IEEE Access
Multidisciplinary : Rapid Review : Open Access Journal

Received May 30, 2019, accepted June 29, 2019, date of publication July 8, 2019, date of current version July 29, 2019.

Digital Object Identifier 10.1109/ACCESS.2019.2927317

A Study of Roughness in Modules of Fractions

ZHIHUA CHEN[1], SABA AYUB[2], WAQAS MAHMOOD[2], ABID MAHBOOB[3], AND CHAHN YONG JUNG[4]

[1]Institute of Computing Science and Technology, Guangzhou University, Guangzhou 510006, China
[2]Department of Mathematics, Quaid-i-Azam University, Islamabad 45320, Pakistan
[3]Department of Mathematics, University of Education, Lahore 54000, Pakistan
[4]Department of Business Administration, Gyeongsang National University, Jinju 52828, South Korea

Corresponding author: Chahn Yong Jung (bb5734@gnu.ac.kr)

ABSTRACT The theory of rough sets is successfully applied in various algebraic systems (e.g. groups, rings, and modules). In this paper, the concept of roughness is introduced in modules of fractions with respect to its submodules. Hence, the notion of the lower and upper approximation spaces based on a submodule of the modules of fractions is introduced. Some fundamental results related to these approximation spaces are examined with examples. Moreover, this paper establishing several connections between the approximation spaces of two different modules of fractions with respect to the image and pre-image under a module homomorphism. This technique of building up a connection among the approximation spaces via module homomorphisms is useful to connect two information systems in the field of information technology.

INDEX TERMS Rough sets, rough modules, module homomorphisms.

I. INTRODUCTION

Data handling is encountered in many daily life problems as well as complex problems of specialized fields including computer sciences, medical sciences and environmental sciences. Many mathematical approaches are proposed in literature to solve such kind of issues, One of the most successful among these is fuzzy set theory, proposed by Zadeh [1] in 1965. Fuzzy set theory is a generalization of the crisp sets. In crisp set theory, a set is uniquely defined by its elements, i.e. an element is either a member of a set or not. So, there is a membership function describing the belongingness of elements of the universe to the set. This function can attain only one value, 0 or 1. In fuzzy set theory membership function assigns the grade of membership to the elements of the universe in the unit interval [0, 1]. For example, in real world we say that a man is young or old, an object is expensive or cheap, a painting is beautiful or not and etc. Let us take a painting as an illustration. We cannot classify all the paintings into two distinct classes, i. e. beautiful or not. Some paintings cannot be decided whether they are beautiful or not. Thus they remain in doubtful area. Similarly, we cannot say confidently that either a person is ill or not. Because a person's disease may on its initial or last stage. In fuzzy set theory, a person who is very sick, could have the degree of sickness near to 0.89. On contrary, a person could have sickness degree of 0.12 indicating that a person has nearly recovered from illness. Likewise, a painting having degree of beauty near to 1 represents that a painting is very beautiful and degree of 0.2 indicates that a painting is somehow beautiful. However, assigning the grade of membership is also sometimes a problem.

In 1982, a computer scientist Zdzislaw I. Pawlak introduced the concept of rough set theory [2] to manage various types of uncertainties and imprecision including the raw data. Rough set theory is mainly concerned with the classification and analysis of imprecise information. This theory is an extension of classical set theory, which is not defined by means of membership function but by two precise sets, called the lower and upper approximations. These approximations are beneficent in the extraction of useful information hidden in data. Rough set theory provides us very simple algorithms to characterize the original objects having the same value of attributes in an information system.

Rough set theory is based on the assumption that we have some additional information (data) about the elements of a set. Consider as an example, a group of some patients suffering from malaria. To diagnose malaria, one must see various symptoms, e.g. headache, fever, fatigue, muscle pain, back pain, chills, sweating, dry cough, enlargement, nausea and vomiting. The patients revealing the same symptoms are indiscernible with respect to the available information and

The associate editor coordinating the review of this manuscript and approving it for publication was Jafar A. Alzubi.

form elementary classes of knowledge. Similarly, two acids with pH level of 4.12 and 4.53 will be in many contexts, be perceived as so equally weak, that they are similar with respect to this attribute. They are part of a rough set ''weak acids'' as compared to ''strong'' or ''medium'' or whatsoever other category are relevant in this context of classification.

Primarily, rough set theory is used to reveal useful information from an information system (*i.e*. data table, where columns are labeled by attributes and rows are labeled by objects) based on the indiscernibility relation which is an equivalence relation. The classes obtained from this relation containing similar elements in view of available information, are the fundamental blocks of knowledge about the objects of the universe. Any union of these classes is a crisp set, and any other set is a rough set. The lower approximation of a set X is the set of all elements that surely belongs to set X, whereas the upper approximation of X is the set of all elements that possibly belongs to X. Thus, based on the lower approximation certain information can be derived, while by using upper approximation partially certain information may be derived. The difference of the lower and upper approximation spaces of X is its boundary region. A set is rough if its boundary region is non-empty, otherwise it is a crisp set.

The theory of rough sets has been demonstrated to be of fundamental importance to numerous fields of computer sciences. For example, data mining, pattern recognition, knowledge acquisition, artificial intelligence, cognitive sciences, machine learning, decision support systems, knowledge discovery from databases, expert systems and inductive reasoning are the most auspicious amongst it's applications. Furthermore, the theory has been applied to solve several real life problems of diverse fields including medicine, engineering, banking, financial and market analysis, pharmacology and others. Hence, this theory grabbed attention of several researchers, scientists, philosophers and mathematicians owed to valuable features.

From the beginning, the applications of rough set theory to various algebraic systems was of great interest to researchers. Hence, many attempts were made to apply the theory of rough sets to a numerous algebraic structures. For instance, Biswas and Nanda [3] was the first to initiate the study of roughness in a special algebraic structure-groups and introduced the notion of rough subgroups. However, the work of Biswas and Nanda [3] was based only on the lower approximation. Keeping this idea in mind, Kuroki and Wang [4] introduced the notion of the lower and upper approximation spaces in groups based on the normal subgroups to study the algebraic properties of rough sets in groups. Moreover, Kuroki [5] elaborated the notion of rough ideals in semigroups [5]. In [6], Mahmood et al. established a relationship between the lower and upper approximation spaces of groups by manoeuvring the group homomorphisms. Later in [7], they also studied the concept of roughness in quotient groups and established several homomorphisms. In [8], Ayub et al. introduced the concept of roughness in soft-intersection groups and developed connection between the approximation spaces via group homomorphisms.

Large number of mathematicians proposed meaningful extensions of Pawlak's rough sets [2]. In this regard, Yao [9] was one amongst those beginners to introduce the notion of rough sets using binary relations. Shabbir *et al.* [10] investigated the notion of modified soft rough sets (MSR-sets) using soft sets to improve the soft rough sets provided by Feng *et al.* [11]. In [12], Davvaz established the notion of $T-$rough sets by utilizing the set-valued homomorphisms. Some properties of generalized rough sets [12] was studied by Ali *et al.* in [13].

Current work is about roughness in modules of fractions. The concept of roughness in modules was first studied by Davvaz and Mahdavipour [14]. Later, this concept has been further studied by Hosseini and Saberifar [15] utilizing the concept of set valued homomorphisms given by Davvaz [12]. In [16], Ayub *et al.* introduced the notion of Fuzzy modules of fractions and defined the notion of fuzzy approximation spaces using soft modules of fractions making use of multi-granulation rough sets defined in [17]. Some authors also established the notion of roughness in hyperstructures. For example, Kanzanci *et al.* [18] introduced the notion of the lower and upper approximations in quotient hypermodules with respect to fuzzy sets. Moreover, the notion of the lower and upper approximations in $Hv-$modules is studied by Davvaz [19]. He also investigated the concept of rough approximation spaces of hyperrings in [20]. Miravakili *et al.* [21] studied the concept of roughness in hypermodules by maneuvering set-valued homomorphisms. Furthermore, Leoreanu and Davvaz [22] and Anvariyeh *et al.* [23] applied the notion of Pawalk's rough sets to $n-$array hypergroups and $\gamma-$semihypergroups, respectively.

In the present paper, an endeavor to investigate a connection between rough set theory and modules of fractions is made. In this regard, the submodules are used to define the rough approximation spaces in modules of fractions. This paper is organized as follows: In Section 2, some basic material related to the modules of fractions is presented. Then, an equivalence relation on it's submodules is defined and the notion of lower and upper rough approximation spaces is introduced. Some important properties related to the proposed objective are discussed with illustrative examples. Moreover, a linkage between the rough approximation spaces of two different modules of fractions via module homomorphisms is established in Section 3. As an application, an example is constructed to create a connection between two information systems using module homomorphisms. This is practical illustration of our work in the field of information technology.

II. ROUGHNESS IN MODULES OF FRACTIONS

In this section, the roughness in modules of fractions will be defined. And some of it's fundamental results will be proved.

First some basic definitions and results on commutative algebra will be presented. For more details, see [24], [25]. In this paper, R denotes a commutative ring with identity 1_R.

Definition 1: [25, Definition 1.1] A commutative group $(M,+)$ is called an $R-$module, if the map:

$$R \times M \longrightarrow M, (r, m) \mapsto r \cdot m,$$

satisfies the following properties:

(1) $r_1 \cdot (m_1 + m_2) = r_1 \cdot m_1 + r_1 \cdot m_2$,
(2) $(r_1 + r_2) \cdot m_1 = r_1 \cdot m_1 + r_2 \cdot m_1$,
(3) $r_1 \cdot (r_2 \cdot m) = (r_1 r_2) \cdot m$,
(4) $1_R \cdot m = m$,

for all $r_1, r_2 \in R$ and $m_1, m_2 \in M$.

Definition 2: [25, Definition 1.3] Let M be an $R-$module. A non-empty subset N of M is called an R-submodule, if it fulfils the following two properties:

(1) N is an additive subgroup of M.
(2) $r \cdot n \in N$, for any $r \in R$ and $n \in N$.

