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Further Stability Analysis of Generalized Neural Networks With Time-Varying Delays Based on a Novel Lyapunov-Krasovskii Functional

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ABSTRACT This paper focuses on the stability analysis of generalized neural networks with time-varying delays. First, a novel augmented Lyapunov-Krasovskii functional (LKF) is constructed by introducing a couple of integral vectors. Second, by utilizing the novel augmented LKF and a generalized free-weighting-matrix integral inequality, two further stability criteria are presented in this paper. Third, a less conservative stability condition by refining the allowable delay set is introduced. Finally, the four well-known numerical examples are given to demonstrate the effectiveness and improvements of the proposed methods.

INDEX TERMS Generalized neural networks, Lyapunov-Krasovskii functional, stability, time-varying delays.

I. INTRODUCTION

In past few decades, neural networks (NNs) have attracted great concerns for their extensive application in signal processing, image processing and pattern recognition, *etc*. [1]–[3]. It is two factors that require extra attention during the implementations of NNs. For one thing, the dynamic behavior of equilibrium point of NNs greatly influences on its applications. Thus, it is a premise and fundamental to ensure the stability of NNs. For another, time delays frequently occur in many NNs due to the finite switching speed of amplifiers and the inevitable communication time among neurons. As we all know, time delays can give rise to poor system performance, such as oscillations and instability [4], [5]. Therefore, stability analysis of NNs, especially stability analysis of delayed neural networks (DNNs), has been a research hotspot in recent decades and a lot of stability results have been obtained [6]–[12].

The existing stability criteria include delay-independent criteria and delay-dependent ones. Compared with the delay-independent criteria, the delay-dependent criteria can derive less conservativeness. Thus, this paper mainly concerns in delay-dependent stability criteria for DNNs. The maximum delay bounds, which is an important performance index to evaluate conservatism, can be used to confirm the effectiveness of stability criteria. It is very meaningful to employ some excellent methods to obtain stability criteria with larger maximum delay bounds. According to Lyapunov stability theory, estimating tight the derivative of LKF is the major way for obtaining stability results with less conservatism. In the past few decades of research, Jensen inequality has been widely applied to the estimation of the single integral term [13]–[16]. Wirtinger inequality which presents a tighter estimation of single integral term than Jensen inequality is proposed in [4] and has been the most popular method in stability analysis of DNNs [17]–[20]. A free-matrix-based inequality (FMBI) is proposed in [21] and applied to reduce conservatism of stability criteria for DNNs [22], [23]. However, some slack matrices in FMBI have no effect on reducing conservatism. In [24], a new integral inequality which is more general than Jensen inequality, Wirtinger inequality, and FMBI is proposed and less conservative stability results are derived. Recently, several improved reciprocally convex inequalities which encompass reciprocally convex inequality [25] are employed in [26]–[28]. In [30], a general free-weighting-matrix-based inequality (GFWMBI) is developed to deal with augmented integral terms. Several new less conservative stability results for the DNNs are presented by applying GFWMBI [31].

The construction of suitable LKFs is another important way to derive less conservative criteria [32]. Most LKFs used for stability analysis of DNNs usually contain

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three terms: the nonintegral quadratic term, the activation function-based term, and the integral quadratic term. In [33], an augmented nonintegral quadratic term including integral of state vectors has been confirmed to achieve the advantage of the Wirtinger-based inequality. It is pointed out [15], [31], [34], [35] that augmented single integral quadratic terms and double integral quadratic terms with necessary state vectors are very important in stability analysis of DNNs. In [23], [36], LKFs with triple integral quadratic terms are constructed and some improved stability/passivity conditions for DNNs are derived. Additionally, the information of activation functions of the NNs plays a crucial role in reducing the conservativeness for stability analysis of DNNs [37], [38]. Recently, a novel LKF is constructed in stability analysis for linear systems by adding state integral vectors and less conservative results are obtained in [29]. In general, the LKF containing more information usually achieves a less conservative result. Thus, constructing a more proper augmented LKF involving more key state information is a very meaning work. This is the main motivation for this paper.

In the previous literature, there is less attention on the definition of the allowable delay sets. Recently, several improved allowable delay sets are introduced in stability analysis of linear systems [39], [40]. In [41], the improved allowable delay sets are first introduced in stability analysis of DNNs, which obviously enlarge the maximum delay bounds. Extending this idea to our paper is another motivation.

In this paper, stability analysis for generalized NNs with time-varying delays is investigated.

1) A novel augmented LKF is constructed by augmenting a couple of integral vectors $\int_{t-h}^{s} x(u)\,du$, $\int_{t-h}^{s} \dot{x}(u)\,du$. The novel LKF involves more key information of state vectors and can obtain less conservative stability criteria.

2) New augmented zero equalities which introduce more cross terms are proposed to reduce conservativeness of stability criteria. Combining the novel augmented LKF with the proposed augmented zero equalities, an improved delay-dependent stability criterion is given in Theorem 1.

3) Based on Theorem 1, a less conservative delay-dependent stability criterion is presented in Theorem 2 by relaxing the positiveness of nonintegral quadratic term.

4) Inspired by the [40], [41], a further improved delay-dependent stability criterion is derived in Theorem 3 by refining allowable delay set.

Finally, the superiority and effectiveness of the developed stability criteria will be shown by comparing with the previous results.

Notations: Throughout this paper, X^T represents the transposition of matrix X; $\mathcal{R}^n$ denotes the n dimensional Euclidean space; $\mathcal{R}^{m\times n}$ represents all $m \times n$ real matrices; $Q > 0$ means that Q is positive definite; Symbol $*$ represents the symmetric terms of a symmetric block matrix; $diag\{\cdots\}$ denotes a block diagonal matrix; $X^{\perp}$ denotes a right orthogonal complement of X, $Sym(Y)$ stands for $(Y + Y^T)$ and $Co\{r_1, r_2\}$ represents a polytope with two vertices r_1 and r_2.

II. PROBLEM FORMULATION

Consider the following generalized NN with time-varying delay. (The equilibrium point is converted into the origin.)

$$\dot{x}(t) = -Ax(t) + W_0 f(W_2 x(t)) + W_1 f(W_2 x(t - h(t))), \quad (1)$$

where $x(\cdot) = [x_1(\cdot), \cdots, x_n(\cdot)]^T \in \mathcal{R}^n$ is the neuron state vector, $f(\cdot) = [f_1(\cdot), \cdots, f_n(\cdot)]^T \in \mathcal{R}^n$ denotes the neuron activation function, $A = diag\{a_1, \cdots, a_n\}$ is positive definite, $W_i \in \mathcal{R}^{n\times n}$ $(i = 0, 1, 2)$ are the interconnection weight matrices between neurons.

The time-varying delay $h(t)$ satisfying

$$0 \le h(t) \le h, \quad \dot{h}(t) \le h_D, \quad (2)$$

where h is a positive scalar and h_D is any scalar.

The activation functions $f_i(\cdot)(i = 1, \cdots, n)$ are continuous and bounded, which satisfy the following inequalities

$$k_i^- \le \frac{f_i(u) - f_i(v)}{u - v} \le k_i^+ \quad u \ne v (i = 1, \cdots, n), \quad (3)$$

where k_i^- and k_i^+ are known constants.

We will utilize the following lemmas to derive the main results.

Lemma 1 ([30]): For a symmetric matrix $R > 0$, a differentiable vector function ω: $[a, b] \to \mathcal{R}^n$, any vector $\xi \in \mathcal{R}^k$, and any appropriate dimensions matrices N_1 and N_2, the following inequality

$$\begin{aligned} -\int_a^b \omega^T(s)R\omega(s)ds &\le \xi^T(b-a)\left(N_1R^{-1}N_1^T + \frac{1}{3}N_2R^{-1}N_2^T\right)\xi \\ &\quad + Sym(\xi^T N_1\xi_1 + \xi^T N_2\xi_2), \end{aligned} \quad (4)$$

is satisfied, where

$$\xi_1 = \int_a^b \omega(s)ds,$$
$$\xi_2 = -\int_a^b \omega(s)ds + \frac{2}{b-a}\int_a^b\int_s^b \omega(u)duds.$$

Lemma 2 ([42]): Let $\xi \in R^n$, symmetric matrix $X \in R^{n\times n}$ and $H \in R^{m\times n}$ such that rank$(H) < n$. The following inequalities

(1) $\xi^T X\xi < 0$,

(2) $(H^{\perp})^T XH^{\perp} < 0$.

are equivalent for all $H\xi = 0$, $\xi \ne 0$.

