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# New QAM Complementary Sequences for Control of Peak Envelope Power of OFDM Signals

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**ABSTRACT** By developing new mathematical descriptions of $4^q$ (integer $q \geq 2$) quadrature amplitude modulation (QAM) constellation, a novel construction producing $4^q$-QAM complementary sequences (CSs) of length $2^m$ (integer $m \geq 2$) is presented. The proposed sequences include the known $4^q$-QAM CSs constructed from Cases I to III constructions, proposed by Li, as special cases. For 16-QAM CSs, the number of the resultant sequences is determined precisely. New sequences have a larger family size so as to increase the code rates. When used in orthogonal frequency-division multiplexing (OFDM) systems, new sequences possess the same peak envelope power (PEP) upper bounds as those of the known sequences referred to above.

**INDEX TERMS** Quadrature amplitude modulation complementary sequences, standard Golay-Davis-Jedwab complementary sequences, generalized Boolean function, code rates, peak envelope power (PEP).

## I. INTRODUCTION

IN contemporary communication, orthogonal frequency-division multiplexing (OFDM) communication systems are preferred, due to their own special advantages. However, high peak envelope power (PEP) of OFDM signals brings great challenge for OFDM systems' implementation. It has been proven that OFDM signals, encoded by complementary sequences (CSs), possess low peak envelope power (PEP) upper bounds [1]–[3]. On the other hand, quadrature amplitude modulation (QAM) symbols are widely applied to high rate OFDM transmissions. Hence, it is natural to explore QAM CSs [4]–[11].

In general, QAM CSs are produced by employing mathematical descriptions of QAM constellation. There exist two such descriptions. One of them is based on quadriphase inputs [4], which is called Q-type description for short (see Section IV for its concept). The other is on basis of binary inputs, which is referred to as B-type description [26]. Q-type description can be divided into Q-type-1 and Q-type-2 cases (see Section IV for their concepts). Based on Q-type-1 description, a large number of QAM CSs are presented [4]–[18]. In 2003, Chong *et al.* [4] constructed 16-QAM CSs by adding an offset and a pair difference (see Theorem 6 in Section V for their definitions). In 2006, Lee and Golomb [5] proposed 64-QAM CSs by adding two offsets and gave some conjectures. In 2008, Li [6] made some comments to both [4] and [5]. In 2010, Chang *et al.* [7] gave new 64-QAM CSs and completed the verification of the conjectures in [5]. Also, Li [8] gave general QAM CSs' construction, called Cases I-III constructions, by adding several offsets, and determined their family size. Interestingly, Cases I-III constructions in [8] includes all previously known QAM CSs as special cases. It should be noted that the family size of this general construction is mainly determined by the number of the offsets, which implies a possible approach of expansion of family size: enlarging the number of the offsets. In 2013, Liu et al. [9] presented Cases IV-V constructions by introducing nonsymmetrical Gaussian integer pair and constructed the larger family size for fairly long QAM CSs (typically, length $> 2^{122}$ for 64-QAM CSs). Notice that all the aforementioned constructions are on the basis of standard

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quadrature phase shift keying (QPSK) Golay-Davis-Jedwab (GDJ) CSs [3]. However, there exist a large number of non-standard QPSK GDJ CSs [19] [20]. In 2014, Zeng *et al.* [10] proposed 16-QAM CSs from non-standard QPSK GDJ CSs. In the following year, Zeng and Zhang [11] gave a brief derivation on Cases I-III constructions. In addition, it should be noted that the lengths of all the QAM CSs referred to above must be a power of 2. Fortunately, many researchers [12]- [18] provided the QAM CSs with more lengths so as to overcome such length confinement.

Family size of QAM CSs is important, due to the fact that code rates are decided by this parameter. In [4], it has been pointed out that in an OFDM system, number of sub-carriers with code rate not excelling one is not fit. As a consequence, expansion of family size of QAM CSs is an efficient approach for improving code rates.

In this paper, by developing new mathematical description Q-type-2 of QAM constellation, a new family of $4^q$-QAM CSs is presented. New sequences include ones from Cases I-III constructions in [8] as special cases, and have lager family size so as to increase the code rates. Besides, [8] and this paper have the same PEP upper bounds.

The rest of this paper is organized as follows. In Section II, we briefly recall definitions of CSs and generalized Boolean functions (GBFs), and relevant conclusions. In Section III, roles of CSs in the control of PEP of OFDM signals are introduced. In the following section, Q-type mathematical description of QAM constellation is investigated. New QAM CSs from Q-type-2 description are presented in Section V. In Section VI and VII, family size and PEP upper bounds of the resultant sequences are discussed, respectively. Comparisons between the existing relevant QAM CSs and two examples follow in next two sections, respectively. Finally, the conclusion remarks appear in Section X.

## II. DEFINITION OF CSs AND STANDARD $2^h$-PSK GDJ CSs

Consider two complex sequences: $\underline{A} = (A_0, A_1, \cdots, A_{N-1})$ and $\underline{B} = (B_0, B_1, \cdots, B_{N-1})$ of length $N$. We define

$$C_{A,B}(\tau) = \begin{cases} \sum_{i=0}^{N-1-\tau} A_i\overline{B_{i+\tau}} & 0 \le \tau \le N-1 \\ \sum_{i=0}^{N-1+\tau} A_{i-\tau}\overline{B_i} & 1-N \le \tau < 0 \\ 0 & |\tau| \ge N. \end{cases} \quad (1)$$

If $\underline{A} = \underline{B}$, we refer to $C_{A,A}(\tau)$ as an aperiodic autocorrelation function, otherwise, we say $C_{A,B}(\tau)$ to be an aperiodic cross-correlation function.

If two sequences $\underline{A}$ and $\underline{B}$ satisfy

$$C_{A,A}(\tau) + C_{B,B}(\tau) = 0 \ \ (\forall\, \tau \ne 0), \quad (2)$$

we say these two sequences to be CSs, and each of them is referred to as a Golay sequence.

Up to now, there have existed many methods producing CSs, therein the methods based on standard GBFs are attractive [3], and the resultant CSs are called standard $2^h$ phase shift keying (PSK) Golay-Davis-Jedweb (GDJ) CSs. Here, we briefly recall standard $2^h$-PSK GDJ CSs and their main conclusions.

Let $Z_{2^h} = \{0, 1, 2, \cdots, 2^h - 1\}$ (integer $h \ge 1$). The following GBFs:

$$f(\underline{x}) = 2^{h-1}\sum_{k=1}^{m-1} x_{\pi(k)}x_{\pi(k+1)} + \sum_{k=1}^{m} c_k x_k + c, \quad (3)$$

are said to be standard GBFs, where $\pi$ means a permutation of the symbol set $\{1, 2, \cdots, m\}$, and $c$, $c_k \in Z_{2^h}$ $(1 \le k \le m)$. Apparently, for given $m$-dimensional vector $\underline{x} = (x_1, x_2, \cdots, x_m) \in Z_2^m$, we have the Boolean function value $f(\underline{x}) \in Z_{2^h}$. When the $m$-dimensional vector $\underline{x}$ ranges over the range from $(0, \cdots, 0)$ to $(1, \cdots, 1)$, $2^m - 1$ function values in $Z_{2^h}$ are educed. Note that the $m$-dimensional vectors from $(0, \cdots, 0)$ to $(1, \cdots, 1)$ are exactly the binary representations of the integers from 0 to $2^m - 1$. Thus, we can obtain a sequence, denoted by $\underline{f} = (f_0, f_1, \cdots, f_{2^m-1})$ regardless of the difference between the integers and the vectors, of length $N = 2^m$ over $Z_{2^h}$, where $f_i = f(i_1, i_2, \cdots, i_m)$, and $i = \sum_{k=1}^{m} i_k 2^{m-k}$. The relevant conclusions of the standard $2^h$-PSK GDJ CSs [3] are stated below.

*Lemma 1 ( [3], Corollary 4):* Let

$$\begin{cases} a(\underline{x}) = f(\underline{x}) \\ b(\underline{x}) = f(\underline{x}) + 2^{h-1}x_{\pi(1)} + c', \end{cases} \quad (4)$$

where $c' \in Z_{2^h}$. Then, the sequences $\underline{a}$ and $\underline{b}$ form $2^h$-PSK CSs over $Z_{2^h}$ with length $N = 2^m$.

*Lemma 2 ( [3], Corollary 4):* There are $2^{h(m+1)}(m!/2)$ standard GBFs over $Z_{2^h}$ or standard $2^h$-PSK Golay sequences $\underline{a}$'s.

