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Efficient Homogeneously Weighted Moving Average Chart for Monitoring Process Mean Using an Auxiliary Variable

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ABSTRACT In this paper, we propose an efficient control chart for monitoring small shifts in a process mean for scenarios where the process variable is observed with a correlated auxiliary variable. The proposed chart, called an auxiliary homogeneously weighted moving average (AHWMA) chart, is a homogeneously weighted moving average type control chart that uses both the process and auxiliary variables in the form of a regression estimator to provide an efficient and unbiased estimate of the mean of the process variable. We provide the design structure of the chart and examine its performance in terms of its run length properties. Using a simulation study, we compare its run length performance with several existing methods for detecting a small shift in the process mean. Our simulation results show that the proposed chart is more efficient in detecting a small shift in the process mean than its competitors. We provide a detailed study of the chart's robustness to non-normal distributions and show that the chart may also be designed to be less sensitive to non-normality. We give some recommendations on the application of the chart when the process parameters are unknown and provide an example to show the implementation of the proposed new technique.

INDEX TERMS Auxiliary variable, average run length, control chart, parameter estimation, robustness.

I. INTRODUCTION

Monitoring programs are designed to detect unnatural changes in process variables for a wide variety of applications, particularly in industrial and manufacturing settings. Control charts are popular tools for tracking processes of interest, ensuring they are kept in control by monitoring essential quality characteristics [1]. To date, several univariate control charts have been proposed in statistical process control (SPC) literature; they are classified into (*i*) memory-less control charts and (*ii*) memory-type control charts for monitoring large and small-to-moderate shifts in the process, respectively. For example, the Shewhart chart is a memory-less control chart that uses only the current process information and not the past behavior of the process. It is very effective for detecting large shifts in the process mean (i.e., $\delta \geq 2$, where δ is the size of the shift in standard deviation units [2]). The homogeneously weighted moving average (HWMA) control chart by [3] is a memory-type chart proposed for efficient monitoring of small (i.e., $\delta \leq 0.5$) to moderate (i.e., $0.5 < \delta < 2$) shifts in the process mean. Other memory-type charts include the EWMA chart by [4], the CUSUM chart by [5], and the mixed EWMA-CUSUM chart proposed by [6].

These univariate classical charts are widely used in most of today's industries; their attractiveness is motivated by the simplicity of their construction, implementation, and interpretation, as well as their prompt detection of small, moderate, or large shifts in a process mean. These techniques have been implemented by [7] to monitor the quality of garments produced on the sewing floor, by [8] to monitor and control steam boiler generation for vacuum degassing processes, and by [9] to evaluate critical control point hygiene data. Also, see [10]–[13] for some other industrial applications of these classical charts.

Several applications of classical charts focus on monitoring the process in situations where the process variable is independent of other variables; however, in some cases,

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the process variable may be observed along with another correlated auxiliary variable. The concept of using supplemental information to provide an efficient estimate of a population parameter is popular in the field of survey sampling [14]. Several researchers have studied and recommended the use of auxiliary variables in the monitoring of a process variable of interest, and have proposed a variety of different control chart tools for this purpose.

For example, [15] proposed a regression control chart, while [16] proposed a cause-selecting control chart. Recently, [17] proposed a Shewhart-type chart in the form of a regression-based estimator, called a V_r chart, for monitoring process variability. He compared the proposed V_r chart with some other existing charts (specifically, R, S and S^2 charts for the same purpose), and showed that the V_r chart was effective in detecting moderate to large shifts in the process variability under certain conditions on the correlation between the process variable and auxiliary variable. Similarly, a Shewhart-type control chart using a regression-based estimator (M_r chart) for monitoring a process mean (proposed by [18]) was shown to be more powerful at detecting shifts in the process mean. This work was later extended to an EWMA chart for detecting small-to-moderate changes in the process mean under different correlation structures between the process and auxiliary variables (see [19]–[24]).

Here, we propose a more efficient control chart for monitoring the process mean when the process variable is observed along with an auxiliary variable. The proposed chart, called an auxiliary homogeneously weighted moving average (AHWMA) chart, is an HWMA-type control chart that uses both the deviation of the process mean from its target value (known apriori or estimated from historical reference samples), as well as a regression estimator for the process mean provided through its relationship (or estimated relationship) with an auxiliary variable with which it is known to be correlated.

The rest of the article is organized as follows: in Section II, we outline the structure of the chart. Section III compares the AHWMA chart (run length) performance in detecting a small shift in the process mean with several other existing charts. Section IV gives a detailed study of the chart's robustness to non-normality. We give recommendations regarding the application of the chart when the process parameters are unknown in Section V. Section VI also provides an example to demonstrate practical implementation of the chart, followed by a conclusion and discussion in Section VII.

II. THE AHWMA CONTROL CHART

Consider a control chart based on observations z_{ij} of the quality characteristics Z_{ij}, for each of $i = 1, \ldots, m$ time-points and $j = 1, \ldots, n$ sampling units per time-point (i.e., n is the sample size). Assume that these quality characteristics (Z_{ij}) are identically distributed as normal random variables with a known in-control mean (μ_Z) and standard deviation (σ_Z), i.e., $Z_{ij} \sim N(\mu_Z, \sigma_Z^2)$ and represents the main process variable. The HWMA statistic, H_i (in Equation (1)), at time-point i, gives a specific weight to the current sample and the remaining weight is equally distributed among the previous samples, and is given by:

$$H_i = w\bar{z}_i + (1-w)\bar{\bar{z}}_{i-1} \tag{1}$$

where $\bar{z}_i$ is the sample average for the *ith* sample, and w is a smoothing constant (also called the sensitivity parameter) selected such that $0 < w \leq 1$. The HWMA structure becomes the Shewhart plotting structure whenever $w = 1$. $\bar{\bar{z}}_{i-1}$ is the average of the sample means of all of the previous samples (i.e., up to and including the $(i-1)th$ sample), and is given by $\bar{\bar{z}}_{i-1} = \frac{1}{n}\sum_{k=1}^{i-1}\bar{z}_k$. The mean and variance of the HWMA statistic in Equation (1) are given by $\mu_H = \mu_Z$, and

$$\sigma_{H_i}^2 = \begin{cases} \frac{1}{n}w^2\sigma_Z^2 & \text{if } i = 1 \\ \frac{1}{n}\left(w^2\sigma_Z^2 + (1-w)^2\frac{\sigma_Z^2}{i-1}\right) & \text{if } i > 1 \end{cases} \tag{2}$$

where $\mu_H = \mu_Z$ and σ_Z^2 are the mean and variance of the normally distributed random variable Z [3].

Let an auxiliary variable, Y_{ij}, be correlated with the main variable of interest, Z_{ij}, with correlation ρ. We assume the observations of Z_{ij} and Y_{ij} are observed in pairs from a bivariate normal distribution, given by $(Z, Y) \sim N_2(\mu_Z, \mu_Y, \sigma_Z^2, \sigma_Y^2, \rho)$, where N_2 is the bivariate normal distribution, and μ_Y and σ_Y^2 are the population mean and variance of Y, respectively. We assume the linear relationship between the variables can be modeled using linear least squares obtained by adjusting the process mean at time i, z_i, to reflect its known relationship with the auxiliary variable. This yields the regression-informed estimator (i.e., R_i) for the process mean given by:

$$R_i = \bar{z}_i + b(\mu_Y - \bar{y}_i) \tag{3}$$

where b (given as $b = \frac{\rho\sigma_Z}{\sigma_Y}$) is the slope of the regression line; given as the change in the process variable, Z, due to a unit change in the auxiliary variable, Y [14]. The mean and variance of R are given as:

$$\mu_R = \mu_Z \text{ and } \sigma_R^2 = \frac{\sigma_Z^2}{n}(1-\rho^2), \tag{4}$$

respectively.