If $\{N_i : 1 \leq i \leq n\}$ is a family of R-submodules of M and I an ideal of R, then the following sets are R-submodules of M:

$$\cap_{i \in I} N_i = \{x \in M : x \in N_i \text{ for all } 1 \leq i \leq n\}.$$
$$\sum_{i=1}^{n} N_i = \left\{ \sum_{i=1}^{n} x_i : x_i \in N_i \text{ for all } 1 \leq i \leq n \right\}.$$
$$IN_1 = \left\{ \sum_{i=1}^{n} a_i \cdot n_i : a_i \in I, n_i \in N_1 \text{ and } n \in \mathbb{N} \right\}.$$

Definition 3: [25, Definition 1.2] Let M and M' be R-modules, then a map $f : M \rightarrow M'$ is called an $R-$linear map (or $R-$module homomorphism), if it satisfies the following conditions:

(1) $f(m_1 + m_2) = f(m_1) + f(m_2)$,
(2) $f(r \cdot m_1) = r \cdot f(m_1)$,

for all $m_1, m_2 \in M$ and $r \in R$.

Definition 4: [25, Definition 3.1] A non-empty subset S of R is called multiplicatively closed, if the following conditions hold:

(1) $1_R \in S$,
(2) $s_1 s_2 \in S$, for any $s_1, s_2 \in S$.

For an R-module M, there exists a well-known equivalence relation on the set $M \times S$, defined by:

$$(m, s) \sim (m', s') \Leftrightarrow t \cdot (s' \cdot m - s \cdot m') = 0, \quad \text{for some } t \in S.$$

The equivalence class of (m, s) is denoted by $\frac{m}{s}$. Consider the following set of equivalence classes:

$$S^{-1}M = \left\{ \frac{m}{s} : m \in M, s \in S \right\}.$$

If $M = R$, then $S^{-1}R$ is a commutative ring with identity under the following addition and multiplication:

$$\frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{r_1 s_2 + r_2 s_1}{s_1 s_2}$$

and

$$\frac{r_1}{s_1} \cdot \frac{r_2}{s_2} = \frac{r_1 r_2}{s_1 s_2},$$

where

$$\frac{r_1}{s_1}, \frac{r_2}{s_2} \in S^{-1}R.$$

Note that $S^{-1}M$ is an $S^{-1}R-$module with the following scalar multiplication:

$$\frac{r}{t} \cdot \frac{m_1}{s_1} = \frac{rm_1}{ts_1}, \quad \text{where } \frac{m_1}{s_1} \in S^{-1}M \text{ and } \frac{r}{t} \in S^{-1}R.$$

In the rest of paper, S and M will be denoting the multiplicative closed subset of R and module over R respectively.

Lemma 1: [25] If $f : M \rightarrow M'$ is an $R-$linear map, then it induces the following $S^{-1}R-$linear map:

$$S^{-1}f : S^{-1}M \rightarrow S^{-1}M', \quad \frac{m}{s} \mapsto \frac{f(m)}{s}.$$

Definition 5: Suppose that $X \subseteq R$, $X_1 \subseteq S^{-1}R$ $Y, Z \subseteq M$ and $W, V \subseteq S^{-1}M$ are non-empty subsets, then define the following sets:

$$XY = \left\{ \sum_{i=1}^{n} x_i y_i : x_i \in X, y_i \in Y \text{ and } n \in \mathbb{N} \right\},$$
$$Y + Z = \{y + z : y \in Y, z \in Z\} \text{ and}$$
$$S^{-1}X = \left\{ \frac{x}{s} : x \in X, s \in S \right\}.$$

Similarly, $X_1 W$ and $V + W$ can be defined. It is clear that the sets XY and $X_1 W$ are closed under addition.

In the following Lemma, some fundamental properties of any non-empty subsets of modules of fractions are given:

Lemma 2: If $\emptyset \neq X_i \subseteq M$ for all $i = 1, 2$ and $\emptyset \neq X \subseteq R$. Then:

(1) $S^{-1}(X_1 \cup X_2) = S^{-1}X_1 \cup S^{-1}X_2$.
(2) $S^{-1}(X_1 \cap X_2) \subseteq S^{-1}X_1 \cap S^{-1}X_2$.
(3) $S^{-1}(X_1 + X_2) \subseteq S^{-1}X_1 + S^{-1}X_2$.
(4) $S^{-1}(XX_1) \subseteq (S^{-1}X)(S^{-1}X_1)$.
Equality holds, if either X is an ideal of R or X_1 is a submodule of M.

Proof: All claims are straightforward. □

The following result is the special case of Lemma 2.

Corollary 1: [26, Corollary 3.4] Let N and P be R-submodules of M. For any ideal I of R, the following conditions hold:

(1 $S^{-1}N$ *and* $S^{-1}(IN)$ *are* $S^{-1}R$*-submodules of* $S^{-1}R$*-module* $S^{-1}M$.
(2) $S^{-1}(N \cap P) = S^{-1}N \cap S^{-1}P$.
(3) $S^{-1}(N + P) = S^{-1}N + S^{-1}P$.
(4) $S^{-1}(IN) = (S^{-1}I)(S^{-1}N)$.

In what follows, a special kind of an equivalence relation on the submodules of $S^{-1}M$ is defined. This gives the notion of approximation spaces in modules of fractions.

Definition 6: For an R-submodule N of M, define a relation $\theta_{S^{-1}N}$ on $S^{-1}N$ as follows:

$$\frac{m}{s} \theta_{S^{-1}N} \frac{n}{t} \Leftrightarrow \frac{m}{s} = \frac{r}{u} \cdot \frac{n}{t} \text{ for some } \frac{r}{u} \in U\left(S^{-1}R\right),$$

where $U\left(S^{-1}R\right)$ denotes the set of all unit elements of $S^{-1}R$. Since R is a commutative ring with identity, then one can verify that $\theta_{S^{-1}N}$ is an equivalence relation on $S^{-1}N$.

For any $m \in N$ and $s \in S$, the equivalence class of $\frac{m}{s} \in S^{-1}N$ will be denoted as:

$$\left[\frac{m}{s}\right]_{\theta_{S^{-1}N}} = \left\{\frac{n}{t} \in S^{-1}N : \frac{m}{s}\theta_{S^{-1}N}\frac{n}{t}\right\} = \left\{\frac{r}{u}\cdot\frac{m}{s} : \frac{r}{u} \in U\left(S^{-1}R\right)\right\}.$$

where $n \in N, t \in S$.

Lemma 3: With the previous notion, assume that I is an ideal of R and P an R-submodule of M. Then, the following assertions hold:

(1)
$$\left[\frac{m_1}{s_1}+\frac{m_2}{s_2}\right]_{\theta_{S^{-1}N+S^{-1}P}} \subseteq \left[\frac{m_1}{s_1}\right]_{\theta_{S^{-1}N}} + \left[\frac{m_2}{s_2}\right]_{\theta_{S^{-1}P}},$$

(2)
$$\frac{r}{t}\cdot\left[\frac{m_1}{s_1}\right]_{\theta_{S^{-1}N}} = \left[\frac{rm_1}{ts_1}\right]_{\theta_{S^{-1}N}},$$

(3)
$$\left[\frac{u\cdot m_1}{vs_1}\right]_{\theta_{S^{-1}(IN)}} \subseteq \left[\frac{u}{v}\right]_{\theta_{S^{-1}I}}\cdot\left[\frac{m_1}{s_1}\right]_{\theta_{S^{-1}N}},$$

(4)
$$\left[\frac{x}{y}\right]_{\theta_{S^{-1}N}} = \left[\frac{x}{y}\right]_{\theta_{S^{-1}P}} = \left[\frac{x}{y}\right]_{\theta_{S^{-1}N}} \cap \left[\frac{x}{y}\right]_{\theta_{S^{-1}P}} = \left[\frac{x}{y}\right]_{\theta_{S^{-1}(N\cap P)}}.$$

for all

$$\frac{r}{t} \in S^{-1}R,\quad \frac{u}{v} \in S^{-1}I,\quad \frac{x}{y} \in S^{-1}(N\cap P),\quad \frac{m_1}{s_1} \in S^{-1}N,$$

and

$$\frac{m_2}{s_2} \in S^{-1}P.$$

Proof: The proof is straightforward in view of Lemma 2. □

In the following Example, it is shown that $\theta_{S^{-1}R}$ is not a congruence relation. Hence, the reverse inclusion in Lemma 3 (1) is not true.

Example 1: Consider $R = \mathbb{Z}_4$ and $S = \{\bar{1}, \bar{3}\}$, then

$$S^{-1}R = \left\{\bar{0}, \frac{\bar{1}}{\bar{1}}, \frac{\bar{2}}{\bar{1}}, \frac{\bar{3}}{\bar{1}}\right\},$$

$$U\left(S^{-1}R\right) = \left\{\frac{\bar{1}}{\bar{1}}, \frac{\bar{3}}{\bar{1}}\right\}.$$

It follows that:

$$[\bar{0}]_{\theta_{S^{-1}R}} = \{\bar{0}\},$$

$$\left[\frac{\bar{1}}{\bar{1}}\right]_{\theta_{S^{-1}R}} = \left[\frac{\bar{3}}{\bar{1}}\right]_{\theta_{S^{-1}R}} = \left\{\frac{\bar{1}}{\bar{1}}, \frac{\bar{3}}{\bar{1}}\right\}$$

and

$$\left[\frac{\bar{2}}{\bar{1}}\right]_{\theta_{S^{-1}R}} = \left\{\frac{\bar{2}}{\bar{1}}\right\}.$$

Note that:

$$\left[\frac{\bar{1}}{\bar{1}}\right]_{\theta_{S^{-1}R}} + \left[\frac{\bar{3}}{\bar{1}}\right]_{\theta_{S^{-1}R}} = \left\{\bar{0}, \frac{\bar{1}}{\bar{1}}, \frac{\bar{2}}{\bar{1}}\right\}$$

and

$$\left[\frac{\bar{1}}{\bar{1}}+\frac{\bar{3}}{\bar{1}}\right]_{\theta_{S^{-1}R}} = \{\bar{0}\}.$$

Definition 7: Let X be a non-empty subset of M. If N is an R-submodule M. Then, by the rough approximation in the approximation space

$$\left(S^{-1}N, \theta_{S^{-1}N}\right),$$

there is a mapping

$$Apr : P\left(S^{-1}N\right) \to P\left(S^{-1}N\right) \times P\left(S^{-1}N\right)$$

such that

$$Apr\left(S^{-1}X\right) = \left(\underline{S^{-1}X}_{S^{-1}N}, \overline{S^{-1}X}_{S^{-1}N}\right),$$

where

$$\underline{S^{-1}X}_{S^{-1}N} = \left\{\frac{n}{s} \in S^{-1}N : \left[\frac{n}{s}\right]_{\theta_{S^{-1}N}} \subseteq S^{-1}X\right\}$$

and

$$\overline{S^{-1}X}_{S^{-1}N} = \left\{\frac{n}{s} \in S^{-1}N : \left[\frac{n}{s}\right]_{\theta_{S^{-1}N}} \cap S^{-1}X \neq \emptyset\right\}.$$

The sets $\underline{S^{-1}X}_{S^{-1}N}$ and $\overline{S^{-1}X}_{S^{-1}N}$ are called lower and upper approximations of $S^{-1}X$ in the approximation space $\left(S^{-1}N, \theta_{S^{-1}N}\right)$ respectively.