*Lemma 3 ( [3]):* Under Lemma 1, consider two non-negative integers $\tau$ $(0 < \tau \le N-1)$ and $i$ $(0 \le i \le N-1)$. Let $\xi = \exp(2\pi j/2^h)$ $(j = \sqrt{-1})$. If $i_{\pi(1)} = (i+\tau)_{\pi(1)}$, for $1 \le k \le m$ and given $\tau$ there must exist two integers $i'$ $(0 \le i' \le N-1)$, and $v$ $(1 \le v \le m)$ satisfying

$$\begin{cases} v = \inf\{k | i_{\pi(k)} \ne (i+\tau)_{\pi(k)}, 1 \le k \le m\} & (5a) \\ i_{\pi(k)} + (i'+\tau)_{\pi(k)} = 1 \quad (k < v) & (5b) \\ (i+\tau)_{\pi(k)} + i'_{\pi(k)} = 1 \quad (k < v) & (5c) \\ i_{\pi(k)} = i'_{\pi(k)} \quad (k \ge v) & (5d) \\ (i+\tau)_{\pi(k)} = (i'+\tau)_{\pi(k)} \quad (k \ge v) & (5e) \\ \xi^{a_{i'} - a_{i'+\tau}} = -\xi^{a_i - a_{i+\tau}}, & (sf) \end{cases}$$

and there exists a 1-1 corresponding between the integer pairs $(i, i+\tau)$ and $(i', i'+\tau)$.

Lemma 3 is educed by the derivation of Case 2 in Theorem 3 in [3]. For the reader who needs more detailed explanations of Lemma 3, please see Case 1 in Theorem 4 in [4].

## III. REDUCTION OF PEP OF ENCODED OFDM SIGNALS

The PEP of OFDM signals, encoded by CSs, can be better controlled, which owes to complementary correlation property of CSs. By employing well-designed CSs of length $N$ over binary, quaternary, or polyphase constellations, upper bound of accompanying PEP does not exceed $2N$.

In an OFDM communication system of $N$ sub-carriers, its $i$-th sub-carrier has the frequency $f_i$, where $f_i = f_0 + i\Delta f$ $(0 \le i \le N-1)$, $f_0$ is the carrier frequency, and $\Delta f$ is the spacing frequency between sub-carriers. A complex signal $s_{u^{(r)}}(t)$ $(0 \le t \le T_s)$, encoded by the sequence $\underline{u}^{(r)} = (u_0^{(r)}, u_1^{(r)}, \cdots, u_{N-1}^{(r)})$ of length $N$, is expressed by

$$s_{u^{(r)}}(t) = \sum_{k=0}^{N-1} u_k^{(r)} e^{2\pi j f_k t}. \tag{6}$$

Thus, the transmitted OFDM signal in the OFDM system is exactly the real part of $s_{u^{(r)}}(t)$. Further, the instantaneous envelope power $P_{u^{(r)}}(t)$ of the transmitted OFDM signal can be calculated by [4]

$$\begin{aligned} P_{u^{(r)}}(t) &= |s_{u^{(r)}}(t)|^2 \\ &= C_{u^{(r)},u^{(r)}}(0) + \sum_{l \neq 0} C_{u^{(r)},u^{(r)}}(l) e^{2\pi j l \Delta f t}. \end{aligned} \tag{7}$$

It has been proved that the instantaneous envelope power of an OFDM system is bounded by [3]

$$P_{u^{(r)}}(t) \le N^2, \tag{8}$$

and the worst case can be attained when this system is uncoded, that is, $\underline{u}^{(r)} = (1, \cdots, 1)$.

Let $\underline{u}^{(r)}$ and $\underline{u}^{(s)}$ form the CSs. By making use of the definition of CSs, we apparently have

$$P_{u^{(r)}}(t) + P_{u^{(s)}}(t) = C_{u^{(r)},u^{(r)}}(0) + C_{u^{(s)},u^{(s)}}(0). \tag{9}$$

Let $C$ be a code whose codewords consist of sequences. We define

$$\begin{aligned} \mathrm{PEP}(\underline{u}_r) &= \sup_{t \in [0, T_s]} P_{u^{(r)}}(t) \\ \mathrm{PEP}(C) &= \max\{\mathrm{PEP}(\underline{u}^{(r)}) | \forall\, \underline{u}^{(r)} \in C\}. \end{aligned} \tag{10}$$

When an OFDM signal is encoded by well-designed CSs over binary, quaternary, or polyphase constellations, the following lemma holds.

*Lemma 4 ( [3]):* Let code $C$ consist of binary, quaternary, or polyphase CSs of length $N$. Then, the PEP of the code $C$ satisfies

$$\mathrm{PEP}(C) \le 2N. \tag{11}$$

Lemma 4 is fit for binary, quaternary, and polyphase CSs [21]. However, for QAM CSs, the conclusion on their PEP upper bound is given by

*Lemma 5 ( [8]):* Let code $C$ consist of $4^q$-QAM general CSs of length $N$ from Cases I-III constructions in [8]. Then, the PEP of the code $C$ satisfies

$$\mathrm{PEP}(C) \le \frac{6N(2^q - 1)}{2^q + 1}. \tag{12}$$

Here is an example to highlight the importance of CSs for reduction of PEP.

*Example 1:* Consider 4 sub-carriers system. Take the 16-QAM CSs $\underline{A} = \frac{1}{\sqrt{10}}(1+j, 1+j, 1+j, 1+j)$ and $\underline{B} = \frac{1}{\sqrt{10}}(1+j, -1+3j, 1-3j, -1-j)$, of length $N = 4$ [23]. Apparently, for the encoded OFDM signal, we have

$$P_A(t) + P_B(t) = C_{A,A}(0) + C_{B,B}(0) = 3.2 = 0.8N.$$

Fig.1 simulates the uncoded and encoded OFDM signals, respectively. Clearly, the uncoded OFDM signal has maximum magnitude 4. Whereas, the encoded OFDM signal is bounded to maximum magnitude 2. Hence, the improvement of PEP of the encoded OFDM system is obvious.

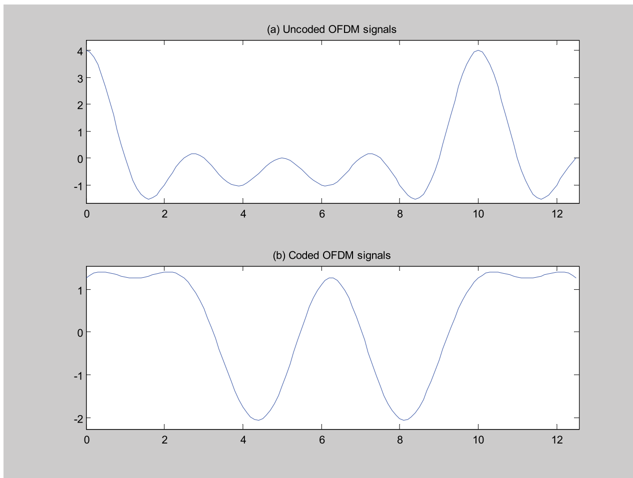


**FIGURE 1.** 4 sub-carriers uncoded and coded OFDM signals.

## IV. Q-TYPE DESCRIPTIONS OF QAM CONSTELLATION

The QAM constellation with order $4^q$ (positive integer $q \ge 2$) is the following symbol set: [22]

$$\Omega_{4^q\text{-QAM}} \overset{def}{=} \{a + bj | -2^q + 1 \le a, b \le 2^q - 1, a, b \text{ odd}\}.$$

There are two methods to produce $4^q$-QAM constellation, therein, one of them is based on quaternary inputs, called Q-type descriptions for short, including two cases: Q-type-1 and Q-type-2.

*Q-type-1 [12], [22]:* The Q-type-1 description consists of $q$ independent quaternary input variables below.

$$\left\{(1+j)\sum_{p=0}^{q-1} 2^p j^{u^{(p)}} | u^{(p)} \in Z_4, 0 \le p \le q-1\right\}. \tag{13}$$

When the $q$-dimensional vector $(u^{(0)}, u^{(1)}, \cdots, u^{(q-1)})$ ranges over the range from $(0, \cdots, 0)$ to $(3, \cdots, 3)$ (total number $4^q$), Q-type-1 is exactly equivalent to $4^q$-QAM constellation. This description is important due to the fact that all the existing QAM CSs in [4]- [18] are educed by it.

*Q-type-2:* This is a new description presented by this paper, and is also the fundamental of the coming QAM CSs' investigation. The Q-type-2, consisting of $2^q - 1$ independent quaternary input variables, is given by

$$\left\{(1+j)\sum_{p=0}^{2^q-2} j^{\nu^{(p)}} | \nu^{(p)} \in Z_4, 0 \le p \le 2^q-2\right\}. \quad (14)$$

*Proof:* We need to verify two cases: (1) each symbol in Q-type-2 must belong to the set $\Omega_{4^q\text{-QAM}}$; (2) each $4^q$-QAM symbol can be produced by Q-type-2.

