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Sliding Mode Observe and Control for the Underactuated Inertia Wheel Pendulum System

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ABSTRACT The inertia wheel pendulum (IWP) is a widely studied nonlinear benchmark underactuated system and its control problem is a challenging task for its underactuated nature. This paper considers the IWP stabilization problem with the classic sliding mode method. The nonlinear canonical model of the underactuated IWP is obtained through a collocated partial feedback linearization and two global changes of coordinates. In order to obtain the unmeasurable states of the newly derived model, two classic sliding mode observers are designed and it is ensured that the observing errors are convergent in finite time to meet the separation principle. In order to reduce the high frequency component of the observing output, the first-order filters are introduced from the view of practical applications. A simple sliding mode controller is proposed with the output of the first-order filters. It is proved that the proposed approach can guarantee semi-global uniform ultimate boundedness of all signals in the closed-loop system and the convergence speed can be improved by appropriately choosing design parameters. The simulation results demonstrate the effectiveness of the proposed approach.

INDEX TERMS Underactuated mechanical systems, inertia wheel pendulum (IWP), sliding mode observe, sliding mode control.

I. INTRODUCTION

The underactuated systems have several advantages: reduction of weight, reduction of the propensity to breakdown or energy cost of the reduced control [1], [2]. As a result, they are now widely employed in many areas. However, their control inputs can only control part of the dynamics while the remaining part is adjusted through the system internal dynamics, and they exhibit a non-zero degree of underactuation and a highly nonlinear dynamic [3]. Therefore, none of the techniques proposed for fully actuated systems can be applied directly and the matured nonlinear control techniques such as feedback linearization [4] and backstepping [5] can also not be applied directly. The advantages and complexity of the underactuated systems have led many researchers devote themselves to the automatic control in underactuated systems field to propose some appropriate techniques to indirectly control the coordinates through the internal dynamics.

As a kind of benchmark underactuated system with an unactuated pendulum and an actuated inertia wheel, the IWP is looked as a test bed to verify the effectiveness of control algorithms for the underactuated mechanical systems in recent years. The control objective is usually to swing the pendulum to the upright position and keep it steady there [6]. To accomplish the objective, the nonlinear coupling between of the pendulum and the wheel is utilized in full. Until now, a plenty of research results proposed for the IWP have been published in the literature [7]–[20]. The main control methods to control the IWP include: interconnection and damping assignment-passivity based control (IDA-PBC) [7]–[9], backstepping [10]–[13] and sliding mode control (SMC) [14]–[18]. IDA-PBC is a kind of passivity-based control methods that capture the essential physical property of energy conservation. A state feedback law is usually obtained by shaping the kinetic and potential energy functions. In the design process, solving PDEs is the main difficulty in application of the IDA-PBC [9]. Lyapunov functions are crucial in the study of stabilization, but are generally hard to construct. To some degree, the backstepping

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technique can reduce the difficulty of constructing Lyapunov functions, since the concerned systems are divided into different subsystems and each time only a lower dimensional subsystem needs to be considered [10]. Some attempts [11]–[13] are made for the IWP. In [11], the IWP is transformed into a cascade interconnection of a nonlinear core subsystem and a linear subsystem. The core subsystem is designed with the nonlinear control proposed in [6], and the linear subsystem is designed with the classic backstepping technique. In [13], a composite block state variable is defined to begin the backstepping process and thus the design process must depend on the block state variable. In order to get satisfactory performance, the sliding mode-based controllers are also widely employed in the stabilizing control of IWP through proposing their different specific sliding mode functions [14]–[16]. In [14] and [15], Sun *et al.* propose a novel sliding mode controller for the IWP system, which achieve superior performance over traditional methods and are robust to external disturbances. Khalid and Memon [16] proposed a state feedback control based on sliding mode control scheme for the IWP and the state feedback controller is extended to an output feedback control with a high gain observer. Some sliding mode tracking controllers are proposed for IWP [17], [18]. A second order sliding mode tracking controller for IWP was presented, where the desired trajectories are generated from a reference model governed by a two-relay controller which introduces periodic motion. In addition, the disturbance observer-based control methods [19], [20] are also successfully applied to the control of IWP. Some reviews are reported in [1] and [2] about the control of the underactuated mechanical systems (include the control of IWP) by Krafes and Liu.

It is well known that almost all existing physical and engineering systems unavoidably include uncertainties and disturbances due to inaccurate modeling, measurement errors, exterior conditions, or parameter variations. The presence of uncertainties may cause instability and bad performances on a controlled system. The capability to control complex nonlinear systems and the robustness to uncertainties and external disturbance are the two main reasons making SMC the first choice to control of nonlinear uncertain systems. Fast response, simple design, and order reduction are extra desirable features [21]. Although great efforts have made on the sliding mode control of IWP, the corresponding research still has a long way to go. If there is no specific sliding mode function, the simple two steps SMC design procedure of: *i*) choosing a stable sliding manifold, and *ii*) finding a control law to enforce sliding mode in the manifold along system dynamics, is not directly applicable to the underactuated IWP because of its complex dynamics. Therefore, a successful application of SMC needs more reconsiderations. In the proposed control design framework, the dynamics of the IWP is transformed into a canonical normal form through partial feedback linearization and two global changes of coordinates. Two classic sliding mode observers are designed to observe the unmeasurable states in the newly derived system dynamics and the observing errors are convergent in finite time. A single sliding surface is proposed with the observed states to ensure: *i*) the control design is simple, *ii*) the selected control law enforces the system dynamics to the sliding mode manifold, and *iii*) the sliding mode manifold is stable. The main contributions and significance of this paper are summarized as follows:

