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Fixed-Time Cluster Synchronization of Discontinuous Directed Community Networks via Periodically or Aperiodically Switching Control

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ABSTRACT The fixed-time cluster synchronization problem for a class of directed community networks with linear couplings and discontinuous nodes is investigated in this paper. First, by means of reduction to absurdity and mathematical induction, two new and vital differential equalities are proposed and proved to study the fixed-time stability. Besides, by designing periodically switching controllers and aperiodically adaptive switching controllers, as well as using the theory of differential inclusions, some fixed-time cluster synchronization criteria are derived in which the settling time function is bounded for any initial values. Finally, the numerical simulations are performed to show the feasibility and effectiveness of the control methodology by comparing with the corresponding finite-time synchronization problem.

INDEX TERMS Fixed-time synchronization, cluster synchronization, directed community networks, switching control, differential inclusions.

I. INTRODUCTION

Complex network is a useful modeling tool in understanding the dynamical behaviors of many natural and artificial systems. A typical feature of many real-world networks is the community structure, such as social relationship networks, scientific cooperation networks, power and traffic flow networks, wired and wireless communication networks, and so on. A community network is made up of many communities, each of them are intra-connected relatively densely while the interconnections among communities are relatively sparse [1]. In general, the nodes from different communities are governed by different self-dynamics. The issues of integrating community structures into complex networks require more complicated analysis. Therefore, it is interesting and challenging to study the dynamics and control for community networks, which can help us to have a better understanding of the structure, modeling, dynamical properties and applications for complex networks.

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As an important collective behavior, synchronization of complex networks has been extensively investigated due to its potential applications in various technological fields. Recently, many scientific and technical workers have been joining the study fields with great interest, and various interesting results for many kinds of synchronization, for example, asymptotic synchronization [2], exponential synchronization [3], lag synchronization [4], projective synchronization [5], cluster synchronization [6], finite-time synchronization [7] and fixed-time synchronization [8], and so on. Different from asymptotic synchronization and exponential synchronization, finite-time and fixed-time synchronization mean that synchronization can be actualized in finite time, in which the settling time is dependent on the initial states or is regardless of the initial states, respectively. Recently, finite-time or fixed-time synchronization problems for complex networks have received increasing attentions due to its faster convergence rate, better robustness against uncertainties and disturbance rejection properties [9].

On the other hand, due to the complexities of node dynamics and topological structure, complex networks are not always able to synchronize by themselves. Therefore, various effective control approaches have been reported to realize fixed-time synchronization. Especially, Khanzadeh and Pourghoil [10] studied the fixed-time synchronization of complex dynamical networks with nonidentical nodes in the presence of bounded uncertainties and disturbances by using sliding mode control technique; in [11], a set of new continuous controller without sign function was developed to realize the fixed-time synchronization for a class of complex networks with stochastic perturbations; Zhang *et al.* [12] proposed a fixed-time controller for a class of complex networks with nonidentical nodes and stochastic noise perturbations by using feedback control; fixed-time synchronization of multi-links complex networks based on feedback control was studied in [13]; under the designed sliding mode controller, the authors in [14] studied the fixed-time synchronization for semi-Markovian switching complex dynamical networks with hybrid couplings and time-varying delays in the presence of disturbances. The controllers for realizing synchronization can be classified into two categories, namely, the continuous controllers and discontinuous ones. The synchronization problems for complex networks via continuous controllers have been investigated. However, the existing methods cannot be used to deal with the fixed-time synchronization problems via periodically or aperiodically switching control. Therefore, comparing with continuous control methods, discontinuous control approaches which include impulsive control, intermittent control, event-triggered control and switching control have attracted more and more interest because these control approaches are more economic and reduce the amount of the transmitted information. Compared with rich results for fixed-time synchronization via continuous control, results for fixed-time synchronization via discontinuous control are much more scare. As far as we know, the fixed-time synchronization problem based on switching control for community networks has not been studied in the literatures and it is interesting to study this problem both in theory and in applications, so there exist open room for further improvement.

Furthermore, the cluster synchronization of complex networks which refers set of nodes is divided into several clusters and only nodes belong to the same cluster are synchronized to a corresponding state, while nodes in different clusters have different synchronization behaviors. Cluster synchronization is a more practical phenomenon than the complete synchronization, which is significant in biological science and communication engineering [15], and so on. Cluster synchronization of community network with hybrid coupling was discussed via adaptive couplings control scheme in [16]; by constructing new piecewise continuous auxiliary functions, Zhou *et al.* investigated the exponential cluster synchronization of directed community networks and colored community networks via adaptive aperiodically intermittent pinning control in [17] and [18], respectively; the cluster synchronization problem for a network of subnetwork with community structure was investigated in [19]; the authors in [20] proposed a new overlapping community model for the coupled complex network, and investigated the finite-time cluster synchronization problem for the addressed community networks based on adaptive control; the fixed-time cluster synchronization problem of coupled complex networks with uncertain disturbances based on adaptive control in [21]; in [22], the finite-time and fixed-time cluster synchronization problem for complex networks with or without pinning control were studied; the authors in [15] investigated the finite-time cluster synchronization problem for a class of Markovian switching complex networks with time delay and stochastic noise perturbations; in [23], the fixed-time cluster lag synchronization for directed heterogeneous community networks was investigated via state feedback controllers.

In fact, discontinuous behaviors of dynamical systems can be found everywhere such as impacting machines, dry friction, and jump discontinuities in the activation functions of neural networks [24], [25]. Therefore, the study of discontinuous complex networks is an important step for practical use. Recently, the investigations of synchronization problems for discontinuous complex networks have attracted numerous scientists. The finite-time synchronization of linearly coupled complex networks with nonidentical nodes was investigated in [26]; in [27], Yang *et al.* studied the finite-time cluster synchronization of a class of discontinuous complex networks with uncertain nonidentical bounded external disturbances, nonlinear coupling strength and random coupling delays. Up to now, the existing results on the fixed-time synchronization mainly focus on complex networks with continuous node dynamics, and it received little attention to investigate the fixed-time especially fixed-time cluster synchronization of community networks with discontinuous dynamical nodes due to the dynamical complexity for integrating discontinuity into complex networks and the lack of correlated theoretical support. In [28], a unified control framework was proposed to discuss the finite-time and fixed-time synchronization problems for a class of discontinuous complex networks by only adjusting the value of a key control parameter; the authors in [29] studied the finite-time and fixed-time synchronization of complex networks with discontinuous node via quantized controllers.

Unfortunately, to the best of our knowledge, there are seldom results, or even no results concerning the fixed-time cluster synchronization of directed community networks with discontinuous nodes via switching control. From the above discussions, we attempt to study the fixed-time cluster synchronization for a class of directed community networks with discontinuous nodes. By introducing and proving two new and important differential inequalities as vital lemmas, this paper is concerned with fixed-time cluster synchronization for the addressed complex networks via designing periodically switching controllers and aperiodically adaptive switching controllers, respectively. Our results effectually complement or improve the previously known results.

The organization of this paper is as follows: in Sec. II, problem statement and preliminaries are presented; in Section III, the periodically switching control and aperiodically adaptive switching control are, respectively, proposed to ensure fixed-time cluster synchronization of the addressed directed community networks; numerical simulations will be given in Section IV to demonstrate the effectiveness and feasibility of our theoretical results; we end this work with a conclusion in Section V.