(1) For $\forall\ (\nu^{(0)}, \nu^{(1)}, \cdots, \nu^{(2^q-2)}) \in Z_4^{2^q-1}$ whose symbol distribution is given by

$$\begin{cases} n_0 = |\{p|\nu^{(p)} = 0, 0 \le p \le 2^q-2\}| \\ n_1 = |\{p|\nu^{(p)} = 1, 0 \le p \le 2^q-2\}| \\ n_2 = |\{p|\nu^{(p)} = 2, 0 \le p \le 2^q-2\}| \\ n_3 = |\{p|\nu^{(p)} = 3, 0 \le p \le 2^q-2\}|, \end{cases} \quad (15)$$

Q-type-2 description produces the symbol:

$$\begin{aligned} A &= (1+j)\sum_{p=0}^{2^q-2} j^{\nu^{(p)}} = (1+j)\Big[(n_0-n_2)+j(n_1-n_3)\Big] \\ &= (n_0-n_2-n_1+n_3)+j(n_0-n_2+n_1-n_3). \end{aligned}$$

Notice that $n_0+n_2+n_1+n_3 = 2^q-1$ (odd number). Hence, in these four integers $n_0$, $n_1$, $n_2$, and $n_3$, only two cases: "one odd-three even" or "one even-three odd" appear. Clearly, both two cases result in the fact that both the integers "$n_0-n_2-n_1+n_3$" and "$n_0-n_2+n_1-n_3$" are odd. On the other hand, apparently, we have

$$\begin{aligned} -(2^q-1) \le n_0-n_2-n_1+n_3, n_0-n_2 \\ +n_1-n_3 \le 2^q-1. \end{aligned}$$

Summarizing the above, we come to the conclusion: $A \in \Omega_{4^q\text{-QAM}}$.

(2) Consider $\forall\ a+jb \in \Omega_{4^q\text{-QAM}}$. We set

$$\sum_{k=0}^{2^q-2} j^{\nu^{(k)}} = \frac{a+jb}{1+j} = \frac{a+b}{2} + j\frac{b-a}{2}. \quad (16)$$

Now, we need to determine the components of $(2^q-1)$-dimensional vector $(\nu^{(0)}, \nu^{(1)}, \cdots, \nu^{(2^q-2)})$ over $Z_4^{2^q-1}$ to guarantee that (16) holds. A direct scheme is given below.

*Step 1:* Arbitrarily choose $|\frac{a+b}{2}|$ "0's" or "2's" in this vector, depending on $\frac{a+b}{2}$ positive or negative.

*Step 2:* Arbitrarily choose $|\frac{b-a}{2}|$ "1's" or "3's" in the other components except those components chosen in Step 1, depending on $\frac{b-a}{2}$ positive or negative.

*Step 3:* Arbitrarily and pairwise choose "0 and 2" or "1 and 3" in the unused components in Steps 1 and 2 so that the sum of powers of $j$ from these unused components vanishes. To clarify this, we set $a = 2a_0+1$ and $b = 2b_0+1$ due to the fact that both $a$ and $b$ are odd, hence we have both $\frac{a+b}{2} = a_0+b_0+1 = (b_0-a_0)+(2a_0+1)$ and $\frac{b-a}{2} = b_0-a_0$, which implies that $|\frac{a+b}{2}|$ and $|\frac{b-a}{2}|$ must be "one odd-one even". Therefore, the number $2^q-1-|\frac{a+b}{2}|-|\frac{b-a}{2}|$ of the unused components must be even, which means that didymous choice operation in Step 3 is feasible.

The above discussions suggest that for each in the set $\Omega_{4^q\text{-QAM}}$, there exists at least a $(2^q-1)$-dimensional vector over $Z_4^{2^q-1}$ so that it is yielded by Q-type-2 description.

Thus, we verify that Q-type-2 description produce all the symbols in the set $\Omega_{4^q\text{-QAM}}$ when the input vector $(\nu^{(0)}, \nu^{(1)}, \cdots, \nu^{(2^q-2)}) \in Z_4^{2^q-1}$ ranges over the range from $(0, \cdots, 0)$ to $(3, \cdots, 3)$. ■

*Remark:* (1) Q-type-1 is a special case of the proposed description.

Apparently, when we set

$$\nu^{(2^p-2+r)} = u^{(p)} \quad (0 \le p \le q-1, 1 \le r \le 2^p), \quad (17)$$

Q-type-2 exactly degenerates to Q-type-1.

(2) A $4^q$-QAM symbol may be mapped by multiple $(2^q-1)$-dimensional vectors in Q-type-2.

**TABLE 1.** Distribution of the symbols in 16-QAM constellation.

| symbol | $\pm1\pm j$ | $\pm1\pm3j$ | $\pm3\pm j$ | $\pm3\pm3j$ |
|---|---|---|---|---|
| freq. | 9 | 3 | 3 | 1 |

**TABLE 2.** Distribution of the symbols in 64-QAM constellation.

| | | | | | | |
|---|---|---|---|---|---|---|
| symbol | $\pm1\pm j$ | $\pm1\pm3j$ | $\pm1\pm5j$ | $\pm1\pm7j$ | $\pm3\pm j$ | $\pm3\pm3j$ |
| freq. | 1125 | 735 | 245 | 35 | 735 | 441 |
| symbol | $\pm3\pm5j$ | $\pm3\pm7j$ | $\pm5\pm j$ | $\pm5\pm3j$ | $\pm5\pm5j$ | $\pm5\pm7j$ |
| freq. | 147 | 21 | 245 | 147 | 49 | 7 |
| symbol | $\pm7\pm j$ | $\pm7\pm3j$ | $\pm7\pm5j$ | $\pm7\pm7j$ | | |
| freq. | 35 | 21 | 7 | 1 | | |

For example, consider the symbol "1+3j" in 16-QAM constellation. Apparently, this symbol can be mapped by three 3-dimensional vectors (0, 0, 1), (0, 1, 0), and (1, 0, 0). Tables 1 and 2 give the distributions of the symbols from Q-type-2 in 16- and 64-QAM constellations, respectively, where the frequency $n$ in the symbol set $\{\pm a \pm bj\}$ means every symbol in this set appears $n$ times when $(2^q-1)$-dimensional vector $(\nu^{(0)}, \nu^{(1)}, \cdots, \nu^{(2^q-2)}) \in Z_4^{2^q-1}$ ranges over the range from $(0, \cdots, 0)$ to $(3, \cdots, 3)$.

## V. NEW QAM CSs FROM Q-TYPE-2

Based on Q-type-2 description, in this section, we will present new QAM CSs. The author's idea is inspired by Chong, *et al.*'s method in [4].

*Theorem 6:* Consider quaternary GBFs $f(\underline{x})$'s with $h = 2$ in (3). Let

$$\begin{cases} a^{(0)}(\underline{x}) = a(\underline{x}) = f(\underline{x}) \\ b^{(0)}(\underline{x}) = a^{(0)}(\underline{x}) + \mu(\underline{x}) = a(\underline{x}) + \mu(\underline{x}) \\ a^{(1)}(\underline{x}) = a(\underline{x}) + s^{(1)}(\underline{x}) \\ b^{(1)}(\underline{x}) = a^{(1)}(\underline{x}) + \mu(\underline{x}) = a(\underline{x}) + s^{(1)}(\underline{x}) + \mu(\underline{x}) \\ \vdots \\ a^{(2^q-2)}(\underline{x}) = a(\underline{x}) + s^{(2^q-2)}(\underline{x}) \\ b^{(2^q-2)}(\underline{x}) = a^{(2^q-2)}(\underline{x}) + \mu(\underline{x}) \\ \qquad = a(\underline{x}) + s^{(2^q-2)}(\underline{x}) + \mu(\underline{x}). \end{cases} \quad (18)$$

By employing Q-type-2 description in (14), we construct $4^q$-QAM sequences $\underline{A} = (A_0, A_1, \cdots, A_{N-1})$ and $\underline{B} = (B_0, B_1, \cdots, B_{N-1})$ with length $N = 2^m$ by

$$\begin{cases} A_i = \delta\gamma \sum_{p=0}^{2^q-2} j^{a_i^{(p)}} \\ B_i = \delta\gamma \sum_{p=0}^{2^q-2} j^{b_i^{(p)}}, \end{cases} \quad (19)$$

where $\delta = \frac{1}{\sqrt{(4^q-1)/3}}$ and $\gamma = e^{\pi j/4}$. In comparison with (14), a constant $\frac{1}{\sqrt{2(4^q-1)/3}}$ in (19) is added as done in [8]. Then, the proposed QAM sequences $\underline{A}$ and $\underline{B}$ are $4^q$-QAM CSs if the offsets $s^{(p)}(\underline{x})$ $(1 \le p \le 2^q - 2)$ and the pairing difference $\mu(\underline{x})$ satisfy one of the following cases, where $d_l^{(p)} \in Z_4$ $(1 \le p \le 2^q - 2, 0 \le l \le 2)$.