Using Equation (3), the plotting statistic (T_i) of the AHWMA chart is given as:

$$T_i = wR_i + (1-w)\bar{R}_{i-1} \tag{5}$$

where w is the smoothing parameter of the chart (selected such that $0 \leq w \leq 1$), R_i is the regression-informed estimate of the process variable, given in Equation (3) for the *ith* sample, and $\bar{R}_{i-1}$ is the average of the sample means of all of the previous samples (i.e., up to and including the $(i-1)th$ sample) of the plotting statistic, and is given as $\bar{R}_{i-1} = \frac{1}{n}\sum_{k=1}^{i-1}R_k$. The mean and variance of the plotting statistic in Equation (5) are given as $\mu_H = \mu_Z$ (also called the center line of the AHWMA chart), and

$$\sigma^2_{T_i} = \begin{cases} \frac{(1-\rho^2)}{n} w^2\sigma^2_Z & \text{if } i = 1 \\ \frac{(1-\rho^2)}{n}\left(w^2\sigma^2_Z + (1-w)^2\frac{\sigma^2_Z}{i-1}\right) & \text{if } i > 1, \end{cases} \tag{6}$$

respectively. The time varying lower (L_i) and upper (U_i) control chart limits of the plotting statistic given in Equation (5) are given as:

$$L_i = \begin{cases} \mu_Z - C\sigma_Z\sqrt{\frac{w^2}{n}(1-\rho^2)} & \text{if } i = 1 \\ \mu_Z - C\sigma_Z\sqrt{\left(\frac{w^2}{n} + \frac{(1-w)^2}{n(i-1)}\right)(1-\rho^2)} & \text{if } i > 1 \end{cases} \tag{7}$$

and

$$U_i = \begin{cases} \mu_Z + C\sigma_Z\sqrt{\frac{w^2}{n}(1-\rho^2)} & \text{if } i = 1 \\ \mu_Z + C\sigma_Z\sqrt{\left(\frac{w^2}{n} + \frac{(1-w)^2}{n(i-1)}\right)(1-\rho^2)} & \text{if } i > 1, \end{cases} \tag{8}$$

respectively, where, C determines the width of the control limits; the values of C and w are chosen to achieve a desired in-control average run length (*ARL*) for the chart. *ARL* is the average number of plotted samples on the control chart before a shift is detected. We provide R-code [25] (in the supplementary material) which practitioners can use to obtain the value of C, given w, that fix the in-control *ARL* of the chart to a desired value. We adopted the ARL numerics algorithm for the EWMA chart [26]; implemented in the spc (R) package [27], to obtain an arbitrary start value (C^{start}) of the AHWMA chart limit, and used a binary search algorithm to determine C for the chart.

III. PERFORMANCE ASSESSMENTS AND COMPARISONS

A. PERFORMANCE ASSESSMENTS

Here, we provide a comprehensive assessment of the AHWMA chart in detecting a shift in the process mean in terms of the chart's *ARL* and standard deviation of run length (SDRL). The value of the *ARL* when a process is in control is denoted by ARL_0, while ARL_1 denotes the value of the *ARL* when the process is out of control. *SDRL* is used to determine the variation of the run length distribution for a given value of shift. Similarly, $SDRL_0$ and $SDRL_1$ can be defined as the *SDRL* for the in-control and out-of-control process, respectively. When comparing two charts, the ARL_0 is fixed to a specific value, and a chart having a smaller value of ARL_1 than another is said to be more efficient in detecting the shift in the process [28]–[31].

To ensure a fair comparison of the AHWMA chart with existing charts of the same ARL_0, we examined the performance of the chart with $w \in \{0.03, 0.05, 0.10, 0.25, 0.5, 0.75\}$, and the corresponding values of C that fix ARL_0 to 500 are used, the R-code provided in the supplementary material finds the value of C (for each value of w), that fixes ARL_0 to 500. We examined the *ARL* performance of the chart under different correlation values between the process and the auxiliary variables. Specifically, we considered $\rho \in \{0.05, 0.25, 0.5, 0.75, 0.95\}$. The *ARL* values of the AHWMA chart are given in Tables 1 - 5. In these tables, δ is the size of shifts, and is calculated as $\delta = \frac{n^{1/2}|\mu_Z - \mu_1|}{\sigma_Z}$, where n is the sample size at each time i (here, we assume $n = 1$ across i), and μ_Z and μ_1 are the in-control and out-of-control mean, respectively.

The main findings of the AHWMA chart (cf. Tables 1 - 5) are:

- For fixed values of δ and ρ, the chart is more efficient for smaller values of w. For example, where $\rho = 0.05$ (Table 1), when $\delta = 0.5$, the values of the ARL_1 when $w = 0.03$ and 0.75 were 20.05 and 132.08, respectively. Thus, the chart detects a shift in the process mean faster when a small value of w is used.
- For fixed values of δ, w and C, the chart is more efficient when large values of ρ are used. For example, when $w = 0.03$, $L = 2.272$, and $\delta = 0.5$, ARL_1 values were 20.05 and 3.43 (in Tables 1 and 5) for $\rho = 0.05$ and $\rho = 0.95$, respectively. Thus, increases in the correlation structure between the process variable and the auxiliary variable lead to an increase in the chart's ability to detect a shift.
- The chart is *ARL* unbiased. That is, the ARL_1 values never exceed the corresponding ARL_0 for any choice of δ examined.
- As δ increases, the ARL_1 and $SDRL_1$ values approach 1 and 0, respectively, especially for large values of ρ; that is, the charts detect large shifts promptly.

B. COMPARISONS

We provide detailed comparisons of the proposed AHWMA chart with some existing control charts: the classical HWMA chart by [3], the classical EWMA chart by [4], the classical CUSUM chart by [4], the auxiliary-based EWMA chart (i.e., M_XEWMA) by [19], and the auxiliary-based CUSUM chart (i.e, $A_{ux}CUSUM_2$ by [22]), in terms of their ARL values. The auxiliary-based EWMA and CUSUM charts are also based on a regression estimator; they provide efficient applications of the classical EWMA and CUSUM charts, respectively, in those situations where the process variable is observed along with a variable. For comparison with the M_XEWMA and $A_{ux}CUSUM_2$ charts, we considered three different values of ρ: namely, $\rho \in \{0.05, 0.5, 0.95\}$. In all cases, the charts' parameters were set to values that fix ARL_0 at 500. We provide the charts' *ARL* results that optimized δ at $w \in \{0.05, 0.1, 0.2\}$.

The results of the comparisons are provided in Table 6. As shown on the table, the AHWMA chart outperformed the classical CUSUM, EWMA and HWMA charts in detecting shifts in the mean, especially when $\rho > 0.05$. For fixed values of w and ρ, the AHWMA chart was more efficient than the $A_{ux}CUSUM_2$ chart, especially for small-to-moderate values

TABLE 1. ***ARL* and *SDRL* values of the AHWMA chart when the correlation between the variables is $\rho = 0.05$. The values of *C* are chosen to fix the chart's ARL_0 to 500 for each chosen value of *w*.**

	w											
	0.03		0.05		0.1		0.25		0.5		0.75	
δ	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$
0.000	502.98	428.85	498.7	371.79	502.95	410.23	501.41	483.52	497.28	495.4	500.09	495.64
0.050	363.45	326.86	382.76	296.57	396.35	325.16	438.68	420.99	478.99	475.17	488.54	490.08
0.075	273.65	250.5	295.14	231.71	317.18	255.34	384.78	368.41	452.28	449.05	476.29	473.72
0.100	206.62	187.81	228.92	180.9	250.84	199.77	326.26	309.14	417.58	417.27	454.29	452.94
0.125	159.27	141.92	180.75	140.23	198.63	154.81	272.98	256.67	383.76	385.05	436.42	435.91
0.150	126.43	111.15	144.81	110.84	162.37	123.83	226.05	209.25	346.85	345.27	411.53	413.32
0.175	102.59	89.67	119.59	89.15	132.35	98.6	185.95	169.49	309.12	304.8	389.93	388.13
0.200	84.61	72.93	100.41	74.29	111.43	80.63	156.47	139.98	276.56	274.03	362.66	359.46
0.250	60.59	51.6	72.93	52.75	81.49	56.88	112.27	97.75	217.59	213.85	313.11	312.89
0.500	20.05	15.71	24.89	17.02	28.54	17.71	33.81	25.21	68.28	64.33	132.08	131.2
0.750	10.31	7.41	12.74	8.25	14.88	8.74	16.13	10.66	27.82	24.51	57.82	56.92
1.000	6.57	4.25	8.01	4.82	9.33	5.16	9.67	5.8	14.09	11.49	28.26	26.74
1.500	3.74	2.13	4.42	2.31	4.96	2.43	4.93	2.54	5.62	3.82	9.19	8.01
2.000	2.55	1.45	2.98	1.52	3.31	1.52	3.19	1.48	3.2	1.82	4.21	3.18
	C											
	2.272		2.608		2.938		3.075		3.089		3.09	