1) To the best of our knowledge, this is the first attempt systematically design the observer and controller with sliding mode and first-order filters, and theoretically prove the stability for the underactuated IWP. It is the first attempt to use the classic method to solving a challenging control problem. Although some SMC for IWP is proposed as stated above, some specific sliding mode functions are required to perform the two steps design procedure of SMC.
2) The proposed approach can stabilize both the pendulum and the wheel at the same time. The first-order filters are introduced to reduce the high frequency component of the observed values from the view of practical applications. The sliding mode observer and controller have robustness to uncertainties and external disturbance unavoidably existing in all physical and engineering systems. In this sense, the proposed approach is close to practical application.
3) The proposed approach is designed for a class of nonlinear canonical system, so it can be used to control all the mechanical systems that can be described with the canonical formation, including the underactuated systems that can be transformed to the cascade system, such as the TORA and the Acrobot. Therefore, the proposed approach indicates a feasible and promising solution for the control of these systems.

The rest of the paper is organized as follows. The section II is the description and preliminary of the IWP, where the system dynamics are transformed to a canonical system; the sliding mode observers and controller are designed and the system stabilization is proved in section III; and some simulations are performed to validate the effectiveness and feasibility of the proposed control approach in section IV.

II. THE IWP SYSTEM MODEL

The IWP is shown in FIGURE.1, which consists of a physical pendulum and a revolving wheel at the end. The motor torque

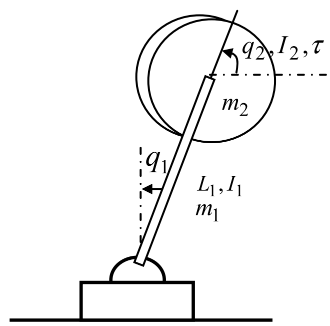

FIGURE 1. IWP system configuration.

produces an angular acceleration for the revolving wheel which generates a coupling torque for the pendulum. The task is to stabilize the pendulum in its upright equilibrium point while the wheel stops rotating and a specific angle is not important. The revolving wheel is actuated and the joint of the pendulum at the base is unactuated. That is to say, it is a benchmark example of the underactuated mechanical system [1], [2], which has one control input and two configuration variables, and its Euler-Lagrange equations of motion [6] can be obtained as

$$\begin{cases} m_{11}\ddot{q}_1 + m_{12}\ddot{q}_2 - m_0\sin(q_1) = 0 \\ m_{21}\ddot{q}_1 + m_{22}\ddot{q}_2 = \tau \end{cases} \tag{1}$$

where, $m_{11} = m_1 l_1^2 + m_2 L_1^2 + I_1 + I_2$, $m_{12} = m_{21} = m_{22} = I_2$, $m_0 = (m_1 l_1 + m_2 L_1)g$ and

m_1, m_2 - the equivalent mass of the pendulum and the wheel (kg)

I_1, I_2 - moment of inertia of the physical pendulum and the revolving wheel (kgm^2)

L_1, l_1 - length of the pendulum and distance to the center of the mass (m)

q_1 - angle that the pendulum makes with the vertical

q_2 - angle of the wheel

τ - input torque applied on the wheel (Nm)

g - acceleration due to gravity (m/sec^2)

The following collocated partial feedback linearization [22] is used

$$\tau = \left(m_{22} - \frac{m_{21}m_{12}}{m_{11}}\right)u + \frac{m_{21}m_0}{m_{11}}\sin(q_1) \tag{2}$$

to reduce the dynamics of the shape variable q_2 to

$$\ddot{q}_2 = u \tag{3}$$

Since that q_2 does not play any important role in the dynamics of the IWP, it is ignored as a state variable.

The following global change of coordinates [23] is designed

$$\begin{cases} x_1 = m_{11}\dot{q}_1 + m_{12}\dot{q}_2 \\ x_2 = q_1 \\ x_3 = \dot{q}_2 \end{cases} \tag{4}$$

to transform the system dynamics into a nonlinear system as

$$\begin{cases} \dot{x}_1 = m_0\sin x_2 \\ \dot{x}_2 = (x_1 - m_{12}x_3)/m_{11} \\ \dot{x}_3 = u \end{cases} \tag{5}$$

From (5), it can be seen that the system model of IWP is a nonlinear feedback cascade model. Unlike available results [10], the controllers to be studied in this paper are derived without using backstepping. To facilitate the control design, a set of new state variables is defined and a coordinate change will be carried out to transform the feedback cascade model (5) into a canonical system.

Another global change of coordinates is defined as

$$\begin{cases} \theta_1 = x_1 \\ \theta_2 = \dot{\theta}_1 = m_0\sin x_2 \\ \theta_3 = \dot{\theta}_2 = \dfrac{m_0 \cos x_2 (x_1 - m_{12}x_3)}{m_{11}} \end{cases} \tag{6}$$

Then system (5) can be represented as

$$\begin{cases} \dot{\theta}_1 = \theta_2 \\ \dot{\theta}_2 = \theta_3 \\ \dot{\theta}_3 = m_0 \sin x_2 (\dfrac{x_1 - m_{12}x_3}{m_{11}})^2 \\ +m_0 \cos x_2 (\dfrac{m_0\sin x_2}{m_{11}} - \dfrac{m_{12}u}{m_{11}}) \end{cases} \tag{7}$$

It is shown that with the newly defined states (6), the nonlinear feedback cascade model (5) is reformulated as a nonlinear canonical system (7). However, in the newly derived system (7), the states θ_2 and θ_3 are not directly measurable, and the functions maybe are unknown or have some uncertainties. In this sense, state-feedback control of the feedback cascade system (5) can be viewed as output-feedback control of canonical system (7).