Lemma 3 ([15]): For a symmetric matrix $R > 0$, a differentiable function ω: $[a, b] \to \mathcal{R}^n$, the following inequalities hold:

$$\int_a^b \omega^T(s)R\omega(s)ds \ge \frac{1}{b-a}\left(\int_a^b \omega(s)ds\right)^T R\left(\int_a^b \omega(s)ds\right), \quad (5)$$

$$\int_a^b\int_s^b \omega^T(u)R\omega(u)duds$$

$$\geq \frac{2}{(b-a)^2}\left(\int_a^b\int_s^b \omega(u)duds\right)^T R\left(\int_a^b\int_s^b \omega(u)duds\right). \tag{6}$$

III. STABILITY ANALYSIS

In this research, three improved delay-dependent stability criteria for generalized NN (1) are presented. And block entry matrices are defined as $e_0 = 0_{14n\times n}$, $e_i = [0_{n\times(i-1)n}\ I_n\ 0_{n\times(14-i)n}]^T (i = 1, \cdots, 14)$. The other related notations are defined as follows:

$$\xi(t) = col\left\{\begin{bmatrix} x(t) \\ x(t-h(t)) \\ x(t-h) \\ \dot{x}(t) \\ \dot{x}(t-h) \\ \int_{t-h(t)}^{t} x(s)ds \\ \int_{t-h}^{t-h(t)} x(s)ds \end{bmatrix}, \begin{bmatrix} \frac{1}{h(t)}\int_{t-h(t)}^{t}\int_s^t x(u)\,duds \\ \frac{1}{h-h(t)}\int_{t-h}^{t-h(t)}\int_s^{t-h(t)} x(u)\,duds \\ \frac{1}{h(t)}\int_{t-h(t)}^{t} x(s)ds \\ \frac{1}{h-h(t)}\int_{t-h}^{t-h(t)} x(s)ds \\ f(W_2x(t)) \\ f(W_2x(t-h(t))) \\ f(W_2x(t-h)) \end{bmatrix}\right\},$$

$$\begin{aligned}
K_p &= diag\left\{k_1^+, k_2^+, \cdots, k_n^+\right\}, \\
K_m &= diag\left\{k_1^-, k_2^-, \cdots, k_n^-\right\}, \\
Q_{aug1} &= Q_1 + Sym\left\{\begin{bmatrix} 0&0&0 \\ I&0&0 \\ 0&I&0 \\ 0&0&I \end{bmatrix} P_1 \begin{bmatrix} I&0&0&0 \\ -I&0&0&0 \\ I&0&0&0 \end{bmatrix}\right\}, \\
Q_{aug2} &= Q_1 + Sym\left\{\begin{bmatrix} 0&0&0 \\ I&0&0 \\ 0&I&0 \\ 0&0&I \end{bmatrix} P_2 \begin{bmatrix} I&0&0&0 \\ -I&0&0&0 \\ I&0&0&0 \end{bmatrix}\right\}, \\
e_{tot} &= [e_1, e_2, e_3, e_4, e_5, e_6, e_7, \\
&\quad e_8, e_9, e_{10}, e_{11}, e_{12}, e_{13}, e_{14}], \\
\Upsilon &= [e_1, e_2, e_6, e_7, e_{10}, e_{11}], \\
\Phi_{1[h(t)]} &= Sym\{[e_1, e_3, e_6 + e_7, \\
&\quad h(t)\,e_8 + (h - h(t))(e_6 + e_9)] \\
&\quad \times R[e_4, e_5, e_1 - e_3, he_1 - e_6 - e_7]^T\}, \\
\Phi_2 &= [e_4, e_1, e_{12}]N[e_4, e_1, e_{12}]^T \\
&\quad - [e_5, e_3, e_{14}]N[e_5, e_3, e_{14}]^T \\
&\quad + Sym\{(e_{12} - e_1W_2^TK_m)D_1W_2e_4^T \\
&\quad + (e_1W_2^TK_p - e_{12})D_2W_2e_4^T\} \\
&\quad + Sym\{(e_{14} - e_3W_2^TK_m)D_3W_2e_5^T \\
&\quad + (e_3W_2^TK_p - e_{14})D_4W_2e_5^T\}, \\
\Phi_3 &= [e_1, e_0, e_0, e_1 - e_3, e_6 + e_7] \\
&\quad \times G_1[e_1, e_0, e_0, e_1 - e_3, e_6 + e_7]^T \\
&\quad - (1 - h_D)[e_2, e_1 - e_2, e_6, e_2 - e_3, e_7] \\
&\quad \times G_1[e_2, e_1 - e_2, e_6, e_2 - e_3, e_7]^T \\
&\quad + [e_1, e_{12}]\,G_2[e_1, e_{12}]^T \\
&\quad - (1 - h_D)[e_2, e_{13}]\,G_2[e_2, e_{13}]^T, \\
\Phi_{31[h(t)]} &= Sym\{[e_6, h(t)e_1 - e_6, h(t)e_8, e_6 - h(t)e_3, \\
&\quad h(t)(e_6 + e_7 - e_8)]G_1[e_0, e_4, e_1, -e_5, -e_3]^T\}, \\
\Phi_4 &= h\,[e_4, e_1, e_0, e_1 - e_3]\,Q_1[e_4, e_1, e_0, e_1 - e_3]^T, \\
\Phi_{41[h(t)]} &= Sym\{e_{tot}L_1[e_1 - e_2, e_6, h(t)e_1 - e_6, \\
&\quad -h(t)e_3 + e_6]^T \\
&\quad + e_{tot}M_1[e_1 + e_2 - 2e_{10}, -e_6 + 2e_8, \\
&\quad e_6 - 2e_8, -e_6 + 2e_8]^T\}, \\
\Phi_{42[h(t)]} &= Sym\{e_{tot}L_2[e_2 - e_3, e_7, (h - h(t))\,e_1 - e_7, \\
&\quad -(h - h(t))\,e_3 + e_7]^T \\
&\quad + e_{tot}M_2[e_2 + e_3 - 2e_{11}, -e_7 + 2e_9, \\
&\quad e_7 - 2e_9, -e_7 + 2e_9]^T\}, \\
\Phi_{43[h(t)]} &= Sym\{[he_1 - e_6 - e_7, \\
&\quad h(t)e_8 + (h - h(t))\,(e_6 + e_9), \\
&\quad (h^2/2)e_1 - h(t)e_8 - (h - h(t))\,(e_6 + e_9), \\
&\quad -(h^2/2)e_3 + h(t)e_8 + (h - h(t))\,(e_6 + e_9)] \\
&\quad \times Q_1[e_0, e_0, e_4, -e_5]^T\}, \\
\Phi_{44} &= [e_1, e_0, e_1 - e_3]\,P_1[e_1, e_0, e_1 - e_3]^T \\
&\quad - [e_2, e_1 - e_2, e_2 - e_3]P_1[e_2, e_1 - e_2, e_2 - e_3]^T \\
&\quad + [e_2, e_1 - e_2, e_2 - e_3]P_2[e_2, e_1 - e_2, e_2 - e_3]^T \\
&\quad - [e_3, e_1 - e_3, e_0]P_2[e_3, e_1 - e_3, e_0]^T, \\
\Theta_{1[h(t)]} &= Sym\{(h(t)e_{10} - e_6)\,\Psi_1\Upsilon^T \\
&\quad + ((h - h(t))\,e_{11} - e_7)\,\Psi_2\Upsilon^T\}, \\
\Theta_{2[h(t)]} &= -Sym\{\left(e_{12} - e_1W_2^TK_m\right) \\
&\quad \times \left(H_1\frac{h(t)}{h} + H_6\frac{h - h(t)}{h}\right) \\
&\quad \times \left(e_{12} - e_1W_2^TK_p\right)^T\} \\
&\quad - Sym\{\left(e_{13} - e_2W_2^TK_m\right) \\
&\quad \times \left(H_2\frac{h(t)}{h} + H_7\frac{h - h(t)}{h}\right) \\
&\quad \times \left(e_{13} - e_2W_2^TK_p\right)^T\} \\
&\quad - Sym\{\left(e_{14} - e_3W_2^TK_m\right) \\
&\quad \times \left(H_3\frac{h(t)}{h} + H_8\frac{h - h(t)}{h}\right) \\
&\quad \times \left(e_{14} - e_3W_2^TK_p\right)^T\} \\
&\quad - Sym\left\{\left(e_{12} - e_{13} - (e_1 - e_2)W_2^TK_m\right)\right. \\
&\quad \times \left(H_4\frac{h(t)}{h} + H_9\frac{h - h(t)}{h}\right)
\end{aligned}$$

$$
\begin{aligned}
&\times \left(e_{12}-e_{13}-(e_1-e_2)W_2^T K_p\right)^T\}\\
&-Sym\{\left(e_{13}-e_{14}-(e_2-e_3)W_2^T K_m\right)\\
&\times \left(H_5\frac{h(t)}{h}+H_{10}\frac{h-h(t)}{h}\right)\\
&\times \left(e_{13}-e_{14}-(e_2-e_3)W_2^T K_p\right)^T\},\\
\Phi_{[h(t)]} &= \Phi_{1[h(t)]}+\Phi_2+\Phi_3+\Phi_{31[h(t)]}+\Phi_4\\
&+\Phi_{41[h(t)]}+\Phi_{42[h(t)]}+\Phi_{43[h(t)]}+\Phi_{44}\\
&+\Theta_{1[h(t)]}+\Theta_{2[h(t)]},\\
\Phi_{tot[h(t)]} &= \Phi_{[h(t)]}+h\,(t)\,e_{tot}\{L_1 Q_{aug1}^{-1}L_1^T\\
&+(1/3)\,M_1 Q_{aug1}^{-1}M_1^T\}e_{tot}^T\\
&+(h-h\,(t))\,e_{tot}\{L_2 Q_{aug2}^{-1}L_2^T\\
&+(1/3)\,M_2 Q_{aug2}^{-1}M_2^T\}e_{tot}^T,\\
B &= [-A,0,0,-I,0,0,0,0,0,0,0,W_0,W_1,0]\,.
\end{aligned}
\tag{7}
$$