*Case I:* $\mu(\underline{x}) = 2x_{\pi(m)}$

$$s^{(p)}(\underline{x}) = d_0^{(p)} + d_1^{(p)} x_{\pi(1)} \ (1 \le p \le 2^q - 2). \quad (20)$$

*Case II:* $\mu(\underline{x}) = 2x_{\pi(1)}$

$$s^{(p)}(\underline{x}) = d_0^{(p)} + d_1^{(p)} x_{\pi(m)} \ (1 \le p \le 2^q - 2). \quad (21)$$

*Case III:* $\mu(\underline{x}) = 2x_{\pi(1)}$ or $2x_{\pi(m)}$

$$s^{(p)}(\underline{x}) = d_0^{(p)} + d_1^{(p)} x_{\pi(w)} + d_2^{(p)} x_{\pi(w+1)}$$
$$\text{with} \quad 2d_0^{(p)} + d_1^{(p)} + d_2^{(p)} \equiv 0 \ (\text{mod } 4)$$
$$(1 \le p \le 2^q - 2, 1 \le w \le m-1). \quad (22)$$

*Proof:* In accordance with (1), for $\forall\ \tau > 0$ the aperiodic autocorrelation function of the sequence $\underline{A}$ in (19) is counted by

$$\begin{aligned} \frac{1}{\delta^2} C_{A,A}(\tau) &= \sum_{i=0}^{N-\tau-1} \Big(\sum_{p=0}^{2^q-2} j^{a_i^{(p)}}\Big)\Big(\sum_{r=0}^{2^q-2} j^{-a_{i+\tau}^{(r)}}\Big) \\ &= \sum_{p=0}^{2^q-2}\sum_{r=0}^{2^q-2} C_{a^{(p)},a^{(r)}}(\tau) \\ &= \sum_{p=0}^{2^q-2} C_{a^{(p)},a^{(p)}}(\tau) + \sum_{\substack{0\le p,r\le 2^q-2 \\ p\ne r}} C_{a^{(p)},a^{(r)}}(\tau). \end{aligned}$$

Similarly, we have

$$\frac{1}{\delta^2} C_{B,B}(\tau) = \sum_{p=0}^{2^q-2} C_{b^{(p)},b^{(p)}}(\tau) + \sum_{\substack{0\le p,r\le 2^q-2 \\ p\ne r}} C_{b^{(p)},b^{(r)}}(\tau).$$

According to Lemma 1, the sequences $\underline{a}^{(p)}$ and $\underline{b}^{(p)}$ $(0 \le p \le 2^q - 2)$ form Golay CSs. Consequently, for the sake of guaranteeing that the sequences $\underline{A}$ and $\underline{B}$ are CSs, it is sufficient for us to have

$$\sum_{\substack{0\le p,r\le 2^q-2 \\ p\ne r}} \Big(C_{a^{(p)},a^{(r)}}(\tau) + C_{b^{(p)},b^{(r)}}(\tau)\Big) = 0 \ (\forall\ \tau > 0). \quad (23)$$

For ease of presentation, we write the left-hand side of (23) as $Lhs(\tau)$. Further, $Lhs(\tau)$ can be equivalently expressed by

$$\begin{aligned} Lhs(\tau) = \sum_{r=1}^{2^q-2} \Big[ & C_{a^{(0)},a^{(r)}}(\tau) + C_{b^{(0)},b^{(r)}}(\tau) \\ & + C_{a^{(r)},a^{(0)}}(\tau) + C_{b^{(r)},b^{(0)}}(\tau)\Big] \\ & + \sum_{1\le p<r\le 2^q-2} \Big[C_{a^{(p)},a^{(r)}}(\tau) \\ & + C_{b^{(p)},b^{(r)}}(\tau) + C_{a^{(r)},a^{(p)}}(\tau) + C_{b^{(r)},b^{(p)}}(\tau)\Big]. \quad (24) \end{aligned}$$

Notice that we have

$$j^{a_i^{(p)} - a_{i+\tau}^{(r)}} = j^{a_i + s_i^{(p)} - a_{i+\tau} - s_{i+\tau}^{(r)}} = j^{a_i - a_{i+\tau}} j^{s_i^{(p)} - s_{i+\tau}^{(r)}}$$

and

$$\begin{aligned} j^{b_i^{(p)} - b_{i+\tau}^{(r)}} &= j^{a_i + s_i^{(p)} + \mu_i - a_{i+\tau} - s_{i+\tau}^{(r)} - \mu_{i+\tau}} \\ &= j^{a_i - a_{i+\tau}} j^{s_i^{(p)} - s_{i+\tau}^{(r)}} j^{\mu_i - \mu_{i+\tau}}, \end{aligned}$$

where $s_i^{(0)} = s_{i+\tau}^{(0)} = 0$.

On the basis of (1), by substituting just above two equations into (24), $Lhs(\tau)$ can be equivalently represented as

$$Lhs(\tau) = \sum_{r=1}^{2^q-2} h_1(\tau, r) + \sum_{1\le p<r\le 2^q-2} h_2(\tau, p, r), \quad (25)$$

where

$$h_1(\tau, r) = \sum_{i=0}^{N-\tau-1} j^{a_i - a_{i+\tau}} \Big[j^{s_i^{(r)}} + j^{-s_{i+\tau}^{(r)}}\Big] \cdot \Big[1 + j^{\mu_i - \mu_{i+\tau}}\Big] \quad (26)$$

and

$$\begin{aligned} h_2(\tau, p, r) = \sum_{i=0}^{N-\tau-1} & j^{a_i - a_{i+\tau}} \Big[j^{s_i^{(p)} - s_{i+\tau}^{(r)}} + j^{s_i^{(r)} - s_{i+\tau}^{(p)}}\Big] \\ & \cdot \Big[1 + j^{\mu_i - \mu_{i+\tau}}\Big]. \quad (27) \end{aligned}$$

*Case III:* Due to $j^{\mu_i - \mu_{i+\tau}} = (-1)^{i_{\pi(1)} - (i+\tau)_{\pi(1)}}$, apparently, whenever $i_{\pi(1)} \ne (i+\tau)_{\pi(1)}$ we have both $h_1(\tau, r) = 0$ and $h_2(\tau, p, r) = 0$ for $\forall\ \tau > 0, p, r$. As a consequence, the key point to verify that (23) holds is that for $\forall\ \tau > 0, p, r$, the functions $h_1(\tau, r)$ and $h_2(\tau, p, r)$ vanish simultaneously whenever $i_{\pi(1)} = (i+\tau)_{\pi(1)}$.

For given $\tau > 0$, we denote

$$I_\tau = \{i | i_{\pi(1)} = (i+\tau)_{\pi(1)}, 0 \le i \le N - \tau - 1\}. \quad (28)$$

In accordance with Lemma 3, we have

$$\begin{cases} I_\tau = I_{1,\tau} \bigcup I_{2,\tau} \\ I_{1,\tau} \bigcap I_{2,\tau} = \phi \ (\text{empty set}), \end{cases} \quad (29)$$

where

$$I_{1,\tau} = \{i | (i, i') \text{ defined by Lemma3}, 0 \le i, i' \le N - \tau - 1\}$$

and

$$I_{2,\tau} = \{i' | (i, i') \text{ defined by Lemma3}, 0 \le i, i' \le N - \tau - 1\}.$$

Note that $1 + j^{\mu_i - \mu_{i+\tau}} = 2$ whenever $i_{\pi(1)} = (i+\tau)_{\pi(1)}$. Hence, the functions $h_1(\tau, r)$ and $h_2(\tau, p, r)$ are respectively reduced to

$$h_1(\tau, r) = 2\sum_{i \in I_{1,\tau}} j^{a_i - a_{i+\tau}} \left[ j^{s_i^{(r)}} + j^{-s_{i+\tau}^{(r)}} \right] + 2\sum_{i' \in I_{2,\tau}} j^{a_{i'} - a_{i'+\tau}} \left[ j^{s_{i'}^{(r)}} + j^{-s_{i'+\tau}^{(r)}} \right] \quad (30)$$

and

$$h_2(\tau, p, r) = 2\sum_{i \in I_{1,\tau}} j^{a_i - a_{i+\tau}} \left[ j^{s_i^{(p)} - s_{i+\tau}^{(r)}} + j^{s_i^{(r)} - s_{i+\tau}^{(p)}} \right] + 2\sum_{i' \in I_{2,\tau}} j^{a_{i'} - a_{i'+\tau}} \left[ j^{s_{i'}^{(p)} - s_{i'+\tau}^{(r)}} + j^{s_{i'}^{(r)} - s_{i'+\tau}^{(p)}} \right]. \quad (31)$$

Consider integers $i$, $i'$, and $v$ defined by Lemma 3. Since the offset $s^{(f)}(\underline{x})$ $(1 \le f \le 2^q - 2)$ in (22) satisfies one of the following two cases.