II. MODELING AND PRELIMINARY

Consider a class of directed community networks consisting of N dynamical nodes and r communities with linearly couplings and discontinuous nodes, which can be described as follows:

$$\dot{x}_i(t) = B_{\mu_i}x_i(t) + f_{\mu_i}(x_i(t)) + c\sum_{k=1}^{r}\sum_{j\in\mathcal{C}_k} a_{ij}\Gamma x_j(t), \quad (1)$$

where $i = 1, 2, \cdots, N$, $k = 1, 2, \cdots, r$; $\mathcal{C}_k$ represents the set of all nodes belonging to the kth community; $x_i(t) = (x_{i1}(t), x_{i2}(t), \cdots, x_{in}(t))^T \in \mathbb{R}^n$; define the function $\mu : \{1, 2, \cdots, N\} \to \{1, 2, \cdots, r\}$, if node $j \in \mathcal{C}_k$, then one has $\mu_j = k$; $B_{\mu_i} = (b_{ij}^{\mu_i})_{n\times n} \in \mathbb{R}^n$; $f_{\mu_i} : \mathbb{R}^n \to \mathbb{R}^n$ is a discontinuous nonlinear vector function; $c > 0$ is the coupling strength; $\Gamma = \text{diag}(\zeta_1, \zeta_2, \cdots, \zeta_n) > 0$ is the inner matrix linking the coupled variables; $A = (a_{ij})_{N\times N}$ denotes the topological structure of the complex networks, in which, $a_{ij} \neq 0$ if there is a connection from node j to node i ($i \neq j$), the diagonal elements of A are defined as

$$a_{ii} = -\sum_{j=1, j\neq i}^{N} a_{ij}, \quad i = 1, 2, \cdots, N.$$

The initial values of (1) are given as $x_i(0) = x_0 \in \mathbb{R}^n$. The nodes can be separated into r nonempty communities, namely,

$$\{1, 2, \cdots, N\} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \cdots \mathcal{C}_r, \quad (2)$$

where $\mathcal{C}_1 = \{1, \cdots, c_1\}$, $\mathcal{C}_2 = \{c_1 + 1, \cdots, c_2\}, \cdots, \mathcal{C}_r = \{c_{r-1} + 1, \cdots, N\}$. Therefore, matrix A can be rewritten as

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1r} \\ A_{21} & A_{22} & \cdots & A_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ A_{r1} & A_{r2} & \cdots & A_{rr} \end{bmatrix}.$$

Suppose that $s_{\mu_i}(t) \in \mathbb{R}^n$ is the state vector of an isolated node which satisfies

$$\dot{s}_{\mu_i}(t) = B_{\mu_i}s_{\mu_i}(t) + f(s_{\mu_i}(t)), \quad i = 1, 2, \cdots, N, \quad (3)$$

with $\lim_{t\to+\infty} \|s_{\mu_i}(t) - s_{\mu_j}(t)\| \neq 0$ ($\mu_i \neq \mu_j$). Here, $s_{\mu_i}(t)$ may be an equilibrium point, a periodic orbit, or a chaotic orbit.

To achieve the fixed-time cluster synchronization, some suitable controllers $u_i(t)$ ($i = 1, 2, \cdots, N$) should be designed onto the network (1).

In order to obtain our main results, the following definitions, assumptions and lemmas are necessary.

Definition 1: The community network (1) is said to achieve cluster synchronization in fixed time under controllers $u_i(t)$ ($i = 1, 2, \cdots, N$), if there exists a fixed settling time T^* which is independent of the initial values, such that $\lim_{t\to T^*} \|e_i(t)\| = 0$ and $\|e_i(t)\| \equiv 0$ for $t > T^*$, $i = 1, 2, \cdots, N$, where $e_i(t) = x_i(t) - s_{\mu_i}(t)$.

In this paper, $f_{\mu_i} : \mathbb{R}^n \to \mathbb{R}^n$ ($i = 1, 2, \cdots, N$) is a discontinuous nonlinear vector function describes the node dynamics. Hence, (1) is a differential equation with discontinuous right-hand side. In this situation, the existence of the solution of (1) may not be guaranteed. Therefore, we should investigate the fixed-time cluster synchronization problem of (1) in the sense of Filippov solution, which is given as follows.

Definition 2: [32] Consider a class of nonlinear dynamical systems described by

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \quad t \geq 0, \quad (4)$$

with discontinuous right-hand side. Vector function $x(t) \in \mathbb{R}^n$ defined on $[0, t^*)$ is said be to a Filippov solution of (4) if it is absolutely continuous and satisfies differential inclusion:

$$\dot{x}(t) \in \mathbb{K}[f](x(t)), \quad t \in [0, t^*),$$

where, the set-valued map $\mathbb{K}[f](\cdot) : \mathbb{R}^n \to \mathbb{B}(\mathbb{R}^n)$ is defined by

$$\mathbb{K}[f](x(t)) = \bigcap_{\varrho>0}\ \bigcap_{\text{meas}(S)=0} \bar{co}\{f(\mathbf{B}(x, \varrho)) \setminus S\},$$

here, $\mathbb{B}(\mathbb{R}^n)$ denotes the collection of all subsets of $\mathbb{R}^n$, $\bar{co}$ represents the convex closure, meas(S) denotes the Lebesgue measure of set S, $\mathbf{B}(x, \varrho)$ is the open ball centered at x with the radius $\varrho > 0$.

Assumption 1: For each $k = 1, 2, \cdots, r$, f_k is continuous except on countable set of isolate points t_l^k, where the right and the left limits $f_k^+(t_l^k)$ and $f_k^-(t_l^k)$ exist. Moreover, f_k has most a finite number of jump discontinuities in every bounded compact interval of $\mathbb{R}$.

It follows from Definition 2 and Assumption 1 that

$$\begin{aligned}\mathbb{K}[f_k](x_i(t)) &= \bar{co}[f_k(x_i(t))] \\ &= (\bar{co}[f_{k1}(x_{i1}(t))], \cdots, \bar{co}[f_{kn}(x_{in}(t))])^T,\end{aligned}$$

where

$$\begin{aligned}\bar{co}[f_{kl}(x_{kl}(t))] = [&\min\{f_{kl}^-(x_{il}), f_{kl}^+(x_{il})\}, \\ &\max\{f_{kl}^-(x_{il}), f_{kl}^+(x_{il})\}],\end{aligned}$$

for $l = 1, 2, \cdots, n$, $k = 1, 2, \cdots, r$, $i = 1, 2, \cdots, N$.

Assumption 2: For $\forall u, v \in \mathbb{R}$, there are two nonnegative constants L_k and N_k satisfying

$$(u - v)^T(\xi_k - \eta_k) \leq L_k\|u - v\| + N_k\sum_{j=1}^{n}|u_j - v_j|,$$

where $\xi_k \in \bar{co}[f_k(u)]$, $\eta_k \in \bar{co}[f_k(v)]$ for $u, v \in \mathbb{R}^n$, $k = 1, 2, \cdots, r$.

Assumption 3: Each block matrix A_{kl} is zero-row-sum-matrix, $k, l = 1, 2, \cdots, r$.

Remark 1: It is easy to see that the coupling configuration matrix A is not necessary to be identical, symmetric or irreducible, that is, the corresponding graphs can be directed, weakly connected and even there is no rooted spanning directed tree. Furthermore, Assumption 3 means that the nodes in the same community only have cooperative relationships, while nodes belonging to different communities can have both cooperative and competitive relationships [17], [30].

Lemma 1: ([31]) Let $a_i \geq 0$ $(i = 1, 2, \cdots, n)$, $0 < p < 1$, $q > 1$. Then, the following inequalities hold:

$$\sum_{i=1}^{n} a_i^p \geq \Big(\sum_{i=1}^{n} a_i\Big)^p, \quad \sum_{i=1}^{n} a_i^q \geq n^{1-q}\Big(\sum_{i=1}^{n} a_i\Big)^q.$$

Lemma 2: Suppose that function $V(t)$ is non-negative when $t \in [0, +\infty)$ and satisfies the following conditions:

$$\begin{cases} \dot{V}(t) \leq -\alpha V^p(t) - \beta V^q(t), & t \in [mT, (m+\theta)T), \\ \dot{V}(t) \leq 0, & t \in [(m+\theta)T, \quad (m+1)T), \end{cases} \tag{5}$$

where $\alpha, \beta, T > 0$, $0 < p, \theta < 1$, $q > 1$, $m = 0, 1, 2, \cdots$. Then $\lim_{t \to T_f} V(t) = 0$ and $V(t) \equiv 0$, if

$$t \geq T_f \doteq \frac{1}{\alpha\theta(1-p)} + \frac{1}{\beta\theta(q-1)}. \tag{6}$$

Proof: The proof is divided in two cases.