Remark 1: Note that $\underline{S^{-1}X}_{S^{-1}N} \subseteq S^{-1}X$. If $X \subseteq N$, then the following inclusion is also true:

$$S^{-1}X \subseteq \overline{S^{-1}X}_{S^{-1}N}.$$

Hence, both the lower and upper approximations of a non-empty set $S^{-1}X$ can be empty sets simultaneously, see Examples 2 and 3.

The following Lemma is the generalized form of [14, Lemma 3.10].

Lemma 4: With the above notion, suppose that $N \subseteq P$ are R-submodules of M and $X \subseteq Y \subseteq M$. Then:

$$\underline{S^{-1}X}_{S^{-1}N} \subseteq \underline{S^{-1}Y}_{S^{-1}P}$$

and

$$\overline{S^{-1}X}_{S^{-1}N} \subseteq \overline{S^{-1}Y}_{S^{-1}P}.$$

Proof: It is an easy consequence of the definition of approximation spaces. □

If X is a submodule of M, then the approximation spaces of $S^{-1}X$ do not yield any new information.

Lemma 5: If X and N are R-submodules of M, then

$$\underline{S^{-1}X}_{S^{-1}N} = S^{-1}X = \overline{S^{-1}X}_{S^{-1}N}.$$

Proof: Let $\frac{n}{s} \in \overline{S^{-1}X}_{S^{-1}N}$. By definition of the upper approximation, $\left[\frac{n}{s}\right]_{\theta_{S^{-1}N}} \cap S^{-1}X \neq \emptyset$. There exists $\frac{n'}{s'} \in S^{-1}N$ such that $\frac{n'}{s'} \in \left[\frac{n}{s}\right]_{\theta_{S^{-1}N}}$ and $\frac{n'}{s'} \in S^{-1}X$. It implies that:

$$\left[\frac{n}{s}\right]_{\theta_{S^{-1}N}} = \left[\frac{n'}{s'}\right]_{\theta_{S^{-1}N}}.$$

Since

$$\frac{r}{s} \cdot \frac{n'}{s'} \in S^{-1}X$$

for all $\frac{r}{s} \in S^{-1}R$, then

$$\left[\frac{n'}{s'}\right]_{\theta_{S^{-1}N}} \subseteq S^{-1}X.$$

Thus,

$$\frac{n}{s} \in \underline{S^{-1}X}_{S^{-1}N}.$$

This completes the proof. □

Proposition 1: Let I be an ideal of R and N an R-submodule of M. For any non-empty subsets X and Y of R and M respectively, we have:

$$\underline{S^{-1}(XY)}_{S^{-1}(IN)} \subseteq \underline{\left(S^{-1}X\right)\left(S^{-1}Y\right)}_{S^{-1}(IN)}.$$

$$\left(\underline{S^{-1}X}_{S^{-1}I}\right)\left(\underline{S^{-1}Y}_{S^{-1}N}\right) \subseteq \underline{\left(S^{-1}X\right)\left(S^{-1}Y\right)}_{S^{-1}(IN)}.$$

In addition, if X is an ideal of R or Y is a submodule of M, then:

$$\underline{S^{-1}(XY)}_{S^{-1}(IN)} = \underline{\left(S^{-1}X\right)\left(S^{-1}Y\right)}_{S^{-1}(IN)}.$$

Proof: We will only prove the second containment, see Lemma 2(4). Suppose that $x \in \left(\underline{S^{-1}X}_{S^{-1}I}\right)\left(\underline{S^{-1}Y}_{S^{-1}N}\right)$. Then, $x = \sum_{i=1}^{n} \frac{a_i}{s_i} \cdot \frac{m_i}{t_i}$, where $\frac{a_i}{s_i} \in \underline{S^{-1}X}_{S^{-1}I}$ and $\frac{m_i}{t_i} \in \underline{S^{-1}Y}_{S^{-1}N}$ for all $i = 1, \ldots, n$. By definition of the lower approximation, $\left[\frac{a_i}{s_i}\right]_{\theta_{S^{-1}I}} \subseteq S^{-1}X$ and $\left[\frac{m_i}{t_i}\right]_{\theta_{S^{-1}N}} \subseteq S^{-1}Y$ for all $i = 1, \ldots, n$. Then, $x \in S^{-1}(IN)$ such that:

$$\begin{aligned}\left[\frac{a_i}{s_i} \cdot \frac{m_i}{t_i}\right]_{\theta_{S^{-1}(IN)}} &\subseteq \left[\frac{a_i}{s_i}\right]_{\theta_{S^{-1}I}} \cdot \left[\frac{m_i}{t_i}\right]_{\theta_{S^{-1}N}} \\ &\subseteq \left(S^{-1}X\right)\left(S^{-1}Y\right),\end{aligned}$$

for all $i = 1, \ldots, n$, see Lemma 3.

Since $\left(S^{-1}X\right)\left(S^{-1}Y\right)$ is closed under addition, then Lemma 3(1) implies that:

$$\begin{aligned}\left[\sum_{i=1}^{n} \frac{a_i}{s_i} \cdot \frac{m_i}{t_i}\right]_{\theta_{S^{-1}(IN)}} &\subseteq \sum_{i=1}^{n} \left[\frac{a_i}{s_i} \cdot \frac{m_i}{t_i}\right]_{\theta_{S^{-1}(IN)}} \\ &\subseteq \left(S^{-1}X\right)\left(S^{-1}Y\right).\end{aligned}$$

Hence,

$$x = \sum_{i=1}^{n} \frac{a_i}{s_i} \cdot \frac{m_i}{t_i} \in \underline{\left(S^{-1}X\right)\left(S^{-1}Y\right)}_{S^{-1}(IN)}.$$

□

Note that in [14, Proposition 3.6], only second inclusion is proved of Proposition 1.

Example 2: (1) *Let $R = \mathbb{Z}_6$, $I = \{\bar{0}, \bar{3}\}$ and $S = \{\bar{1}, \bar{2}, \bar{4}\}$. Then*

$$S^{-1}R = \left\{\bar{0}, \frac{\bar{1}}{\bar{1}}, \frac{\bar{1}}{\bar{2}}\right\} \quad \textit{and} \quad IR = I.$$

Also, $S^{-1}I = S^{-1}(IR) = \left(S^{-1}I\right)\left(S^{-1}R\right) = \{\bar{0}\}$. Since $U\left(S^{-1}R\right) = \left\{\frac{\bar{1}}{\bar{1}}, \frac{\bar{1}}{\bar{2}}\right\}$, then:

$$[\bar{0}]_{\theta_{S^{-1}R}} = [\bar{0}]_{\theta_{S^{-1}I}} = \{\bar{0}\}$$

and

$$\left[\frac{\bar{1}}{\bar{1}}\right]_{\theta_{S^{-1}R}} = \left[\frac{\bar{1}}{\bar{2}}\right]_{\theta_{S^{-1}R}} = \left\{\frac{\bar{1}}{\bar{1}}, \frac{\bar{1}}{\bar{2}}\right\}.$$

If $X = \{\bar{2}\}$ and $Y = \{\bar{3}\}$, then

$$\begin{aligned}S^{-1}X &= \left\{\frac{\bar{1}}{\bar{1}}, \frac{\bar{1}}{\bar{2}}\right\}, \\ S^{-1}Y &= \{\bar{0}\}, \\ XY = S^{-1}(XY) &= \{\bar{0}\}\end{aligned}$$

and

$$\left(S^{-1}X\right)\left(S^{-1}Y\right) = \{\bar{0}\}.$$

This implies that:

$$\begin{aligned}\underline{S^{-1}X}_{S^{-1}I} &= \left(\underline{S^{-1}X}_{S^{-1}I}\right)\left(\underline{S^{-1}Y}_{S^{-1}R}\right) = \emptyset, \\ \underline{S^{-1}Y}_{S^{-1}R} &= \{\bar{0}\}\end{aligned}$$

and

$$\underline{\left(S^{-1}X\right)\left(S^{-1}Y\right)}_{S^{-1}(IR)} = \{\bar{0}\}.$$

This proves that $\underline{\left(S^{-1}X\right)\left(S^{-1}Y\right)}_{S^{-1}(IR)}$ is not a subset of $\left(\underline{S^{-1}X}_{S^{-1}I}\right)\left(\underline{S^{-1}Y}_{S^{-1}R}\right)$.

Proposition 2: With the same assumptions as in Proposition 1, the following inclusion hold:

$$\overline{S^{-1}(XY)}_{S^{-1}(IN)} \subseteq \overline{\left(S^{-1}X\right)\left(S^{-1}Y\right)}_{S^{-1}(IN)}.$$

If we assume in addition that $X \subseteq I$, $Y \subseteq N$ and either X is an ideal of R or Y is a submodule of M, then:

$$\begin{aligned}\overline{S^{-1}(XY)}_{S^{-1}(IN)} &= \overline{\left(S^{-1}X\right)\left(S^{-1}Y\right)}_{S^{-1}(IN)} \\ &\subseteq \left(\overline{S^{-1}X}_{S^{-1}I}\right)\left(\overline{S^{-1}Y}_{S^{-1}N}\right).\end{aligned}$$

Proof: By Lemma 2, it follows that

$$\overline{S^{-1}(XY)}_{S^{-1}(IN)} \subseteq \overline{\left(S^{-1}X\right)\left(S^{-1}Y\right)}_{S^{-1}(IN)}. \tag{1}$$

Now, let $X \subseteq I$ and $Y \subseteq N$. Also, assume that either X is an ideal of R or Y is a submodule of M. Then,

$$\overline{S^{-1}(XY)}_{S^{-1}(IN)} = \overline{\left(S^{-1}X\right)\left(S^{-1}Y\right)}_{S^{-1}(IN)},$$

see Lemma 2. To prove the other inclusion, suppose that $x \in \overline{S^{-1}(XY)}_{S^{-1}(IN)}$. Then, there exists $y \in S^{-1}(IN)$ such that $y \in [x]_{\theta_{S^{-1}(IN)}} \cap S^{-1}(XY)$.