In the following, the output-feedback control design for (7) will be considered to stably control the system (5), and finally achieve the purpose to stably control system (1). The sliding mode observers will be introduced and a sliding mode controller will be proposed.

III. SLIDING MODE OBSERVE AND CONTROL DESIGN

A. SLIDING MODE OBSERVE

To derive a suitable observer of θ_2, i.e. $\dot{\theta}_1$ for the nonlinear canonical system (7), an auxiliary variable [24] is considered as follows:

$$\dot{\xi}_1 = \alpha_1(\theta_1 - \xi_1) + \gamma_1 \mathrm{sign}(\theta_1 - \xi_1) \quad \xi_1(0) = \theta_1(0) \tag{8}$$

where,

$$\alpha_1 > 0,\ \gamma_1 = m_0 + \frac{\beta_1}{2},\ \beta_1 > 0,\ \mathrm{sign}(\alpha) = \begin{cases} -1 & \alpha < 0 \\ 0 & \alpha = 0 \\ 1 & \alpha > 0. \end{cases}$$

Subtracting (8) from the second equation of (6), the error model can be obtained as

$$\dot{\theta}_1 - \dot{\xi}_1 = -\alpha_1(\theta_1 - \xi_1) - \gamma_1 \mathrm{sign}(\theta_1 - \xi_1) + m_0\sin x_2 \tag{9}$$

To ensure a sliding motion on the surface $\theta_1 - \xi_1 = 0$, a Lyapunov function candidate

$$V_1 = (\theta_1 - \xi_1)^2 \tag{10}$$

is selected. Its derivative is from (9)

$$\begin{aligned} \dot{V}_1 &= 2(\theta_1 - \xi_1)[-\alpha_1(\theta_1 - \xi_1) - \gamma_1 \mathrm{sign}(\theta_1 - \xi_1) + m_0\sin x_2] \\ &\leq -\beta_1\sqrt{V_1} \end{aligned} \tag{11}$$

and

$$\sqrt{V_1(t)} \leq \sqrt{V_1(0)} - \frac{\beta_1}{2}t \tag{12}$$

Therefore, the Lyapunov function $V_1 = 0$ and the surfaces $\theta_1 - \xi_1 = 0$ can be reached in finite time from (10), the following equation holds from (9)

$$\gamma_1 \text{sign}(x_1 - \xi_1) = m_0 \sin x_2 \tag{13}$$

where, $\gamma_1 \text{sign}(x_1 - \xi_1)$ is the observable output of θ_2 or $\dot{\theta}_1$.

In order to reduce the high frequency component of the nonlinear signal, a first-order filter is introduced

$$\tau_1 \dot{z}_1 + z_1 = \gamma_1 \text{sign}(x_1 - \xi_1) \tag{14}$$

take

$$\hat{\theta}_2 = \hat{\dot{\theta}}_1 = z_1 \tag{15}$$

as the estimate of θ_2,

Define the filter error as

$$e_1 = \hat{\theta}_2 - \gamma_1 \text{sign}(x_1 - \xi_1) = \hat{\theta}_2 - m_0 \sin x_2 \tag{16}$$

Its derivative is

$$\begin{aligned} \dot{e}_1 &= \dot{z}_1 - m_0 \frac{x_1 - m_{12}x_3}{m_{11}} \cos x_2 \\ &= -\frac{e_1}{\tau_1} - m_0 \frac{x_1 - m_{12}x_3}{m_{11}} \cos x_2 \end{aligned} \tag{17}$$

The above procedure is repeated to estimate θ_3 i.e. $\dot{\theta}_2$, and the following auxiliary variable is introduced:

$$\dot{\xi}_2 = \alpha_2(\theta_2 - \xi_2) + \gamma_2 \text{sign}(\theta_2 - \xi_2) + x_3 \quad \xi_2(0) = \theta_2(0) \tag{18}$$

where,

$$\alpha_2 > 0,\ \gamma_2 = \frac{m_0(|x_1| + m_{12}\,|x_3|)}{m_{11}} + |x_3| + \frac{\beta_2}{2},\ \beta_2 > 0.$$

Subtracting (18) from the third equation of (6), the error model can be obtained as

$$\begin{aligned} \dot{\theta}_2 - \dot{\xi}_2 = -\alpha_2(\theta_2 - \xi_2) - \gamma_2 \text{sign}(\theta_2 - \xi_2) \\ + \frac{m_0 \cos x_2 (x_1 - m_{12}x_3)}{m_{11}} - x_3 \end{aligned} \tag{19}$$

To ensure a sliding motion on the surface $\theta_2 - \xi_2 = 0$, a Lyapunov function candidate

$$V_2 = (\theta_2 - \xi_2)^2 \tag{20}$$

is selected. Its derivative is

$$\begin{aligned} \dot{V}_2 &= 2(\theta_2 - \xi_2)[-\alpha_2(\theta_2 - \xi_2) - \gamma_2 \text{sign}(\theta_2 - \xi_2) \\ &\quad + \frac{m_0 \cos x_2 (x_1 - m_{12}x_3)}{m_{11}} - x_3] \\ &\leq -\beta_2 \sqrt{V_2} \end{aligned} \tag{21}$$

and

$$\sqrt{V_2(t)} \leq \sqrt{V_2(0)} - \frac{\beta_2}{2} t \tag{22}$$

Therefore, the Lyapunov function $V_2 = 0$ and the surfaces $\theta_2 - \xi_2 = 0$ can be reached in finite time from (20), the following equation holds from (19)

$$\gamma_2 \text{sign}(\theta_2 - \xi_2) = \frac{m_0 \cos x_2 (x_1 - m_{12}x_3)}{m_{11}} - x_3 \tag{23}$$

The following first-order filter is introduced

$$\tau_2 \dot{z}_2 + z_2 = \gamma_2 \text{sign}(\theta_2 - \xi_2) \tag{24}$$

where, $\gamma_2 \text{sign}(\dot{x}_1 - \xi_2)$ is the observable output of $\dot{\theta}_2 - x_3$.