*Case A: $w \ge v$.*

According to (5d) and (5e) in Lemma 3, we have

$$\begin{cases} s_i^{(f)} = s_{i'}^{(f)} \\ s_{i+\tau}^{(f)} = s_{i'+\tau}^{(f)}, \end{cases} \quad (32)$$

which results in that in (30) and (31), the following two equations:

$$j^{s_i^{(r)}} + j^{-s_{i+\tau}^{(r)}} = j^{s_{i'}^{(r)}} + j^{-s_{i'+\tau}^{(r)}} \quad (33)$$

and

$$j^{s_i^{(p)} - s_{i+\tau}^{(r)}} + j^{s_i^{(r)} - s_{i+\tau}^{(p)}} = j^{s_{i'}^{(p)} - s_{i'+\tau}^{(r)}} + j^{s_{i'}^{(r)} - s_{i'+\tau}^{(p)}} \quad (34)$$

hold.

Combining (5f), (33), and (34), we come to the conclusion that both integers $i$ and $i'$ contribute zero to both the functions $h_1(\tau, r)$ and $h_2(\tau, p, r)$. After all the integer pairs $(i, i')$'s defined by Lemma 3 are used, the functions $h_1(\tau, r)$ and $h_2(\tau, p, r)$ vanish synchronously.

*Case B: $w < v$.*

According to (5b) and (5c) in Lemma 3, and (22), we have

$$\begin{cases} s_i^{(f)} + s_{i'+\tau}^{(f)} = 2d_0^{(f)} + d_1^{(f)}[i_{\pi(w)} + (i'+\tau)_{\pi(w)}] + \\ d_2^{(f)}[i_{\pi(w+1)} + (i'+\tau)_{\pi(w+1)}] = 2d_0^{(f)} + d_1^{(f)} + d_2^{(f)} \equiv 0 \\ s_{i+\tau}^{(f)} + s_{i'}^{(f)} = 2d_0^{(f)} + d_1^{(f)}[(i+\tau)_{\pi(w)} + i'_{\pi(w)}] + \\ d_2^{(f)}[(i+\tau)_{\pi(w+1)} + i'_{\pi(w+1)}] = 2d_0^{(f)} + d_1^{(f)} + d_2^{(f)} \equiv 0. \end{cases} \quad (35)$$

Thus, from (35) we obtain

$$\begin{cases} j^{s_i^{(r)}} = j^{-s_{i'+\tau}^{(r)}} \\ j^{-s_{i+\tau}^{(r)}} = j^{s_{i'}^{(r)}} \\ j^{s_i^{(p)} - s_{i+\tau}^{(r)}} = j^{s_{i'}^{(r)} - s_{i'+\tau}^{(p)}} \\ j^{s_i^{(r)} - s_{i+\tau}^{(p)}} = j^{s_{i'}^{(p)} - s_{i'+\tau}^{(r)}}, \end{cases} \quad (36)$$

which results in that (33) and (34) hold synchronously. With the same reasons in Case A, thus, the functions $h_1(\tau, r)$ and $h_2(\tau, p, r)$ must vanish synchronously.

By summarizing the above, the conclusion in Case III follows immediately.

*Cases I and II:* Employ the technique used in [4]. For Case II, (33) and (34) must hold due to $w = m \ge v$. Go through the same discussions as Case A in Case III, the conclusion in Case II holds. For Case I, by employing the mapping $\pi'(w) = m - 1 - \pi(w)$, the conclusion in Case I is direct. ■

Finally, all the solutions of the congruence equation in (22) are given in Table 3 [8].

**TABLE 3. All the solutions of congruence equation in (22).**

| $(d_0^{(p)}, (d_1^{(p)}, (d_2^{(p)})$ with $2d_0^{(p)} + (d_1^{(p)} + (d_2^{(p)} \equiv 0 \pmod 4)$ | | | | | | | |
|---|---|---|---|---|---|---|---|
| (0,0,0) | (0,1,3) | (0,2,2) | (0,3,1) | (1,0,2) | (1,1,1) | (1,2,0) | (1,3,3) |
| (2,0,0) | (2,1,3) | (2,2,2) | (2,3,1) | (3,0,2) | (3,1,1) | (3,2,0) | (3,3,3) |

## VI. FAMILY SIZE OF NEW QAM CSs

The family size of CSs is important, due to the fact that the family size influences the choice of sub-carrier number of an OFDM system. In [4], the code rate is introduced to guide the choice of sub-carrier number. Let a code $C$ consist of sequences of length $N$. The code rate of the code $C$ is given by [4]

$$R(C) = \frac{\log_2^{|C|}}{N}, \quad (37)$$

where $|C|$ stands for cardinality of the code $C$.

It has been known that such a choice of sub-carrier number with code rate not excelling 1 is not fit, and high code rates are always anticipated [4].

In order to investigate the improvement of the code rate of new QAM CSs, their family size needs to be calculated. For the sake of easy discussion, we consider two cases: 16-QAM CSs and higher-order QAM CSs.

*Case i:* 16-QAM CSs.

In this case, our construction in (19) is simplified as

$$\begin{cases} A_i = \delta\gamma\left(j^{a_i^{(0)}} + j^{a_i^{(0)} + s_i^{(1)}} + j^{a_i^{(0)} + s_i^{(2)}}\right) \\ B_i = \delta\gamma\left(j^{a_i^{(0)} + \mu_i} + j^{a_i^{(0)} + s_i^{(1)} + \mu_i} + j^{a_i^{(0)} + s_i^{(2)} + \mu_i}\right), \end{cases} \quad (38)$$

which implies that for given the GBF $f(\underline{x})$ and the pair difference $\mu(\underline{x})$, the 16-QAM CS pair $(\underline{A}, \underline{B})$ is determined, only depending on the offset pair $(s^{(1)}(\underline{x}), s^{(2)}(\underline{x}))$. As a result, it is sufficient to calculate the number of the offset pairs for the family size of the proposed 16-QAM CSs. Notice that for two arbitrary offset pairs $(g(\underline{x}), k(\underline{x}))$ and $(u(\underline{x}), v(\underline{x}))$, when they satisfy both $g(\underline{x}) = v(\underline{x})$ and $k(\underline{x}) = u(\underline{x})$, the two 16-QAM CS pairs, educed by these two offset pairs referred to above, are identical. We call such two offset pairs *overlapping*. Hence, the crux to calculate the family size is to count the number of non-overlapping offset pairs. We consider two cases below.

*(1).* $s^{(1)}(\underline{x}) = s^{(2)}(\underline{x})$.

In this case, our construction degenerates to the one in [8]. By using the results in Corollary 3 in [8], we have $12m + 14$ non-overlapping offset pairs.

*(2).* $s^{(1)}(\underline{x}) \neq s^{(2)}(\underline{x})$.

Employing the method in [4] so as to divide possible offsets into the following five sets with empty pairwise intersection. Combining the possible offset coefficients $(d_0^{(p)}, d_1^{(p)}, d_2^{(p)})$'s which are listed in Table 1 in [8], we have

$$S_1 = \{d_0 | d_0 = 0, 1, 2, 3\};$$

$$S_2 = \{d_0 + d_1 x_{\pi(1)} | d_0, d_1 \in Z_4, d_1 \neq 0\};$$

$$S_3 = \{d_0 + d_1 x_{\pi(m)} | d_0, d_1 \in Z_4, d_1 \neq 0\};$$

$$S_4 = \{d_0 + d_1 x_{\pi(w)} | (d_0, d_1) = (1, 2), (3, 2)\} \ (2 \leq w \leq m-1);$$

$$S_5 = \{d_0 + d_1 x_{\pi(w)} + d_2 x_{\pi(w+1)} | (d_0, d_1, d_2) = (0, 1, 3), (0, 2, 2), (0, 3, 1), (1, 1, 1), (1, 3, 3), (2, 1, 3), (2, 2, 2), (2, 3, 1), (3, 1, 1), (3, 3, 3)\} \ (1 \leq w \leq m-1),$$

where the sets $S_2$ and $S_3$ belong to Cases 1 and 2, respectively, and the sets $S_1$, $S_4$, and $S_5$ are in Case 3 in Theorem 6 synchronously.