Note that y can be written as $y = \sum_{i=1}^{n}\left[\frac{x_i}{s_i}\right]\left[\frac{y_i}{t_i}\right]$ with $x_i \in X$ and $y_i \in Y$.

Also, $x = \frac{a}{b}y$, for some $\frac{a}{b} \in U\left(S^{-1}R\right)$. It follows that

$$\begin{aligned}x &= \sum_{i=1}^{n}\left[\frac{a}{b}\frac{x_i}{s_i}\right]\left[\frac{y_i}{t_i}\right] \\ &= \sum_{i=1}^{n}\left[\frac{x_i}{s_i}\right]\left[\frac{a}{b}\frac{y_i}{t_i}\right] \\ &\in \left(S^{-1}X\right)\left(S^{-1}Y\right) \\ &\subseteq \left(\overline{S^{-1}X}_{S^{-1}I}\right)\left(\overline{S^{-1}Y}_{S^{-1}N}\right).\end{aligned}$$

(see Remark 1). Hence,

$$\overline{S^{-1}(XY)}_{S^{-1}(IN)} \subseteq \left(\overline{S^{-1}X}_{S^{-1}I}\right)\left(\overline{S^{-1}Y}_{S^{-1}N}\right),$$

see Equation (1). □

The following Example shows that the inclusion in Proposition 2 is strict.

Example 3: Suppose that R, I, S, X and Y are same as in Example 2. It can be seen that:

$$\begin{aligned}\overline{S^{-1}X}_{S^{-1}I} &= \left(\overline{S^{-1}X}_{S^{-1}I}\right)\left(\overline{S^{-1}Y}_{S^{-1}R}\right) = \emptyset, \\ \overline{S^{-1}Y}_{S^{-1}R} &= \{\bar{0}\}\end{aligned}$$

and

$$\overline{\left(S^{-1}X\right)\left(S^{-1}Y\right)}_{S^{-1}(IR)} = \{\bar{0}\}.$$

So, it follows that:

$$\overline{\left(S^{-1}X\right)\left(S^{-1}Y\right)}_{S^{-1}(IR)} \nsubseteq \left(\overline{S^{-1}X}_{S^{-1}I}\right)\left(\overline{S^{-1}Y}_{S^{-1}R}\right).$$

The following result is same as [14, Proposition 3.2 (11) and (12)].

Proposition 3: Suppose that N is an R-submodule of M. For any non-empty subsets X_1 and X_2 of M, the following conditions are true:

(1)

$$\begin{aligned}\underline{S^{-1}(X_1 \cup X_2)}_{S^{-1}N} &= \underline{S^{-1}X_1 \cup S^{-1}X_2}_{S^{-1}N} \\ &\supseteq \underline{S^{-1}X_1}_{S^{-1}N} \cup \underline{S^{-1}X_2}_{S^{-1}N}.\end{aligned}$$

(2)

$$\begin{aligned}\overline{S^{-1}(X_1 \cup X_2)}_{S^{-1}N} &= \overline{S^{-1}X_1 \cup S^{-1}X_2}_{S^{-1}N} \\ &= \overline{S^{-1}X_1}_{S^{-1}N} \cup \overline{S^{-1}X_2}_{S^{-1}N}.\end{aligned}$$

Proof: By Lemmas 2 and 4, the following results are true:

$$\begin{aligned}\underline{S^{-1}(X_1 \cup X_2)}_{S^{-1}N} &= \underline{S^{-1}X_1 \cup S^{-1}X_2}_{S^{-1}N} \\ &\supseteq \underline{S^{-1}X_1}_{S^{-1}N} \cup \underline{S^{-1}X_2}_{S^{-1}N}\end{aligned}$$

and

$$\begin{aligned}\overline{S^{-1}(X_1 \cup X_2)}_{S^{-1}N} &= \overline{S^{-1}X_1 \cup S^{-1}X_2}_{S^{-1}N} \\ &\supseteq \overline{S^{-1}X_1}_{S^{-1}N} \cup \overline{S^{-1}X_2}_{S^{-1}N}.\end{aligned}$$

Now, we prove that $\overline{S^{-1}X_1 \cup S^{-1}X_2}_{S^{-1}N}$ is a subset of $\overline{S^{-1}X_1}_{S^{-1}N} \cup \overline{S^{-1}X_2}_{S^{-1}N}$. Consider the element

$$\frac{n}{s} \in \overline{S^{-1}X_1 \cup S^{-1}X_2}_{S^{-1}N}.$$

Then,

$$\left[\frac{n}{s}\right]_{\theta_{S^{-1}N}} \cap \left(S^{-1}X_1 \cup S^{-1}X_2\right) \neq \emptyset.$$

It follows that:

$$\left(\left[\frac{n}{s}\right]_{\theta_{S^{-1}N}} \cap S^{-1}X_1\right) \cup \left(\left[\frac{n}{s}\right]_{\theta_{S^{-1}N}} \cap S^{-1}X_2\right) \neq \emptyset.$$

Hence,

$$\frac{n}{s} \in \overline{S^{-1}X_1}_{S^{-1}N} \cup \overline{S^{-1}X_2}_{S^{-1}N}.$$

□

The following Example shows that $\underline{S^{-1}X_1 \cup S^{-1}X_2}_{S^{-1}N}$ is not a subset of $\underline{S^{-1}X_1}_{S^{-1}N} \cup \underline{S^{-1}X_2}_{S^{-1}N}$.

Example 4: Let $R = \mathbb{Z}_8$ and $S = \left\{\bar{1}, \bar{3}\right\}$, then

$$S^{-1}R = \left\{\bar{0}, \frac{\bar{1}}{\bar{1}}, \frac{\bar{1}}{\bar{3}}, \frac{\bar{2}}{\bar{1}}, \frac{\bar{2}}{\bar{3}}, \frac{\bar{4}}{\bar{1}}, \frac{\bar{5}}{\bar{1}}, \frac{\bar{5}}{\bar{3}}\right\}$$

and

$$U\left(S^{-1}R\right) = \left\{\frac{\bar{1}}{\bar{1}}, \frac{\bar{1}}{\bar{3}}, \frac{\bar{5}}{\bar{1}}, \frac{\bar{5}}{\bar{3}}\right\}.$$

The equivalence classes with respect to $\theta_{S^{-1}R}$ are:

$$\begin{aligned}\left[\bar{0}\right]_{\theta_{S^{-1}R}} &= \left\{\bar{0}\right\}, \\ \left[\frac{\bar{2}}{\bar{1}}\right]_{\theta_{S^{-1}R}} &= \left\{\frac{\bar{2}}{\bar{1}}, \frac{\bar{2}}{\bar{3}}\right\}, \\ \left[\frac{\bar{4}}{\bar{1}}\right]_{\theta_{S^{-1}R}} &= \left\{\frac{\bar{4}}{\bar{1}}\right\}\end{aligned}$$

and

$$\left[\frac{\overline{1}}{\overline{3}}\right]_{\theta_{S^{-1}R}} = \left\{\frac{\overline{1}}{\overline{1}}, \frac{\overline{1}}{\overline{3}}, \frac{\overline{5}}{\overline{1}}, \frac{\overline{5}}{\overline{3}}\right\}.$$

Take $X_1 = \{\overline{3}, \overline{4}\}$ *and* $X_2 = \{\overline{5}\}$, *then*

$$S^{-1}X_1 = \left\{\frac{\overline{1}}{\overline{1}}, \frac{\overline{3}}{\overline{1}}, \frac{\overline{4}}{\overline{1}}\right\},$$

$$S^{-1}X_1 \cup S^{-1}X_2 = \left\{\frac{\overline{1}}{\overline{1}}, \frac{\overline{1}}{\overline{3}}, \frac{\overline{4}}{\overline{1}}, \frac{\overline{5}}{\overline{1}}, \frac{\overline{5}}{\overline{3}}\right\}$$

and

$$S^{-1}X_2 = \left\{\frac{\overline{5}}{\overline{1}}, \frac{\overline{5}}{\overline{3}}\right\}.$$

By Lemma 2, it can be obtained:

$$\underline{S^{-1}X_1}_{S^{-1}R} = \left\{\frac{\overline{4}}{\overline{1}}\right\},$$

$$\underline{S^{-1}X_2}_{S^{-1}R} = \emptyset$$

and

$$\underline{S^{-1}X_1 \cup S^{-1}X_2}_{S^{-1}R} = \underline{S^{-1}(X_1 \cup X_2)}_{S^{-1}R} = \left\{\frac{\overline{1}}{\overline{1}}, \frac{\overline{1}}{\overline{3}}, \frac{\overline{4}}{\overline{1}}, \frac{\overline{5}}{\overline{1}}, \frac{\overline{5}}{\overline{3}}\right\}.$$

Therefore,

$$\underline{S^{-1}X_1 \cup S^{-1}X_2}_{S^{-1}R} \nsubseteq \underline{S^{-1}X_1}_{S^{-1}R} \cup \underline{S^{-1}X_2}_{S^{-1}R}.$$

The following result provides the generalized form of [14, Propositions 3.2, 3.12 and Corollary 3.1].