Take

$$\hat{\theta}_3 = \hat{\dot{\theta}}_2 = z_2 + x_3 \tag{25}$$

as the estimate of θ_3,

Define the filter error

$$\begin{aligned} e_2 &= \hat{\theta}_3 - m_0 \frac{x_1 - m_{12}x_3}{m_{11}} \cos x_2 \\ &= z_2 + x_3 - m_0 \frac{x_1 - m_{12}x_3}{m_{11}} \cos x_2 \end{aligned} \tag{26}$$

Its derivative is

$$\begin{aligned} \dot{e}_2 &= \dot{z}_2 + \dot{x}_3 + m_0 (\frac{x_1 - m_{12}x_3}{m_{11}})^2 \sin x_2 \\ &\quad - m_0 \cos x_2 (\frac{m_0 \sin x_2 - m_{12}\dot{x}_3}{m_{11}}) \\ &= -\frac{e_2}{\tau_2} + u + m_0 (\frac{x_1 - m_{12}x_3}{m_{11}})^2 \sin x_2 \\ &\quad - m_0 \cos x_2 (\frac{m_0 \sin x_2 - m_{12}u}{m_{11}}) \end{aligned} \tag{27}$$

B. SLIDING MODE CONTROL

Define a sliding surface

$$s = \sigma_1 \theta_1 + \sigma_2 \hat{\theta}_2 + \hat{\theta}_3 \tag{28}$$

where σ_1 and σ_2 are design parameters. Evaluating $\dot{s}$ have

$$\begin{aligned} \dot{s} &= \sigma_1 \dot{\theta}_1 + \sigma_2 \dot{\hat{\theta}}_2 + \dot{\hat{\theta}}_3 \\ &= \sigma_1 m_0 \sin x_2 + \sigma_2 \dot{z}_1 + \dot{z}_2 + u \end{aligned} \tag{29}$$

Selecting the control action as

$$u = -\sigma_2 \dot{z}_1 - \dot{z}_2 - \gamma_3 sign(s) \tag{30}$$

where $\gamma_3 = m_0 \sigma_1 + \frac{\beta_3}{2}$, $\beta_3 > 0$.

C. STABILITY ANALYSIS

Although the stability analysis for the proposed approach is complicated due to the introduction of the first-order filters, the semi-global boundedness of all the signals in the closed-loop system will be proven with the following theorem.

Theorem 1: Consider the underactuated IWP system described by equation (1) under the control input (30) and (2) with the global change of coordinates (4) and (6). Given any positive number λ, there exist σ_1, σ_2, α_1, α_2, τ_1, τ_2, β_1, β_2, β_3 and positive definite symmetry matrices P, Q, for all initial conditions satisfying $\theta_{12}^T P \theta_{12} + e_1^2 + e_2^2 \leq 2\lambda$, such that the overall closed-loop control system is semi-globally stable in the sense that all closed-loop signals are uniformly bounded convergence under the proposed sliding mode controller (30) based on the sliding mode observers (15) and (25).

Proof: The design process of the sliding mode observers has demonstrated finite time convergence of the observation errors and the finite time convergence can ensure the separation principle hold [26]. To make the stability analyzing process clearer, it can be separated into two parts. The first part is to prove that the system state can reach the designed manifold in finite time, and the second part is to demonstrate that the state variables converge to their desired values fast as long as they are on the manifold.

1) THE REACHABILITY OF THE SLIDING MODE SURFACE

The following Lyapunov function candidate is selected

$$V_3 = s^2 \tag{31}$$

Its derivative is calculated from (29) and (30)

$$\dot{V}_3 \leq -\beta_3\sqrt{V_3} \tag{32}$$

and

$$\sqrt{V_3(t)} \leq \sqrt{V_3(0)} - \frac{\beta_3}{2}t \tag{33}$$

Therefore, the Lyapunov function candidate $V_3 = 0$ and the surfaces $s = 0$ can be reached in finite time.

2) THE STABILITY OF THE SLIDING MODE SURFACE

Now our aim is to prove that the convergence of s to zero implies the convergence of θ_1, θ_2 and θ_3 to a neighborhood of zero.

$$s = \sigma_1\theta_1 + \sigma_2\hat{\theta}_2 + \hat{\theta}_3 = \sigma_1\theta_1 + \sigma_2(\theta_2 + e_1) + \theta_3 + e_2 = 0 \tag{34}$$

i.e.

$$\sigma_1\theta_1 + \sigma_2(\theta_2 + e_1) + m_0 \cos x_2(x_1 - m_{12}x_3)/m_{11} + e_2 = 0 \tag{35}$$

In view of the above equation, x_3 can be described as a function of $(\theta_1, \theta_2, e_1, e_2)$ and $x_2$can be described as a function of $\dot{x}_1$ or θ_2 from (6). Thus, the error equation (17) can be rewritten as

$$\dot{e}_1 == -\frac{e_1}{\tau_1} + B_1(\theta_1, \theta_2, e_1, e_2) \tag{36}$$

where,

$$B_1(\theta_1, \theta_2, e_1, e_2) = -m_0\frac{\theta_1 - m_{12}x_3}{m_{11}}\cos x_2 \tag{37}$$

is a continuous function. At this time ($s = 0$), the reaching phase $-\gamma sign(s)$ in the control action (30) can be neglected

$$u = -\sigma_2\dot{z}_1 - \dot{z}_2 = \sigma_2\frac{e_1}{\tau_1} + \frac{e_2}{\tau_2} \tag{38}$$