Case 1: Suppose that $V(0) \leq 1$. From condition (5), one has

$$\begin{cases} \dot{V}(t) \leq -\alpha V^p(t), & t \in [mT, (m+\theta)T), \\ \dot{V}(t) \leq 0, & t \in [(m+\theta)T, (m+1)T). \end{cases} \tag{7}$$

Let $Q(t) = H(t) - hM_0$, where

$$H(t) = V^{1-p}(t) + \alpha(1-p)t, \quad M_0 = V^{1-p}(0), \quad h > 1, t \geq 0.$$

It is easy to see that

$$Q(0) < 0. \tag{8}$$

In the following, we will prove that

$$Q(t) < 0, \quad \forall t \in [0, \theta T). \tag{9}$$

If this is not true, by (8) and the continuity of $V(t)$ as $t \in [0, \infty)$, then there exists a $t_0 \in [0, \theta T)$ such that

$$Q(t_0) = 0, \quad \dot{Q}(t)|_{t=t_0} > 0, \quad Q(t) < 0, \quad t \in [0, t_0). \tag{10}$$

It follows from (7) that

$$\dot{Q}(t)|_{t=t_0} = (1-p)V^{-p}(t)\dot{V}(t)|_{t=t_0} + \alpha(1-p) \leq 0, \tag{11}$$

which contradicts (10). Hence, we have

$$H(t) < hM_0, \quad \forall t \in [0, \theta T). \tag{12}$$

In the following, we will prove that for $t \in [\theta T, T)$,

$$\tilde{Q}(t) = H(t) - hM_0 - \alpha(1-p)(t - \theta T) < 0. \tag{13}$$

Otherwise, there exists a $t_1 \in [\theta T, T)$ such that

$$\tilde{Q}(t_1) = 0, \quad \dot{\tilde{Q}}(t)|_{t=t_1} > 0, \quad \tilde{Q}(t) < 0, \quad t \in [\theta T, t_1). \tag{14}$$

From (12), (13) and the continuity of $H(t)$, it is easy to see that $\tilde{Q}(\theta T) = H(\theta T) - hM_0 < 0$. It follows from (7) and (13) that

$$\begin{aligned} \dot{\tilde{Q}}(t)|_{t=t_1} &= (1-p)V^{-p}(t_1)\dot{V}(t)|_{t=t_1} + \alpha(1-p) \\ &\quad -\alpha(1-p) \\ &\leq 0, \end{aligned}$$

which contradicts (14). Hence, (15) holds. That is, for $t \in [\theta T, T)$,

$$H(t) < hM_0 + \alpha(1-p)(t - \theta T) < hM_0 + \alpha(1-p)(1-\theta)T. \tag{15}$$

Similarly, with the same approach, one can prove that: for $t \in [T, (1+\theta)T)$,

$$H(t) < hM_0 + \alpha(1-p)(1-\theta)T,$$

and for $t \in [(1+\theta)T, 2T)$,

$$H(t) < hM_0 + \alpha(1-p)(t - 2\theta T).$$

By mathematical induction, we can derive the following estimations of $H(t)$ for any integer m.

For $t \in [mT, (m+\theta)T)$,

$$H(t) < hM_0 + \alpha(1-p)(1-\theta)mT, \tag{16}$$

and for $(t, x) \in [(m+\theta)T, (m+1)T) \times \Omega$,

$$H(t) < hM_0 + \alpha(1-p)(t - (m+1)\theta T). \tag{17}$$

If $t \in [mT, (m+\theta)T)$, we have $m \leq t/T$, then it follows from (16) that

$$H(t) < hM_0 + \alpha(1-p)(1-\theta)t. \tag{18}$$

Similarly, if $t \in [(m+\theta)T, (m+1)T)$, then it follows from (17) that (18) holds. Hence, for any $t \in [0, +\infty)$, (18) always holds. Let $h \to 1$, from the definition of $H(t)$, we can get

$$V^{1-p}(t) \leq M_0 - \alpha\theta(1-p)t \leq 1 - \alpha\theta(1-p)t, \quad t \in [0, +\infty). \tag{19}$$

Let $\varphi(t) = 1 - \alpha\theta(1-p)t$. It is easy to see that $\varphi(t)$ is a strictly decreasing continuous function of t. Set the right side of the inequality (19) to zero, we can obtain

$$T_1 \doteq \frac{1}{\alpha\theta(1-p)}, \tag{20}$$

and $\lim_{t \to T_1} V^{1-q}(t) = 0$. Correspondingly, it follows from (19) and (20) and the monotonicity of $\varphi(t)$ that $\lim_{t \to T_1} V(t) = 0$ and $V(t) \equiv 0$ for all $t \geq T_1$.

Case 2: Suppose that $V(0) > 1$. From condition (5), one has

$$\begin{cases} \dot{V}(t) \leq -\beta V^q(t), & t \in [mT, (m+\theta)T), \\ \dot{V}(t) \leq 0, & t \in [(m+\theta)T, (m+1)T). \end{cases} \tag{21}$$

Let $R(t) = S(t) - \tilde{h}\tilde{M}_0$, where

$$S(t) = V^{1-q}(t) + \beta(1-q)t,$$
$$\tilde{M}_0 = V^{1-q}(0), 0 < \tilde{h} < 1, t \geq 0.$$

It is easy to see that

$$R(0) > 0. \tag{22}$$

In the following, we will prove that

$$R(t) > 0, \quad \forall t \in [0, \theta T). \tag{23}$$

If this is not true, by (22) and the continuity of $V(t)$ as $t \in [0, \infty)$, then there exists a $\tilde{t}_0 \in [0, \theta T)$ such that

$$R(\tilde{t}_0) = 0, \quad \dot{R}(t)|_{t=\tilde{t}_0} < 0, \quad R(t) > 0, \quad t \in [0, \tilde{t}_0). \tag{24}$$

It follows from (21) that

$$\dot{R}(t)|_{t=\tilde{t}_0} = (1-q)V^{-q}(t)\dot{V}(t)|_{t=\tilde{t}_0} + \beta(1-q) \geq 0, \tag{25}$$

which contradicts (24). Hence, we have

$$S(t) > \tilde{h}\tilde{M}_0, \quad \forall t \in [0, \theta T). \tag{26}$$

In the following, we will prove that for $t \in [\theta T, T)$,

$$\tilde{R}(t) = S(t) - \tilde{h}\tilde{M}_0 - \beta(1-q)(t - \theta T) > 0. \tag{27}$$

Otherwise, there exists a $\tilde{t}_1 \in [\theta T, T)$ such that

$$\tilde{R}(\tilde{t}_1) = 0, \quad \dot{\tilde{R}}(t)|_{t=\tilde{t}_1} < 0, \quad \tilde{R}(t) > 0, \quad t \in [\theta T, \tilde{t}_1). \tag{28}$$

From (26), (27) and the continuity of $S(t)$, it is easy to see that $\tilde{R}(\theta T) = S(\theta T) - \tilde{h}\tilde{M}_0 > 0$. It follows from (21) and (23) that

$$\begin{aligned}\dot{\tilde{R}}(t)|_{t=\tilde{t}_1} &= (1-q)V^{-q}(\tilde{t}_1)\dot{V}(t)|_{t=\tilde{t}_1} + \beta(1-q) \\ &\quad -\beta(1-q) \\ &\geq 0,\end{aligned}$$

which contradicts (28). Hence, (27) holds. That is, for $t \in [\theta T, T)$,

$$S(t) > \tilde{h}\tilde{M}_0 + \beta(1-q)(t - \theta T) > \tilde{h}\tilde{M}_0 + \beta(1-q)(1-\theta)T. \tag{29}$$

Similarly, with the same approach, one can prove that: for $t \in [T, (1+\theta)T)$,

$$S(t) > \tilde{h}\tilde{M}_0 + \beta(1-q)(1-\theta)T,$$

and for $t \in [(1+\theta)T, 2T)$,

$$S(t) > \tilde{h}\tilde{M}_0 + \beta(1-q)(t - 2\theta T).$$

By mathematical induction, we can derive the following estimations of $S(t)$ for any integer m.