Proposition 4: With the previous notion, suppose that P is an R-submodules of M. Then:

(1)

$$\underline{S^{-1}(X_1 \cap X_2)}_{S^{-1}(N\cap P)} \subseteq \underline{S^{-1}X_1 \cap S^{-1}X_2}_{S^{-1}(N\cap P)} = \underline{S^{-1}X_1}_{S^{-1}N} \cap \underline{S^{-1}X_2}_{S^{-1}P}.$$

(2)

$$\overline{S^{-1}(X_1 \cap X_2)}_{S^{-1}(N\cap P)} \subseteq \overline{S^{-1}X_1 \cap S^{-1}X_2}_{S^{-1}(N\cap P)} \subseteq \overline{S^{-1}X_1}_{S^{-1}N} \cap \overline{S^{-1}X_2}_{S^{-1}P}.$$

Proof: Note that $S^{-1}(X_1 \cap X_2) \subseteq S^{-1}X_1 \cap S^{-1}X_2 \subseteq S^{-1}X_i$, for all $i = 1, 2$ (see Lemma 2). By Lemma 4, the following containments are easy to prove:

$$\underline{S^{-1}(X_1 \cap X_2)}_{S^{-1}(N\cap P)} \subseteq \underline{S^{-1}X_1 \cap S^{-1}X_2}_{S^{-1}(N\cap P)} \subseteq \underline{S^{-1}X_1}_{S^{-1}N} \cap \underline{S^{-1}X_2}_{S^{-1}P}. \quad (2)$$

$$\overline{S^{-1}(X_1 \cap X_2)}_{S^{-1}(N\cap P)} \subseteq \overline{S^{-1}X_1 \cap S^{-1}X_2}_{S^{-1}(N\cap P)} \subseteq \overline{S^{-1}X_1}_{S^{-1}N} \cap \overline{S^{-1}X_2}_{S^{-1}P}.$$

Now, suppose that $x \in \underline{S^{-1}X_1}_{S^{-1}N} \cap \underline{S^{-1}X_2}_{S^{-1}P}$. Then, $x \in S^{-1}N$ and $x \in S^{-1}P$ such that $[x]_{\theta_{S^{-1}N}} \subseteq S^{-1}X_1$ and $[x]_{\theta_{S^{-1}P}} \subseteq S^{-1}X_2$. By Lemma 3 and Corollary 1, it implies that:

$$x \in S^{-1}N \cap S^{-1}P = S^{-1}(N \cap P)$$

such that

$$[x]_{\theta_{S^{-1}(N\cap P)}} \subseteq S^{-1}X_1 \cap S^{-1}X_2.$$

Consequently, we get

$$x \in \underline{S^{-1}X_1 \cap S^{-1}X_2}_{S^{-1}(N\cap P)}.$$

Form Equation (2), the following equality holds:

$$\underline{S^{-1}X_1 \cap S^{-1}X_2}_{S^{-1}(N\cap P)} = \underline{S^{-1}X_1}_{S^{-1}N} \cap \underline{S^{-1}X_2}_{S^{-1}P}.$$

□

In general,

$$\underline{S^{-1}(X_1 \cap X_2)}_{S^{-1}(N\cap P)} \neq \underline{S^{-1}X_1 \cap S^{-1}X_2}_{S^{-1}(N\cap P)}$$

and

$$\overline{S^{-1}(X_1 \cap X_2)}_{S^{-1}(N\cap P)} \neq \overline{S^{-1}X_1 \cap S^{-1}X_2}_{S^{-1}(N\cap P)}.$$

Example 5: Consider the ring R with multiplicative subset S of Example 2. Let $X_1 = \{\overline{0}, \overline{1}\}$ and $X_2 = \{\overline{0}, \overline{2}, \overline{3}, \overline{4}\}$, then $X_1 \cap X_2 = \{\overline{0}\}$. It implies that

$$S^{-1}X_1 = S^{-1}X_2 = S^{-1}R.$$

This proves that

$$S^{-1}X_1 \cap S^{-1}X_2 = S^{-1}R$$

and

$$S^{-1}(X_1 \cap X_2) = \{\overline{0}\}.$$

The following results can be easily deduced:

$$\underline{S^{-1}X_1 \cap S^{-1}X_2}_{S^{-1}R} = \overline{S^{-1}X_1 \cap S^{-1}X_2}_{S^{-1}R} = S^{-1}R$$

and

$$\underline{S^{-1}(X_1 \cap X_2)}_{S^{-1}R} = \overline{S^{-1}(X_1 \cap X_2)}_{S^{-1}R} = \{\overline{0}\}.$$

It is worthy to note that the following Proposition is a generalized form of [14, Propositions 3.8, 3.9 and 3.13].

Proposition 5: With the same notion as in Proposition 4, the following statements hold:

(1) $\underline{S^{-1}(X_1 + X_2)}_{S^{-1}(N+P)} \subseteq \underline{S^{-1}X_1 + S^{-1}X_2}_{S^{-1}(N+P)}$.

(2) $\underline{S^{-1}X_1}_{S^{-1}N} + \underline{S^{-1}X_2}_{S^{-1}P} \subseteq \underline{S^{-1}X_1 + S^{-1}X_2}_{S^{-1}(N+P)}$.

(3) $\overline{S^{-1}(X_1 + X_2)}_{S^{-1}(N+P)} \subseteq \overline{S^{-1}X_1 + S^{-1}X_2}_{S^{-1}(N+P)}$.

Proof: By Lemmas 2 and 4, the claims in (1) and (3) are obvious. Now, we prove (2).

Suppose that

$$x \in \underline{S^{-1}X_1}_{S^{-1}N} + \underline{S^{-1}X_2}_{S^{-1}P},$$

then

$$x = \frac{n}{s} + \frac{p}{t}, \text{ for some } \frac{n}{s} \in \underline{S^{-1}X_1}_{S^{-1}N}$$

and

$$\frac{p}{t} \in \underline{S^{-1}X_2}_{S^{-1}P}.$$

Note that $\frac{n}{s} \in S^{-1}N$ such that $\left[\frac{n}{s}\right]_{\theta_{S^{-1}N}} \subseteq S^{-1}X_1$ and $\frac{p}{t} \in S^{-1}P$ such that $\left[\frac{p}{t}\right]_{\theta_{S^{-1}P}} \subseteq S^{-1}X_2$. Consequently, $x = \frac{n}{s} + \frac{p}{t} \in S^{-1}N + S^{-1}P$ such that $\left[\frac{n}{s}\right]_{\theta_{S^{-1}N}} + \left[\frac{p}{t}\right]_{\theta_{S^{-1}P}} \subseteq S^{-1}X_1 + S^{-1}X_2$ (see Corollary 1). Using Lemma 3(1), we obtain:

$$[x]_{\theta_{S^{-1}N+S^{-1}P}} = \left[\frac{n}{s} + \frac{p}{t}\right]_{\theta_{S^{-1}N+S^{-1}P}} \subseteq S^{-1}X_1 + S^{-1}X_2.$$

Hence, $x \in \underline{S^{-1}X_1 + S^{-1}X_2}_{S^{-1}(N+P)}$. □

In following Example, the inclusions in Proposition 5 are proved strict. Also, it is proved that $\overline{S^{-1}X_1}_{S^{-1}N} + \overline{S^{-1}X_2}_{S^{-1}P} \nsubseteq \overline{S^{-1}X_1 + S^{-1}X_2}_{S^{-1}(N+P)}$.

Example 6: Suppose R and S as used in Example 4. Assume that $I = R$ and $J = \{\bar{0}, \bar{4}\}$ then $I + J = I$. Then:

$$S^{-1}J = \left\{\bar{0}, \frac{\bar{4}}{\bar{1}}\right\}, \quad S^{-1}I = S^{-1}R \text{ and } S^{-1}(I+J) = S^{-1}I.$$

Suppose that $X_1 = \{\bar{2}\}$, $X_2 = \{\bar{3}, \bar{4}\}$, then $X_1 + X_2 = \{\bar{5}, \bar{6}\}$. It implies that:

$$S^{-1}X_1 = \left\{\frac{\bar{2}}{\bar{1}}, \frac{\bar{2}}{\bar{3}}\right\}, \quad S^{-1}X_2 = \left\{\frac{\bar{1}}{\bar{1}}, \frac{\bar{1}}{\bar{3}}, \frac{\bar{4}}{\bar{1}}\right\}$$

$$S^{-1}(X_1 + X_2) = \left\{\frac{\bar{2}}{\bar{1}}, \frac{\bar{2}}{\bar{3}}, \frac{\bar{5}}{\bar{1}}, \frac{\bar{5}}{\bar{3}}\right\}$$

and

$$S^{-1}X_1 + S^{-1}X_2 = \left\{\frac{\bar{1}}{\bar{1}}, \frac{\bar{2}}{\bar{1}}, \frac{\bar{2}}{\bar{3}}, \frac{\bar{1}}{\bar{3}}, \frac{\bar{5}}{\bar{1}}, \frac{\bar{5}}{\bar{3}}\right\}$$

By definition of the approximation spaces, it can be seen that:

$$\begin{aligned}
\underline{S^{-1}X_2}_{S^{-1}J} &= \overline{S^{-1}X_2}_{S^{-1}J} = \left\{\frac{\bar{4}}{\bar{1}}\right\}, \\
\underline{S^{-1}X_1}_{S^{-1}I} &= \overline{S^{-1}X_1}_{S^{-1}I} \\
&= \underline{S^{-1}(X_1 + X_2)}_{S^{-1}I} \\
&= \left\{\frac{\bar{2}}{\bar{1}}, \frac{\bar{2}}{\bar{3}}\right\}, \\
\underline{S^{-1}X_1 + S^{-1}X_2}_{S^{-1}I} &= \overline{S^{-1}X_1 + S^{-1}X_2}_{S^{-1}I} \\
&= \left\{\frac{\bar{1}}{\bar{1}}, \frac{\bar{1}}{\bar{3}}, \frac{\bar{2}}{\bar{1}}, \frac{\bar{2}}{\bar{3}}, \frac{\bar{5}}{\bar{1}}, \frac{\bar{5}}{\bar{3}}\right\}
\end{aligned}$$

and

$$\begin{aligned}
\underline{S^{-1}X_1}_{S^{-1}I} + \underline{S^{-1}X_2}_{S^{-1}J} &= \overline{S^{-1}X_1}_{S^{-1}I} + \overline{S^{-1}X_2}_{S^{-1}J} \\
&= \left\{\frac{\bar{2}}{\bar{1}}, \frac{\bar{2}}{\bar{3}}\right\}.
\end{aligned}$$

Hence, $\underline{S^{-1}(X_1 + X_2)}_{S^{-1}I} \nsupseteq \underline{S^{-1}X_1 + S^{-1}X_2}_{S^{-1}I}$, $\overline{S^{-1}X_1 + S^{-1}X_2}_{S^{-1}I} \nsubseteq \overline{S^{-1}X_1}_{S^{-1}I} + \overline{S^{-1}X_2}_{S^{-1}J}$ *and* $\underline{S^{-1}X_1}_{S^{-1}I} + \underline{S^{-1}X_2}_{S^{-1}J} \nsupseteq \underline{S^{-1}X_1 + S^{-1}X_2}_{S^{-1}I}$.