It can be seen from (38) that u can be described as a function of e_1 and e_2. As stated before that x_3 can be described as a function of (θ_1, θ_2, e_1, e_2) and x_2 can be described as a function of θ_2, the error equation (27) can be written as

$$\dot{e}_2 = -\frac{e_2}{\tau_2} + B_2(\theta_1, \theta_2, e_1, e_2) \tag{39}$$

where,

$$B_2(\theta_1, \theta_2, e_1, e_2) = u + m_0\left(\frac{x_1 - m_{12}x_3}{m_{11}}\right)^2 \sin x_2 - m_0 \cos x_2\left(\frac{m_0 \sin x_2}{m_{11}} - \frac{m_{12}}{m_{11}}u\right) \tag{40}$$

is a continuous function.

The sliding mode surface

$$\sigma_1\theta_1 + \sigma_2(\theta_2 + e_1) + \dot{\theta}_2 + e_2 = 0 \tag{41}$$

can be written as

$$\begin{cases} \dot{\theta}_1 = \theta_2 \\ \dot{\theta}_2 = -\sigma_1\theta_1 - \sigma_2\theta_2 - \sigma_2 e_1 - e_2 \end{cases} \tag{42}$$

Now combine $s = 0$ with the e_1 error model (36) and the e_2 error model (39) as follows.

$$\begin{cases} \dot{\theta}_{12} = A\theta_{12} + B(-\sigma_2 e_1 - e_2) \\ \dot{e}_1 = -\frac{e_1}{\tau_1} + B_1(\theta_1, \theta_2, e_1, e_2) \\ \dot{e}_2 = -\frac{e_2}{\tau_2} + B_2(\theta_1, \theta_2, e_1, e_2) \end{cases} \tag{43}$$

where $\theta_{12} = [\theta_1\theta_2]^T$, $A = \begin{bmatrix} 01 \\ -\sigma_1 - \sigma_2 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

According to the control theory for the linear time-invariant system [25], the design constants σ_1 and σ_2 chosen such that the following polynomial

$$\sigma_1 + \sigma_2 p + p^2 = 0 \tag{44}$$

is Hurwitz, can ensure the existence of two positive definite symmetry matrices

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \quad Q = \begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix}$$

such that

$$A^T P + PA = -Q \tag{45}$$

Consider the set

$$\Omega := \left\{\theta_{12}^T P\theta_{12} + e_1^2 + e_2^2 \leq 2\lambda\right\} \tag{46}$$

where, Ω is a compact set in R^4, which contains variables θ_1, θ_2, e_1 and e_2. As mentioned earlier, $x_3$can be described as a function of $(\theta_1, \theta_2, e_1, e_2)$, x_2 can be described as a function of θ_2, and u can be described as a function of e_1 and e_2. Both $B_1(\cdot)$ and $B_2(\cdot)$ are continuous functions with θ_1, θ_2, e_1 and e_2 as independent variables and there is no singularity in (37) and (40). Moreover, all the variables θ_1, θ_2, e_1 and e_2 are bounded in Ω from (46). Therefore, $|B_1(\cdot)|$ exists a maximum M_1 and $|B_2(\cdot)|$ exists a maximum M_2 when all independent variables of the function $B_1(\cdot)$ and $B_2(\cdot)$ are in the compact Ω.

For all conditions satisfying $\theta_{12}^T P\theta_{12} + e_1^2 + e_2^2 \leq 2\lambda$, i.e. θ_1, θ_2, e_1 and e_2 being in the compact set Ω, consider the Lyapunov function candidate

$$V_4 = \frac{1}{2}\theta_{12}^T P\theta_{12} + \frac{1}{2}e_1^2 + \frac{1}{2}e_2^2 \tag{47}$$