For $t \in [mT, (m+\theta)T)$,

$$S(t) > \tilde{h}\tilde{M}_0 + \beta(1-q)(1-\theta)mT, \tag{30}$$

and for $(t, x) \in [(m+\theta)T, (m+1)T) \times \Omega$,

$$S(t) > \tilde{h}\tilde{M}_0 + \beta(1-q)(t - (m+1)\theta T). \tag{31}$$

If $t \in [mT, (m+\theta)T)$, we have $m \leq t/T$, then it follows from (30) that

$$S(t) > \tilde{h}\tilde{M}_0 + \beta(1-q)(1-\theta)t. \tag{32}$$

Similarly, if $t \in [(m+\theta)T, (m+1)T)$, then it follows from (31) that (32) holds. Hence, for any $t \in [0, +\infty)$, (32) always holds. Let $\tilde{h} \to 1$, from the definition of $S(t)$, we can get

$$V^{1-q}(t) \geq \tilde{M}_0 - \beta\theta(1-q)t \geq \beta\theta(q-1)t, \quad t \in [0, +\infty), \tag{33}$$

which implies that

$$V^{q-1}(t) \leq \frac{1}{\beta\theta(q-1)t}, \quad t \in [0, +\infty). \tag{34}$$

Let $\tilde{\varphi}(t) = 1/\beta\theta(q-1)t$. It is easy to see that $\tilde{\varphi}(t)$ is a strictly decreasing continuous function of t. Set the right side of the inequality (33) to one, we can get

$$T_2 \doteq \frac{1}{\beta\theta(q-1)}, \tag{35}$$

and $\lim_{t \to T_2} V^{q-1}(t) = 1$. Correspondingly, it follows from (34), (35) and the monotonicity of $\tilde{\varphi}(t)$ that $\lim_{t \to T_2} V(t) = 1$ and $V(t) \leq 1$ for all $t \geq T_2$.

Applying the result obtained in Case 1, for any $V(0) \in \mathbb{R}$, we can derive that $\lim_{t \to T_f} V(t) = 0$ and $V(t) \equiv 0$ for all $t \geq T_f$. This completes the proof. □

Similar to the proof of Lemma 2, the following lemma can be given directly.

Lemma 3: Suppose that function $V(t)$ is non-negative when $t \in [0, +\infty)$ and satisfies the following conditions:

$$\begin{cases} \dot{V}(t) \leq -\alpha V^p(t) - \beta V^q(t), & t \in [t_m, s_m), \\ \dot{V}(t) \leq 0, & t \in [s_m, t_{m+1}), \end{cases} \tag{36}$$

where $\alpha, \beta > 0$, $0 < p < 1$, $q > 1$, $m = 1, 2, \cdots$, $t_1 = 0$. Then $\lim_{t \to T_f^*} V(t) = 0$ and $V(t) \equiv 0$, if

$$t \geq T_f^* \doteq \frac{1}{\alpha(1-\vartheta)(1-p)} + \frac{1}{\beta(1-\vartheta)(q-1)},$$

where $\vartheta = \lim_{t \to +\infty} \sup \frac{t_{m+1} - s_m}{t_{m+1} - t_m}$.

III. MAIN RESULTS

In this section, we will discuss the fixed-time cluster synchronization problem for the directed community networks with linearly couplings and discontinuous nodes under periodically switching control and aperiodically adaptive switching control, respectively. The main results are stated as follows.

Theorem 1: Suppose that Assumptions 1-3 hold. The directed community network (1) can achieve fixed-time cluster synchronization under the periodically switching controllers:

$$u_i(t) = \begin{cases} -d_i e_i(t) - \gamma \operatorname{sign}(e_i(t)) - \alpha e_i^p(t) - \beta e_i^q(t), \\ \quad mT \leq t < (m+\theta)T, \\ -d_i e_i(t) - \gamma \operatorname{sign}(e_i(t)), \\ \quad (m+\theta)T \leq t < (m+1)T, \end{cases} \tag{37}$$

if the following conditions hold:

$$\begin{aligned}&\tilde{B}+c\zeta\tilde{A}+L-D\leq 0, \quad l=1,2,\cdots,n,\\&N_k-\gamma\leq 0, \quad k=1,2,\cdots,r,\end{aligned}\tag{38}$$

where $L=\mathrm{diag}(\tilde{L}_1,\cdots,\tilde{L}_r)$, $\tilde{L}_k=\mathrm{diag}(L_{c_{k-1}+1},\cdots,L_{c_k})$ $(k=1,\cdots,r)$, $D=\mathrm{diag}(d_1,\cdots,d_N)$, $\zeta=\|\Gamma\|$, $\tilde{B}=\mathrm{diag}(B_1,B_2,\cdots,B_r)$, $\tilde{A}=(\tilde{a}_{ij})_{N\times N}$, $\tilde{a}_{ij}=a_{ij}$ for $i\neq j$, $\tilde{a}_{ii}=\zeta_{\min}a_{ii}/\zeta$, $\zeta_{\min}$ is the minimum eigenvalue of Γ; $m=0,1,2,\cdots$, $0<p<1$, $q>1$, α, β, γ and d_i $(i=1,2,\cdots,N)$ are positive constants denoting the control strengths; $T>0$ denotes the control period, $0<\theta\leq 1$ is the ratio of the control width to the control period called control rate. Moreover, the fixed settling time T_f can be estimated by

$$T_f\leq\frac{1}{2^{\frac{1+p}{2}}\alpha\theta(1-p)}+\frac{1}{2^{\frac{1+q}{2}}(Nn)^{\frac{q-1}{2}}\beta\theta(q-1)}.\tag{39}$$

Proof: Construct the following Lyapunov function

$$V(e(t))=\frac{1}{2}\sum_{k=1}^{r}\sum_{i\in\mathcal{C}_k}e_i^T(t)e_i(t).\tag{40}$$

When $mT\leq t<(m+\theta)T$ $(m=0,1,2,\cdots)$, it follows from the theory of differential inclusion [32], [33] that

$$\begin{aligned}\dot{x}_i(t)\in{}&B_{\mu_i}x_i(t)+\bar{co}[f_{\mu_i}(x_i(t))]+c\sum_{k=1}^{r}\sum_{j\in\mathcal{C}_k}a_{ij}\Gamma x_j(t)\\&-d_ie_i(t)-\gamma\mathrm{SIGN}(e_i(t))-\alpha e_i^p(t)-\beta e_i^q(t),\end{aligned}\tag{41}$$

and

$$\dot{s}_{\mu_i}(t)\in B_{\mu_i}s_{\mu_i}(t)+\bar{co}[f_{\mu_i}(s_{\mu_i}(t))],\tag{42}$$

where $\bar{co}[f_{\mu_i}(x_i(t))]=(\bar{co}[f_{\mu_i,1}(x_i(t))],\cdots\bar{co}[f_{\mu_i,n}(x_i(t))])^T$, $\bar{co}[f_{\mu_i}(s_{\mu_i}(t))]=(\bar{co}[f_{\mu_i,1}(s_{\mu_i}(t))],\cdots\bar{co}[f_{\mu_i,n}(s_{\mu_i}(t))])^T$ and $\mathrm{SIGN}(e_i(t))=(\mathrm{SIGN}(e_{i1}(t)),\cdots,\mathrm{SIGN}(e_{in}(t)))^T$ with

$$\mathrm{SIGN}(x)=\begin{cases}-1, & x<0,\\ [-1,1], & x=0,\\ 1, & x>0.\end{cases}$$