Theorem 1: For given scalars h and h_D, the generalized NN (1) with $h(t)$ satisfying (2) is asymptotically stable, if there exist positive diagonal matrices $D_i = diag\{d_{1i},\cdots,d_{ni}\}(i=1,2,3,4)$, $H_i(i=1,\cdots,10)$, positive symmetric matrices $G_1 \in \mathcal{R}^{5n\times 5n}$, $R, Q_1 \in \mathcal{R}^{4n\times 4n}$, $N \in \mathcal{R}^{3n\times 3n}$, $G_2 \in \mathcal{R}^{2n\times 2n}$, symmetric matrices $P_i(i=1,2) \in \mathcal{R}^{3n\times 3n}$, and any matrices $\Psi_i(i=1,2)\in\mathcal{R}^{n\times 6n}$, $M_i, L_i \in \mathcal{R}^{14n\times 4n}(i=1,2)$ satisfying the following LMIs:

$$
\begin{bmatrix}
(B^{\perp})^T\Phi_h(B^{\perp}) & (B^{\perp})^T(he_{tot}L_1) & (B^{\perp})^T(he_{tot}M_1)\\
* & -hQ_{aug1} & 0\\
* & * & -3hQ_{aug1}
\end{bmatrix}<0,
\tag{8}
$$

$$
\begin{bmatrix}
(B^{\perp})^T\Phi_0(B^{\perp}) & (B^{\perp})^T(he_{tot}L_2) & (B^{\perp})^T(he_{tot}M_2)\\
* & -hQ_{aug2} & 0\\
* & * & -3hQ_{aug2}
\end{bmatrix}<0,
\tag{9}
$$

where $\Phi_h = \Phi_{[h(t)=h]}$, $\Phi_0 = \Phi_{[h(t)=0]}$, B, e_{tot}, Q_{aug1} and Q_{aug2} are given in (7).

Proof: We choose the following LKF

$$
V(t)=\sum_{i=1}^{4}V_i(t),
\tag{10}
$$

where

$$
\begin{aligned}
V_1(t) &= \begin{bmatrix} x\,(t)\\ x\,(t-h)\\ \int_{t-h}^{t}x\,(s)\,ds\\ \int_{t-h}^{t}\int_s^t x\,(u)\,duds\end{bmatrix}^T R \begin{bmatrix} x\,(t)\\ x\,(t-h)\\ \int_{t-h}^{t}x\,(s)\,ds\\ \int_{t-h}^{t}\int_s^t x\,(u)\,duds\end{bmatrix},\\
V_2(t) &= \int_{t-h}^{t}\begin{bmatrix}\dot x\,(s)\\ x\,(s)\\ f\,(W_2x\,(s))\end{bmatrix}^T N \begin{bmatrix}\dot x\,(s)\\ x\,(s)\\ f\,(W_2x\,(s))\end{bmatrix}ds\\
&+2\sum_{i=1}^{n}\left(d_{i1}\int_0^{W_{2i}x(t)}(f_i(s)-k_i^- s)ds\right.\\
&\left.+d_{i2}\int_0^{W_{2i}x(t)}(k_i^+ s-f_i(s))ds\right)\\
&+2\sum_{i=1}^{n}\left(d_{i3}\int_0^{W_{2i}x(t-h)}(f_i(s)-k_i^- s)ds\right.\\
&\left.+d_{i4}\int_0^{W_{2i}x(t-h)}(k_i^+ s-f_i(s))ds\right),\\
V_3(t) &= \int_{t-h(t)}^{t}\begin{bmatrix} x\,(s)\\ \int_s^t\dot x\,(u)\,du\\ \int_s^t x\,(u)\,du\\ \int_{t-h}^{s}\dot x\,(u)\,du\\ \int_{t-h}^{s}x\,(u)\,du\end{bmatrix}^T G_1 \begin{bmatrix} x\,(s)\\ \int_s^t\dot x\,(u)\,du\\ \int_s^t x\,(u)\,du\\ \int_{t-h}^{s}\dot x\,(u)\,du\\ \int_{t-h}^{s}x\,(u)\,du\end{bmatrix}ds\\
&+\int_{t-h(t)}^{t}\begin{bmatrix} x\,(s)\\ f\,(W_2x\,(s))\end{bmatrix}^T G_2\begin{bmatrix} x\,(s)\\ f\,(W_2x\,(s))\end{bmatrix}ds,\\
V_4(t) &= \int_{t-h}^{t}\int_s^t\begin{bmatrix}\dot x\,(u)\\ x\,(u)\\ \int_u^t\dot x\,(v)\,dv\\ \int_{t-h}^{u}\dot x\,(v)\,dv\end{bmatrix}^T Q_1\begin{bmatrix}\dot x\,(u)\\ x\,(u)\\ \int_u^t\dot x\,(v)\,dv\\ \int_{t-h}^{u}\dot x\,(v)\,dv\end{bmatrix}duds,
\end{aligned}
$$

where W_{2i} is ith row vector of W_2.

The $\dot V_1(t)$ is given as

$$
\begin{aligned}
\dot V_1(t) &= 2\begin{bmatrix} x\,(t)\\ x\,(t-h)\\ \int_{t-h(t)}^{t}x\,(s)\,ds+\int_{t-h}^{t-h(t)}x\,(s)\,ds\\ \left(\begin{array}{c}\int_{t-h(t)}^{t}\int_s^t x\,(u)\,duds\\ +(h-h\,(t))\int_{t-h(t)}^{t}x\,(s)\,ds\\ +\int_{t-h}^{t-h(t)}\int_s^{t-h(t)}x\,(u)\,duds\end{array}\right)\end{bmatrix}^T\\
&\times R\begin{bmatrix}\dot x\,(t)\\ \dot x\,(t-h)\\ x\,(t)-x\,(t-h)\\ \left(\begin{array}{c}hx\,(t)-\int_{t-h(t)}^{t}x\,(s)\,ds\\ -\int_{t-h}^{t-h(t)}x\,(s)\,ds\end{array}\right)\end{bmatrix}\\
&=\xi^T\,(t)\,\Phi_{1[h(t)]}\xi\,(t)\,,
\end{aligned}
\tag{11}
$$

where $\Phi_{1[h(t)]}$ can be found in (7). Similarly, we get

$$
\begin{aligned}
\dot V_2(t) &= \begin{bmatrix}\dot x\,(t)\\ x\,(t)\\ f\,(W_2x\,(t))\end{bmatrix}^T N\begin{bmatrix}\dot x\,(t)\\ x\,(t)\\ f\,(W_2x\,(t))\end{bmatrix}\\
&-\begin{bmatrix}\dot x\,(t-h)\\ x\,(t-h)\\ f\,(W_2x\,(t-h))\end{bmatrix}^T N\begin{bmatrix}\dot x\,(t-h)\\ x\,(t-h)\\ f\,(W_2x\,(t-h))\end{bmatrix}\\
&+2\{f\,(W_2x\,(t))-K_mW_2x\,(t)\}^T D_1W_2\dot x\,(t)\\
&+2\{K_pW_2x\,(t)-f\,(W_2x\,(t))\}^T D_2W_2\dot x\,(t)\\
&+2\{f\,(W_2x\,(t-h))-K_mW_2x\,(t-h)\}^T\\
&\times D_3W_2\dot x\,(t-h)\\
&+2\{K_pW_2x\,(t-h)-f\,(W_2x\,(t-h))\}^T\\
&\times D_4W_2\dot x\,(t-h)\\
&=\xi^T\,(t)\,\Phi_2\xi\,(t)\,,
\end{aligned}
\tag{12}
$$

where Φ_2 is defined in (7). $\dot{V}_3(t)$ is calculated as

$$\begin{aligned}\dot{V}_3(t) =& \begin{bmatrix} x(t) \\ 0 \\ 0 \\ x(t)-x(\text{t-h}) \\ \int_{t-h}^{t} x(s)\,ds \end{bmatrix}^T G_1 \begin{bmatrix} x(t) \\ 0 \\ 0 \\ x(t)-x(\text{t-h}) \\ \int_{t-h}^{t} x(s)\,ds \end{bmatrix} \\ &- \left(1-\dot{h}(t)\right) \begin{bmatrix} x(t-h(t)) \\ x(t)-x(t-h(t)) \\ \int_{t-h(t)}^{t} x(s)\,ds \\ x(t-h(t))-x(t-h) \\ \int_{t-h}^{t-h(t)} x(s)\,ds \end{bmatrix}^T \\ &\times G_1 \begin{bmatrix} x(t-h(t)) \\ x(t)-x(t-h(t)) \\ \int_{t-h(t)}^{t} x(s)\,ds \\ x(t-h(t))-x(t-h) \\ \int_{t-h}^{t-h(t)} x(s)\,ds \end{bmatrix} \\ &+2 \begin{bmatrix} \int_{t-h(t)}^{t} x(s)\,ds \\ h(t)x(t)-\int_{t-h(t)}^{t} x(s)\,ds \\ \int_{t-h(t)}^{t}\int_{s}^{t} x(u)\,duds \\ \int_{t-h(t)}^{t} x(s)\,ds - h(t)x(\text{t-h}) \\ \begin{pmatrix} h(t)\int_{t-h}^{t} x(s)\,ds \\ -\int_{t-h(t)}^{t}\int_{s}^{t} x(u)\,duds \end{pmatrix} \end{bmatrix}^T \\ &\times G_1 \begin{bmatrix} 0 \\ \dot{x}(t) \\ x(t) \\ -\dot{x}(t-h) \\ -x(\text{t-h}) \end{bmatrix} \\ &+ \begin{bmatrix} x(t) \\ f(W_2x(s)) \end{bmatrix}^T G_2 \begin{bmatrix} x(t) \\ f(W_2x(s)) \end{bmatrix} \\ &- \left(1-\dot{h}(t)\right) \begin{bmatrix} x(t-h(t)) \\ f(W_2x(t-h(t))) \end{bmatrix}^T \\ &\times G_2 \begin{bmatrix} x(t-h(t)) \\ f(W_2x(t-h(t))) \end{bmatrix} \\ \le& \xi^T(t)\left(\Phi_3+\Phi_{31[h(t)]}\right)\xi(t), \end{aligned} \tag{13}$$