Its time derivative $\dot{V}_4$ is given by

$$\begin{aligned}
\dot{V}_4 &= -\frac{1}{2}\theta_{12}^T Q\theta_{12} + \theta_{12}^T PB(-\sigma_2 e_1 - e_2) + e_1\dot{e}_1 + e_2\dot{e}_2 \\
&= -\frac{1}{2}q_{11}\theta_1^2 - \frac{1}{2}q_{22}\theta_2^2 - p_{12}\theta_1(\sigma_2 e_1 + e_2) \\
&\quad - p_{22}\theta_2(\sigma_2 e_1 + e_2) + e_1(-\frac{e_1}{\tau_1} + B_1) + e_2(-\frac{e_2}{\tau_2} + B_2) \\
&= -\frac{1}{2}q_{11}\theta_1^2 - \frac{1}{2}q_{22}\theta_2^2 - \frac{e_1^2}{\tau_1} - \frac{e_2^2}{\tau_2} - p_{12}\sigma_2\theta_1 e_1 - p_{12}\theta_1 e_2 \\
&\quad - p_{22}\sigma_2\theta_2 e_1 - p_{22}\theta_2 e_2 + e_1 B_1 + e_2 B_2 \\
&\leq -\frac{1}{2}q_{11}\theta_1^2 - \frac{1}{2}q_{22}\theta_2^2 - \frac{e_1^2}{\tau_1} - \frac{e_2^2}{\tau_2} + \frac{1}{2}p_{12}\sigma_2\theta_1^2 + \frac{1}{2}p_{12}\sigma_2 e_1^2 \\
&\quad + \frac{1}{2}p_{12}\theta_1^2 + \frac{1}{2}p_{12}e_2^2 + \frac{1}{2}p_{22}\sigma_2\theta_2^2 + \frac{1}{2}p_{22}\sigma_2 e_1^2 + \frac{1}{2}p_{22}\theta_2^2 \\
&\quad + \frac{1}{2}p_{22}e_2^2 + \frac{1}{2}e_1^2 B_1^2 + \frac{1}{2} + \frac{1}{2}e_2^2 B_2^2 + \frac{1}{2} \\
&\leq \frac{1}{2}(-q_{11} + p_{12}\sigma_2 + p_{12})\theta_1^2 + \frac{1}{2}(-q_{22} + p_{22}\sigma_2 + p_{22})\theta_2^2 \\
&\quad + \frac{1}{2}(-\frac{2}{\tau_1} + p_{12}\sigma_2 + p_{22}\sigma_2 + B_1^2)e_1^2 \\
&\quad + \frac{1}{2}(-\frac{2}{\tau_2} + p_{12} + p_{22} + B_2^2)e_2^2 + 1 \\
&\leq -\gamma(\frac{1}{2}p_{11}\theta_1^2 + \frac{1}{2}p_{22}\theta_2^2 + p_{12}\theta_1\theta_2) - \frac{1}{2}\gamma e_1^2 - \frac{1}{2}\gamma e_2^2 \\
&\quad + \frac{1}{2}(-q_{11} + p_{12}\sigma_2 + p_{12} + \gamma p_{11} + \gamma p_{12})\theta_1^2 \\
&\quad + \frac{1}{2}(-q_{22} + p_{22}\sigma_2 + p_{22} + \gamma p_{22} + \gamma p_{12})\theta_2^2 \\
&\quad + \frac{1}{2}(-\frac{2}{\tau_1} + p_{12}\sigma_2 + p_{22}\sigma_2 + B_1^2 + \gamma)e_1^2 \\
&\quad + \frac{1}{2}(-\frac{2}{\tau_2} + p_{12} + p_{22} + B_2^2 + \gamma)e_2^2 + 1
\end{aligned}$$

Therefore

$$\dot{V}_4 \leq -\gamma V_4 + 1 \tag{48}$$

where, γ is given by

$$\gamma := \min\left\{\begin{array}{l} \dfrac{q_{11} - p_{12}\sigma_2 - p_{12}}{p_{11} + p_{12}}, \\ \dfrac{q_{22} - p_{22}\sigma_2 - p_{22}}{p_{12} + p_{22}}, \\ \dfrac{2}{\tau_1} - p_{12}\sigma_2 - p_{22}\sigma_2 - M_1^2, \\ \dfrac{2}{\tau_2} - p_{12} - p_{22} - M_2^2 \end{array}\right\} \tag{49}$$

To ensure the closed-loop system stability, the corresponding design parameters $\sigma_1, \sigma_2, \tau_1, \tau_2$ and matrices P, Q should be chosen to make the following inequalities hold:

$$\begin{aligned}
& q_{11} - p_{12}\sigma_2 - p_{12} > 0 \\
& q_{22} - p_{22}\sigma_2 - p_{22} > 0
\end{aligned} \tag{50}$$

$$\begin{aligned}
& \frac{2}{\tau_1} - p_{12}\sigma_2 - p_{22}\sigma_2 - M_1^2 > 0 \\
& \frac{2}{\tau_2} - p_{12} - p_{22} - M_2^2 > 0
\end{aligned} \tag{51}$$

Let

$$\gamma > \frac{1}{\lambda} \tag{52}$$

where λ is positive. Then $\dot{V}_4 \leq 0$ on $V_4 = \lambda$. Thus, $V_4 \leq \lambda$ is an invariant set, i.e. if $V_4(0) \leq \lambda$, then $V_4(t) \leq \lambda$ for all $t \geq 0$. In another words, for all initial conditions satisfying $\theta_{12}^T P\theta_{12} + e_1^2 + e_2^2 \leq 2\lambda$, i.e. θ_1, θ_2, e_1 and e_2 being in the compact set Ω when $t = 0$, θ_1, θ_2, e_1 and e_2 are also in the compact set Ω when $t > 0$. Solving the above inequality (48) gives

$$0 \leq V_4(t) \leq \frac{1}{\gamma} + \left(V_4(0) - \frac{1}{\gamma}\right)e^{-\gamma t} \tag{53}$$

From (53), we can know that $V_4(t)$ eventually enters a range that

$$0 \leq V_4(\infty) \leq \frac{1}{\gamma} \tag{54}$$

According to (47) and (54), it may directly show that all the signals $\theta_1, \theta_2, e_1, e_2$ are semi-globally uniformly ultimately bounded when $t \to \infty$. As stated before, $x_3$can be described as a function of $(\theta_1, \theta_2, e_1, e_2)$ and x_2 can be described as a function of θ_2. Therefore, both x_2 and x_3 are semi-globally uniformly ultimately bounded. Moreover, it can be deduced that θ_3 also is semi-globally uniformly ultimately bounded from the last equation of (6). Since the state transformations (4) and (6) are invertible, $q_1, \dot{q}_1, q_2$ are semi-globally uniformly ultimately bounded. This concludes the proof. ■

Remark 1: It should be stated that the sliding mode observers can ensure the observing errors convergence in finite time. This property makes the sliding mode observer attractive in the control design and synthesis. As pointed out in [26], The finite-time-convergence makes the separation principle hold. That is to say, the controller and the observer can be designed separately, and the observer-controller-combined feedback preserves the main features of the controller with the full state available.

