \phi)^\top \in D(\mathbb{A}), \\
 D(\mathbb{A}) = \{(\varphi, \phi)^\top \in H^1(0, 1) \times H^1(0, 1) | \\
 \varphi(1) = \int_0^1 k(1, y)[(I + \mathbb{P})^{-1}\phi](y)dy, \phi(1) = 0\}.
 \end{cases} \quad (50)$$

Now we show that \mathbb{A} is dissipative in \mathbb{H} . Actually, for any $(\varphi, \phi)^\top \in D(\mathbb{A})$, a straightway computation shows that

$$\begin{aligned} & \operatorname{Re}\langle \mathbb{A}(\varphi, \phi)^\top, (\varphi, \phi)^\top \rangle \\ &= \frac{e}{2} \left(\int_0^1 k(1, y)(I + \mathbb{P})^{-1}\phi(y)dy \right)^2 - \frac{1}{2}\varphi^2(0) \\ &\quad - \frac{1}{2} \int_0^1 e^x \varphi^2(x)dx \\ &\quad + \kappa \left(-\frac{1}{2}\phi^2(0) - \frac{1}{2} \int_0^1 e^x \phi^2(x)dx \right) \\ &\leq \frac{e}{2} \int_0^1 k^2(1, y)dy \int_0^1 [(I + \mathbb{P})^{-1}\phi(y)]^2 dy \\ &\quad - \frac{\kappa}{2} \int_0^1 e^x \phi^2(x)dx \\ &\leq \frac{e}{2} \|(I + \mathbb{P})^{-1}\|^2 \int_0^1 k^2(1, y)dy \int_0^1 \phi^2(y)dy \\ &\quad - \frac{\kappa}{2} \int_0^1 e^x \phi^2(x)dx \\ &\leq \frac{e}{2} \|(I + \mathbb{P})^{-1}\|^2 \int_0^1 k^2(1, y)dy \int_0^1 e^y \phi^2(y)dy \\ &\quad - \frac{\kappa}{2} \int_0^1 e^x \phi^2(x)dx \\ &= -\frac{1}{2} \left(\kappa - e \int_0^1 k^2(1, y)dy \|(I + \mathbb{P})^{-1}\|^2 \right) \\ &\quad \times \int_0^1 e^x \phi^2(x)dx \leq 0. \end{aligned} \tag{51}$$

In the last step of (51), (44) is used. Thus \mathbb{A} is dissipative in \mathbb{H} . Next, we show that $R(\lambda I - \mathbb{A}) = \mathbb{H}$ for any $\lambda > 0$, where $R(\lambda I - \mathbb{A})$ represents the range of $\lambda I - \mathbb{A}$. That is, for any $\lambda > 0$ and $(\widehat{\varphi}, \widehat{\phi})^\top \in \mathbb{H}$, we need to show that there exists a $(\varphi, \phi)^\top \in D(\mathbb{A})$ such that $(\lambda I - \mathbb{A})(\varphi, \phi)^\top = (\widehat{\varphi}, \widehat{\phi})^\top$, which amounts to

$$\begin{cases} \lambda\varphi(x) - \varphi'(x) = \widehat{\varphi}(x), & 0 \leq x \leq 1, \\ \lambda\phi(x) - \phi'(x) = \widehat{\phi}(x), & 0 \leq x \leq 1, \\ \varphi(1) = \int_0^1 k(1, y)[(I + \mathbb{P})^{-1}\phi](y)dy, \\ \phi(1) = 0. \end{cases} \tag{52}$$

A simple computation shows that the solution of (52) is given by

$$\begin{cases} \varphi(x) = - \int_0^1 k(1, y) \left[(I + \mathbb{P})^{-1} \int_y^1 e^{\lambda(y-\xi)} \widehat{\varphi}(\xi) d\xi \right] dy \\ \quad \times e^{\lambda(x-1)} - \int_x^1 e^{\lambda(x-y)} \widehat{\varphi}(y) dy, \\ \phi(x) = - \int_x^1 e^{\lambda(x-y)} \widehat{\phi}(y) dy. \end{cases}$$

Thus, $R(\lambda I - \mathbb{A}) = \mathbb{H}$ for any $\lambda > 0$. Finally, it is easy to verify that $D(\mathbb{A}) = \mathbb{H}$. It follows from the semigroup theory [34, Theorem 1.4.3] that \mathbb{A} generates a C_0 -semigroup of contractions on \mathbb{H} . Therefore, the existence of solution of (49) follows from Lemma 1.