By the measurable selection theorem [32], [33], there exist measurable functions $\varphi_{\mu_i}(t)\in\bar{co}[f_{\mu_i}(x_i(t))]$, $\tilde{\varphi}_{\mu_i}(t)\in\bar{co}[f_{\mu_i}(s_{\mu_i}(t))]$ and $\omega_i(t)\in\mathrm{SIGN}(e_i(t))$ such that

$$\begin{aligned}\dot{x}_i(t)={}&B_{\mu_i}x_i(t)+\varphi_{\mu_i}(t)+c\sum_{k=1}^{r}\sum_{j\in\mathcal{C}_k}a_{ij}\Gamma x_j(t)\\&-d_ie_i(t)-\gamma\omega_i(t)-\alpha e_i^p(t)-\beta e_i^q(t),\\\dot{s}_{\mu_i}(t)={}&B_{\mu_i}s_{\mu_i}(t)+\tilde{\varphi}_{\mu_i}(t).\end{aligned}\tag{43}$$

According to Assumption 3, we know that $\sum_{j\in\mathcal{C}_k}a_{ij}\Gamma s_k(t)=0$. Based on (43), it is easy to see that

$$\begin{aligned}\dot{e}_i(t)={}&B_{\mu_i}e_i(t)+\varphi_{\mu_i}(t)-\tilde{\varphi}_{\mu_i}(t)+c\sum_{k=1}^{r}\sum_{j\in\mathcal{C}_k}a_{ij}\Gamma e_j(t)\\&-d_ie_i(t)-\gamma\omega_i(t)-\alpha e_i^p(t)-\beta e_i^q(t).\end{aligned}\tag{44}$$

Taking the derivative of $V(e(t))$ with respect to time t along the solutions of (44), we can get

$$\begin{aligned}\dot{V}(e(t))={}&\sum_{i=1}^{N}e_i^T(t)\Big(B_{\mu_i}e_i(t)+\varphi_{\mu_i}(t)-\tilde{\varphi}_{\mu_i}(t)\\&+c\sum_{j=1}^{N}a_{ij}\Gamma e_j(t)-d_ie_i(t)-\gamma\omega_i(t)\\&-\alpha e_i^p(t)-\beta e_i^q(t)\Big).\end{aligned}\tag{45}$$

It is easy to see that

$$\begin{aligned}c\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}e_i^T(t)\Gamma e_j(t)\leq{}&c\sum_{i,j=1,j\neq i}^{N}\zeta a_{ij}\|e_i(t)\|\|e_j(t)\|\\&+\sum_{i=1}^{N}\zeta_{\min}a_{ii}e_i^T(t)e_i(t)\\={}&c\zeta\tilde{e}^T(t)\tilde{A}\tilde{e}(t),\end{aligned}\tag{46}$$

where $\tilde{e}(t)=(\|e_1(t)\|,\|e_2(t)\|),\cdots,\|e_N(t)\|)^T$.

It follows from Assumption 2 that

$$\begin{aligned}&\sum_{i=1}^{N}e_i^T(t)(\varphi_{\mu_i}(t)-\tilde{\varphi}_{\mu_i}(t))\\&\leq\sum_{i=1}^{N}L_{\mu_i}\|e_i(t)\|^2+\sum_{i=1}^{N}\sum_{j=1}^{n}N_{\mu_i}|e_{ij}(t)|\\&=\tilde{e}^T(t)L\tilde{e}(t)+\sum_{k=1}^{r}\sum_{i=c_{k-1}+1}^{c_k}\sum_{l=1}^{n}N_k|e_{il}(t)|,\end{aligned}\tag{47}$$

which implies that

$$\begin{aligned}&\sum_{i=1}^{N}e_i^T(t)\big(B_{\mu_i}e_i(t)+\varphi_{\mu_i}(t)-\tilde{\varphi}_{\mu_i}(t)\\&\quad+c\sum_{j=1}^{N}a_{ij}\Gamma e_j(t)-d_ie_i(t)-\gamma\omega_i(t)\big)\\&\leq\tilde{e}^T(t)(\tilde{B}+c\zeta\tilde{A}+L-D)\tilde{e}(t)\\&\quad+\sum_{k=1}^{r}\sum_{i=c_{k-1}+1}^{c_k}\sum_{l=1}^{n}(N_k-\gamma)|e_{il}(t)|\\&\leq\tilde{e}^T(t)(\tilde{B}+c\zeta\tilde{A}+L-D)\tilde{e}(t).\end{aligned}\tag{48}$$

From Lemma 1, we have

$$\begin{aligned}-\alpha\sum_{i=1}^{N}e_i^T(t)e_i^p(t)={}&-\alpha\sum_{i=1}^{N}\sum_{l=1}^{n}e_{il}^{1+p}(t)\\\leq{}&-\alpha\Big(\sum_{i=1}^{N}e_i^T(t)e_i(t)\Big)^{\frac{1+p}{2}},\end{aligned}\tag{49}$$

and

$$\begin{aligned}-\beta\sum_{i=1}^{N}e_i^T(t)e_i^q(t)={}&-\beta\sum_{i=1}^{N}\sum_{l=1}^{n}e_{il}^{1+q}(t)\\\leq{}&-\beta(Nn)^{\frac{1-q}{2}}\Big(\sum_{i=1}^{N}e_i^T(t)e_i(t)\Big)^{\frac{1+q}{2}}.\end{aligned}\tag{50}$$

Substituting (46)-(50) to (45), for $mT \leq t < (m+\theta)T$ ($m = 0, 1, 2, \cdots$), we obtain

$$\begin{aligned}\dot{V}(e(t)) &\leq \tilde{e}^T(t)(\tilde{B} + c\zeta\tilde{A} + L - D)\tilde{e}(t)\\ &\quad -2^{\frac{1+p}{2}}\alpha V^{\frac{1+p}{2}}(e(t))\\ &\quad -2^{\frac{1+q}{2}}\beta(Nn)^{\frac{1-q}{2}}V^{\frac{1+q}{2}}(e(t))\\ &\leq -2^{\frac{1+p}{2}}\alpha V^{\frac{1+p}{2}}(e(t))\\ &\quad -2^{\frac{1+q}{2}}\beta(Nn)^{\frac{1-q}{2}}V^{\frac{1+q}{2}}(e(t)). \end{aligned} \quad (51)$$

Similarly, for $(m+\theta)T \leq t < (1+\theta)T$ ($m = 0, 1, 2, \cdots$), we have

$$\dot{V}(e(t)) \leq \tilde{e}^T(t)(\tilde{B} + c\zeta\tilde{A} + L - D)\tilde{e}(t) \leq 0. \quad (52)$$

From Lemma 2, the community network (1) under controllers (37) is said to achieve cluster synchronization in fixed time.

The proof of Theorem 1 is completed. □

Remark 2: Note that the dynamic behaviors of the nodes in community network (1) is discontinuous, which are different from the continuous dynamics of traditional community networks in [16]–[19]. In addition, although the fixed-time cluster synchronization (see [22], [23] and references therein) and the leader-follower fixed-time group consensus control of multi-agent systems (see [34] and references therein) have been intensively studied. However, the existing methods cannot be used to deal with the fixed-time cluster synchronization problems via periodically or aperiodically switching control. In this paper, it is the first time to establish the fixed-time cluster synchronization criteria for directed community networks with discontinuous nodes by designing periodically or aperiodically switching controllers. Hence, our results are more general and they effectually complement or improve the previously known results.

Remark 3: When $\theta = 1$, the periodically switching control (37) becomes the common feedback control. Based on Theorem 1, the following result can be obtained easily.