Remark 2: According to the Lyapunov theory, for a stable linear time-invariant system, given an arbitrary positive definite matrix Q, there exists a positive definite symmetric matrix P that makes (45) hold. The (45) can be rewritten as

$$\begin{cases} q_{11} - 2\sigma_1 p_{12} = 0 \\ p_{11} - \sigma_2 p_{12} - \sigma_1 p_{22} = 0 \\ q_{22} - 2p_{12}2\sigma_2 p_{22} = 0 \end{cases}$$

The inequality (50) can be transformed as

$$\begin{aligned}
2\sigma_1 &> \sigma_2 + 1 \\
2\sigma_1(\sigma_2 - 1)q_{22} &> (\sigma_1 + 1)q_{11}
\end{aligned} \tag{55}$$

Thus, the process to design the appropriate γ in equation (48) can be summarized as:

i) choose σ_1 and σ_2 to make the first inequality of (55) hold and make (44) be Hurwitz;

ii) choose q_{11} and q_{22} (i.e. Q) to make the second inequality of (55) hold;

iii) p_{11}, p_{12}, and p_{22} (i.e. P) can be calculated as

$$\begin{cases} p_{11} = \dfrac{(\sigma_2^2 + \sigma_1)q_{11} + \sigma_1^2 q_{22}}{2\sigma_1\sigma_2} \\ p_{12} = \dfrac{q_{11}}{2\sigma_1} \\ p_{22} = \dfrac{q_{11} + \sigma_1 q_{22}}{2\sigma_1\sigma_2} \end{cases}$$

iv) choose τ_1 and τ_2 to make (51) hold;

v) γ can be obtained with (49).

Remark 3: Although there are many parameters to be preset in the proposed approach, most of these parameters have their own role and there is no strong connection between them. From the stability analyzing process, it seems that reducing the value of τ_1 and τ_2, or adjusting the values of σ_1, σ_2 and positive definite symmetry matrices P, Q, i.e. increasing the value of γ, the convergence speed can be improved. In fact, reducing the value of τ_1 and τ_2 mainly fast reduce the filter errors and adjusting the values of σ_1, σ_2 mainly change the convergence speed. How to adjust σ_1 and σ_2 can be analyzed with the control theory for the linear time-invariant system [25]. As long as the matrices P, Q, and parameters M_1, M_2, γ, λ exist to make (50), (51) and (52) hold, the system stability can be assured. Therefore, it is not so difficult to adjust the parameters to improve the performance of the proposed control algorithm.

Remark 4: The introduction of the first-order filters makes the stability proving difficult, but it makes the proposed algorithm be closer to real world applications. The sliding mode observe and control have some robustness to the uncertainties and disturbances due to inaccurate modeling, measurement errors, exterior conditions, or parameter variations. In the sense, the propose approach is suitable for the actual production system.

Remark 5: Both the collocated partial feedback linearization (2) and the global changes of coordinates (3) and (6) in the II section are invertible transformation, which is

$$\begin{cases} q_1 = x_2 \\ \dot{q}_1 = \dfrac{1}{m_{11}}(x_1 - m_{12}x_3) \\ \dot{q}_2 = x_3 \end{cases} \tag{56}$$

It can be seen from equation (56) that (x_1, x_2, x_3) approach to $(0, 0, 0)$ implies that $(q_1, \dot{q}_1, \dot{q}_2)$ approach to $(0, 0, 0)$. The control input τ can be calculated with equation (2), (30) and (56).

Remark 6: The proposed approach is designed for the nonlinear canonical system (7), so it can be used to control all the mechanical systems that can be described with the formation as (7), including the underactuated systems that can be transformed to the cascade system (5), such as the TORA [27] and the Acrobot [28].

IV. SIMULATION STUDIES

In order to test the proposed nonlinear control algorithm, the same system parameters are used as [6], [11]: $m_{11} = 4.83 \times 10^{-3}$, $m_{12} = m_{21} = m_{22} = 32 \times 10^{-6}$, $m_0 = 379.26 \times 10^{-3}$. The parameters of the nonlinear controller are chosen as $\sigma_1 = 10$, $\sigma_2 = 10$, $\tau_1 = 0.01$, $\tau_2 = 0.01$, $\alpha_1 = 100$, $\alpha_2 = 100$, $\beta_1 = 20$, $\beta_2 = 20$, $\beta_3 = 2000$. The matrices P, Q, and parameters M_1, M_2, γ, λ are only use to prove the system stability.

The simulation results are shown in FIGURE 2-6. The simulation is performed under the initial state $(x_1, x_2, x_3) = (0, 90°, 0)$ i.e. $(q_1, \dot{q}_1, q_2) = (90°, 0, 0)$ FIGURE.2 is the time responses of q_1 and $\dot{q}_1$, FIGURE.3 is the time response of q_2 and $\dot{q}_2$, FIGURE.4 is the control torque τ for the IWP. The FIGURE.5 and FIGURE.6 show the performance of the sliding mode observer and correspondingly are for θ_2 and θ_3.

It can be seen form the simulation results that the IWP system can be fleetly driven to converge to the equilibrium

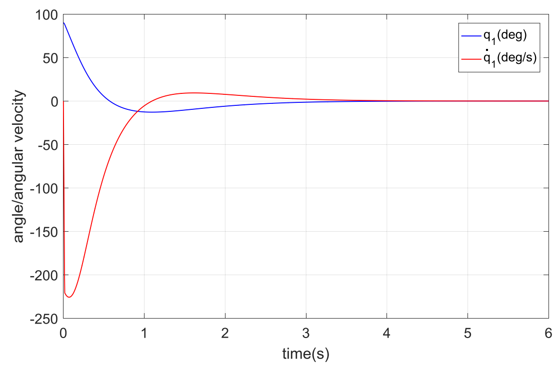

FIGURE 2. The time responses of q_1 and $\dot{q}_1$.