TABLE 2. ***ARL* and *SDRL* values of the AHWMA chart when the correlation between the variables is $\rho = 0.25$. The values of *C* are chosen to fix the chart's ARL_0 to 500 for each chosen value of *w*.**

	w											
	0.03		0.05		0.1		0.25		0.5		0.75	
δ	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$
0	502.5	429.42	498.92	372.53	502.19	411.15	501.01	488.82	499.07	492.39	500.98	496.82
0.050	358.21	321.07	378.19	292.9	391.86	321.41	437.22	418.88	476.3	469.57	489.51	490.58
0.075	264.91	242.15	288.01	226.75	311.24	252.59	376.49	360.73	451.69	446.04	471.6	472.85
0.100	199.08	180.8	221.19	173.23	243.62	193.54	317.17	302.38	413.08	413.82	457.88	455.84
0.125	153.2	136.33	174.67	135.14	191.55	148.78	263.92	246.52	379.53	378.61	432.72	430.93
0.150	121.43	106.87	139.66	106.8	155.22	117.75	218.69	201.72	338.72	336.89	409.57	409.29
0.175	98.06	85.24	114.52	85.74	128.19	94.54	179.98	164.52	304.51	300.64	381.68	379.18
0.200	81.18	69.62	95.62	70.28	106.39	77.07	148.91	133.53	268.09	263.87	356.45	358.04
0.250	58.11	49.05	69.54	50.26	77.88	54.35	106.9	92.35	208.71	206.16	305	303.75
0.500	19.02	15	23.88	16.33	27.25	16.85	31.91	23.63	64.32	60.14	125.44	123.43
0.750	9.74	6.92	12.21	7.83	14.16	8.25	15.32	9.99	25.75	22.63	53.68	52.13
1.000	6.29	4	7.65	4.53	8.85	4.84	9.16	5.39	13	10.42	26.05	24.65
1.500	3.57	2.02	4.19	2.17	4.72	2.3	4.68	2.37	5.27	3.49	8.38	7.23
2.000	2.45	1.4	2.86	1.46	3.18	1.47	3.03	1.4	3.03	1.69	3.93	2.91
	C											
	2.272		2.608		2.938		3.075		3.089		3.09	

of δ (i.e., $\delta < 2$). For fixed values of w and ρ, the AHWMA chart was less efficient than the M_XEWMA chart in detecting moderate-to-large shifts (i.e., $\delta > 0.5$) in the process mean. However, the AHWMA chart shows greater efficiency than the M_XEWMA chart in detecting small shifts (i.e., $\delta \leq 0.5$) in the mean.

IV. ROBUSTNESS TO NON-NORMALITY OF THE CHART

The AHWMA chart described in Section II relies on the assumption that the process variable and the auxiliary variable are bivariate normally distributed. In practice, this assumption does not always hold. Non-normality is not a major concern with a large sample size because the central limit theorem warrants that the sample mean will be approximately normally distributed for any continuous variables [32]. When $n = 1$, however, it is important to check the sensitivities of control charts to departures from normality [2]. We refer readers to [33]–[35], and [36] for detailed studies on the robustness of the EWMA control chart to non-normality.

Here, we investigate the robustness of the AHWMA chart to non-normality. As mentioned by [33] "a control chart is robust if its in-control run-length distribution remains stable (unchanged or nearly unchanged) when the underlying distributional assumption(s) (e.g. normality) are violated". Following previous investigators [33]–[37], we considered a heavy-tailed bivariate distribution (the bivariate Student's t-distribution), and a skewed distribution (the bivariate gamma distribution). We denote the bivariate t-distribution with v degrees of freedom by $t_2(v)$. The probability density

TABLE 3. ***ARL*** **and** ***SDRL*** **values of the AHWMA chart when the correlation between the variables is $\rho = 0.5$. The values of** ***C*** **are chosen to fix the chart's** ARL_0 **to 500 for each chosen value of** w**.**

	w											
	0.03		0.05		0.1		0.25		0.5		0.75	
δ	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$
0	502.09	428.04	498.36	371.62	501.47	406.7	504.88	486.43	498.55	497.01	496.55	498.91
0.050	333.66	301.2	354.43	276.58	373.92	302.68	425.24	405.64	471.88	466.32	487.38	486.56
0.075	240.95	220.02	263.31	207.91	285.65	231.4	354.88	339.89	435.63	429.98	469.55	466.55
0.100	174.83	156.92	197.66	153.49	216.45	170.81	290.14	273.78	397.54	393.99	445.34	446.92
0.125	132.93	117.26	151.91	115.44	168.43	129.25	234.95	218.49	357.17	354.94	419.21	419.54
0.150	104.06	90.67	120.7	91.47	135.03	100.63	190.51	174.88	312.66	310.1	390.07	387.53
0.175	83.71	71.86	98.45	72.9	110.08	80.03	153.84	138.3	275.08	270.88	363.64	363.96
0.200	69.32	59.15	81.65	59.65	91.18	65.05	127.81	112.89	239.37	235.41	331.84	333.34
0.250	49.08	41.15	59.23	42.45	65.89	44.93	89.86	76.3	180.91	176.47	276	276.42
0.500	15.75	12.08	19.73	13.38	22.63	13.72	25.77	18.3	50.43	46.59	101.74	100.72
0.750	8.23	5.64	10.12	6.38	11.79	6.7	12.52	7.87	19.87	16.95	40.79	39.3
1.000	5.35	3.28	6.44	3.67	7.43	3.97	7.59	4.31	10.06	7.75	19.11	17.68
1.500	3.1	1.73	3.61	1.84	4.04	1.89	3.95	1.91	4.21	2.63	6.12	4.98
2.000	2.1	1.23	2.43	1.3	2.72	1.29	2.61	1.19	2.5	1.29	3	2.03
	C											
	2.272		2.608		2.938		3.075		3.089		3.09	