Corollary 1: Suppose that Assumptions 1-3 hold. The directed community network (1) can achieve fixed-time cluster synchronization under the feedback controllers:

$$u_i(t) = -d_i e_i(t) - \gamma \mathrm{SIGN}(e_i(t)) - \alpha e_i^p(t) - \beta e_i^q(t), \quad (53)$$

if the following conditions hold:

$$\begin{aligned}&\tilde{B} + c\zeta\tilde{A} + L - D \leq 0, \quad l = 1, 2, \cdots, n,\\ &N_k - \gamma \leq 0, \quad k = 1, 2, \cdots, r, \end{aligned} \quad (54)$$

where $L = \mathrm{diag}(\tilde{L}_1, \cdots, \tilde{L}_r)$, $\tilde{L}_k = \mathrm{diag}(L_{c_{k-1}+1}, \cdots, L_{c_k})$ ($k = 1, \cdots, r$), $D = \mathrm{diag}(d_1, \cdots, d_N)$, $\zeta = \|\Gamma\|$, $\tilde{B} = \mathrm{diag}(B_1, B_2, \cdots, B_r)$, $\tilde{A} = (\tilde{a}_{ij})_{N\times N}$, $\tilde{a}_{ij} = a_{ij}$ for $i \neq j$, $\tilde{a}_{ii} = \zeta_{\min} a_{ii}/\zeta$, $\zeta_{\min}$ is the minimum eigenvalue of Γ; $m = 0, 1, 2, \cdots$, $0 < p < 1$, $q > 1$, α, β, γ and d_i ($i = 1, 2, \cdots, N$) are positive constants denoting the control strengths. Moreover, the fixed settling time $\tilde{T}_f$ can be estimated by

$$\tilde{T}_f \leq \frac{1}{2^{\frac{1+p}{2}}\alpha(1-p)} + \frac{1}{2^{\frac{1+q}{2}}(Nn)^{\frac{q-1}{2}}\beta(q-1)}. \quad (55)$$

Remark 4: Each part of (37) has the unique contribution for fixed-time cluster synchronization. The discontinuous dynamics and coupled configuration are compensated by the terms $-d_i e_i(t)$ and $-\gamma \mathrm{sign}(e_i(t))$; the term $-\alpha e_i^p(t)$ plays a key role for fixed-time synchronization when $V(t) > 1$, while the term $-\beta e_i^q(t)$ plays a key role for fixed-time synchronization when $V(t) \leq 1$. Let $\beta = 0$, the results for finite-time cluster synchronization of directed community networks with discontinuous nodes based on periodically switching control can be derived as:

Corollary 2: Suppose that Assumptions 1-3 hold. The directed community network (1) can achieve finite-time cluster synchronization under the periodically switching controllers:

$$u_i(t) = \begin{cases} -d_i e_i(t) - \gamma \mathrm{sign}(e_i(t)) - \alpha e_i^p(t), \\ \quad mT \leq t < (m+\theta)T, \\ -d_i e_i(t) - \gamma \mathrm{sign}(e_i(t)), \\ \quad (m+\theta)T \leq t < (m+1)T, \end{cases} \quad (56)$$

if the following conditions hold:

$$\begin{aligned}&\tilde{B} + c\zeta\tilde{A} + L - D \leq 0, \quad l = 1, 2, \cdots, n,\\ &N_k - \gamma \leq 0, \quad k = 1, 2, \cdots, r, \end{aligned} \quad (57)$$

where $L = \mathrm{diag}(\tilde{L}_1, \cdots, \tilde{L}_r)$, $\tilde{L}_k = \mathrm{diag}(L_{c_{k-1}+1}, \cdots, L_{c_k})$ ($k = 1, \cdots, r$), $D = \mathrm{diag}(d_1, \cdots, d_N)$, $\zeta = \|\Gamma\|$, $\tilde{B} = \mathrm{diag}(B_1, B_2, \cdots, B_r)$, $\tilde{A} = (\tilde{a}_{ij})_{N\times N}$, $\tilde{a}_{ij} = a_{ij}$ for $i \neq j$, $\tilde{a}_{ii} = \zeta_{\min} a_{ii}/\zeta$, $\zeta_{\min}$ is the minimum eigenvalue of Γ; $m = 0, 1, 2, \cdots$, $0 < p < 1$, α, γ and d_i ($i = 1, 2, \cdots, N$) are positive constants denoting the control strengths; $T > 0$ denotes the control period, $0 < \theta \leq 1$ is the ratio of the control width to the control period called control rate. Moreover, the settling time T_s can be estimated by

$$T_s \leq \frac{V^{1-p}(0)}{2^{\frac{1+p}{2}}\alpha\theta(1-p)}. \quad (58)$$

It is easy to see that, the control gains in (37) are fixed. However, they usually give much larger feedback strengths than those needed in practice, which means a kind of waste in practice. A better way is a to use adaptive method to tune the feedback strengths. On the other hand, the limitation of periodicity for the switching control is quite restricted and may not be realistic in many real applications. Therefore, in the following, we will propose the results for fixed-time cluster synchronization of directed community networks with discontinuous nodes based on aperiodically adaptive switching control.

Theorem 2: Suppose that Assumptions 1-3 hold. The directed community network (1) can achieve fixed-time cluster synchronization under the aperiodically adaptive switching controllers:

$$u_i(t) = \begin{cases} -d_i(t)e_i(t) - \gamma_i(t)\mathrm{sign}(e_i(t)) - \alpha e_i^p(t) \\ -\beta e_i^q(t), \quad t_m \leq t < s_m, \\ -d_i(t)e_i(t) - \gamma_i(t)\mathrm{sign}(e_i(t)), \\ \quad s_m \leq t < t_{m+1}, \end{cases} \quad (59)$$

where $m = 1, 2, \cdots, t_1 = 0, 0 < p < 1, q > 1, \alpha, \beta > 0$; the feedback strengths $d_i(t)$ and $\gamma_i(t)$ are adapted according to the following update laws:

$$\dot{d}_i(t) = \lambda_i e_i^T(t)e_i(t), \tag{60}$$
$$\dot{\gamma}_i(t) = \epsilon_i \mathrm{sign}(e_i(t)), \tag{61}$$

respectively, where λ_i and ϵ_i are arbitrary positive constants. Moreover, the fixed settling time T_f^* can be estimated by

$$T_f^* \le \frac{1}{2^{\frac{1+p}{2}}\alpha\vartheta(1-p)} + \frac{1}{2^{\frac{1+q}{2}}(Nn)^{\frac{q-1}{2}}\beta\vartheta(q-1)}, \tag{62}$$

where $\vartheta = \lim_{t\to+\infty} \sup \frac{t_{m+1}-s_m}{t_{m+1}-t_m}$.

Proof: Construct the following Lyapunov function

$$V(e(t)) = \frac{1}{2}\sum_{k=1}^{r}\sum_{i\in\mathcal{C}_k} e_i^T(t)e_i(t) + \frac{1}{2}\sum_{i=1}^{N}\frac{1}{\lambda_i}(d_i(t)-d^*)^2 + \frac{1}{2}\sum_{i=1}^{N}\frac{1}{\epsilon_i}(\gamma_i(t)-\gamma^*)^2, \tag{63}$$

where d^* and γ^* is the positive constants to be determined below.