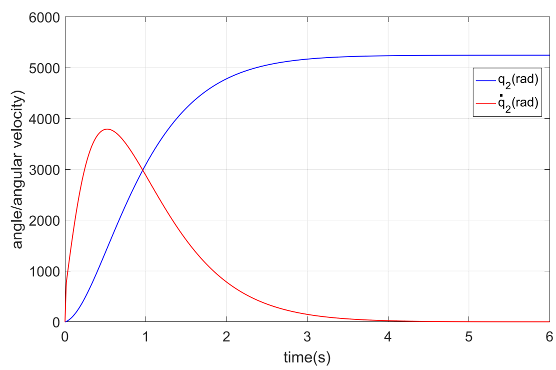

FIGURE 3. The time response of q_2 and $\dot{q}_2$.

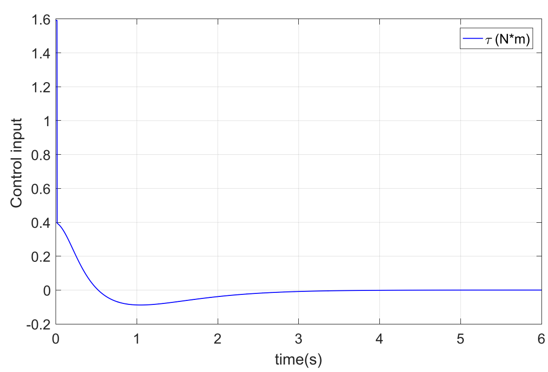

FIGURE 4. The control torque τ.

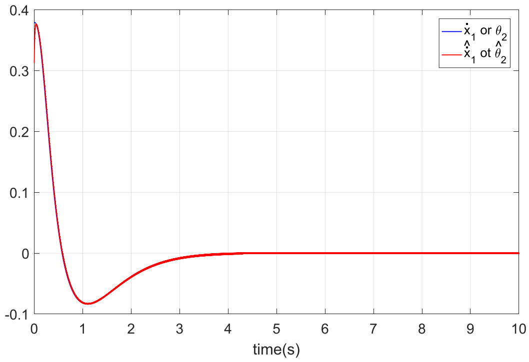

FIGURE 5. The time response of θ_2 and $\hat{\theta}_2$.

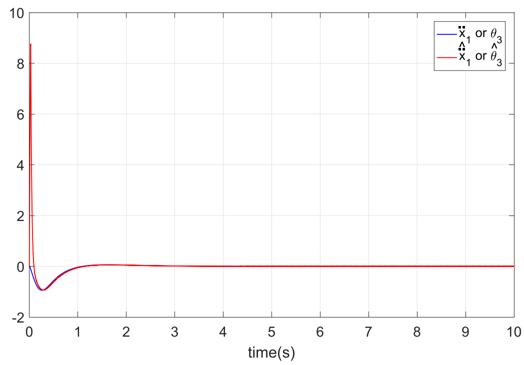

FIGURE 6. The time response of θ_3 and $\hat{\theta}_3$.

point under different initial states with the proposed control algorithm. The observed values can approach the corresponding variable in finite time. On the other hand, the control performance can be improved through adjusting the parameters of the proposed controller. From amounts of simulation results, it is found that the controller is asymptotically stable and the system state can be stabilized to the zero state, i.e. $(q_1, \dot{q}_1, \dot{q}_2) = (0, 0, 0)$, but a feasible Lyapunov function cannot be found to prove the asymptotic stability. Therefore, it is only proved that the system asymptotically approaches to a small neighborhood of (0, 0, 0) and all signals of the close-loop system are semi-globally uniformly ultimately bounded. Moreover, the control performance can be improved through adjusting the parameters of the proposed control algorithm as stated in Remark 2 and 3.

By comparing with the control algorithms of some the literature [7], [10], [14], it can be found from the view of theoretical design that the proposed control scheme does not need complex analysis and design process, the obtained controller is simpler and the proposed approach can be understood more easily and intuitively. However, the stability analysis is a bit complicated due to the introduction of the first-order filters. From the view of the performance comparisons with the simulation results, a faster and smoother time response can be obtained with the proposed approach. From the view of practical applications, the first-order filters are introduced to reduce the high frequency component of the observed values. Moreover, the sliding mode observer and controller have robustness to uncertainties and external disturbance unavoidably existing in all physical and engineering systems. In this sense, the proposed approach is close to practical application.

The control of the benchmark underactuated IWP is not easy for its underactuated nature and the weak, sinusoid-type nonlinear interaction in (5). Our proposed approach uses the classic sliding mode observers to obtain the unmeasurable states and then use the classic sliding mode function to obtain a simple controller to semi-globally stabilize the IWP system. It is interesting to solve a challenging control problem with the classic sliding mode method and the simulation results have demonstrate the feasibility of the proposed algorithm.

V. CONCLUSION

This paper proposed a sliding mode observe and control framework for the inertial wheel pendulum system that can be transformed to the feedback cascade nonlinear systems through some coordinate state transforms. The sliding mode observers are employed to estimate unknown states of the transformed system, and a sliding mode controller is developed to guarantee the closed-loop system stability. The simplicity in the control implementation renders the developed methods attractive for industrial applications. Simulation are performed to demonstrate the effectiveness. Future work will be focused on the generalization of this idea for a class of underactuated system or other nonlinear systems.

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