TABLE 4. ***ARL*** **and** ***SDRL*** **values of the AHWMA chart when the correlation between the variables is $\rho = 0.75$. The values of** ***C*** **are chosen to fix the chart's** ARL_0 **to 500 for each chosen value of** w**.**

	w											
	0.03		0.05		0.1		0.25		0.5		0.75	
δ	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$
0	501.65	430.15	498.97	374.39	499.57	404.52	504.97	490.17	493.17	493.87	497.6	500.41
0.050	271.72	248.22	297.52	234.08	315.26	256.33	385.37	369.96	450.49	449.65	476.24	477.56
0.075	178.17	160.77	201.34	156.45	222	174.13	296.36	278.87	398.8	400.65	450.12	449.58
0.100	125.55	110.22	144.11	109.47	160.49	122.96	223.57	207	343.09	341.7	410.87	406.97
0.125	91.66	79.47	108.24	79.75	120.11	88.2	171.01	155.54	290.88	287.62	375.59	373.21
0.150	70.54	60.2	84	61.41	93.52	66.57	131.78	115.65	245.44	239.44	336.11	332.07
0.175	55.94	47.27	67.61	49.08	75.41	52.08	103.9	89.92	201.67	198.62	298.92	297.8
0.200	45.3	37.77	54.88	39.25	61.84	41.94	83.18	70.42	169.04	164.22	262.48	262.96
0.250	31.95	25.93	39.23	27.61	44.38	28.8	55.98	44.86	117.61	113.89	203.97	203.6
0.500	10.24	7.26	12.66	8.17	14.71	8.57	15.96	10.46	27.41	24.28	56.59	55.52
0.750	5.5	3.42	6.68	3.83	7.64	4.08	7.79	4.42	10.53	8.18	19.97	18.55
1.000	3.69	2.11	4.35	2.28	4.91	2.39	4.86	2.49	5.53	3.72	8.94	7.78
1.500	2.16	1.26	2.52	1.33	2.8	1.31	2.67	1.21	2.57	1.35	3.11	2.14
2.000	1.45	0.85	1.65	0.96	1.86	0.99	1.76	0.84	1.66	0.73	1.76	0.92
	C											
	2.272		2.608		2.938		3.075		3.089		3.09	

function of a bivariate t-distribution is given by

$$f(x) = \frac{[\Gamma(v+2)/2]}{\Gamma(v/2)v\pi|\Sigma|^{1/2}} \left[1 + \frac{1}{v}(\boldsymbol{x}-\boldsymbol{\mu})^{\boldsymbol{T}}\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right]^{-(v+2)/2} \quad (9)$$

where $\boldsymbol{x} \in R^2$, $\boldsymbol{\mu} = [\mu_1, \mu_2]^T$ is the 2×1 vector of location parameters, $\boldsymbol{\Sigma}$ is a 2×2 positive-definite (or covariance) matrix, v is the number of the degrees of freedom, and $\Gamma(n) = (n-1)!$ for $n = 1, 2, \ldots$. The mean vector and covariance matrix are given as $\boldsymbol{\mu}$ (if $v > 1$, else, undefined), and $\frac{v}{(v-2)}\boldsymbol{\Sigma}$ (when $v > 2$, else, undefined), respectively.

We denote the bivariate gamma distribution with shape parameter, $\alpha\mathbf{1}_p$, and scale parameter, $\beta\mathbf{1}_p$, by $G_2(\alpha\mathbf{1}_p, \beta\mathbf{1}_p, \boldsymbol{\Sigma})$, where $\mathbf{1}_p$ is a column vector of ones of size $p = 2$. The probability density function of the bivariate gamma distribution is given as:

$$f(x) = \frac{|\boldsymbol{\Sigma}|^{-\alpha}}{\beta^{2\alpha}\Gamma_2(\alpha)}|\boldsymbol{x}|^{\alpha-3/2}\exp\left(tr\left(\frac{1}{\beta}\boldsymbol{\Sigma}^{-1}\boldsymbol{x}\right)\right) \quad (10)$$

where $\alpha > 0$ is the scale parameter, $\beta > 0$ is the shape parameter, and Γ_2 is the bivariate gamma function given as $\Gamma_2(\alpha) = \pi^{1/2}\Gamma(\alpha)\Gamma(\alpha - 1/2)$. See [38] and [39] for detailed information on the bivariate gamma distribution and its properties.

We studied the chart's robustness under a large range of degrees of freedom (v) for the bivariate t-distribution; namely, $v \in \{4, 6, 8, 10, 15, 20, 30, 40, 50, 100, 1000\}$. For the bivariate gamma distribution, without loss of generality, we considered the scale parameter, $\beta = 1$, and a range of values of the shape parameter,

TABLE 5. ***ARL* and *SDRL* values of the AHWMA chart when the correlation between the variables is $\rho = 0.95$. The values of *C* are chosen to fix the chart's ARL_0 to 500 for each chosen value of *w*.**

	w											
	0.03		0.05		0.1		0.25		0.5		0.75	
δ	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$
0	502.44	429.19	499.63	372.3	501.08	409.78	501.37	483.01	504.2	502.97	503.71	503.15
0.050	115.73	101.52	133.96	101.16	148.56	112.37	209.06	193.55	332.14	328.99	401.56	398.62
0.075	65.13	55.07	77.2	55.89	86.73	61.75	120.3	105.64	228.82	223.07	322.7	323.39
0.100	41.61	34.64	50.65	35.8	56.68	37.78	74.83	62.18	155.76	152.02	247.89	248.76
0.125	28.84	23.43	35.68	25.06	40.43	26.11	50.99	40.33	105.88	102	187.84	185.45
0.150	21.43	17.04	26.85	18.37	30.69	19.2	36.25	27.29	75.21	71.06	140.91	139.68
0.175	16.54	12.67	20.62	13.88	23.92	14.67	27.41	19.86	54.38	50.75	107.39	106.69
0.200	13.35	10.06	16.58	11.01	19.31	11.63	21.51	14.81	39.96	36.15	81.86	80.94
0.250	9.32	6.55	11.5	7.33	13.41	7.75	14.43	9.32	23.77	20.57	49.83	48.49
0.500	3.43	1.91	4.01	2.07	4.5	2.16	4.45	2.23	4.93	3.23	7.68	6.53
0.750	1.99	1.18	2.33	1.25	2.59	1.24	2.46	1.12	2.34	1.19	2.76	1.8
1.000	1.33	0.75	1.52	0.88	1.69	0.92	1.62	0.77	1.53	0.66	1.6	0.79
1.500	1.01	0.12	1.02	0.19	1.04	0.24	1.04	0.21	1.04	0.2	1.04	0.21
2.000	1	0	1	0.01	1	0.02	1	0.02	1	0.02	1	0.02
	C											
	2.272		2.608		2.938		3.075		3.089		3.09	

i.e., $\alpha \in \{1, 2, 3, 4, 5, 10, 50, 100, 1000\}$. Hence, we denote the bivariate gamma distribution as $G_2(\alpha)$, for short. The ARL_0 values of the chart for $\rho \in \{0.25, 0.5, 0.95\}$, and $w \in \{0.03, 0.05, 0.25, 0.75\}$ for the bivariate t and bivariate gamma distributions are given in Tables 7 and 8, respectively.

The ARL_0 results in Tables 7 and 8 are summarized below:

- For a fixed value of w, the ARL_0 values of the bivariate t-distributions are the same for all the correlation values (i.e., $\rho_{ZY} = 0.25, 0.5$, or 0.95) examined. This result is due to the symmetry of the t-distribution.
- However, for a fixed value of w, the ARL_0 of the bivariate gamma distributions differ across all the values of ρ_{ZY} examined. Here, the chart appears to be more robust to non-normality only for smaller values of ρ_{ZY}.
- For both non-normal distributions, as expected, the ARL_0 value increases, and tends to converge to the required nominal ARL_0 of the AHWMA chart, for large degrees of freedom (v) or larger values of the shape parameter (i.e., $\alpha \geq 50$), especially when $w = 0.3$ or 0.05 is used.
- Importantly, the chart's ARL_0 value is more robust to non-normality only when a small value of w (i.e., $w = 0.03$ or 0.05) is used. This implies that small values of w (i.e., $w = 0.03$ and 0.05) are useful when the underlying distribution is not normal.

Table 9 displays the ARL_1 values for the AHWMA chart under bivariate normal, t and gamma distributions for various values of δ when $w = 0.03$, or 0.75, and $\rho_{ZY} = 0.25$. The results in Table 9 indicate that the chart's ARL_1 values tend to approach values obtained for bivariate normal data when a smaller value of w (i.e., $w = 0.03$) is used. For example, when $w = 0.03$, $v = 50$, $\beta = 50$, and $\delta = 0.5$, the ARL_1 for the AHWMA were 19.02 (normal distribution), 19.65 (t-distribution), and 19.18 (gamma distribution). The percentage deviation of the ARL_1 values obtained under the t or gamma distributions from ARL_1 values obtained under normal distribution are 3.13% and 0.84%, respectively. On the other hand, when w is large (i.e., $w = 0.75$), and other parameters are unchanged (i.e., $v = 50$, $\beta = 50$, and $\delta = 0.5$), the ARL_1 for the AHWMA under normal, t and gamma distributions were 125.44, 118.63, and 68.10, respectively; the percentage deviation of these ARL_1 values from obtained under the normal distribution were -5.43% and -45.71% for the t and gamma distributions, respectively.