Next, we show that the solution of (49) is Mittag-Leffler stable. Consider the Lyapunov function

$$V(t) = \frac{1}{2} \int_0^1 e^x z^2(x, t)dx + \frac{\kappa}{2} \int_0^1 e^x \widetilde{v}^2(x, t)dx.$$

where κ satisfies (44). By Lemma 2, we can obtain

$$\begin{aligned} {}_0^C D_t^\alpha V(t) &= \frac{1}{2} \int_0^1 e^x {}_0^C D_t^\alpha z^2(x, t)dx \\ &\quad + \frac{\kappa}{2} \int_0^1 e^x {}_0^C D_t^\alpha \widetilde{v}^2(x, t)dx \\ &\leq \int_0^1 e^x z(x, t)z_x(x, t)dx \\ &\quad + \kappa \int_0^1 e^x \widetilde{v}(x, t)\widetilde{v}_x(x, t)dx \\ &= \frac{e}{2} \left(\int_0^1 k(1, y)[(I + \mathbb{P})^{-1}\widetilde{v}(y, t)]dy \right)^2 \\ &\quad - \frac{1}{2}z^2(0, t) - \frac{1}{2} \int_0^1 e^x z^2(x, t)dx \\ &\quad + \kappa \left(-\frac{1}{2}\widetilde{v}^2(0, t) - \frac{1}{2} \int_0^1 e^x \widetilde{v}^2(x, t)dx \right) \\ &\leq \frac{e}{2} \|(I + \mathbb{P})^{-1}\|^2 \int_0^1 k^2(1, y)dy \int_0^1 \widetilde{v}^2(x, t)dx \\ &\quad - \frac{1}{2} \int_0^1 e^x z^2(x, t)dx - \frac{\kappa}{2} \int_0^1 e^x \widetilde{v}^2(x, t)dx \\ &\leq -\frac{1}{2} \left(\kappa - e \int_0^1 k^2(1, y)dy \|(I + \mathbb{P})^{-1}\|^2 \right) \\ &\quad \times \int_0^1 e^x \widetilde{v}^2(x, t)dx - \frac{1}{2} \int_0^1 e^x z^2(x, t)dx \\ &\leq -CV(t), \end{aligned} \tag{53}$$

where

$$C = 1 - \frac{e}{\kappa} \int_0^1 k^2(1, y)dy \|(I + \mathbb{P})^{-1}\|^2.$$

It then follows from Lemma 3 and (53) that $V(t) \leq V(0)E_\alpha(-Ct^\alpha)$, which implies that (45) holds.

V. AN ILLUSTRATIVE EXAMPLE AND SIMULATION

In this section, we design an output feedback controller and present some numerical simulations for example 1. Consider the following fractional hyperbolic equation:

$$\begin{cases} {}_0^C D_t^\alpha w(x, t) = w_x(x, t) + e^{\lambda x} w(0, t), \\ w(1, 0) = u(t), \quad t \geq 0, \\ y_o(t) = w(0, t), \quad t \geq 0, \\ w(x, 0) = w_0(x), \quad 0 \leq x \leq 1. \end{cases} \tag{54}$$

From example 1, if $u(t) \equiv 0$ and $w_0(x) = e^{\lambda x}(1 - x)$, (54) has an unstable solution. The observer for (54) is designed as follows:

$$\begin{cases} {}_0^C D_t^\alpha \widehat{w}(x, t) = \widehat{w}_x(x, t) + e^{\lambda x} y_o(t), \\ \widehat{w}(1, t) = u(t), \quad t \geq 0, \\ \widehat{w}(x, 0) = \widehat{w}_0(x), \quad 0 \leq x \leq 1, \end{cases} \tag{55}$$

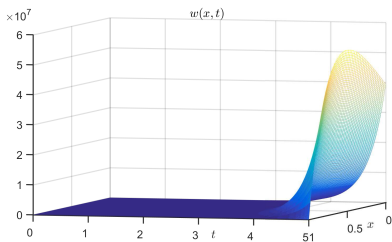


FIGURE 1. The state $w(x, t)$ without control.