By using a similar argument as the proof of Theorem 1, it can be shown that, for $t_m \le t < s_m$ $(m = 1, 2, \cdots)$,

$$\begin{aligned}
\dot{V}(e(t)) \le\ & \tilde{e}^T(t)(\tilde{B} + c\zeta\tilde{A} + L - D(t))\tilde{e}(t) \\
&+ \sum_{k=1}^{r}\sum_{i=c_{k-1}+1}^{c_k}\sum_{l=1}^{n} N_k|e_{il}(t)| \\
&- \sum_{i=1}^{N}\gamma_i(t)\mathrm{sign}(e_i(t)) - 2^{\frac{1+p}{2}}\alpha V^{\frac{1+p}{2}}(e(t)) \\
&- 2^{\frac{1+q}{2}}\beta(Nn)^{\frac{1-q}{2}}V^{\frac{1+q}{2}}(e(t)) \\
&+ \sum_{i=1}^{N} e_i^T(t)(d_i(t)-d_i^*)e_i(t) \\
&+ \sum_{i=1}^{N}(\gamma_i(t)-\gamma_i^*)\mathrm{sign}(e_i(t)) \\
\le\ & \tilde{e}^T(t)(\tilde{B} + c\zeta\tilde{A} + L - d^*I_N)\tilde{e}(t) \\
&+ \sum_{k=1}^{r}\sum_{i=c_{k-1}+1}^{c_k}\sum_{l=1}^{n}(N_k-\gamma^*)|e_{il}(t)| \\
&- 2^{\frac{1+p}{2}}\alpha V^{\frac{1+p}{2}}(e(t)) \\
&- 2^{\frac{1+q}{2}}\beta(Nn)^{\frac{1-q}{2}}V^{\frac{1+q}{2}}(e(t)),
\end{aligned} \tag{64}$$

where $D(t) = \mathrm{diag}(d_1(t), \cdots, d_N(t))$.

Let $\gamma^* = \max_{1\le k\le r}\{N_k\}$, $d^* = \lambda_{\max}(\tilde{B} + c\zeta\tilde{A} + L)$, where $\lambda_{\max}(\cdot)$ is the maximum eigenvalue of matrix, we have

$$V(e(t)) \le -2^{\frac{1+p}{2}}\alpha V^{\frac{1+p}{2}}(e(t)) - 2^{\frac{1+q}{2}}\beta(Nn)^{\frac{1-q}{2}}V^{\frac{1+q}{2}}(e(t)), \tag{65}$$

for $t_m \le t < s_m$ $(m = 1, 2, \cdots)$.

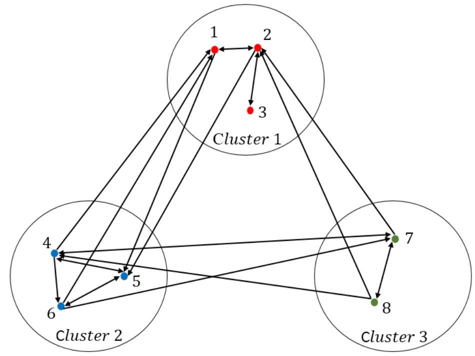

FIGURE 1. A directed community network with 3 clusters consisting of 8 nodes.

Similarly, for $s_m \le t < t_{m+1}$ $(m = 1, 2, \cdots)$, we get

$$\dot{V}(e(t)) \le 0. \tag{66}$$

From Lemma 3, the community network (1) under controllers (59) is said to achieve cluster synchronization in fixed time. This completes the proof. □

IV. NUMERICAL SIMULATIONS

In this section, in order to illustrate the effectiveness of the proposed methods in fixed-time cluster synchronization obtained above, we consider a directed community network of 8 dynamical nodes with 3 communities $\mathcal{C}_1 = \{1, 2, 3\}$, $\mathcal{C}_2 = \{4, 5, 6\}$, $\mathcal{C}_3 = \{7, 8\}$. Fig. 1 shows the topological structure of the proposed directed community network.

The node dynamics of the 3 communities are, respectively, chosen as discontinuous Chen system [35], Chua circuit [17], [26] and cellular neural networks [28]:

$$\dot{s}_1(t) = B_1 s_1(t) + f_1(s_1(t)), \tag{67}$$
$$\dot{s}_2(t) = B_2 s_2(t) + f_2(s_2(t)), \tag{68}$$
$$\dot{s}_3(t) = B_3 s_3(t) + f_3(s_3(t)), \tag{69}$$

where $s_i(t) = (s_{i1}(t), s_{i2}(t), s_{i3}(t))^T \in \mathbb{R}^3$ $(i = 1, 2, 3)$, $f_1(s_1(t)) = (0, \mathrm{sign}(s_{11})(5.82 - s_{13}), \mathrm{sign}(s_{12})s_{11})^T$, $f_2(s_2(t)) = (3.86\mathrm{sign}(s_{21}), 0, 0)^T$, $f_3(s_3(t)) = G(g(s_{31}(t)), g(s_{32}(t)), g(s_{33}(t)))^T$ and

$$g(v) = \begin{cases} 0.5(|v+1| - |v-1|) - 0.002, & v \le 0, \\ 0.5(|v+1| - |v-1|) + 0.001, & v > 0, \end{cases}$$

$$G = \begin{bmatrix} 1.25 & -3.2 & -3.2 \\ -3.2 & 1.1 & -4.4 \\ -3.2 & 4.4 & 1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} -1.18 & 1.18 & 0 \\ 0 & 0.7 & 0 \\ 0 & 0 & -0.168 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} -2.57 & 9 & 0 \\ 1 & -1 & 1 \\ 0 & -17 & 0 \end{bmatrix},$$

$$B_3 = -I_2.$$

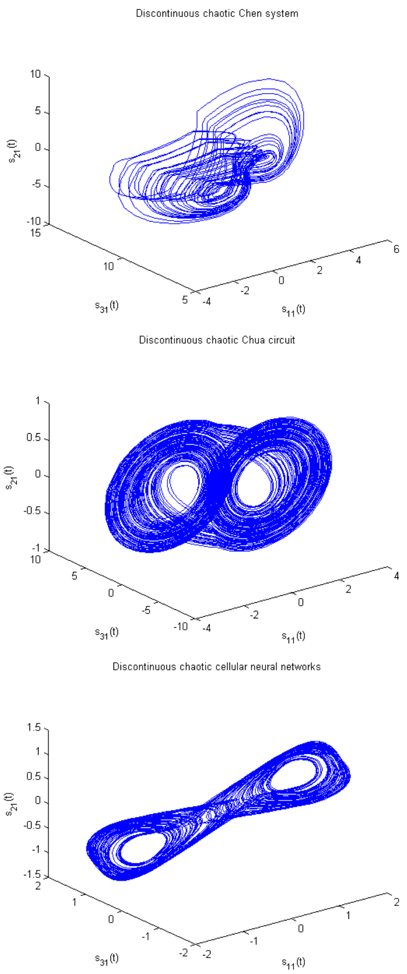

FIGURE 2. Chaotic trajectories of discontinuous Chen system, Chua circuit and cellular neural networks, respectively.

By simple computation, it is easy to know $L_1 = 0.5$, $N_1 = 36.0078$, $L_2 = 0$, $N_2 = 3.72$, $L_3 = 7$, $N_3 = 0.0364$. Fig. 2 presents the chaotic trajectories of (67)-(69) with initial values $s_1(0) = (0.6, 0.25, 8)^T$, $s_2(0) = (0.5, 0.3, 1)^T$ and $s_3(0) = (0.1, 0.1, 0.1)^T$, respectively.

For brevity, taking the coupling strength and inner connecting matrix as $c = 1$ and $\Gamma = I_3$, respectively. The outer connecting matrix A is given by:

$$A = \begin{bmatrix} -1 & 1 & 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

Choosing $\alpha = 5, \beta = 10, T = 0.2, \theta = 0.8, \gamma = 36.0078$, $d_i = 6.10$ $(i = 1, 2, \cdots, 8)$, $p = 0.5$ and $q = 1.5$ in the controller (37). The above parameters are substituted into the conditions (38), which are met after calculating.

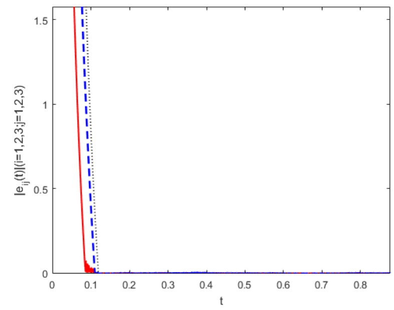

FIGURE 3. Trajectories of the synchronization error for cluster 1 under controllers (37).