V. STEP-BY-STEP ALGORITHM FOR CONSTRUCTING THE AHWMA CHART WHEN PARAMETERS ARE UNKNOWN

The AHWMA chart in Section II was formulated based on the assumption that the parameters associated with the process variable and auxiliary variable are all known. However, these parameters are generally unknown in practice and need to be estimated. In this case, the regression model in Equation (3) would be based on estimated parameters, and is given as:

$$\hat{R}_i = \bar{z}_i + \hat{b}(\hat{\mu}_Y - \bar{y}_i) \tag{11}$$

where $\hat{b}$ is the estimated slope of the regression line; given as the estimated change in the process variable Z due to a unit change in the auxiliary variable Y [14], and $\hat{\mu}_Y$ ($\hat{\mu}_Y = \frac{1}{m}\sum_{i=1}^{m}\bar{Y}_i$) is the unbiased estimate of the mean of the auxiliary variable (i.e., μ_Y). The estimated mean and variance of $\hat{R}$ are given as $\bar{\hat{R}} = \hat{\mu}_Z$, and $S^2_{\hat{R}} = \frac{\hat{\sigma}^2_Z}{n}(1 - r^2)$, where r is the estimated value of the correlation size between the variables, $\hat{\mu}_Z$ and $\hat{\sigma}^2_Z$ are the unbiased estimates of μ_Z and σ^2_Z, respectively. The $\hat{\mu}_Z$ and $\hat{\sigma}^2_Z$ are calculated from a specified set of sample values measured when the process was known to be in control. They are given as $\hat{\mu}_Z = \frac{1}{m}\sum_{i=1}^{m}\bar{Z}_i$ and $\hat{\sigma}^2_Z = \frac{s_p}{c_{4,m}}$, where $s_p = \left(\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(Z_{ij} - \bar{Z}_i)^2}{m(n-1)}\right)^{1/2}$,

TABLE 6. ***ARL*** **comparisons of the charts.**

					$\mathrm{A}_{ux}\mathrm{CUSUM}_2$				$\mathrm{M}_X\mathrm{EWMA}$				AHWMA			
		Classical chart			ρ				ρ				ρ			
w	δ	CUSUM	EWMA	HWMA	0.05	0.5	0.75	0.95	0.05	0.5	0.75	0.95	0.05	0.5	0.75	0.95
0.03	0	497.95	500.3	501.2	498.9	497.43	501.98	501.22	502.7	500.82	503.52	498.71	502.98	502.09	501.65	502.44
	0.050	399.33	389.49	365.09	400.21	374.31	319.5	149.88	388.63	362.25	304.3	131.56	363.45	333.66	271.72	115.73
	0.075	320.31	304.9	273.63	322.98	287.25	222.65	88.09	303.73	271.43	205.42	70.58	273.65	240.95	178.17	65.13
	0.100	254.31	233.62	205.84	252.59	219.7	161.3	60.28	236.25	201.31	143.61	44.75	206.62	174.83	125.55	41.61
	0.125	201.09	181.26	159.07	200.39	170.07	121.51	45.42	182.06	151.38	103.07	30.82	159.27	132.93	91.66	28.84
	0.150	161.81	143.7	125.81	162.17	135.08	95	36.36	144.63	117.7	77.65	22.71	126.43	104.06	70.54	21.43
	0.175	133.45	116.37	102.08	133.18	110.83	77.63	30.2	114.6	93.03	60.51	17.48	102.59	83.71	55.94	16.54
	0.200	111.78	94.55	84.62	112.05	92.38	65.02	25.94	94.25	75.25	49.1	14.07	84.61	69.32	45.3	13.35
	0.250	83.19	66.23	61.21	83.33	69.13	48.66	20.16	66.71	52.69	33.91	9.66	60.59	49.08	31.95	9.32
	0.500	34.63	21.26	20.01	34.65	29.26	21.5	9.59	21.42	16.69	10.62	3.12	20.05	15.75	10.24	3.43
	0.750	21.67	10.75	10.29	21.62	18.45	13.76	6.37	10.71	8.44	5.4	1.75	10.31	8.23	5.5	1.99
	1.000	15.75	6.63	6.61	15.74	13.5	10.16	4.82	6.63	5.23	3.4	1.26	6.57	5.35	3.69	1.33
	1.500	10.27	3.44	3.72	10.24	8.85	6.75	3.3	3.44	2.78	1.88	1.01	3.74	3.1	2.16	1.01
	2.000	7.64	2.25	2.55	7.64	6.62	5.1	2.67	2.23	1.85	1.33	1	2.55	2.1	1.45	1
0.05	0	499.43	497.84	497.55	498.96	497.14	500.21	497.54	499.34	499.04	496.68	499.39	498.7	498.36	500.95	499.63
	0.050	425.38	412.14	380.67	420.29	404.36	355.01	175.78	410.96	388.75	334.74	154.71	382.76	354.43	297.52	133.96
	0.075	355.89	336.92	297.09	355.33	326.01	260.3	97.17	335.44	302.76	236.23	83.31	295.14	263.31	201.34	77.2
	0.100	291.22	266.23	230.76	289.35	253.87	187.5	61.51	267.56	231.69	168.05	51.09	228.92	197.66	144.11	50.65
	0.125	234.83	213.89	180.59	235.22	198.79	139.53	43.36	210.55	177.37	121.77	34.75	180.75	151.91	108.24	35.68
	0.150	189.47	168.75	146.07	190.51	157.49	106.35	33.11	168.53	137.9	91.73	25.44	144.81	120.7	84	26.85
	0.175	154.72	136.38	119.93	154.38	126.25	83.45	26.58	134.81	108.29	70.33	19.44	119.59	98.45	67.61	20.62
	0.200	128.86	110.55	100.46	127.41	103.56	67.15	22.14	110.86	88.48	56.2	15.57	100.41	81.65	54.88	16.58
	0.250	91.21	77.73	73.05	90.94	72.8	47.31	16.57	77.52	60.93	38.44	10.65	72.93	59.23	39.23	11.5
	0.500	31.14	23.73	24.86	31.28	25.52	17.83	7.39	23.73	18.45	11.72	3.4	24.89	19.73	12.66	4.01
	0.750	18.06	11.91	12.81	17.9	15.03	10.9	4.83	11.83	9.31	5.96	1.87	12.74	10.12	6.68	2.33
	1.000	12.63	7.28	8.06	12.59	10.66	7.87	3.65	7.27	5.75	3.73	1.32	8.01	6.44	4.35	1.52
	1.500	7.94	3.77	4.42	7.93	6.78	5.12	2.52	3.76	3	2.03	1.02	4.42	3.61	2.52	1.02
	2.000	5.85	2.42	2.98	5.82	5.03	3.86	2.01	2.43	1.98	1.4	1	2.98	2.43	1.65	1
0.1	0	501.58	501.67	500.39	501.92	502.21	501.54	498.63	500.99	497.33	499.62	502.92	502.95	501.47	499.57	501.08
	0.050	471.36	437.93	397.64	469.78	461.18	437.92	298.48	439.33	420.52	377.94	199.61	396.35	373.92	315.26	148.56
	0.075	438.67	380.9	318.89	438.02	422.52	374.37	190.06	377.32	350.23	287.74	110.71	317.18	285.65	222	86.73
	0.100	397.64	318.94	249.81	398.67	372.84	311.88	119.6	318.25	281.56	213.89	66.65	250.84	216.45	160.49	56.68
	0.125	354.81	262.15	200.21	356.1	320.97	253.42	78.24	260.83	227.52	159.78	43.91	198.63	168.43	120.11	40.43
	0.150	312.61	215.56	161.03	315.18	277.04	204.92	53.32	216.54	181.54	121.34	30.94	162.37	135.03	93.52	30.69
	0.175	273.9	178.62	133.14	273.86	233.99	164.5	37.95	177.41	145.27	94.14	23.26	132.35	110.08	75.41	23.92
	0.200	237.27	146.13	111.78	235.96	200.4	132.38	28.07	147.45	118.36	73.99	18.16	111.43	91.18	61.84	19.31
	0.250	179.15	102.97	81.64	178.76	143	88.48	17.08	103.48	80.95	48.77	12.15	81.49	65.89	44.38	13.41
	0.500	48.81	28.7	28.51	48.63	35.23	19.31	5.01	28.64	21.95	13.41	3.75	28.54	22.63	14.71	4.5
	0.750	19.62	13.6	14.87	19.66	14.55	8.64	2.95	13.59	10.52	6.59	2.03	14.88	11.79	7.64	2.59
	1.000	10.93	8.21	9.34	10.93	8.43	5.43	2.17	8.25	6.42	4.14	1.4	9.33	7.43	4.91	1.69
	1.500	5.5	4.17	4.96	5.47	4.48	3.15	1.46	4.15	3.32	2.22	1.02	4.96	4.04	2.8	1.04
	2.000	3.69	2.65	3.32	3.67	3.09	2.3	1.04	2.65	2.15	1.5	1	3.31	2.72	1.86	1
0.25	0	503.87	501.19	498.41	499.91	498.65	496.72	499.94	499.65	500.9	497.9	497.83	501.41	504.88	502.97	501.37
	0.050	473.65	464.89	440.22	476.76	467.82	443.37	316.9	469.38	454.7	429.18	284.8	438.68	425.24	385.37	209.06
	0.075	447.21	433.74	382.85	440.66	430.85	390.87	211.47	433.41	413.71	368.59	179.31	384.78	354.88	296.36	120.3
	0.100	411.5	390.42	324.23	412.17	387.72	334.12	138.29	387.97	361.12	298.88	114.37	326.26	290.14	223.57	74.83
	0.125	374.92	344.94	271.63	372.42	343.61	277.01	91.75	342.65	310.56	241.79	75.25	272.98	234.95	171.01	50.99
	0.150	335.01	300.26	224.11	333.73	297.96	228.79	62.32	301.25	265.16	194.2	51.62	226.05	190.51	131.78	36.25
	0.175	296.57	263.35	187.63	295.29	258.58	184.72	44.02	260.14	224.89	156.58	37.1	185.95	153.84	103.9	27.41
	0.200	260.28	225.42	157.86	261.6	221.65	152.31	31.91	226.33	190.14	126.5	27.39	156.47	127.81	83.18	21.51
	0.250	199.46	168.99	113.14	200.01	162.65	101.91	19.06	169.1	136.54	83.92	16.88	112.27	89.86	55.98	14.43
	0.500	57.14	47.4	33.79	56.76	40.9	21.68	4.89	47.02	34.6	18.99	4.24	33.81	25.77	15.96	4.45
	0.750	22.16	19.34	16.25	21.95	15.9	8.94	2.81	19.19	14.13	8.1	2.22	16.13	12.52	7.79	2.46
	1.000	11.57	10.45	9.71	11.56	8.69	5.36	2.04	10.39	7.77	4.73	1.49	9.67	7.59	4.86	1.62
	1.500	5.42	4.77	4.93	5.42	4.34	3	1.3	4.75	3.72	2.42	1.04	4.93	3.95	2.67	1.04
	2.000	3.55	2.93	3.18	3.54	2.93	2.16	1.02	2.94	2.34	1.6	1	3.19	2.61	1.76	1
	C	13.15029	2.483	2.272	13.15029	13.15029	13.15029	13.15029	2.483	2.483	2.483	2.483	2.272	2.272	2.272	2.272