Now, if we take $X_1 = \{\bar{1}\}$ and continuing with same X_2, then $S^{-1}X_1 = \left\{\frac{\bar{1}}{\bar{1}}, \frac{\bar{1}}{\bar{3}}\right\}$, $X_1 + X_2 = \{\bar{4}, \bar{5}\}$ and hence:

$$S^{-1}(X_1 + X_2) = \left\{\frac{\bar{4}}{\bar{1}}, \frac{\bar{5}}{\bar{1}}, \frac{\bar{5}}{\bar{3}}\right\} \text{ and}$$

$$S^{-1}X_1 + S^{-1}X_2 = \left\{\frac{\bar{2}}{\bar{1}}, \frac{\bar{2}}{\bar{3}}, \frac{\bar{4}}{\bar{1}}, \frac{\bar{5}}{\bar{1}}, \frac{\bar{5}}{\bar{3}}\right\}$$

It follows us that:

$$\overline{S^{-1}(X_1 + X_2)}_{S^{-1}I} = \left\{\frac{\bar{1}}{\bar{1}}, \frac{\bar{1}}{\bar{3}}, \frac{\bar{4}}{\bar{1}}, \frac{\bar{5}}{\bar{1}}, \frac{\bar{5}}{\bar{3}}\right\} \text{ and}$$

$$\overline{S^{-1}X_1 + S^{-1}X_2}_{S^{-1}I} = \left\{\frac{\bar{1}}{\bar{1}}, \frac{\bar{1}}{\bar{3}}, \frac{\bar{2}}{\bar{1}}, \frac{\bar{2}}{\bar{3}}, \frac{\bar{4}}{\bar{1}}, \frac{\bar{5}}{\bar{1}}, \frac{\bar{5}}{\bar{3}}\right\}$$

Therefore, $\overline{S^{-1}(X_1 + X_2)}_{S^{-1}I} \nsupseteq \overline{S^{-1}X_1 + S^{-1}X_2}_{S^{-1}I}$.

Finally assume that $X_1 = \{\bar{1}\}$ and $X_2 = \{\bar{5}\}$, then:

$$S^{-1}X_1 = \left\{\frac{\bar{1}}{\bar{1}}, \frac{\bar{1}}{\bar{3}}\right\}, \quad S^{-1}X_2 = \left\{\frac{\bar{5}}{\bar{1}}, \frac{\bar{5}}{\bar{3}}\right\} \text{ and}$$

$$S^{-1}X_1 + S^{-1}X_2 = \left\{\bar{0}, \frac{\bar{2}}{\bar{1}}, \frac{\bar{2}}{\bar{3}}\right\}$$

Consequently, we have:

$$\overline{S^{-1}X_1 + S^{-1}X_2}_{S^{-1}I} = \left\{\bar{0}, \frac{\bar{2}}{\bar{1}}, \frac{\bar{2}}{\bar{3}}\right\}, \text{ and}$$

$$\overline{S^{-1}X_1}_{S^{-1}I} = \overline{S^{-1}X_2}_{S^{-1}I} = \left\{\frac{\bar{1}}{\bar{1}}, \frac{\bar{1}}{\bar{3}}, \frac{\bar{5}}{\bar{3}}, \frac{\bar{5}}{\bar{1}}\right\}$$

Note that $\frac{\bar{4}}{\bar{1}} = \frac{\bar{1}}{\bar{1}} + \frac{\bar{1}}{\bar{3}} \in \overline{S^{-1}X_1}_{S^{-1}I} + \overline{S^{-1}X_2}_{S^{-1}I}$ but $\frac{\bar{4}}{\bar{1}} \notin \overline{S^{-1}X_1 + S^{-1}X_2}_{S^{-1}I}$.

III. LOWER AND UPPER APPROXIMATIONS VIA $S^{-1}R$-LINEAR MAPS

Let $f : M \longrightarrow M'$ be an $R-$linear map. By Lemma 1, the induced map $S^{-1}f : S^{-1}M \longrightarrow S^{-1}M'$; $\frac{m}{s} \mapsto \frac{f(m)}{s}$ is an $S^{-1}R-$linear map. If M_1 and M_1' are R-submodules of M and M' respectively. Then, it is well-known that $f(M_1)$ and $f^{-1}(M_1')$ are R-submodules of M' and M respectively.

Lemma 6: With the same notion followed and $\frac{m}{s} \in S^{-1}M_1$, the following statement is true:

$$\frac{m'}{s'} \in \left[\frac{m}{s}\right]_{\theta_{S^{-1}M_1}} \Longrightarrow S^{-1}f\left(\frac{m'}{s'}\right) \in \left[S^{-1}f\left(\frac{m}{s}\right)\right]_{\theta_{S^{-1}f(S^{-1}M_1)}}.$$

In addition, if $S^{-1}f$ is one-one, then the converse is also true.

Proof: The following implication can be proved in view of linear property of $S^{-1}f$:

$$\frac{m'}{s'} \in \left[\frac{m}{s}\right]_{\theta_{S^{-1}M_1}} \Longrightarrow S^{-1}f\left(\frac{m'}{s'}\right) \in \left[S^{-1}f\left(\frac{m}{s}\right)\right]_{\theta_{S^{-1}f(S^{-1}M_1)}}.$$

Conversely, assume that $S^{-1}f$ is one-one and $S^{-1}f\left(\frac{m'}{s'}\right) \in \left[S^{-1}f\left(\frac{m}{s}\right)\right]_{\theta_{S^{-1}f(S^{-1}M_1)}}$. By definition of $S^{-1}f$, it implies that:

$$\begin{aligned} S^{-1}f\left(\frac{m}{s}\right) &= \frac{f(m)}{s} = \frac{r}{u}\cdot\frac{f(m')}{s'} = \frac{f(r\cdot m')}{us'} \\ &= S^{-1}f\left(\frac{r\cdot m'}{us'}\right), \end{aligned}$$

for some $\frac{r}{u} \in U\left(S^{-1}R\right)$.

Since $S^{-1}f$ is injective, it follows that $\frac{m}{s} = \frac{r}{u}\cdot\frac{m'}{s'}$. This proves the required result. □

The following Example shows that the converse of Lemma 6 is not true, if $S^{-1}f$ is not one-one.

Example 7: Let us consider $R = \mathbb{Z}_4$. Define $f : R \longrightarrow R$ as follows:

$$f(x) = \begin{cases} \bar{0}, & \text{if } x = \bar{0}, \bar{2} \\ \bar{2}, & \text{if } x = \bar{1}, \bar{3} \end{cases}$$

Then, f is an $R-$linear map. Take $S = \{\bar{1}, \bar{3}\}$, then $S^{-1}R = \left\{\bar{0}, \frac{\bar{1}}{\bar{1}}, \frac{\bar{2}}{\bar{1}}, \frac{\bar{3}}{\bar{1}}\right\}$. By definition of $S^{-1}f$, we obtain:

$$S^{-1}f\left(\frac{r}{s}\right) = \begin{cases} \bar{0}, & \text{if } \frac{r}{s} = \bar{0}, \frac{\bar{2}}{\bar{1}} \\ \frac{\bar{2}}{\bar{1}}, & \text{if } \frac{r}{s} = \frac{\bar{1}}{\bar{1}}, \frac{\bar{3}}{\bar{1}} \end{cases}$$

Since, $S^{-1}f\left(S^{-1}R\right) = \left\{\bar{0}, \frac{\bar{2}}{\bar{1}}\right\}$. By Example 1, it follows that:

$$S^{-1}f\left(\bar{0}\right) \in \left[S^{-1}f\left(\frac{\bar{2}}{\bar{1}}\right)\right]_{\theta_{S^{-1}f(S^{-1}R)}} \quad \text{and } \bar{0} \notin \left[\frac{\bar{2}}{\bar{1}}\right]_{\theta_{S^{-1}R}}.$$

Example 8: Since there exists a one-one ring homomorphism $f : \mathbb{Z}_5 \to \mathbb{Z}_{10}; \bar{x} \mapsto \widehat{6x}$. Then $\mathbb{Z}_{10}$ becomes a $\mathbb{Z}_5-$module under the scalar multiplication defined as follows:

$$\bar{r}.\hat{x} = f(\bar{r}).\hat{x} = \widehat{6rx}, \quad \text{for all } \bar{r} \in \mathbb{Z}_5 \text{ and } \hat{x} \in \mathbb{Z}_{10}.$$

Assume that $S = \{\bar{1}, \bar{2}, \bar{3}, \bar{4}\}$. Let $U = S^{-1}\mathbb{Z}_5 = \{v_1, v_2, v_3, v_4, v_5\}$ and $U' = S^{-1}\mathbb{Z}_{10} = \{u_1, u_2, u_3, u_4, u_5\}$ be two universe sets of stores, where

$$v_1 = \bar{0},\ v_2 = \frac{\bar{1}}{\bar{1}},\ v_3 = \frac{\bar{1}}{\bar{2}},\ v_4 = \frac{\bar{1}}{\bar{3}},\ v_5 = \frac{\bar{1}}{\bar{4}} \text{ and}$$
$$u_1 = \hat{0},\ u_2 = \frac{\hat{1}}{\bar{1}},\ u_3 = \frac{\hat{1}}{\bar{2}},\ u_4 = \frac{\hat{1}}{\bar{3}},\ u_5 = \frac{\hat{1}}{\bar{4}}.$$

Suppose that $a, b \in U$ (resp. $a', b' \in U'$) have the same value of attributes, if $(a, b) \in \theta_U$ (resp. $\left(a', b'\right) \in \theta_{U'}$). Let $A = \{E, Q, L, P\}$ be the subset of attributes, where $E =$ Empowerment of sales personnel, $Q =$ Perceived quality of merchandisers, $L =$ High traffic location, $P =$ Store profit or loss. Consider the following information system $\left(U', A\right)$: Since, $S^{-1}f$ is one-one. By Lemma 6, the following information system (U, A) can be deduced from Table 1:

TABLE 1. Information system (U', A).

	E	Q	L	P
u_1	*med.*	*good*	*yes*	*loss*
u_2	*high*	*good*	*no*	*profit*
u_3	*high*	*good*	*no*	*profit*
u_4	*high*	*good*	*no*	*profit*
u_5	*high*	*good*	*no*	*profit*

TABLE 2. Information system (U, A).