where Φ_3 and $\Phi_{31[h(t)]}$ are defined in (7). Calculating the upper bounds of $\dot{V}_4(t)$ leads to

$$\begin{aligned}\dot{V}_4(t) =& h \begin{bmatrix} \dot{x}(t) \\ x(t) \\ 0 \\ x(t)-x(t-h) \end{bmatrix}^T Q_1 \begin{bmatrix} \dot{x}(t) \\ x(t) \\ 0 \\ x(t)-x(t-h) \end{bmatrix} \\ &+2 \begin{bmatrix} hx(t)-\int_{t-h}^{t} x(s)\,ds \\ \int_{t-h}^{t}\int_{s}^{t} x(u)\,duds \\ \frac{h^2}{2}x(t)-\int_{t-h}^{t}\int_{s}^{t} x(u)\,duds \\ -\frac{h^2}{2}x(t-h)+\int_{t-h}^{t}\int_{s}^{t} x(u)\,duds \end{bmatrix}^T \\ &\times Q_1 \begin{bmatrix} 0 \\ 0 \\ \dot{x}(t) \\ -\dot{x}(\text{t-h}) \end{bmatrix} \\ &- \int_{t-h}^{t} \begin{bmatrix} \dot{x}(s) \\ x(s) \\ \int_{s}^{t} \dot{x}(u)\,du \\ \int_{t-h}^{s} \dot{x}(u)\,du \end{bmatrix}^T Q_1 \begin{bmatrix} \dot{x}(s) \\ x(s) \\ \int_{s}^{t} \dot{x}(u)\,du \\ \int_{t-h}^{s} \dot{x}(u)\,du \end{bmatrix} ds \\ =& \xi^T(t)\left(\Phi_4+\Phi_{43[h(t)]}\right)\xi(t) \\ &- \int_{t-h}^{t} \begin{bmatrix} \dot{x}(s) \\ x(s) \\ \int_{s}^{t} \dot{x}(u)\,du \\ \int_{t-h}^{s} \dot{x}(u)\,du \end{bmatrix}^T Q_1 \begin{bmatrix} \dot{x}(s) \\ x(s) \\ \int_{s}^{t} \dot{x}(u)\,du \\ \int_{t-h}^{s} \dot{x}(u)\,du \end{bmatrix} ds. \end{aligned} \tag{14}$$

Based on the work of [31], [43], the following new zero equalities are introduced for any symmetric matrices $P_i (i = 1, 2)$.

$$\begin{aligned} 0 =& \begin{bmatrix} x(t) \\ 0 \\ x(t)-x(\text{t-h}) \end{bmatrix}^T P_1 \begin{bmatrix} x(t) \\ 0 \\ x(t)-x(\text{t-h}) \end{bmatrix} \\ &- \begin{bmatrix} x(t-h(t)) \\ x(t)-x(t-h(t)) \\ x(t-h(t))-x(\text{t-h}) \end{bmatrix}^T P_1 \\ &\times \begin{bmatrix} x(t-h(t)) \\ x(t)-x(t-h(t)) \\ x(t-h(t))-x(\text{t-h}) \end{bmatrix} \\ &-2\int_{t-h(t)}^{t} \begin{bmatrix} \dot{x}(s) \\ x(s) \\ \int_{s}^{t} \dot{x}(u)\,du \\ \int_{t-h}^{s} \dot{x}(u)\,du \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \\ &\times P_1 \begin{bmatrix} I & 0 & 0 & 0 \\ -I & 0 & 0 & 0 \\ I & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}(s) \\ x(s) \\ \int_{s}^{t} \dot{x}(u)\,du \\ \int_{t-h}^{s} \dot{x}(u)\,du \end{bmatrix} ds, \end{aligned} \tag{15}$$

$$\begin{aligned} 0 =& \begin{bmatrix} x(t-h(t)) \\ x(t)-x(t-h(t)) \\ x(t-h(t))-x(\text{t-h}) \end{bmatrix}^T P_2 \\ &\times \begin{bmatrix} x(t-h(t)) \\ x(t)-x(t-h(t)) \\ x(t-h(t))-x(\text{t-h}) \end{bmatrix} \\ &- \begin{bmatrix} x(t-h) \\ x(t)-x(t-h) \\ 0 \end{bmatrix}^T P_2 \begin{bmatrix} x(t-h) \\ x(t)-x(t-h) \\ 0 \end{bmatrix} \\ &-2\int_{t-h}^{t-h(t)} \begin{bmatrix} \dot{x}(s) \\ x(s) \\ \int_{s}^{t} \dot{x}(u)\,du \\ \int_{t-h}^{s} \dot{x}(u)\,du \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \\ &\times P_2 \begin{bmatrix} I & 0 & 0 & 0 \\ -I & 0 & 0 & 0 \\ I & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}(s) \\ x(s) \\ \int_{s}^{t} \dot{x}(u)\,du \\ \int_{t-h}^{s} \dot{x}(u)\,du \end{bmatrix} ds. \end{aligned} \tag{16}$$

Summing up the two zero equalities presented at (15) and (16) leads to

$$
\begin{aligned}
0 &= \xi^T(t)\,\Phi_{44}\xi(t) \\
&\quad - 2\int_{t-h(t)}^{t}\begin{bmatrix}\dot{x}(s)\\ x(s)\\ \int_s^t \dot{x}(u)\,du\\ \int_{t-h}^{s}\dot{x}(u)\,du\end{bmatrix}^T\begin{bmatrix}0&0&0\\ I&0&0\\ 0&I&0\\ 0&0&I\end{bmatrix}P_1 \\
&\quad \times\begin{bmatrix}I&0&0&0\\ -I&0&0&0\\ I&0&0&0\end{bmatrix}\begin{bmatrix}\dot{x}(s)\\ x(s)\\ \int_s^t \dot{x}(u)\,du\\ \int_{t-h}^{s}\dot{x}(u)\,du\end{bmatrix}ds \\
&\quad - 2\int_{t-h}^{t-h(t)}\begin{bmatrix}\dot{x}(s)\\ x(s)\\ \int_s^t \dot{x}(u)\,du\\ \int_{t-h}^{s}\dot{x}(u)\,du\end{bmatrix}^T\begin{bmatrix}0&0&0\\ I&0&0\\ 0&I&0\\ 0&0&I\end{bmatrix}P_2 \\
&\quad \times\begin{bmatrix}I&0&0&0\\ -I&0&0&0\\ I&0&0&0\end{bmatrix}\begin{bmatrix}\dot{x}(s)\\ x(s)\\ \int_s^t \dot{x}(u)\,du\\ \int_{t-h}^{s}\dot{x}(u)\,du\end{bmatrix}ds. \qquad (17)
\end{aligned}
$$

By adding (17) into (14), we have

$$
\begin{aligned}
\dot{V}_4(t) &\le \xi^T(t)\left(\Phi_4 + \Phi_{43[h(t)]} + \Phi_{44}\right)\xi(t) \\
&\quad - \int_{t-h}^{t}\begin{bmatrix}\dot{x}(s)\\ x(s)\\ \int_s^t \dot{x}(u)\,du\\ \int_{t-h}^{s}\dot{x}(u)\,du\end{bmatrix}^T Q_1\begin{bmatrix}\dot{x}(s)\\ x(s)\\ \int_s^t \dot{x}(u)\,du\\ \int_{t-h}^{s}\dot{x}(u)\,du\end{bmatrix}ds \\
&\quad - 2\int_{t-h(t)}^{t}\begin{bmatrix}\dot{x}(s)\\ x(s)\\ \int_s^t \dot{x}(u)\,du\\ \int_{t-h}^{s}\dot{x}(u)\,du\end{bmatrix}^T\begin{bmatrix}0&0&0\\ I&0&0\\ 0&I&0\\ 0&0&I\end{bmatrix} \\
&\quad \times P_1\begin{bmatrix}I&0&0&0\\ -I&0&0&0\\ I&0&0&0\end{bmatrix}\begin{bmatrix}\dot{x}(s)\\ x(s)\\ \int_s^t \dot{x}(u)\,du\\ \int_{t-h}^{s}\dot{x}(u)\,du\end{bmatrix}ds \\
&\quad - 2\int_{t-h}^{t-h(t)}\begin{bmatrix}\dot{x}(s)\\ x(s)\\ \int_s^t \dot{x}(u)\,du\\ \int_{t-h}^{s}\dot{x}(u)\,du\end{bmatrix}^T\begin{bmatrix}0&0&0\\ I&0&0\\ 0&I&0\\ 0&0&I\end{bmatrix} \\
&\quad \times P_2\begin{bmatrix}I&0&0&0\\ -I&0&0&0\\ I&0&0&0\end{bmatrix}\begin{bmatrix}\dot{x}(s)\\ x(s)\\ \int_s^t \dot{x}(u)\,du\\ \int_{t-h}^{s}\dot{x}(u)\,du\end{bmatrix}ds \\
&= \xi^T(t)\left(\Phi_4 + \Phi_{43[h(t)]} + \Phi_{44}\right)\xi(t) \\
&\quad - \int_{t-h(t)}^{t}\begin{bmatrix}\dot{x}(s)\\ x(s)\\ \int_s^t \dot{x}(u)\,du\\ \int_{t-h}^{s}\dot{x}(u)\,du\end{bmatrix}^T Q_{aug1}\begin{bmatrix}\dot{x}(s)\\ x(s)\\ \int_s^t \dot{x}(u)\,du\\ \int_{t-h}^{s}\dot{x}(u)\,du\end{bmatrix}ds \\
&\quad - \int_{t-h}^{t-h(t)}\begin{bmatrix}\dot{x}(s)\\ x(s)\\ \int_s^t \dot{x}(u)\,du\\ \int_{t-h}^{s}\dot{x}(u)\,du\end{bmatrix}^T Q_{aug2}\begin{bmatrix}\dot{x}(s)\\ x(s)\\ \int_s^t \dot{x}(u)\,du\\ \int_{t-h}^{s}\dot{x}(u)\,du\end{bmatrix}ds, \qquad (18)
\end{aligned}
$$

where Q_{aug1} and Q_{aug2} are given in (7). Then, the integral terms in (18) are estimated by using Lemma 1.