TABLE 7. $\boldsymbol{ARL_0}$ **with bivariate** $\boldsymbol{t}$**−distirbution.**

w	ρ_{ZY}	AHWMA	$t_2(4)$	$t_2(8)$	$t_2(10)$	$t_2(15)$	$t_2(30)$	$t_2(50)$	$t_2(100)$	$t_2(1000)$
0.03	0.25	502.5	268.66	346.71	372.71	413.4	454.92	472.15	485.14	499.74
	0.5	502.09	268.16	346.42	372.37	413.24	454.77	472.99	485.93	499.36
	0.95	502.44	268.4	346.48	372.04	413.12	454.27	472.85	485.4	499.98
0.05	0.25	498.92	223.83	306.24	332.44	375.3	433.23	454.27	476.72	494.77
	0.5	498.36	223.1	305.14	332.2	375.7	432.25	455.71	477.37	494.7
	0.95	498.97	223.74	305.58	332.1	375.69	432.53	455.09	477.4	494.65
0.25	0.25	504.01	103.84	163.35	188.54	237.89	328.6	381.05	433.02	489.41
	0.5	504.88	103.87	163.21	188.52	237.59	328.94	381.32	433.3	489.25
	0.95	504.97	104.53	163.6	188.42	237.36	328.48	381.82	433.04	489.66
0.75	0.25	497.98	84.55	137.14	163.15	213.72	307.88	366.82	429.93	485.26
	0.5	496.55	84.37	137.01	163.27	213.11	307.99	366.91	429.73	485.26
	0.95	497.6	84.29	137.96	163	213.55	307.67	366.43	429.51	485.68

and $c_{4,m} = \dfrac{2^{1/2}\Gamma\left(\dfrac{m(n-1)+1}{2}\right)}{(m(n-1))^{1/2}\left(\dfrac{m(n-1)}{2}\right)}$ is an unbiasing constant [3], [40].

Using Equation (11), the plotting statistic for the AHWMA control chart based on estimated parameters is given as:

$$\hat{T}_i = w\hat{R}_i + (1-w)\bar{\hat{R}}_{i-1} \tag{12}$$

TABLE 8. ARL_0 with bivariate gamma distribution.

w	ρ_{ZY}	AHWMA	$G(1)$	$G(2)$	$G(3)$	$G(4)$	$G(5)$	$G(50)$	$G(100)$	$G(1000)$
0.03	0.25	502.5	318.96	375.58	403.3	423	429.81	490.83	499.57	501.11
	0.5	502.09	276.22	331.92	366.38	388.89	402.29	486.64	495.02	499.28
	0.95	502.44	203.56	260.01	301	326.33	345.62	474.14	486.89	499.52
0.05	0.25	498.92	260.97	316.67	348.37	377.24	391.84	484.87	496.06	498.53
	0.5	498.36	228.92	281.5	321.79	345.27	366.05	482.82	482.17	496.23
	0.95	498.97	169.00	222.22	260.75	285.58	308.43	457.36	480.23	495.58
0.25	0.25	504.01	91.82	120.34	139.28	159.94	174.75	394.77	444.34	492.63
	0.5	504.88	87.9	117.06	139.89	159.19	175.89	398.25	438.8	492.79
	0.95	504.97	65.44	95.52	117.63	139.88	160.81	386.1	436.85	485.02
0.75	0.25	497.98	61.61	82.27	98.38	115.33	127.7	352.01	410.57	491.43
	0.5	496.55	59.7	82.53	96.01	114.57	125.31	351.67	409.13	484.35
	0.95	497.6	46.52	71.75	92.58	113.61	122.13	344.77	410.59	480.14

TABLE 9. ARL_1 with bivariate t and gamma distributions.