In order to obtain non-overlapping offset pairs $(s^{(1)}(\underline{x}), s^{(2)}(\underline{x}))$'s, we use the following selection strategy. Consider selection of offset pairs in $S_i$ $(1 \leq i \leq 5)$. Step 1: arbitrarily choose an offset in $S_i$, written by $\Xi_1$, as $s^{(1)}(\underline{x})$, and arbitrarily take an offset in $S_i - \Xi_1$ for $s^{(2)}(\underline{x})$, which can produce $|S_i| - 1$ possible offset pairs. Step 2: in $S_i - \Xi_1$, arbitrarily choose an offset (written by $\Xi_2$), and arbitrarily take an offset in $S_i - \Xi_1 - \Xi_2$ for $s^{(2)}(\underline{x})$, which can produce $|S_i| - 2$ possible offset pairs. So continue, until we have $|S_i - \Xi_1 - \Xi_2 - \cdots| \leq 1$. Summarizing up the above, we obtain $(|S_i| - 1) + (|S_i| - 2) + \cdots + 1$ possible offset pairs with no overlaps to one another.

According to the above strategy, (a) in $S_1$, we have $3+2+1 = 6$ possible offset pairs; (b) in $S_2$, we have $11 + 10 + \cdots + 1 = 66$ possible offsets pairs; (c) in $S_3$, this case is the same as Case (b); (d) in $S_4$, only one pair exists. However, the parameter $w$ can vary from 2 to $m-1$. Hence, we have $m-2$ possible offset pairs; (e) in $S_5$, there are $9+8+\cdots+1 = 45$ possible offset pairs, in which each has the parameter $w$ that can vary from 1 to $m-1$. Thus, this case has $45(m-1)$ pairs in total. Summarizing up (a)-(e), we totally have $46m + 91$ possible offset pairs.

Summing up Cases (1) and (2), we come to the conclusion that the total number of non-overlapping offset pairs is $(12m+14)+(46m+91) = 58m+105$. According to Lemma 2, hence, the family size of new sequences is given by the following theorem.

*Theorem 7:* The number of 16-QAM CSs, educed by Theorem 6, of length $2^m$ is

$$(58m + 105)(m!/2)4^{m+1} \quad (m \geq 2, q \geq 2). \tag{39}$$

**TABLE 4.** Comparison of the code rates and family sizes of 16-QAM CSs.

| Ref. | [8] | this paper |
|---|---|---|
| size $\lvert C\rvert$ | $(12m + 14)(m!/2)4^{m+1}$ | $(58m + 105)(m!/2)4^{m+1}$ |
| Rates | $\log_a \lvert C\rvert/2^m$ | $\log_a \lvert C\rvert/2^m$ |
| $m = 2$ | 2.8123 | 3.4457 |
| $m = 3$ | 1.9038 | 2.2132 |
| $m = 4$ | 1.2213 | 1.3997 |
| $m = 5$ | 0.7537 | 0.8291 |
| $m = 6$ | 0.4519 | 0.4893 |

Table 4 gives the comparison of the code rates and family sizes between [8] and this paper, in which it is intuitive that this paper results in the increase of code rates.

*Case ii:* Higher-order QAM CSs.

Due to the fact that the number of the offsets increases exponentially, the enumeration of the proposed QAM CSs becomes difficult, since the method in Case i does not work well. The authors would like to invite the reader to join us so as to solve such an enumeration.

*Open Problem:* How many are the resultant higher-order QAM CSs?

## VII. PEP UPPER BOUNDS

In this section, the PEP of new QAM CSs will be investigated.

*Theorem 8:* Let the code $C$ consist of $4^q$-QAM CSs from Theorem 6 with length $N = 2^m$. Then, the PEP of the code $C$ is bounded by

$$\mathrm{PEP}(C) \leq \frac{6N(2^q - 1)}{2^q + 1}. \tag{40}$$

*Proof:* For $\forall \underline{A} \in C$, let $(\underline{A}, \underline{B})$ is a pair of $4^q$-QAM CSs from Theorem 6 with length $N = 2^m$. The PEP of the sequence $\underline{A}$ satisfies $\mathrm{PEP}(\underline{A}) \leq 2 \sum_{i=0}^{N-1} |A_i|^2$ [8]. From (19), we have

$$|A_i|^2 = \left|\delta \sum_{p=0}^{2^q-2} j^{a_i^{(p)}}\right|^2 \leq \delta^2 \sum_{p=0}^{2^q-2} 1 = \frac{3(2^q - 1)}{2^q + 1}. \tag{41}$$

According to (10), this theorem follows immediately. ■

Compared with Lemma 5, $4^q$-QAM CSs in [8] and this paper have the same PEP upper bounds.

## VIII. RELEVANT COMPARISONS

Only consider the existing $4^q$-QAM CSs based on the standard GBFs, which can be divided into quaternary input and binary input. The former includes the references [4], [5], [6], [7], [8], [9], [11], and this paper, and the latter is rare, only including [24], [25], and [26].

### A. NEW $4^q$-QAM CSs INCLUDE ONES FROM CASES I-III CONSTRUCTION AS SPECIAL CASES

Notice that [8] includes the references [4], [5], and [7] as special cases. Consequently, it is sufficient to compare [8] and this paper.

[8] is based on Q-type-1 and this paper is on basis of Q-type-2. Apparently, when we set

$$\begin{cases} s^{(0)}(\underline{x}) = s^{(1)}(\underline{x}) = \cdots = s^{(2^{q-1}-1)}(\underline{x}) = 0 \\ s^{(2^{q-1})}(\underline{x}) = s^{(2^{q-1}+1)}(\underline{x}) = \cdots = s^{(2^{q-1}+2^{q-2}-1)}(\underline{x}) \\ s^{(2^{q-1}+2^{q-2})}(\underline{x}) = s^{(2^{q-1}+2^{q-2}+1)}(\underline{x}) = \cdots \\ \quad = s^{(2^{q-1}+2^{q-2}+2^{q-3}-1)}(\underline{x}) \\ \vdots \\ s^{(2^{q-1}+2^{q-2}+\cdots+2^{q-(q-2)})}(\underline{x}) \\ \quad = s^{(2^{q-1}+\cdots+2^{q-(q-2)}+2^{q-(q-1)}-1)}(\underline{x}), \end{cases} \tag{42}$$

new construction in Theorem 6 in this paper is exactly the same as the construction in (6) in Theorem 2 in [8], which implies that [8] is a special case of this paper.

In order to clarify the aforementioned conclusion, we consider 16-QAM and 64-QAM CSs of length $2^m$, respectively, as follows.

### 1) 16-QAM CSs

In this case, [8] has the construction to produce 16-QAM CSs below.

$$\begin{cases} A_i = \frac{1+j}{\sqrt{10}}\left[2j^{a_i^{(0)}} + j^{a_i^{(1)}}\right] \\ B_i = \frac{1+j}{\sqrt{10}}\left[2j^{b_i^{(0)}} + j^{b_i^{(1)}}\right], \end{cases} \tag{43}$$

which can be produced by (19) under the condition $a_i^{(0)} = a_i^{(1)}$.

Apparently, if the following condition

$$a^{(k)}(\underline{x}) \neq a^{(r)}(\underline{x}) \ (0 \leq k, r \leq 2 \text{ and } k \neq r), \tag{44}$$

holds, new construction from (19) is

$$\begin{cases} A_i = \frac{1+j}{\sqrt{10}}\left[j^{a_i^{(0)}} + j^{a_i^{(1)}} + j^{a_i^{(2)}}\right] \\ B_i = \frac{1+j}{\sqrt{10}}\left[j^{b_i^{(0)}} + j^{b_i^{(1)}} + j^{b_i^{(2)}}\right], \end{cases} \tag{45}$$

which is fairly different with (43).