$$
\begin{aligned}
&-\int_{t-h(t)}^{t}\begin{bmatrix}\dot{x}(s)\\ x(s)\\ \int_s^t \dot{x}(u)\,du\\ \int_{t-h}^{s}\dot{x}(u)\,du\end{bmatrix}^T Q_{aug1}\begin{bmatrix}\dot{x}(s)\\ x(s)\\ \int_s^t \dot{x}(u)\,du\\ \int_{t-h}^{s}\dot{x}(u)\,du\end{bmatrix}ds \\
&\le Sym\left\{\xi^T(t)\,e_{tot}L_1\eta_1(t) + \xi^T(t)\,e_{tot}M_1\eta_2(t)\right\} \\
&\quad + h(t)\,\xi^T(t)\,e_{tot} \\
&\quad \times \left\{L_1Q_{aug1}^{-1}L_1^T + (1/3)\,M_1Q_{aug1}^{-1}M_1^T\right\}e_{tot}^T\xi(t) \\
&= \xi^T(t)\,\{\Phi_{41[h(t)]} + h(t)\,e_{tot}(L_1Q_{aug1}^{-1}L_1^T \\
&\quad + (1/3)\,M_1Q_{aug1}^{-1}M_1^T)e_{tot}^T\}\xi(t), \qquad (19)
\end{aligned}
$$

$$
\begin{aligned}
&-\int_{t-h}^{t-h(t)}\begin{bmatrix}\dot{x}(s)\\ x(s)\\ \int_s^t \dot{x}(u)\,du\\ \int_{t-h}^{s}\dot{x}(u)\,du\end{bmatrix}^T Q_{aug2}\begin{bmatrix}\dot{x}(s)\\ x(s)\\ \int_s^t \dot{x}(u)\,du\\ \int_{t-h}^{s}\dot{x}(u)\,du\end{bmatrix}ds \\
&\le Sym\left\{\xi^T(t)\,e_{tot}L_2\eta_3(t) + \xi^T(t)\,e_{tot}M_2\eta_4(t)\right\} \\
&\quad + (h - h(t))\,\xi^T(t)\,e_{tot} \\
&\quad \times \left\{L_2Q_{aug2}^{-1}L_2^T + (1/3)\,M_2Q_{aug2}^{-1}M_2^T\right\}e_{tot}^T\xi(t) \\
&= \xi^T(t)\,\{\Phi_{42[h(t)]} + (h - h(t))\,e_{tot} \\
&\quad \times \left(L_2Q_{aug2}^{-1}L_2^T + (1/3)\,M_2Q_{aug2}^{-1}M_2^T\right)e_{tot}^T\}\xi(t), \qquad (20)
\end{aligned}
$$

where

$$
\begin{aligned}
\eta_1(t) &= [e_1 - e_2, e_6, h(t)\,e_1 - e_6, -h(t)\,e_3 + e_6]^T\xi(t), \\
\eta_2(t) &= [e_1 + e_2 - 2e_{10}, -e_6 + 2e_8, e_6 - 2e_8, -e_6 + 2e_8]^T\xi(t), \\
\eta_3(t) &= [e_2 - e_3, e_7, (h - h(t))\,e_1 - e_7, \\
&\qquad - (h - h(t))\,e_3 + e_7]^T\xi(t), \\
\eta_4(t) &= [e_2 + e_3 - 2e_{11}, -e_7 + 2e_9, e_7 - 2e_9, -e_7 + 2e_9]^T\xi(t).
\end{aligned}
$$

From (14) to (20), we give an estimation of $\dot{V}_4(t)$ as follows:

$$
\begin{aligned}
\dot{V}_4(t) &\le \xi^T(t)\,\{\Phi_4 + \Phi_{41[h(t)]} + \Phi_{42[h(t)]} + \Phi_{43[h(t)]} \\
&\quad + \Phi_{44}\}\xi(t) + h(t)\,\xi^T(t)\,e_{tot}\{L_1Q_{aug1}^{-1}L_1^T \\
&\quad + (1/3)\,M_1Q_{aug1}^{-1}M_1^T\}e_{tot}^T\xi(t) \\
&\quad + (h - h(t))\,\xi^T(t)\,e_{tot}\{L_2Q_{aug2}^{-1}L_2^T \\
&\quad + (1/3)\,M_2Q_{aug2}^{-1}M_2^T\}e_{tot}^T\xi(t), \qquad (21)
\end{aligned}
$$

where Φ_4, $\Phi_{41[h(t)]}$, $\Phi_{42[h(t)]}$, $\Phi_{43[h(t)]}$, and Φ_{44} are given in (7).

Inspired by the work of [31] and [43], we consider the following two zero equalities in terms of the relationship among some vectors of $\xi(t)$.

$$
0 = \xi^T(t)\,Sym\left\{((h - h(t))\,e_{11} - e_7)\,\Psi_2\Upsilon^T\right\}\xi(t), \qquad (22)
$$

$$0 = \xi^T(t)\, Sym\left\{(h(t)\, e_{10} - e_6)\, \Psi_1 \Upsilon^T\right\} \xi(t). \quad (23)$$

Summing up the zero equalities above leads to

$$0 = \xi^T(t)\, \Theta_{1[h(t)]} \xi(t), \quad (24)$$

where $\Theta_{1[h(t)]}$ is defined in (7).

From (3), for any diagonal matrices $H_i > 0 (i = 1, \cdots, 10)$, we consider following inequality

$$\xi^T(t)\, \Theta_{2[h(t)]} \xi(t) \geq 0, \quad (25)$$

where $\Theta_{2[h(t)]}$ is given in (7). From (11) to (25), we give an estimation of $\dot{V}(t)$ as follows:

$$\begin{aligned}\dot{V}(t) \leq\ & \xi^T(t)\Phi_{[h(t)]}\xi(t) + h(t)\, \xi^T(t)\, e_{tot}\{L_1 Q_{aug1}^{-1} L_1^T \\ & + (1/3)\, M_1 Q_{aug1}^{-1} M_1^T\} e_{tot}^T \xi(t) \\ & + (h - h(t))\, \xi^T(t)\, e_{tot}\{L_2 Q_{aug2}^{-1} L_2^T \\ & + (1/3)\, M_2 Q_{aug2}^{-1} M_2^T\} e_{tot}^T \xi(t), \quad (26)\end{aligned}$$

where $\Phi_{[h(t)]}$ is given in (7). Then, a stability condition for generalized NN (1) is given as

$$\begin{aligned}& \xi^T(t)\Phi_{[h(t)]}\xi(t) \\ & \quad + h(t)\, \xi^T(t)\, e_{tot}\{L_1 Q_{aug1}^{-1} L_1^T \\ & \quad + (1/3)\, M_1 Q_{aug1}^{-1} M_1^T\} e_{tot}^T \xi(t) \\ & \quad + (h - h(t))\, \xi^T(t)\, e_{tot}\{L_2 Q_{aug2}^{-1} L_2^T \\ & \quad + (1/3)\, M_2 Q_{aug2}^{-1} M_2^T\} e_{tot}^T \xi(t) < 0. \quad (27)\end{aligned}$$

Since $0 = B\xi(t)$, the inequality (27) is equivalent to following inequality by using Lemma 2.

$$\begin{aligned}& (B^{\perp})^T \Phi_{[h(t)]}(B^{\perp}) \\ & \quad + (B^{\perp})^T h(t)\, e_{tot}\{L_1 Q_{aug1}^{-1} L_1^T \\ & \quad + (1/3)\, M_1 Q_{aug1}^{-1} M_1^T\} e_{tot}^T (B^{\perp}) \\ & \quad + (B^{\perp})^T (h_M - h(t))\, e_{tot}\{L_2 Q_{aug2}^{-1} L_2^T \\ & \quad + (1/3)\, M_2 Q_{aug2}^{-1} M_2^T\} e_{tot}^T (B^{\perp}) < 0. \quad (28)\end{aligned}$$