			δ					
w	ρ		0.1	0.2	0.25	0.5	1	2
0.03	0.25	AHWMA	199.08	81.18	58.11	19.02	3.57	2.45
		$t_2(4)$	208.67	121.2	92.89	33.63	10.79	3.92
		$t_2(8)$	207.97	95.66	71.22	24.01	7.83	2.97
		$t_2(10)$	206.5	92.86	68.27	22.93	7.47	2.83
		$t_2(15)$	204.56	88.88	63.92	21.42	7.05	2.71
		$t_2(30)$	202.67	84.32	61.18	20.23	6.64	2.57
		$t_2(50)$	201.41	83.31	59.57	19.65	6.54	2.53
		$t_2(100)$	201.33	82.01	59.37	19.39	6.44	2.49
		$t_2(1000)$	201	81.45	58.84	18.91	6.32	2.45
		$G(1,1)$	145.21	71.65	54.11	19.54	7.07	2.98
		$G(2)$	154.77	73.77	54.64	19.53	6.79	2.84
		$G(3)$	162.67	74.64	55	19.39	6.63	2.77
		$G(4)$	163.66	75.48	56.17	19.33	6.62	2.72
		$G(5)$	169.73	75.68	56.08	19.23	6.62	2.7
		$G(50)$	186.67	80.17	57.93	19.18	6.39	2.55
		$G(100)$	187.92	80.29	57.8	19.2	6.36	2.49
		$G(1000)$	196.47	80.52	58.36	19.09	6.39	2.46
0.75	0.25	AHWMA	457.88	356.45	305	125.44	8.38	3.93
		$t_2(4)$	84.1	81.67	80.5	71.4	46.83	13.02
		$t_2(8)$	134.59	125.99	121.83	87.94	35.57	6.16
		$t_2(10)$	157.11	147.49	140.22	92.84	32.98	5.48
		$t_2(15)$	205.46	187.12	171.97	103.76	30.63	4.85
		$t_2(30)$	291.42	246.38	224.29	114.43	28.34	4.28
		$t_2(50)$	344.22	285.82	251.86	118.63	27.08	4.17
		$t_2(100)$	391.79	312.18	274.8	120.82	26.45	4.05
		$t_2(1000)$	448.32	354.74	299.77	124.34	25.83	3.89
		$G(1)$	47.75	38.57	34.32	21.5	9.77	3.68
		$G(2)$	63.26	49.67	44.14	26.01	11.04	3.68
		$G(3)$	75.12	57.82	52.45	29.26	11.86	3.7
		$G(4)$	87.47	66.01	57.66	32.31	12.48	3.7
		$G(5)$	95.62	72.83	63.7	34.49	13.01	3.66
		$G(50)$	250.86	177.39	150.4	68.1	18.63	3.77
		$G(100)$	303.73	215.39	179.48	79.03	20.29	3.8
		$G(1000)$	407.08	296.01	258.56	106.41	23.41	3.87

The estimated mean and variance of the plotting statistic in Equation (12) are given by $\hat{\mu_T} = \hat{\mu_Z}$, and

$$\hat{\sigma}^2_{\hat{T}_i} = \begin{cases} \dfrac{(1-r^2)}{n} w^2 \hat{\sigma}^2_Z & \text{if } i=1 \\ \dfrac{(1-r^2)}{n}\left(w^2\hat{\sigma}^2_Z + (1-w)^2\dfrac{\hat{\sigma}^2_Z}{i-1}\right) & \text{if } i>1 \end{cases} \quad (13)$$

The upper and lower control limits for the (plotting statistic given in Equation (12)) estimated time varying control chart are given as:

$$\hat{L}_i = \begin{cases} \hat{\mu}_Z - C'\hat{\sigma}_Z\sqrt{\dfrac{w^2}{n}(1-r^2)} & \text{if } i=1 \\ \hat{\mu}_Z - C'\hat{\sigma}_Z\sqrt{\left(\dfrac{w^2}{n} + \dfrac{(1-w)^2}{n(i-1)}\right)(1-r^2)} & \text{if } i>1 \end{cases} \quad (14)$$

$$\hat{U}_i = \begin{cases} \hat{\mu}_Z + C'\hat{\sigma}_Z\sqrt{\dfrac{w^2}{n}(1-r^2)} & \text{if } i=1 \\ \hat{\mu}_Z + C'\hat{\sigma}_Z\sqrt{\left(\dfrac{w^2}{n} + \dfrac{(1-w)^2}{n(i-1)}\right)(1-r^2)} & \text{if } i>1 \end{cases} \quad (15)$$

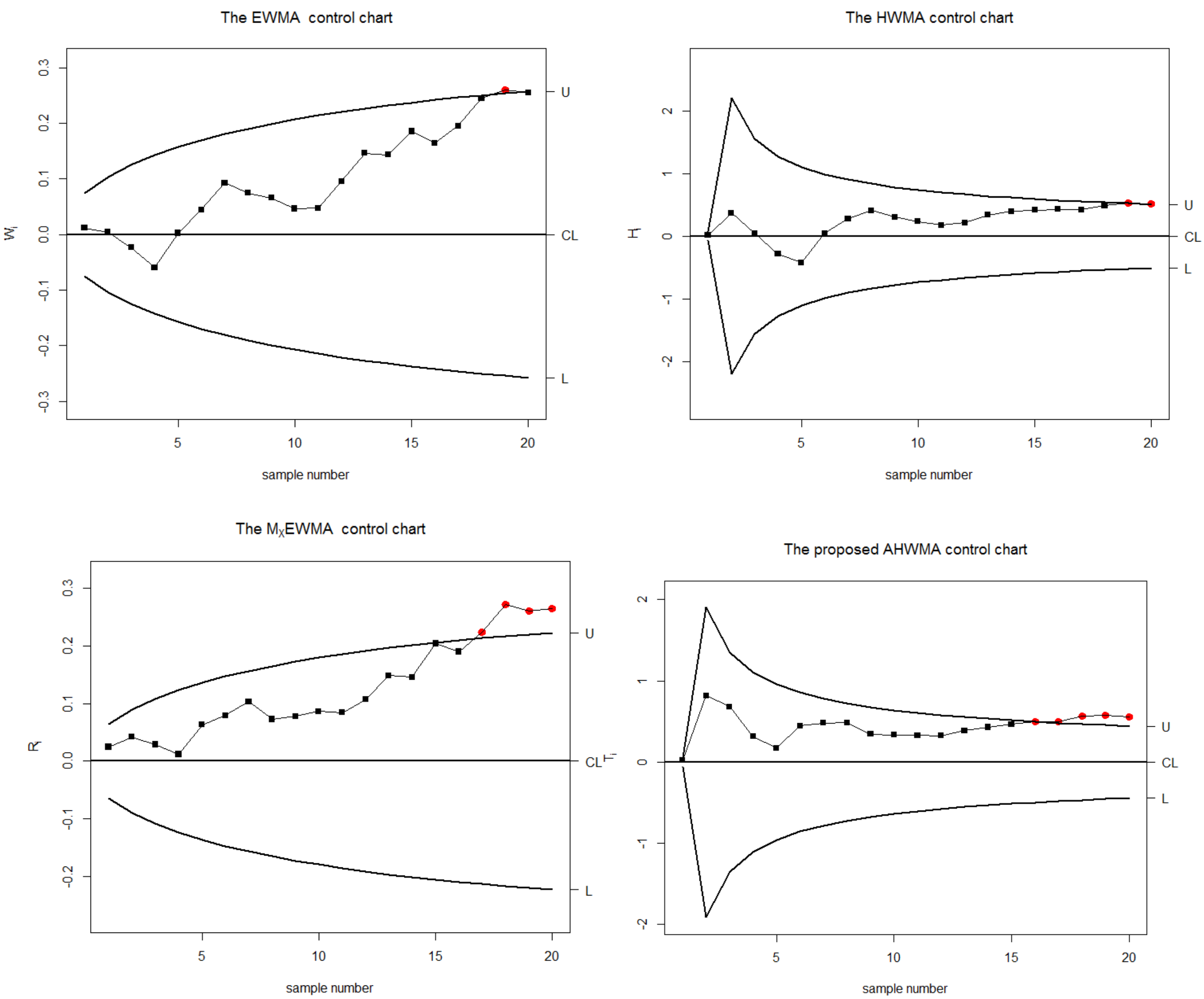

FIGURE 1. **Application of the EWMA, HWMA, M_XEWMA, and AHWMA charts.**

where C' determines the width of the estimated control limits. Also, the estimated center line (CL) of the AHWMA chart is given by:

$$\hat{CL} = \hat{\mu}_Z \tag{16}$$

When the chart is based on estimated parameters, implementation occurs in two phases. In phase I (retrospective phase), a historical reference sample is studied to establish the in-control state and to evaluate the stability of the process [41], [42]. Once the in-control reference sample is characterized, the process parameters are estimated from phase I and control chart limits are obtained for use in phase II. Phase II involves regular monitoring of the process. If successive observed values obtained at the beginning of Phase II fall within the in-control limits calculated from Phase I, the process is considered to be in control. In contrast, any observed values during Phase II that fall outside the control limits indicate that the process may be out of control, and remedial responses are then required [43], [44]. A shift in a process parameter needs to be detected quickly so that corrective actions can be taken as early as possible.

We give below a step-by-step algorithm to implement the chart in phase I and phase II [3], [45].

- Phase I

    1. Simulate m bivariate samples (Z, Y) each of size n from the in-control bivariate (normally distributed) process.
    2. Calculate the sample means, $(\bar{Z}_i, \bar{Y}_i)$, and the sample variances (S^2_{Zi}, S^2_{Yi}), where $S^2_{Zi} = \dfrac{\sum_{j=1}^{n}(Z_{ij} - \bar{Z}_i)^2}{(n-1)}$, and $S^2_{Yi} = \dfrac{\sum_{j=1}^{n}(Y_{ij} - \bar{Y}_i)^2}{(n-1)}$, for each sample $i = 1, 2, \ldots, m$.

TABLE 10. Calculation of the AHWMA chart statistic and its limits.

i	Z_i	Y_i	R_i	T_i	LCL_i	UCL_i
1	0.39	-0.865	0.8225	0.0247	-0.059	0.059
2	-0.242	-1.686	0.601	0.8159	-1.9095	1.9095
3	-0.919	-1.046	-0.396	0.6785	-1.3509	1.3509
4	-1.22	-1.366	-0.537	0.3161	-1.1035	1.1035
5	2.01	0.574	1.723	0.1706	-0.9561	0.9561
6	1.395	1.61	0.59	0.4471	-0.8556	0.8556
7	1.66	1.542	0.889	0.4799	-0.7814	0.7814
8	-0.514	0.816	-0.922	0.484	-0.7238	0.7238
9	-0.213	-0.907	0.2405	0.3431	-0.6774	0.6774
10	-0.588	-1.923	0.3735	0.3357	-0.6389	0.6389
11	0.074	0.132	0.008	0.3285	-0.6064	0.6064
12	1.673	1.64	0.853	0.3247	-0.5785	0.5785
13	1.765	0.575	1.4775	0.3875	-0.5541	0.5541
14	0.061	-0.008	0.065	0.429	-0.5326	0.5326
15	1.537	-1.084	2.079	0.4634	-0.5135	0.5135
16	-0.519	-0.52	-0.259	0.5010*	-0.4963	0.4963
17	1.198	-0.246	1.321	0.5009*	-0.4808	0.4808
18	1.853	0.028	1.839	0.5646*	-0.4666	0.4666
19	0.733	1.715	-0.1245	0.5765*	-0.4537	0.4537
20	0.108	-0.6	0.408	0.5556*	-0.4418	0.4418

$*$ Out-of-control signal

3. Repeat steps 1 and 2, many times, and compute estimates for the means, $(\hat{\mu}_Z, \hat{\mu}_Y)$ and variances $(\hat{\sigma}_Z^2, \hat{\sigma}_Y^2)$. These are used to set the control limits in phase II.

- Phase II

4. At each time i, simulate a bivariate sample (Z, Y) of size n from the process.
5. Compute the estimated regression estimator in Equation (11), and use this to compute the chart's plotting statistic, $\hat{T}_i$, in Equation (12).
6. Use the estimated parameters from phase I (from step 3 above) to construct the estimated control limits given in Equation (14)- (15). Compared $\hat{T}_i$, against these control limits.
7. If $\hat{T}_i$ falls within the control limits, the process is considered to be in control. Alternatively, if $\hat{T}_i$ falls outside the control limits, the process is declared to be in an out-of-control state.

VI. ILLUSTRATIVE EXAMPLE

In this section, we provide an example to illustrate the implementation of the AHWMA chart, using the simulated dataset provided in [19]. The data were obtained by simulating $m = 20$ bivariate samples, each of size $n = 1$, from $(Z, Y) \sim N_2(\mu_Z + \delta\sigma_Z, \mu_Y, \sigma_Z^2, \sigma_Y^2, \rho)$. The values of the parameters used for the simulation were: $\mu_Z = 0$, $\mu_Y = 0$, $\sigma_Z^2 = 1$, $\sigma_Y^2 = 1$, $\rho = 0.5$, and $\delta = 0.5$, where δ is the size of the shift applied to the in-control mean, μ_Z, of the process variable of interest, and ρ is the correlation between the process variable and the auxiliary variable. The bivariate dataset is given in the first two columns of Table 10. We examined the ability of the AHWMA chart to detect a shift in the process variable and compared this to the M_XEWMA, the classical EWMA and the HWMA charts. In all cases, the chart parameters: w and C, were chosen to fix the ARL_0 to 500. The parameters for the classical EWMA and M_XEWMA were $w = 0.03$ and $C = 2.483$ (see Table 6). For the classical HWMA and AHWMA charts, we used $w = 0.03$ and $C = 2.272$ (see Table 6). We give the calculations for the AHWMA chart in Table 10, and the results for all the control charts are shown graphically in Figure 1.

The AHWMA chart detected the shift in the process mean faster than any of the other methods. In particular, it detected the shift after the $14th$ sample, whereas the M_XEWMA chart detected the shift after the $15th$ sample, and the classical EWMA, and HWMA charts both detected the shift only after the $18th$ sample.

VII. CONCLUSION AND DISCUSSION

We propose here a new efficient control-chart method, for monitoring small shifts in the process mean where the process variable of interest is correlated with and observed alongside an auxiliary variable. Based on the homogeneously weighted moving average, the proposed chart uses both the process and auxiliary variable to form a regression estimator that yields an efficient and unbiased estimate of the mean of the process variable. We provided the design structure of the chart and examined its performance in terms of its run length properties. Our simulation results showed that the chart detects a shift in the process mean more rapidly than other methods. Also, the *ARL* comparisons showed that the proposed chart is more efficient than existing control charts used for the same purpose, especially when interest lies in detecting a small shift in the process mean. We provided a detailed study of the chart's robustness to non-normality. The chart's *ARL* values showed that the chart is more robust to non-normality when a smaller value of w is used. In particular, when a small value is chosen for the chart's smoothing parameter (for example $w \leq 0.05$), the proposed chart can be designed to have an in-control *ARL* that is reasonably close to the *ARL* for the chart under a normally distributed process. We gave some recommendations on the application of the chart when the process parameters are unknown, and provided a step-by-step algorithm to construct the chart for phase I and phase II of SPC. Also, we applied the chart to a simulated dataset and showed that it detected a small shift in the process mean faster than other examined charts including EWMA, HWMA, M_XEWMA, and A_{ux}CUSUM$_2$ methods. We consider that the effect of estimating parameters during phase I of the process on subsequent performance of the AHWMA chart warrants further study.

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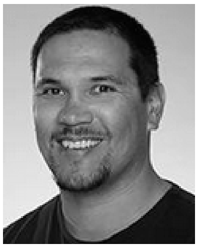

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