### 2) 64-QAM CSs

In this case, [8] has the construction to produce 64-QAM CSs below.

$$\begin{cases} A_i = \frac{1+j}{\sqrt{42}}\left[4j^{a_i^{(0)}} + 2j^{a_i^{(1)}} + j^{a_i^{(2)}}\right] \\ B_i = \frac{1+j}{\sqrt{42}}\left[4j^{b_i^{(0)}} + 2j^{b_i^{(1)}} + j^{b_i^{(4)}}\right], \end{cases} \tag{46}$$

Clearly, (19) includes lots of new constructions which do not exist in Cases I-III constructions in [8]. Typically, two examples are given below. If we set the following condition:

$$\begin{cases} a^{(1)}(\underline{x}) = a^{(2)}(\underline{x}) = a^{(3)}(\underline{x}) \\ a^{(4)}(\underline{x}) = a^{(5)}(\underline{x}) = a^{(6)}(\underline{x}) \end{cases} \tag{47}$$

to be held, new construction from (19) is

$$\begin{cases} A_i = \frac{1+j}{\sqrt{42}}\left[j^{a_i^{(0)}} + 3j^{a_i^{(1)}} + 3j^{a_i^{(4)}}\right] \\ B_i = \frac{1+|}{\sqrt{42}}\left[j^{b_i^{(0)}} + 3j^{b_i^{(1)}} + 3j^{b_i^{(4)}}\right], \end{cases} \tag{48}$$

and if we let

$$a^{(2)}(\underline{x}) = a^{(3)}(\underline{x}) = a^{(4)}(\underline{x}) = a^{(5)}(\underline{x}) = a^{(6)}(\underline{x}), \tag{49}$$

new construction educed by (19) is given by

$$\begin{cases} A_i = \frac{1+j}{\sqrt{42}}\left[j^{a_i^{(0)}} + j^{a_i^{(1)}} + 5j^{a_i^{(2)}}\right] \\ B_i = \frac{1+j}{\sqrt{42}}\left[j^{b_i^{(0)}} + j^{b_i^{(1)}} + 5j^{b_i^{(2)}}\right]. \end{cases} \tag{50}$$

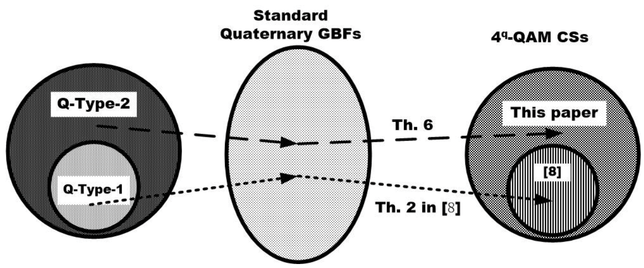


**FIGURE 2. Relationship between [8] and this paper.**

Fig. 2 clearly depicts the relationship between [8] and this paper. Table 5 compares the relevant parameters in the existing $4^q$-QAM CSs from standard GBFs. Due to lack of family size for higher-order QAM CSs in this paper, only 16-QAM CSs are compared. The comparison suggests that the resultant sequences in this paper have good performance.

## B. A RELATIONSHIP BETWEEN Q-TYPE AND B-TYPE CONSTRUCTIONS

In order to avoid symbol confusion in the comparison, the construction of $4^q$-QAM CSs, based on B-type-2, in [26] is rewritten by new symbols below.

$$\begin{cases} A_i = \frac{1}{\sqrt{2(4^q-1)/3}}\left[\sum_{p=0}^{2^q-2}(-1)^{u_i^{(p)}} + j\sum_{p=0}^{2^q-2}(-1)^{u_i^{(2^q-1+p)}}\right] \\ B_i = \frac{1}{\sqrt{2(4^q-1)/3}}\left[\sum_{p=0}^{2^q-2}(-1)^{v_i^{(p)}} + j\sum_{p=0}^{2^q-2}(-1)^{v_i^{(2^q-1+p)}}\right], \end{cases} \tag{51}$$

where $u_i^{(p)}$'s and $v_i^{(p)}$'s $(0 \leq p \leq 2(2^q-1))$ are binary GBFs' values. In fact, in [26] the symbols $u_i^{(p)}$ and $v_i^{(p)}$ are written by $a_i^{(p)}$ and $b_i^{(p)}$, respectively.

It is not difficult for the careful reader to find that (19) and (51) are identical under the following conditions.

$$\begin{cases} a^{(p)}(\underline{x}) \in \{\text{binary GBFs}\} \ (0 \leq p \leq 2^q-2) \\ b^{(p)}(\underline{x}) \in \{\text{binary GBFs}\} \ (0 \leq p \leq 2^q-2) \\ u_i^{(p)} = u_i^{(2^q-2+p)} = a_i^{(p)} \ (0 \leq p \leq 2^q-2) \\ v_i^{(p)} = v_i^{(2^q-2+p)} = b_i^{(p)} \ (0 \leq p \leq 2^q-2). \end{cases} \tag{52}$$

**TABLE 5. Comparison of relevant parameters in $4^q$-QAM CSs with length $N = 2^m$.**

| Refs. | Type | Family size | | PEP upper bound | Code rates | |
|---|---|---|---|---|---|---|
| | | 16-QAM | higher-order QAM | | 16-QAM | higher-order QAM |
| this paper | Q-type-2 | $(58m+105)(m!/2)4^{m+1}$ | to be solved | $\frac{6N(2^q-1)}{2^q+1}$ | high | - |
| [4],[7],[8] | Q-type-1 | $(12m+14)(m!/2)4^{m+1}$ | $\left[(m+1)4^{2(q-1)}-(m+1)4^{q-1}+2^{q-1}\right](m!/2)4^{m+1}$ | | low | - |
| [9] | Q-type-1 | no existence | $\mathcal{N}_q\frac{(m-2)(m+1)}{2}(m!/2)4^{m+1}$ | | - | - |
| [24] | B-type-1 | $(480m-448)(m!/2)2^{m+1}$ | no existence | | lowest | - |
| [26] | B-type-2 | $(992m-1024)(m!/2)2^{m+1}$ | $\left[(m+1)4^{2\cdot 2^q-3}-m2^{2\cdot 2^q-3}\right]\cdot(m!/2)2^{m+1}$ | | lower | - |

Here "-" stands for being currently difficult for comparison, and $\mathcal{N}_q$ is given in [9].

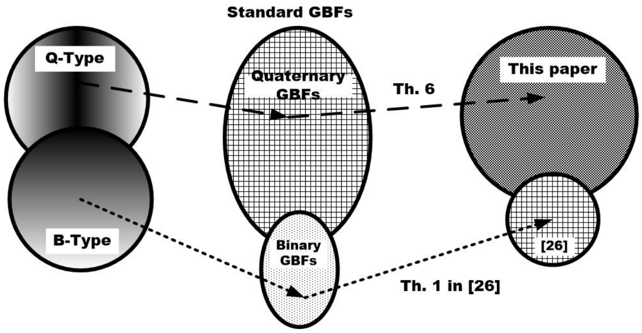


**FIGURE 3. Relation between [26] and this paper.**

Whereas, when the conditions in (52) are broken, (19) and (51) result in their own $4^q$-QAM CSs in which both are different at all. Hence, [26] and this paper have a small overlapping area.

Fig. 3 clearly depicts the relationship between [26] and this paper.

## IX. TWO EXAMPLES

Here are two examples to highlight the effects of Theorem 6.

*Example 2:* Consider 16-QAM CSs of length $N = 2^4 = 16$. Let $\pi(i) = i$ $(1 \le i \le 5)$, $h = 2$, and $(c, c_1, c_2, c_3, c_4) = (3, 2, 1, 3, 1)$. Consequently, the standard GBFs are

$$f(\underline{x}) = 2\sum_{k=1}^{4} x_k x_{k+1} + 2x_1 + x_2 + 3x_3 + x_4 + 3.$$

Consider new construction in (45). Take $\mu(\underline{x}) = 2x_1$, $w = 1$, $(d_0^{(1)}, d_0^{(1)}, d_0^{(1)}) = (1, 1, 1)$, and $(d_0^{(2)}, d_0^{(2)}, d_0^{(2)}) = (0, 1, 3)$. Consequently, the offsets are given by

$$\begin{cases} s^{(1)}(\underline{x}) = 1 + x_1 + x_2 \\ s^{(2)}(\underline{x}) = x_1 + 3x_2. \end{cases}$$

Thus, we have

$$\begin{cases} a^{(0)}(\underline{x}) = 2(x_1x_2 + x_2x_3 + x_3x_4) + 2x_1 + x_2 \\ \qquad\qquad +3x_3 + x_4 + 3 \\ b^{(0)}(\underline{x}) = a^{(0)}(\underline{x}) + 2x_1 \\ a^{(1)}(\underline{x}) = a^{(0)}(\underline{x}) + 1 + x_1 + x_2 \\ b^{(1)}(\underline{x}) = a^{(1)}(\underline{x}) + 2x_1 \\ a^{(2)}(\underline{x}) = a^{(0)}(\underline{x}) + x_1 + 3x_2 \\ b^{(2)}(\underline{x}) = a^{(2)}(\underline{x}) + 2x_1, \end{cases}$$

which implies

$$\begin{cases} a^{(0)}(\underline{x}) \neq a^{(1)}(\underline{x}) \\ a^{(0)}(\underline{x}) \neq a^{(2)}(\underline{x}) \\ a^{(1)}(\underline{x}) \neq a^{(2)}(\underline{x}), \end{cases}$$

in other words, (45) holds. Hence, the resultant 16-QAM CSs can be not produced by [8]. After calculation, the resultant 16-QAM CSs of length $N = 16$ are

$$\begin{aligned} \underline{A} = (&3-j, 1+3j, -1-3j, -3+j, 1-j, 1+j, 1+j, \\ &1-j, -1-j, 1-j, -1+j, 1+j, 3+j, -1+3j, \\ &-1+3j, 3+j) \end{aligned}$$

and

$$\begin{aligned} \underline{B} = (&3-j, 1+3j, -1-3j, -3+j, 1-j, 1+j, 1+j, \\ &1-j, 1+j, -1+j, 1-j, -1-j, -3-j, 1-3j, \\ &1-3j, -3-j), \end{aligned}$$

the sum of whose aperiodic autocorrelation functions is

$$\begin{aligned} &C_{A,A}(\tau) + C_{B,B}(\tau) \\ &\quad = (19.2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), \end{aligned}$$