The inequality above depends on $h(t)$. Therefore, the inequality above is equivalent to following inequalities,

$$\begin{aligned}& (B^{\perp})^T \Phi_h(B^{\perp}) \\ & \quad + h\, (B^{\perp})^T e_{tot}\{L_1 Q_{aug1}^{-1} L_1^T \\ & \quad + (1/3)\, M_1 Q_{aug1}^{-1} M_1^T\} e_{tot}^T (B^{\perp}) < 0, \quad (29) \\ & (B^{\perp})^T \Phi_0(B^{\perp}) \\ & \quad + h\, (B^{\perp})^T e_{tot}\{L_2 Q_{aug2}^{-1} L_2^T \\ & \quad + (1/3)\, M_2 Q_{aug2}^{-1} M_2^T\} e_{tot}^T (B^{\perp}) < 0. \quad (30)\end{aligned}$$

Based on Schur complement, if inequalities (29) and (30) hold, then (8) and (9) hold. Therefore, if (8) and (9) hold, the generalized NN (1) with $h(t)$ satisfying (2) is asymptotically stable, which allows concluding the proof. ■

Remark 1: The novel LKF (10) contains the augmented nonintegral terms, the activation function-based terms, the augmented single integral terms and the augmented double integral terms. The novelty of LKF (10) in this paper is a couple of integral vectors $\int_{t-h}^{s} x(u)\, du$, $\int_{t-h}^{s} \dot{x}(u)\, du$ are introduced in G_1-dependent integral term and integral vector $\int_{t-h}^{s} \dot{x}(u)\, du$ is introduced in Q_1-dependent integral term. These points are main contributions in this paper. Because of the novel LKF (10), some new cross terms among state vectors $\dot{x}(t-h)$, $x(t-h)$ and other vectors are introduced in the proposed stability conditions. More state marginal information is utilized to derive the stability criterion with less conservatism.

Remark 2: In the previous literature such as [31], [43], and [35], the proposed LKFs included the form of $\int_{t-h(t)}^{t} \begin{bmatrix} x(s) \\ \int_s^t \dot{x}(u)\, du \\ \int_s^t x(u)\, du \end{bmatrix}^T G \begin{bmatrix} x(s) \\ \int_s^t \dot{x}(u)\, du \\ \int_s^t x(u)\, du \end{bmatrix} ds$, the couple of integral vectors $\int_s^t x(u)\, du$ and $\int_s^t \dot{x}(u)\, du$ are very effective in reducing the conservatism. As a result, the time derivative of the LKFs involves many cross terms about vector $\int_{t-h(t)}^{t} x(s)\, ds$. However, some cross terms about $\int_{t-h}^{t-h(t)} x(s)\, ds$ are ignored, which may bring conservativeness. In this paper, some new cross terms among vectors $\int_{t-h(t)}^{t} x(s)\, ds$, $\int_{t-h}^{t-h(t)} x(s)\, ds$, and other vectors are included in $\dot{V}_3(t)$ by introducing the new couple of integral vectors $\int_{t-h}^{s} x(u)\, du$ and $\int_{t-h}^{s} \dot{x}(u)\, du$ into $V_3(t)$. We will demonstrate the effectiveness of the new couple of integral vectors in numerical examples.

Remark 3: The conservation of stability condition is reduced by using the new augmented zero equalities (15) and (16) in Theorem 1. Note that the new augmented zero equalities (15) and (16) encompasses several existing zero equalities of [43], [44].

Remark 4: In the previous researches about stability analysis for DNNs, the presented LKFs usually involve $\int_{t-h}^{t}\int_s^t \begin{bmatrix} \dot{x}(u) \\ x(u) \end{bmatrix}^T Q_1 \begin{bmatrix} \dot{x}(u) \\ x(u) \end{bmatrix} duds$ or $\int_{t-h}^{t}\int_s^t \begin{bmatrix} \dot{x}(u) \\ x(u) \\ \int_u^t \dot{x}(v)\, dv \end{bmatrix}^T Q_1 \begin{bmatrix} \dot{x}(u) \\ x(u) \\ \int_u^t \dot{x}(v)\, dv \end{bmatrix} duds$ which have effect on reducing the conservatism of stability conditions. Unlike previous works, the integral vector $\int_{t-h}^{s} \dot{x}(u)\, du$ is introduced in double integral quadratic term $V_4(t)$. Based on the new zero equality (17) and Lemma 1, $\dot{V}_4(t)$ is estimated as shown in (21), which contains more cross terms and helps the stability conditions increase maximum delay bounds.

IV. RELAXATION THE POSITIVENESS OF NONINTEGRAL QUADRATIC TERM

In Theorem 1, in order to make sure that LKF (10) is positive definite, conditions $D_i > 0 (i = 1, 2, 3, 4)$, $G_i > 0 (i = 1, 2)$, $R > 0$, $N > 0$, and $Q_1 > 0$ are required. However, by combining the nonintegral term and some integral terms in LKF (10), the requirement of the condition $R > 0$ can be relaxed and less conservative stability criterion will be presented in

Theorem 2. To simplify computing, block entry matrices are defined as $\tilde{e}_0 = 0_{5n\times n}$, $\tilde{e}_i = [0_{n\times(i-1)n} \quad I_n \quad 0_{n\times(5-i)n}]^T (i = 1, \cdots, 5)$. Assume that $D_i > 0 (i = 1, 2, 3, 4)$, $G_i > 0 (i = 1, 2)$, $N > 0$, $Q_1 > 0$. Then we obtain

$$V(t) > \begin{bmatrix} x(t) \\ x(t-h) \\ \int_{t-h}^{t} x(s)\,ds \\ \int_{t-h}^{t}\int_{s}^{t} x(u)\,duds \end{bmatrix}^T R \begin{bmatrix} x(t) \\ x(t-h) \\ \int_{t-h}^{t} x(s)\,ds \\ \int_{t-h}^{t}\int_{s}^{t} x(u)\,duds \end{bmatrix} + \int_{t-h}^{t} \begin{bmatrix} \dot{x}(s) \\ x(s) \\ f(W_2 x(s)) \end{bmatrix}^T N \begin{bmatrix} \dot{x}(s) \\ x(s) \\ f(W_2 x(s)) \end{bmatrix} ds + \int_{t-h}^{t}\int_{s}^{t} \begin{bmatrix} \dot{x}(u) \\ x(u) \\ \int_{u}^{t} \dot{x}(v)\,dv \\ \int_{t-h}^{u} \dot{x}(v)\,dv \end{bmatrix}^T Q_1 \begin{bmatrix} \dot{x}(u) \\ x(u) \\ \int_{u}^{t} \dot{x}(v)\,dv \\ \int_{t-h}^{u} \dot{x}(v)\,dv \end{bmatrix} duds. \tag{31}$$

The estimation of the integral terms in (31) can be given by applying Lemma 3:

$$\begin{aligned} &\int_{t-h}^{t} \begin{bmatrix} \dot{x}(s) \\ x(s) \\ f(W_2 x(s)) \end{bmatrix}^T N \begin{bmatrix} \dot{x}(s) \\ x(s) \\ f(W_2 x(s)) \end{bmatrix} ds + \int_{t-h}^{t}\int_{s}^{t} \begin{bmatrix} \dot{x}(u) \\ x(u) \\ \int_{u}^{t} \dot{x}(v)\,dv \\ \int_{t-h}^{u} \dot{x}(v)\,dv \end{bmatrix}^T Q_1 \begin{bmatrix} \dot{x}(u) \\ x(u) \\ \int_{u}^{t} \dot{x}(v)\,dv \\ \int_{t-h}^{u} \dot{x}(v)\,dv \end{bmatrix} duds \\ &\geq \frac{1}{h} \begin{bmatrix} x(t) - x(\text{t-h}) \\ \int_{t-h}^{t} x(s)\,ds \\ \int_{t-h}^{t} f(W_2 x(s))\,ds \end{bmatrix}^T N \begin{bmatrix} x(t) - x(\text{t-h}) \\ \int_{t-h}^{t} x(s)\,ds \\ \int_{t-h}^{t} f(W_2 x(s))\,ds \end{bmatrix} \\ &+ \frac{2}{h^2} \begin{bmatrix} hx(t) - \int_{t-h}^{t} x(s)\,ds \\ \int_{t-h}^{t}\int_{s}^{t} x(u)duds \\ \frac{h^2}{2} x(t) - \int_{t-h}^{t}\int_{s}^{t} x(u)duds \\ \int_{t-h}^{t}\int_{s}^{t} x(u)duds - \frac{h^2}{2} x(\text{t-h}) \end{bmatrix}^T \times Q_1 \begin{bmatrix} hx(t) - \int_{t-h}^{t} x(s)\,ds \\ \int_{t-h}^{t}\int_{s}^{t} x(u)duds \\ \frac{h^2}{2} x(t) - \int_{t-h}^{t}\int_{s}^{t} x(u)duds \\ \int_{t-h}^{t}\int_{s}^{t} x(u)duds - \frac{h^2}{2} x(\text{t-h}) \end{bmatrix} \\ &= \theta^T(t)\,\Pi_1 \theta(t), \end{aligned} \tag{32}$$

where

$$\theta(t) = \left[x^T(t), x^T(t-h), \int_{t-h}^{t} x^T(s)\,ds, \int_{t-h}^{t}\int_{s}^{t} x^T(u)\,duds, \int_{t-h}^{t} f^T(W_2 x(s))\,ds\right]^T,$$

$$\Pi_1 = \frac{1}{h}\left[\tilde{e}_1 - \tilde{e}_2, \tilde{e}_3, \tilde{e}_5\right] N \left[\tilde{e}_1 - \tilde{e}_2, \tilde{e}_3, \tilde{e}_5\right]^T + \frac{2}{h^2}\left[h\tilde{e}_1 - \tilde{e}_3, \tilde{e}_4, \frac{h^2}{2}\tilde{e}_1 - \tilde{e}_4, \tilde{e}_4 - \frac{h^2}{2}\tilde{e}_2\right] \times Q_1 \left[h\tilde{e}_1 - \tilde{e}_3, \tilde{e}_4, \frac{h^2}{2}\tilde{e}_1 - \tilde{e}_4, \tilde{e}_4 - \frac{h^2}{2}\tilde{e}_2\right]^T.$$

We can easily know that if the following inequality holds

$$\left[\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4\right] R \left[\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4\right]^T + \Pi_1 > 0, \tag{33}$$

then the $V(t)$ is positive definite. Therefore, by replacing the condition $R > 0$ in Theorem 1 with the inequality (33), we obtain the following theorem.