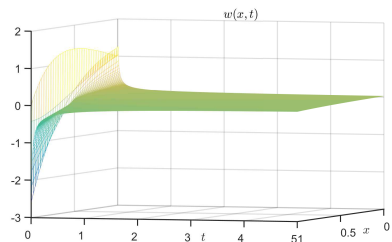


FIGURE 2. The state $w(x, t)$ with the control.

where the initial state \widehat{w}_0 of observer can be taken any value in $L^2(0, 1)$. For the case of (54), the equation (20) becomes

$$\begin{cases} k_x(x, y) + k_y(x, y) = 0, \\ k(x, 0) = \int_0^x k(x, y)e^{\lambda y} dy - e^{\lambda x}. \end{cases} \quad (56)$$

A simple computation shows that the solution of (56) is given by

$$k(x, y) = -e^{(\lambda+1)(x-y)}.$$

Hence, under the feedback control (41), the closed-loop system of (54) reads as

$$\begin{cases} {}^C_0 D_t^\alpha w(x, t) = w_x(x, t) + e^{\lambda x} w(0, t), \\ w(1, t) = -\int_0^1 e^{(\lambda+1)(1-y)} \widehat{w}(y, t) dy, \\ {}^C_0 D_t^\alpha \widehat{w}(x, t) = \widehat{w}_x(x, t) + e^{\lambda x} w(0, t), \\ \widehat{w}(1, t) = -\int_0^1 e^{(\lambda+1)(1-y)} \widehat{w}(y, t) dy, \\ w(x, 0) = w_0(x), \widehat{w}(x, 0) = \widehat{w}_0(x), \quad 0 \leq x \leq 1. \end{cases} \quad (57)$$

By Theorem 2, the closed-loop system (57) admits a unique solution $(w(\cdot, t), \widehat{w}(\cdot, t)) \in C(0, +\infty; L^2(0, 1) \times L^2(0, 1))$ and there exist two positive constants $M, \mu > 0$ such that

$$\|(w(\cdot, t), \widehat{w}(\cdot, t))\|_{\mathbb{H}}^2 \leq ME_\alpha(-\mu t^\alpha) \|(w_0, \widehat{w}_0)\|_{\mathbb{H}}^2.$$

For numerical simulation, the fractional order is taken as $\alpha = 0.6$, the parameter is taken as $\lambda = 2$, and the initial state is taken as $w_0(x) = e^{\lambda x}(1-x)$, $\widehat{w}_0(x) = 0$. Figure 1 shows that the state w is not convergent without control. Figure 2 and Figure 3 show the state w and \widehat{w} . It is clearly seen that the convergence of w and \widehat{w} is satisfactory. Figure 4 displays the feedback control in time.

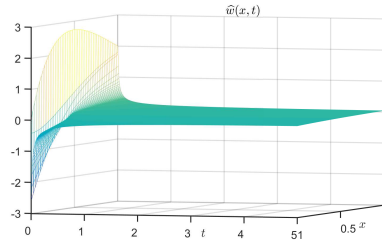


FIGURE 3. The state $\widehat{w}(x, t)$ with the control.

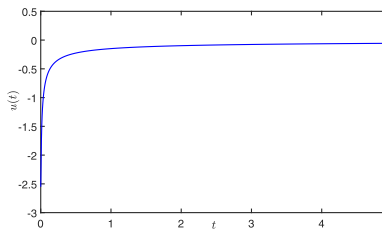


FIGURE 4. The control $u(t)$.

VI. CONCLUDING REMARKS

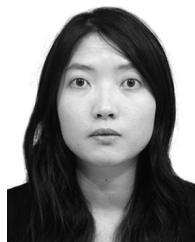
This paper puts an effort to obtain output feedback stabilization for time fractional hyperbolic partial differential equation system that might be potentially unstable without control. The backstepping transformation is used to design of the state feedback. The observer is proposed and the observer-based feedback control is obtained based on the state feedback. The closed-loop system is shown to admit a unique solution and to be Mittag-Leffler stable. The idea is potentially promising for treating other fractional partial differential equations. Finally, an example and numerical simulations are presented to confirm the effectiveness of the theoretical results.

In future works, dealing with the disturbance to obtain the stability of fractional system is very interesting since the uncertainty and the disturbance widely exist in control systems. In addition, a future research direction may be to use adaptive control method to solve the stabilization for uncertain fractional partial differential equation systems, such as the system with the boundary control matched the disturbance or the system with disturbance suffered from the boundary observation [37].

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