Thus, we have

$$P_A(t) + P_B(t) = 19.2 = 1.2N.$$

*Example 3:* Consider 64-QAM CSs of length $N = 2^5 = 32$. Let $\pi(i) = i$ $(1 \le i \le 5)$, $h = 2$, and $(c, c_1, c_2, c_3, c_4, c_5) = (1, 3, 0, 1, 2, 0)$. Consequently, the standard GBFs are

$$f(\underline{x}) = 2\sum_{k=1}^{4} x_k x_{k+1} + 3x_1 + x_3 + 2x_4 + 1.$$

Consider new construction in (50). Take $\mu(\underline{x}) = 2x_1$, $w = 3$, $(d_0^{(1)}, d_0^{(1)}, d_0^{(1)}) = (2, 3, 1)$, and $(d_0^{(p)}, d_0^{(p)}, d_0^{(p)}) = (3, 0, 2)$ $(p = 2, 3, 4, 5, 6)$. As a result, the offsets are given by

$$\begin{cases} s^{(1)}(\underline{x}) = 2 + 3x_3 + x_4 \\ s^{(p)}(\underline{x}) = 3 + 2x_4 \ (2 \le p \le 6). \end{cases}$$

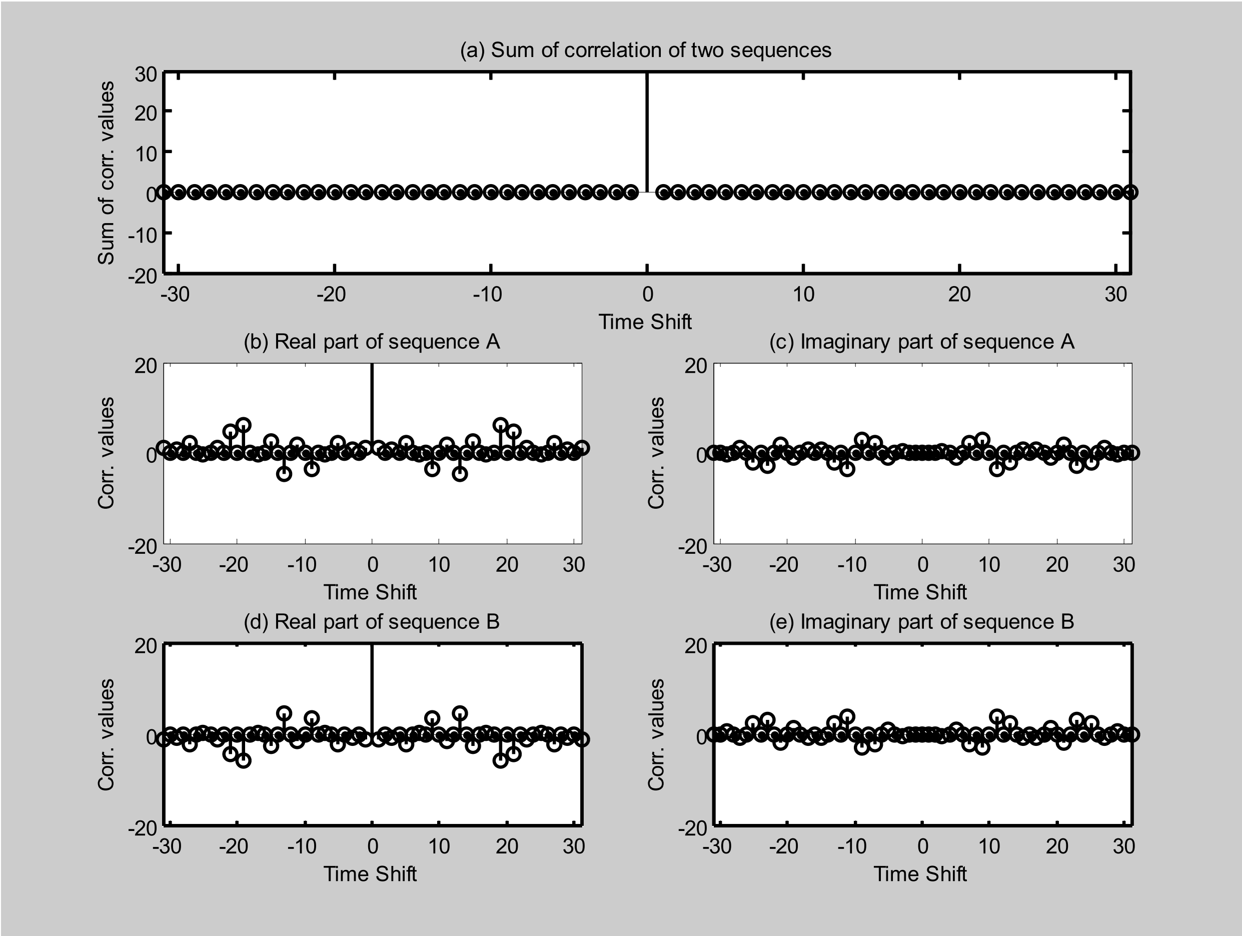


**FIGURE 4.** **Sum of aperiodic autocorrelation functions of the sequences $\underline{A}$ and $\underline{B}$ in Example 2.**

Thus, we have

$$\begin{cases} a^{(0)}(\underline{x}) = 2(x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5) + 3x_1 + x_3 + 2x_4 + 1 \\ b^{(0)}(\underline{x}) = a^{(0)}(\underline{x}) + 2x_1 \\ a^{(1)}(\underline{x}) = a^{(0)}(\underline{x}) + 2 + 3x_3 + x_4 \\ b^{(1)}(\underline{x}) = a^{(1)}(\underline{x}) + 2x_1 \\ a^{(p)}(\underline{x}) = a^{(0)}(\underline{x}) + 3 + 2x_4 \ (2 \leq p \leq 6) \\ b^{(p)}(\underline{x}) = a^{(p)}(\underline{x}) + 2x_1 \ (2 \leq p \leq 6). \end{cases}$$

Then, the resultant 64-QAM CSs of length $N = 32$ are

$$\begin{aligned} \underline{A} = (&5+5j, 5+5j, 5+3j, -5-3j, -5+3j, -5+3j, \\ &5-5j, -5+5j, 5+5j, 5+5j, 5+3j, -5-3j, 5-3j, \\ &5-3j, -5+5j, 5-5j, 5-5j, 5-5j, 3-5j, -3+5j, \\ &3+5j, 3+5j, -5-5j, 5+5j, -5+5j, -5+5j, -3+5j, \\ &3-5j, 3+5j, 3+5j, -5-5j, 5+5j) \end{aligned}$$

and

$$\begin{aligned} \underline{B} = (&5+5j, 5+5j, 5+3j, -5-3j, -5+3j, -5+3j, \\ &5-5j, -5+5j, 5+5j, 5+5j, 5+3j, -5-3j, 5-3j, \\ &5-3j, -5+5j, 5-5j, -5+5j, -5+5j, -3+5j, 3-5j, \\ &-3-5j, -3-5j, 5+5j, -5-5j, 5-5j, 5-5j, 3-5j, \\ &-3+5j, -3-5j, -3-5j, 5+5j, -5-5j), \end{aligned}$$

the sum of whose aperiodic autocorrelation functions is

$$\begin{aligned} C_{A,A}(\tau) + C_{B,B}(\tau) = (&64, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \\ &0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), \end{aligned}$$

which is depicted in Fig.4. Apparently, the sum of aperiodic autocorrelation functions of the sequences $\underline{A}$ and $\underline{B}$ is impulse-like. Besides, these two sequences satisfy

$$P_A(t) + P_B(t) = 64 = 2N.$$

Clearly, the resulting PEP is well controlled.

## X. CONCLUSION

From a novel mathematical description of QAM constellation, this paper presents the new construction producing $4^q$-QAM CSs of length $2^m$ with larger family size. However, the enumeration of higher-order is unknown, which is a coming work.

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