	E	Q	L	P
v_1	*med.*	*good*	*yes*	*loss*
v_2	*high*	*good*	*no*	*profit*
v_3	*high*	*good*	*no*	*profit*
v_4	*high*	*good*	*no*	*profit*
v_5	*high*	*good*	*no*	*profit*

Theorem 1: With the above notion, suppose that $X \subseteq M$, $X' \subseteq M'$ and $N = \left(S^{-1}f\right)^{-1}\left(S^{-1}M_1'\right)$. Then the following implications are true:

$$\frac{m}{s} \in \overline{S^{-1}X}_N \Longrightarrow S^{-1}f\left(\frac{m}{s}\right) \in \overline{S^{-1}f\left(S^{-1}X\right)}_{S^{-1}M_1'}$$
$$\frac{m}{s} \in \overline{\left(S^{-1}f\right)^{-1}\left(S^{-1}X'\right)}_N \Longrightarrow S^{-1}f\left(\frac{m}{s}\right) \in \overline{S^{-1}X'}_{S^{-1}M_1'}$$

Proof: Let $\frac{m}{s} \in \overline{S^{-1}X}_N$. Then, $\frac{m}{s} \in N$ such that $\left[\frac{m}{s}\right]_{\theta_N} \cap S^{-1}X \neq \emptyset$. There exists $\frac{m'}{s'} \in N$ such that $\frac{m'}{s'} \in \left[\frac{m}{s}\right]_{\theta_N}$ and $\frac{m'}{s'} \in S^{-1}X$. By Lemma 6, $S^{-1}f\left(\frac{m'}{s'}\right) \in \left[S^{-1}f\left(\frac{m}{s}\right)\right]_{\theta_{S^{-1}f(N)}}$ and $S^{-1}f\left(\frac{m'}{s'}\right) \in S^{-1}f\left(S^{-1}X\right)$. Since, $S^{-1}f\left(\frac{m}{s}\right) \in S^{-1}M_1'$. By Lemma 3(4), it follows that:

$$\left[S^{-1}f\left(\frac{m}{s}\right)\right]_{\theta_{S^{-1}f(N)}} = \left[S^{-1}f\left(\frac{m}{s}\right)\right]_{\theta_{S^{-1}M_1'}} \quad \text{and}$$
$$S^{-1}f\left(\frac{m'}{s'}\right) \in \left[S^{-1}f\left(\frac{m}{s}\right)\right]_{\theta_{S^{-1}M_1'}} \cap S^{-1}f\left(S^{-1}X\right).$$

Hence, $S^{-1}f\left(\frac{m}{s}\right) \in \overline{S^{-1}f\left(S^{-1}X\right)}_{S^{-1}M_1'}$. This proves the first implication. The second implication can be proved on the same lines. □

In following Theorem, the converse of above result is proved.

Theorem 2: With the same assumptions as in Theorem 1, we have:

$$S^{-1}f\left(\frac{m}{s}\right) \in \overline{S^{-1}X'}_{S^{-1}M_1'} \Longrightarrow \frac{m}{s} \in \overline{\left(S^{-1}f\right)^{-1}\left(S^{-1}X'\right)}_N$$

In addition, if $S^{-1}X$ is an additive subgroup of $S^{-1}M$ with $Ker\left(S^{-1}f\right) \subseteq S^{-1}X$, then

$$S^{-1}f\left(\frac{m}{s}\right) \in \overline{S^{-1}f\left(S^{-1}X\right)}_{S^{-1}M_1'} \Longrightarrow \frac{m}{s} \in \overline{S^{-1}X}_N$$

where $N = \left(S^{-1}f\right)^{-1}\left(S^{-1}M_1'\right)$.

Proof: Let $S^{-1}f\left(\frac{m}{s}\right) \in \overline{S^{-1}X'}_{S^{-1}M_1'}$. By definition of the upper approximation $S^{-1}f\left(\frac{m}{s}\right) \in S^{-1}M_1'$ such that $\left[S^{-1}f\left(\frac{m}{s}\right)\right]_{\theta_{S^{-1}M_1'}} \cap S^{-1}X' \neq \emptyset$. It implies that $\frac{n}{t} \in$

$\left[S^{-1}f\left(\frac{m}{s}\right)\right]_{\theta_{S^{-1}M_1'}}$ and $\frac{n}{t} \in S^{-1}X'$, for some $\frac{n}{t} \in S^{-1}M_1'$. Then,

$$\frac{n}{t} = \frac{r}{u}S^{-1}f\left(\frac{m}{s}\right) = S^{-1}f\left(\frac{rm}{us}\right), \quad \text{for some } \frac{r}{u} \in U\left(S^{-1}R\right).$$

Hence, $\frac{rm}{us} \in \left(S^{-1}f\right)^{-1}\left(S^{-1}X'\right)$. Note that $\frac{m}{s} \in N$ and $\frac{rm}{us} \in \left[\frac{m}{s}\right]_{\theta_N}$, where $N = \left(S^{-1}f\right)^{-1}\left(S^{-1}M_1'\right)$. Thus, $\frac{rm}{us} \in \left[\frac{m}{s}\right]_{\theta_N} \cap \left(S^{-1}f\right)^{-1}\left(S^{-1}X'\right)$. This proves the first implication.

Now, assume that $S^{-1}X$ is an additive subgroup of $S^{-1}M$ with $Ker\left(S^{-1}f\right) \subseteq S^{-1}X$ and $S^{-1}f\left(\frac{m}{s}\right) \in \overline{S^{-1}f\left(S^{-1}X\right)}_{S^{-1}M_1'}$, then $S^{-1}f\left(\frac{m}{s}\right) \in S^{-1}M_1'$ such that $\left[S^{-1}f\left(\frac{m}{s}\right)\right]_{\theta_{S^{-1}M_1'}} \cap S^{-1}f\left(S^{-1}X\right) \neq \emptyset$. There exists $\frac{n}{t} \in S^{-1}M_1'$ such that $\frac{n}{t} \in \left[S^{-1}f\left(\frac{m}{s}\right)\right]_{\theta_{S^{-1}M_1'}}$ and $\frac{n}{t} \in S^{-1}f\left(S^{-1}X\right)$. So,

$$\frac{n}{t} = \frac{r}{u}S^{-1}f\left(\frac{m}{s}\right) = S^{-1}f\left(\frac{rm}{us}\right), \quad \text{for some } \frac{r}{u} \in U\left(S^{-1}R\right).$$

Assume that $S^{-1}f\left(\frac{rm}{us}\right) = S^{-1}f\left(\frac{x}{t}\right)$, for some $\frac{x}{t} \in S^{-1}X$. It implies that $\frac{rm}{us} - \frac{x}{t} \in Ker\left(S^{-1}f\right) \subseteq S^{-1}X$. By the assumption on $S^{-1}X$, we have $\frac{rm}{us} \in \left[\frac{m}{s}\right]_{\theta_N} \cap S^{-1}X$. Hence, $\frac{m}{s} \in \overline{S^{-1}X}_N$. □

An illustration of the second implication in Theorem 2 is made in following Example.

Example 9: Let $R = M = \mathbb{Z}_{24}$, $M' = \mathbb{Z}_{10}$. Define a map $f : \mathbb{Z}_{24} \to \mathbb{Z}_{10}; \bar{x} \mapsto \widehat{5x}$, i.e.,

$$f(x) = \begin{cases} \widehat{0}, & \text{if } \bar{x} = \bar{0}, \bar{2}, \bar{4}, \bar{6}, \ldots, \overline{22} \\ \widehat{5}, & \text{if } \bar{x} = \bar{1}, \bar{3}, \bar{5}, \bar{7}, \ldots, \overline{23} \end{cases}$$

Then f is a ring homomorphism. Hence, $\mathbb{Z}_{10}$ is a $\mathbb{Z}_{24}$−module over the scalar multiplication defined as:

$$\bar{r}.\widehat{x} = f(\bar{r}).\widehat{x} = \widehat{5r}.\widehat{x}$$

for all $\bar{r} \in \mathbb{Z}_{24}, \widehat{x} \in \mathbb{Z}_{10}$. For $S = \left\{\bar{1}, \bar{9}\right\}$, we have:

$$S^{-1}\mathbb{Z}_{24} = \left\{\bar{0}, \frac{\bar{1}}{\bar{1}}, \frac{\bar{2}}{\bar{1}}, \frac{\bar{3}}{\bar{1}}, \frac{\bar{4}}{\bar{1}}, \frac{\bar{5}}{\bar{1}}, \frac{\bar{6}}{\bar{1}}, \frac{\bar{7}}{\bar{1}}\right\} \text{ and } S^{-1}\mathbb{Z}_{10} = \left\{\widehat{0}, \frac{\widehat{1}}{\bar{1}}\right\}$$

By Lemma 1, the map $S^{-1}f : S^{-1}\mathbb{Z}_{24} \to S^{-1}\mathbb{Z}_{10}$ is defined as follows:

$$S^{-1}f\left(\frac{m}{s}\right) = \begin{cases} \widehat{0}, & \text{if } \frac{m}{s} = \bar{0}, \frac{\bar{2}}{\bar{1}}, \frac{\bar{4}}{\bar{1}}, \frac{\bar{6}}{\bar{1}} \\ \frac{\widehat{1}}{\bar{1}}, & \text{if } \frac{m}{s} = \frac{\bar{1}}{\bar{1}}, \frac{\bar{3}}{\bar{1}}, \frac{\bar{5}}{\bar{1}}, \frac{\bar{7}}{\bar{1}} \end{cases}$$

for all $\frac{m}{s} \in S^{-1}\mathbb{Z}_{24}$. Let $M_1' = \left\{\widehat{0}, \widehat{2}, \widehat{4}, \widehat{8}\right\}$, then $S^{-1}M_1' = \left\{\widehat{0}\right\}$ and $\left(S^{-1}f\right)^{-1}\left(S^{-1}M_1'\right) = \left\{\bar{0}, \frac{\bar{2}}{\bar{1}}, \frac{\bar{4}}{\bar{1}}, \frac{\bar{6}}{\bar{1}}\right\} = N$ (say). Since $U\left(S^{-1}\mathbb{Z}_{24}\right) = \left\{\frac{\bar{1}}{\bar{1}}, \frac{\bar{3}}{\bar{1}}, \frac{\bar{5}}{\bar{1}}, \frac{\bar{7}}{\bar{1}}\right\}$. Then,

$$\left[\widehat{0}\right]_{\theta_{S^{-1}M_1'}} = \left\{\widehat{0}\right\}, \quad \left[\bar{0}\right]_{\theta_N} = \left\{\bar{0}\right\}, \quad \left[\frac{\bar{4}}{\bar{1}}\right]_{\theta_N} = \left\{\frac{\bar{4}}{\bar{1}}\right\} \text{ and}$$
$$\left[\frac{\bar{2}}{\bar{1}}\right]_{\theta_N} = \left[\frac{\bar{6}}{\bar{1}}\right]_{\theta_N} = \left\{\frac{\bar{2}}{\bar{1}}, \frac{\bar{6}}{\bar{1}}\right\}$$

Assume that $X = \left\{\bar{1}, \bar{2}, \bar{3}\right\}$, then $S^{-1}X = \left\{\frac{\bar{1}}{\bar{1}}, \frac{\bar{2}}{\bar{1}}, \frac{\bar{3}}{\bar{1}}\right\}$ and $\left(S^{-1}f\right)\left(S^{-1}X\right) = \left\{\widehat{0}, \frac{\widehat{1}}{\bar{1}}\right\}$. Hence:

$$\overline{S^{-1}X}_N = \left\{\frac{\bar{2}}{\bar{1}}, \frac{\bar{6}}{\bar{1}}\right\} \text{ and } \overline{\left(S^{-1}f\right)\left(S^{-1}X\right)}_{S^{-1}M_1'} = \left\{\widehat{0}\right\}$$

Note that $S^{-1}f\left(\bar{0}\right) = S^{-1}f\left(\frac{\bar{4}}{\bar{1}}\right) = \widehat{0} \in \overline{\left(S^{-1}f\right)\left(S^{-1}X\right)}_{S^{-1}M_1'}$ and $\bar{0}, \frac{\bar{4}}{\bar{1}} \notin \overline{S^{-1}X}_N$.