Theorem 2: For given scalars h and h_D, the generalized NN (1) with $h(t)$ satisfying (2) is asymptotically stable, if there exist positive diagonal matrices $D_i = diag\{d_{1i}, \cdots, d_{ni}\} (i = 1, 2, 3, 4)$, $H_i (i = 1, \cdots, 10)$, positive symmetric matrices $G_1 \in \mathcal{R}^{5n\times 5n}$, $Q_1 \in \mathcal{R}^{4n\times 4n}$, $N \in \mathcal{R}^{3n\times 3n}$, $G_2 \in \mathcal{R}^{2n\times 2n}$, symmetric matrices $R \in \mathcal{R}^{4n\times 4n}$, $P_i (i = 1, 2) \in \mathcal{R}^{3n\times 3n}$, and any matrices $\Psi_i (i = 1, 2) \in \mathcal{R}^{n\times 6n}$, $M_i, L_i \in \mathcal{R}^{14n\times 4n} (i = 1, 2)$, such that LMIs (8), (9), and (33) are satisfied.

Remark 5: The positive definiteness of the LKF (10) is relaxed by (33), which is different from Theorem 1. Therefore, the positive definiteness of the LKF (10) does not require the condition $R > 0$, which helps to reduce the conservativeness of stability criterion. We will demonstrate the improvements of Theorem 2 in numerical examples.

V. REFINEMENT OF ALLOWABLE DELAY SET

In Theorem 1 and 2, the resulting conditions are affinely on the independent delay parameters $\dot{h}(t)$ and $h(t)$. Actually, inspired of [40], [41], the delay parameters $\dot{h}(t)$ and $h(t)$ are interrelated in the real world. Thus, it is necessary to introduce a refinement allowable delay set. In Theorem 1, 2, and other stability criteria of previous literature, the general formula of the delay set for (2) is as follows:

$$\left(h(t), \dot{h}(t)\right) \in \mathcal{H}_1 := [0, h] \times (-\infty, h_D], \tag{34}$$

where $\mathcal{H}_1$ is a polytope with four vertices. Let $d > 0$ be a scalar, then $\mathcal{H}_1$ can be represented as

$$\mathcal{H}_1 = \lim_{d\to\infty} Co\{(0, -d), (0, h_D), (h, -d), (h, h_D)\}. \tag{35}$$

If $h_D > 0$, it should be pointed out that the vertices (h, h_D) and $(0, -d)$ in $\mathcal{H}_1$ are unreasonable because $\dot{h}(t)$ cannot be positive when $h(t) = h$ and negative when $h(t) = 0$. The details can be found in [40] and [41]. Considering these cases, the allowable delay set $\mathcal{H}_1$ can be revised as

$$\widehat{\mathcal{H}}_1 = \lim_{d\to\infty} Co\{(0, 0), (0, h_D), (h, -d), (h, 0)\}. \tag{36}$$

The delay set $\widehat{\mathcal{H}}_1$ provides a more realistic presentation of time delay function which can enlarges maximum delay bounds. Hence, the following theorem for the delay set $\widehat{\mathcal{H}}_1$ is derived.

Theorem 3: For given scalars h and h_D, the generalized NN (1) with $\left(h(t), \dot{h}(t)\right) \in \widehat{\mathcal{H}}_1$ is asymptotically stable, if there exist positive diagonal matrices $D_i = diag\{d_{1i}, \cdots, d_{ni}\} (i = 1, 2, 3, 4)$, $H_i (i = 1, \cdots, 10)$, positive symmetric matrices $G_1 \in \mathcal{R}^{5n\times 5n}$, $Q_1 \in \mathcal{R}^{4n\times 4n}$,

$N \in \mathcal{R}^{3n\times 3n}$, $G_2 \in \mathcal{R}^{2n\times 2n}$, symmetric matrices $R \in \mathcal{R}^{4n\times 4n}$, $P_i(i = 1, 2) \in \mathcal{R}^{3n\times 3n}$, and any matrices $\Psi_i(i = 1, 2) \in \mathcal{R}^{n\times 6n}$, M_i, $L_i \in \mathcal{R}^{14n\times 4n}(i = 1, 2)$ satisfying (33) and the following LMIs:

$$\begin{bmatrix} (B^{\perp})^T \tilde{\Phi}_h(B^{\perp}) & (B^{\perp})^T (he_{tot}L_1) & (B^{\perp})^T (he_{tot}M_1) \\ * & -hQ_{aug1} & 0 \\ * & * & -3hQ_{aug1} \end{bmatrix} < 0, \tag{37}$$

$$\begin{bmatrix} (B^{\perp})^T \Phi_0(B^{\perp}) & (B^{\perp})^T (he_{tot}L_2) & (B^{\perp})^T (he_{tot}M_2) \\ * & -hQ_{aug2} & 0 \\ * & * & -3hQ_{aug2} \end{bmatrix} < 0, \tag{38}$$

where $\tilde{\Phi}_h = \Phi_{[h(t)=h]} - h_D\tilde{\Pi}_1$, $\Phi_0 = \Phi_{[h(t)=0]}$, B, e_{tot}, Q_{aug1} and Q_{aug2} are defined in (7).

Proof: Based on (10)-(25), we give an estimation of $\dot{V}(t)$ as follows:

$$\dot{V}(t) \leq \xi^T(t)\left\{\Phi_{tot[h(t)]} + \left(\dot{h}(t) - h_D\right)\tilde{\Pi}_1\right\}\xi(t), \tag{39}$$

where

$$\begin{aligned}\tilde{\Pi}_1 &= [e_2, e_1 - e_2, e_6, e_2 - e_3, e_7] \\ &\quad \times G_1[e_2, e_1 - e_2, e_6, e_2 - e_3, e_7]^T \\ &\quad \times +[e_2, e_{13}]\, G_2[e_2, e_{13}]^T,\end{aligned}$$

and $\Phi_{tot[h(t)]}$ is defined in (7).

From (39), a stability condition for the generalized NN (1) with $(h(t), \dot{h}(t)) \in \widehat{\mathcal{H}}_1$ is given as

$$(B^{\perp})^T\left\{\Phi_{tot[h(t)=0]} - h_D\tilde{\Pi}_1\right\}(B^{\perp}) < 0, \tag{40}$$

$$(B^{\perp})^T\Phi_{tot[h(t)=0]}(B^{\perp}) < 0, \tag{41}$$

$$\lim_{d\to\infty}(B^{\perp})^T\left\{\Phi_{tot[h(t)=h]} - (d + h_D)\,h_D\tilde{\Pi}_1\right\}(B^{\perp}) < 0, \tag{42}$$

$$(B^{\perp})^T\left\{\Phi_{tot[h(t)=h]} - h_D\tilde{\Pi}_1\right\}(B^{\perp}) < 0. \tag{43}$$

If $h_D > 0$ and $\tilde{\Pi}_1 > 0$, we can clearly see that if (41) holds, so is (40); and if (43) holds, so is (42). Based on Schur complement, the inequality (41) and (43) are equivalent to (37) and (38), respectively. Therefore, if inequalities (37), (38), and (33) hold, the generalized NN (1) with $\left(h(t), \dot{h}(t)\right) \in \widehat{\mathcal{H}}_1$ is asymptotically stable, which allows concluding the proof. ■

Remark 6: In most of previous literature, stability conditions of DNNs are affinely on independent delay informations $h(t)$ and $\dot{h}(t)$. Actually, the delay parameters $h(t)$ and $\dot{h}(t)$ are interrelated in the real world. The allowable of delay set (36) which provides a more realistic presentation of the time-varying delay function is more reasonable than (35). Compared to Theorem 2, Theorem 3 replaces condition (8) with the condition (37) by refining the allowable delay set. As a result, Theorem 3 relaxes the stability conditions of Theorem 1 and 2. The effectiveness of refining the allowable delay set will be demonstrated in numerical examples.

Remark 7: In the case that $\dot{h}(t)$ is unknown, the Theorems 1, 2, and 3 are still feasible by setting $G_i = 0(i = 1, 2)$.

TABLE 1. Maximum delay bounds h for different h_D (Example 1).

methods	$h_D = 0.8$	$h_D = 0.9$	$unknown\ h_D$
[15]	3.7174	2.8339	2.8222
[46](Theorem 3)	4.0918	2.5895	-
[46](Theorem 2)	4.5940	3.4671	3.4504
[22]	4.8167	3.4245	3.4011
[18](Theorem 9)	5.0945	3.4978	3.4506
[47]	5.3611	3.6708	3.5724
[41](Proposition 1)	6.2889	3.7523	-
[31](Theorem 3)	6.7001	4.0707	3.6066
Theorem 1	7.1699	4.2040	3.6146
Theorem 2	7.1718	4.2979	3.7615
Theorem 3	20.4357	7.8157	3.7615

VI. NUMERICAL EXAMPLES

In this part, we give four numerical examples for verifying the improvements and effectiveness of the proposed stability criteria.