Theorem 3: Suppose that $\emptyset \neq X' \subseteq M'$, M_1' is a submodule of M' and $N = \left(S^{-1}f\right)^{-1}\left(S^{-1}M_1'\right)$. Then the following claims are true:

$$\frac{m}{s} \in \underline{\left(S^{-1}f\right)^{-1}\left(S^{-1}X'\right)}_N \Longrightarrow S^{-1}f\left(\frac{m}{s}\right) \in \underline{S^{-1}X'}_{S^{-1}M_1'}.$$
$$\frac{m}{s} \in \underline{S^{-1}X}_N \Longrightarrow S^{-1}f\left(\frac{m}{s}\right) \in \underline{S^{-1}f\left(S^{-1}X\right)}_{S^{-1}M_1'}.$$

Proof: Let $\frac{m}{s} \in N$. Note that $S^{-1}f\left(\frac{m}{s}\right) \in S^{-1}M_1'$. If $\frac{n}{t}$ is an arbitrary element of $\left[S^{-1}f\left(\frac{m}{s}\right)\right]_{\theta_{S^{-1}M_1'}}$. Then there exists $\frac{r}{u} \in U\left(S^{-1}R\right)$ such that

$$\frac{n}{t} = \frac{r}{u} \cdot S^{-1}f\left(\frac{m}{s}\right) = S^{-1}f\left(\frac{rm}{us}\right). \tag{3}$$

Firstly, suppose that $\frac{m}{s} \in \underline{\left(S^{-1}f\right)^{-1}\left(S^{-1}X'\right)}_N$. Since, $\frac{rm}{us} \in \left[\frac{m}{s}\right]_{\theta_N} \subseteq \left(S^{-1}f\right)^{-1}\left(S^{-1}X'\right)$. It follows that $\frac{n}{t} = S^{-1}f\left(\frac{rm}{us}\right) \in S^{-1}X'$ (see Equation 3). Thus, $\left[S^{-1}f\left(\frac{m}{s}\right)\right]_{\theta_{S^{-1}M_1'}} \subseteq S^{-1}X'$. The proof of the other claim is on the similar lines. □

In the next result, the converse of Theorem 3 is proved.

Theorem 4: With the previous notion, the following statement holds:

$$S^{-1}f\left(\frac{m}{s}\right) \in \underline{S^{-1}X'}_{S^{-1}M_1'} \Longrightarrow \frac{m}{s} \in \underline{\left(S^{-1}f\right)^{-1}\left(S^{-1}X'\right)}_N$$

In addition, if $S^{-1}X$ is an additive subgroup of $S^{-1}M$ with $Ker\left(S^{-1}f\right) \subseteq S^{-1}X$, then:

$$S^{-1}f\left(\frac{m}{s}\right) \in \underline{S^{-1}f\left(S^{-1}X\right)}_{S^{-1}M_1'} \Longrightarrow \frac{m}{s} \in \underline{S^{-1}X}_N$$

where $N = \left(S^{-1}f\right)^{-1}\left(S^{-1}M_1'\right)$.

Proof: Let $\frac{m}{s} \in N$. Suppose that $\frac{m'}{s'} \in \left[\frac{m}{s}\right]_{\theta_N}$ is an arbitrary element. From Lemma 6, it follows that $S^{-1}f\left(\frac{m'}{s'}\right) \in \left[S^{-1}f\left(\frac{m}{s}\right)\right]_{\theta_{S^{-1}f(N)}}$. Since, $S^{-1}f\left(\frac{m}{s}\right) \in S^{-1}M_1' \cap S^{-1}f\left(N\right)$. It implies that

$$S^{-1}f\left(\frac{m'}{s'}\right) \in \left[S^{-1}f\left(\frac{m}{s}\right)\right]_{\theta_{S^{-1}f(N)}} = \left[S^{-1}f\left(\frac{m}{s}\right)\right]_{\theta_{S^{-1}M_1'}}, \tag{4}$$

see Lemma 3. Now, assume that $S^{-1}f\left(\frac{m}{s}\right) \in \underline{S^{-1}X'}_{S^{-1}M_1'}$. By definition of the lower approximation, $\left[S^{-1}f\left(\frac{m}{s}\right)\right]_{\theta_{S^{-1}M_1'}}$

is a subset of $S^{-1}X'$. By Equation (4), it follows that $\frac{m'}{s'} \in \left(S^{-1}f\right)^{-1}\left(S^{-1}X'\right)$. This proves that

$$\left[\frac{m}{s}\right]_{\theta_N} \subseteq \left(S^{-1}f\right)^{-1}\left(S^{-1}X'\right) \text{ and } \frac{m}{s} \in \underline{\left(S^{-1}f\right)^{-1}\left(S^{-1}X'\right)}_N.$$

The second implication can be proved using same methodology as used in proof of Theorem 2. □

Theorem 5: Let $\emptyset \neq X \subseteq M$ and $\emptyset \neq X' \subseteq M'$. If M_1 is a submodule of M and $\frac{m}{s} \in S^{-1}M_1$, then following statements hold:

$$\begin{aligned} \frac{m}{s} &\in \overline{\left(S^{-1}f\right)^{-1}\left(S^{-1}X'\right)}_{S^{-1}M_1} \\ &\Longleftrightarrow S^{-1}f\left(\frac{m}{s}\right) \in \overline{S^{-1}X'}_{S^{-1}f\left(S^{-1}M_1\right)} \\ \frac{m}{s} &\in \overline{S^{-1}X}_{S^{-1}M_1} \\ &\Longrightarrow S^{-1}f\left(\frac{m}{s}\right) \in \overline{S^{-1}f\left(S^{-1}X\right)}_{S^{-1}f\left(S^{-1}M_1\right)} \end{aligned}$$

Further, if $S^{-1}X$ is an additive subgroup of $S^{-1}M$ with Ker $\left(S^{-1}f\right) \subseteq S^{-1}X$. Then, the converse of second statement can also be proved.

Proof: This proof is analogous to the proof of Theorems 1 and 2. □

Theorem 6: With the same notion as in Theorem 5, the following assertions hold:

$$\begin{aligned} \frac{m}{s} &\in \underline{\left(S^{-1}f\right)^{-1}\left(S^{-1}X'\right)}_{S^{-1}M_1} \\ &\Longleftrightarrow S^{-1}f\left(\frac{m}{s}\right) \in \underline{S^{-1}X'}_{S^{-1}f\left(S^{-1}M_1\right)} \\ \frac{m}{s} &\in \underline{S^{-1}X}_{S^{-1}M_1} \\ &\Longrightarrow S^{-1}f\left(\frac{m}{s}\right) \in \underline{S^{-1}f\left(S^{-1}X\right)}_{S^{-1}f\left(S^{-1}M_1\right)} \end{aligned}$$

Moreover, if $S^{-1}X$ is an additive subgroup of $S^{-1}M$ with Ker $\left(S^{-1}f\right) \subseteq S^{-1}X$. Then, the converse of second assertion hold.

Proof: This proof is parallel to the proof of Theorems 3 and 4. □

IV. CONCLUSION

Rough sets are technical tool for modeling an incomplete information system. The module theory is largely employed in Algebra, Geometry and Physics. In the present work, the lower and upper approximations between modules of fractions are established with respect to its submodules. Some significant properties related to these notions are studied with illustrative examples. The rough modules of fractions presented here may contribute significantly in advancement and implementation of this theory. We refer [28] for insight about applications.

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ZHIHUA CHEN received the Ph.D. degree in system analysis and integration from the Huazhong University of Science and Technology (HUST), Wuhan, China, in 2009. Since 2017, she has been a Professor with the Institute of Computing Science and Technology, Guangzhou University. Her research interests include the modeling, control, and optimization strategy of complex systems.

SABA AYUB received the M.Sc. degree in mathematics from the Government College University and the Ph.D. degree from Quaid-i-Azam University, Islamabad, Pakistan. She has published over 20 research articles in different international journals. Her main research interest includes fuzzy soft set theory.

WAQAS MAHMOOD received the Ph.D. degree in mathematics from the Abdus Salam School of Mathematical Sciences, Government College University, Lahore, Pakistan. During his studies, he was funded by the Higher Education Commission of Pakistan. He is currently an Assistant Professor with the Quaid-i-Azam University, Islamabad, Pakistan. He is also a Mathematician from Pakistan. He has published over 30 research articles in different international journals. His research interests include algebra and fuzzy set theory.

ABID MAHBOOB was born in Okara, Pakistan. He graduated at Okara. He received the master's degree from Lahore, Pakistan, the M.Phil. degree in mathematics from the Government College University Lahore, and the Ph.D. degree in algebra from the University of Science and Technology of China, Hefei, China, in 2012. He joined the University of Education as a Tutor in mathematics. In 2015, he joined the University of Education, Lahore, Pakistan, where he has been an Assistant Professor Coordinator with the Department of Mathematics.

CHAHN YONG JUNG received the Ph.D. degree in quantitative management from Pusan National University, Pusan, South Korea. He is currently a Professor with Gyeongsang National University, Jinju, South Korea. He has published over 50 research articles in different international journals. His research interest includes business mathematics.