Example 1: we consider the generalized NN (1), where

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad W_0 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix},$$
$$W_1 = \begin{bmatrix} 0.88 & 1 \\ 1 & 1 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$K_p = diag\{0.4, 0.8\}, \quad K_m = diag\{0, 0\}.$$

Under different h_D, the maximum delay bounds derived by Theorems 1-3 are shown in Table 1. Also, the results of [15], [18], [22], [31], [41], [46], and [47] are included in Table 1. It is clearly confirmed that our methods are significantly improved, compared with the methods provided in previous literature, which shows advantage of the novel LKF and proposed zero equalities. Furthermore, compared with Theorem 1, Theorem 2 increases maximum delay bounds of stability condition by relaxing the positiveness of nonintegral quadratic term. Lastly, we can clearly see that the maximum bounds derived by Theorem 3 are obviously larger than those derived by Theorems 1 and 2. This means that the refined allowable delay set $\widehat{\mathcal{H}}_1$ is very useful to reduce the conservatism.

Example 2: we consider the generalized NN (1), where

$$A = \begin{bmatrix} 1.5 & 0 \\ 0 & 0.7 \end{bmatrix}, \quad W_0 = \begin{bmatrix} 0.0503 & 0.0454 \\ 0.0987 & 0.2075 \end{bmatrix},$$
$$W_1 = \begin{bmatrix} 0.2381 & 0.9320 \\ 0.0388 & 0.5062 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$K_m = diag\{0, 0\}, \quad K_p = diag\{0.3, 0.8\}.$$

when h_D is 0.4, 0.45, 0.5, and 0.55, the comparison results on maximum delay bounds are listed in Table 2. One can see that our results are superior to those obtained in [17], [18], [30], [31], [35], and [47]. This means that the proposed methods are efficient in reducing the conservativeness of stability criteria.

Example 3: we consider the generalized NN (1), where

$$A = \begin{bmatrix} 7.3458 & 0 & 0 \\ 0 & 6.9987 & 0 \\ 0 & 0 & 5.5949 \end{bmatrix}, \quad W_0 = 0, \ W_1 = I,$$

TABLE 2. Maximum delay bounds h for different h_D (Example 2).

methods	$h_D = 0.4$	$h_D = 0.45$	$h_D = 0.5$	$h_D = 0.55$
[17](Corollary 3)	7.4203	6.6190	6.3428	6.2095
[30](Theorem 1)	8.3498	7.3817	7.0219	6.8156
[18] (Theorem 6)	9.3397	8.2439	7.7893	7.5068
[18] (Theorem 9)	9.6800	8.5192	8.0535	7.7707
[47]	10.2367	9.0586	8.5986	8.3181
[31](Theorem 3)	16.8020	11.6745	9.9098	9.0062
[35](Theorem 3)	18.8262	12.8039	10.6230	9.5038
Theorem 1	20.4781	13.9052	11.5450	10.2434
Theorem 2	20.8249	14.0082	11.5834	10.2690
Theorem 3	38.3045	25.1413	19.9996	17.1362

TABLE 3. Maximum delay bounds h for different h_D (Example 3).

methods	$h_D = 0$	$h_D = 0.1$	$h_D = 0.5$	$h_D = 0.9$
[19](Corollary 1)	1.5575	0.9430	0.4417	0.3632
[48]	1.6409	0.9962	0.4470	0.3803
[22]	1.7250	1.0408	0.4480	0.3946
[30](Theorem 1)	1.7302	1.0453	0.4486	0.3938
[31](Theorem 3)	1.8899	1.1194	0.4599	0.4027
Theorem 1	1.8899	1.1202	0.4605	0.4032
Theorem 2	1.8899	1.1205	0.4605	0.4032
Theorem 3	1.8899	1.8899	1.1461	0.6989

$$W_2 = \begin{bmatrix} 13.6014 & -2.9616 & -0.6936 \\ 7.4736 & 21.6810 & 3.2100 \\ 0.7920 & -2.6334 & -20.1300 \end{bmatrix},$$
$$K_m = diag\{0, 0, 0\}, \quad K_p = diag\{0.3680, 0.1795, 0.2867\}.$$

This example is frequently used to confirm effectiveness of the stability conditions in [19], [22], [30], [31], and [48]. In Table 3, for various h_D, we show the corresponding results obtained by our methods and the results of other literature. Obviously, our methods are better than those in [19], [22], [30], [31], and [48]. When the delay set $\mathcal{H}_1$ is refined as $\widehat{\mathcal{H}}_1$, the corresponding result outperforms the results of Theorem 1 and 2. This means that the delay set $\widehat{\mathcal{H}}_1$ is very helpful to reduce the conservativeness.

Example 4: we consider the generalized NN (1), where

$$A = \begin{bmatrix} 1.2769 & 0 & 0 & 0 \\ 0 & 0.6231 & 0 & 0 \\ 0 & 0 & 0.9230 & 0 \\ 0 & 0 & 0 & 0.4480 \end{bmatrix},$$
$$W_0 = \begin{bmatrix} -0.0373 & 0.4852 & -0.3351 & 0.2336 \\ -1.6033 & 0.5988 & -0.3224 & 1.2352 \\ 0.3394 & -0.0860 & -0.3824 & -0.5785 \\ -0.1311 & 0.3253 & -0.9534 & -0.5015 \end{bmatrix},$$
$$W_1 = \begin{bmatrix} 0.8674 & -1.2405 & -0.5325 & 0.0220 \\ 0.0474 & -0.9164 & 0.0360 & 0.9816 \\ 1.8495 & 2.6117 & -0.3788 & 0.8428 \\ -2.0413 & 0.5179 & 1.1734 & -0.2775 \end{bmatrix},$$
$$W_2 = diag\{1, 1, 1, 1\},$$
$$K_m = diag\{0, 0, 0, 0\},$$
$$K_p = diag\{0.1137, 0.1279, 0.7994, 0.2368\}.$$

when h_D is 0.1, 0.5, and 0.9, the comparison results on maximum delay bounds are listed in Table 4.

TABLE 4. Maximum delay bounds h for different h_D (Example 4).

methods	$h_D = 0.1$	$h_D = 0.5$	$h_D = 0.9$
[15]	3.7857	3.0546	2.6703
[22]	4.1903	3.0779	2.8268
[30] (Theorem 1)	4.2778	3.2152	2.9361
[17]	4.2993	3.1577	2.8371
[18] (Theorem 6)	4.3711	3.2891	3.0482
[18] (Theorem 9)	4.3726	3.2899	3.0557
Theorem 1	4.4525	3.4855	3.1964
Theorem 2	4.4557	3.4906	3.2000
Theorem 3	4.7763	4.7763	4.7341

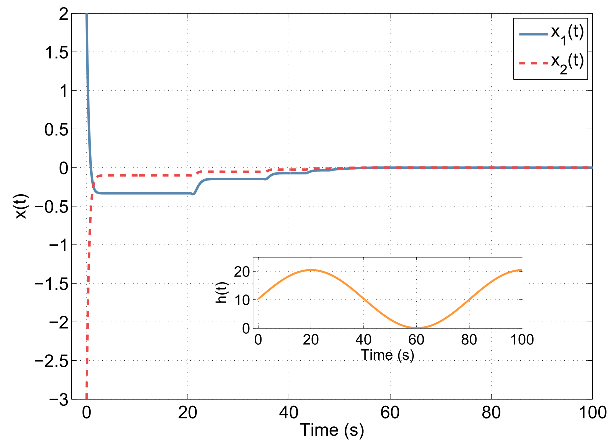

FIGURE 1. State trajectories of the generalized NN (1) in Example 1.

We can see that our results outperform those obtained in [15], [17], [17], [18], [22], and [30], which indicates the improvements of the proposed criteria.

Finally, by setting $x(0) = [2, -3]^T$, $f(x(t)) = [0.4tanh(x_1(t)), 0.8tanh(x_2(t))]^T$, $h(t) = (20.4357/2)(1 + sin(1.6t/20.4357))$, the state responses of the generalized NN (1) described in Example 1 is shown in Figure 1. We clearly see that the generalized NN (1) described in Example 1 is asymptotic stability by the resulting responses.

VII. CONCLUSION

The stability analysis of generalized NNs with time-varying delays is revisited in this paper. First, we provide an improved delay-dependent stability criterion of generalized NNs in Theorem 1 by constructing a novel augmented LKF. Based on Theorem 1, we present a less conservative delay-dependent stability criterion in Theorem 2 by relaxing the positiveness of the nonintegral quadratic term. Then, a further improved delay-dependent stability criterion is derived in Theorem 3 by refining allowable delay set. Finally, four numerical examples are utilized to clearly demonstrate improvements.

In the future, we will further study the state estimation problems [49], [50], exponential stability analysis, and passivity analysis [51] for various NNs by using our methods. Besides, the methods proposed in the paper can be extended to analyze the control synthesis problem for many industrial control systems, such as offshore platforms [52], linear motors [53] and other nonlinear systems.

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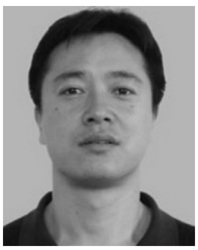

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