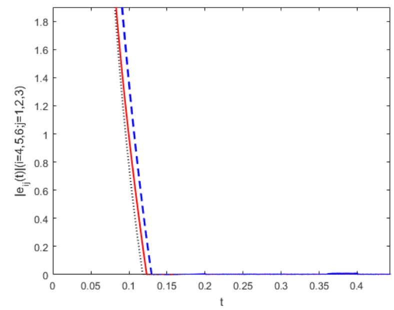

FIGURE 4. Trajectories of the synchronization error for cluster 2 under controllers (37).

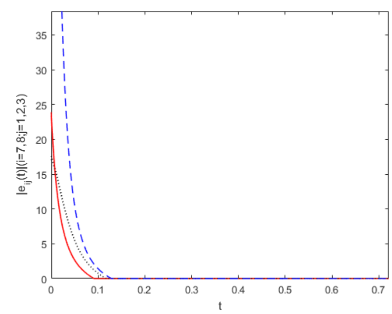

FIGURE 5. Trajectories of the synchronization error for cluster 3 under controllers (37).

From Theorem 1, the community network (1) can achieve cluster synchronization in fixed time under the periodically switching control (37) as shown in Figs. 3-8, and the settling time can be estimated as $T_f \leq 0.3448$.

It is shown from inequality (39) that the periodically switching control rate θ heavily influences the estimating for the upper bound of the convergence time of synchronization state. Figs. 9-11 show that the larger periodically switching control rate is, the faster the convergence rate of network can be obtained.

When $\theta = 1$, the periodically switching controllers (37) become the common feedback controllers. As shown in the Figs. 12-14, the community network (1) can achieve cluster synchronization in fixed time under the feedback controllers (53) and we obtain that the settling time satisfies $\tilde{T}_f \leq 0.2758$ by primitive calculation.

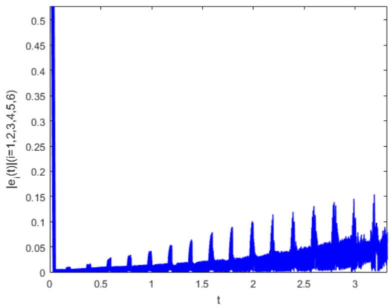

FIGURE 6. Trajectories of the synchronization error between clusters 1 and 2 under controllers (37).

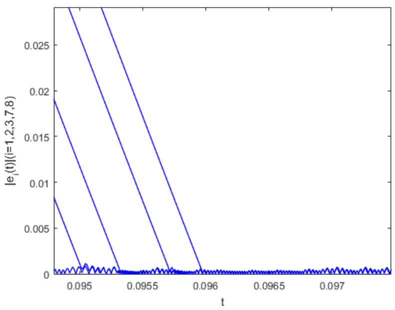

FIGURE 7. Trajectories of the synchronization error between clusters 1 and 3 under controllers (37).

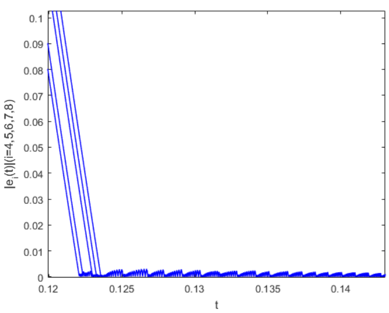

FIGURE 8. Trajectories of the synchronization error between clusters 2 and 3 under controllers (37).

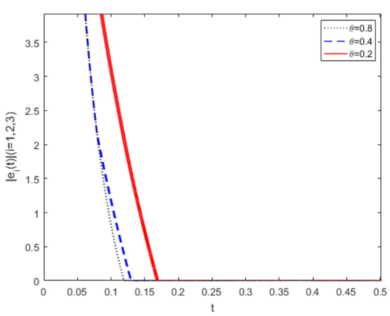

FIGURE 9. Trajectories of the synchronization error for cluster 1 with different control rates.

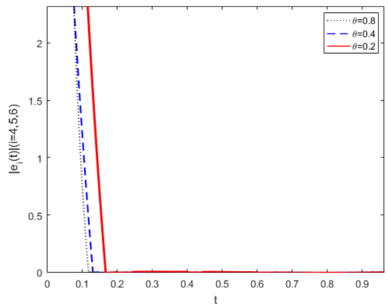

FIGURE 10. Trajectories of the synchronization error for cluster 2 with different control rates.

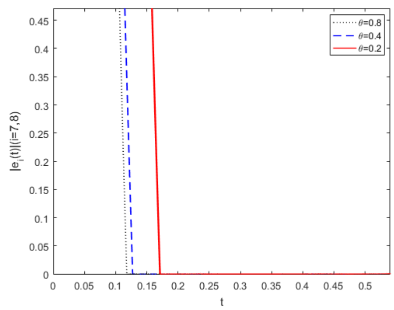

FIGURE 11. Trajectories of the synchronization error for cluster 3 with different control rates.

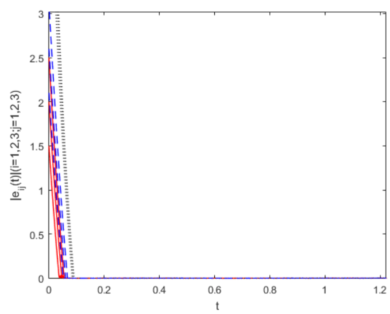

FIGURE 12. Trajectories of the synchronization error for cluster 1 under controllers (53).

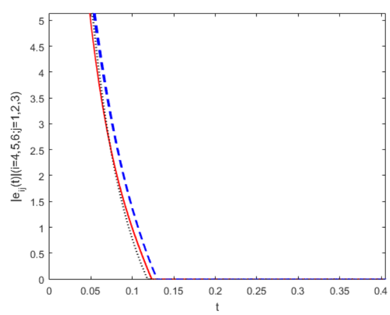

FIGURE 13. Trajectories of the synchronization error for cluster 2 under controllers (53).

Comparing with the corresponding finite-time synchronization, fixed-time synchronization can be achieved in a settling time, which is bounded and independent of the initial states. Figs. 15-17 shows the comparisons of finite-time cluster synchronization and fixed-time cluster synchronization, which proved that the convergence rate of

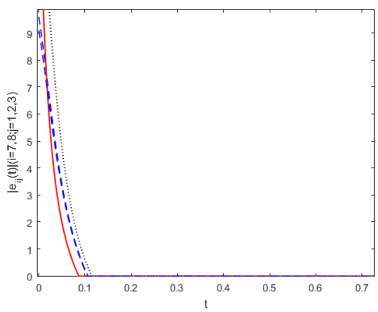

FIGURE 14. Trajectories of the synchronization error for cluster 3 under controllers (53).

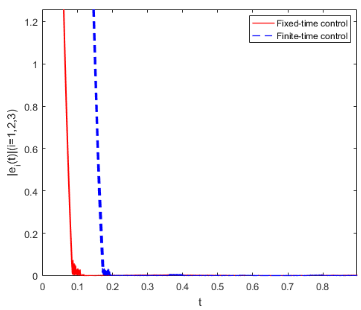

FIGURE 15. Trajectories of the synchronization error for cluster 1 under controllers (37) and (56), respectively.

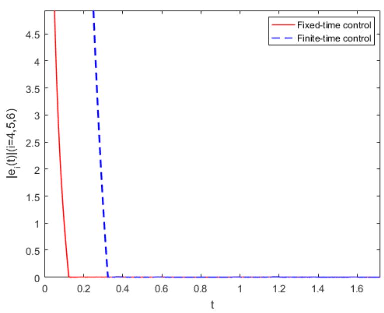

FIGURE 16. Trajectories of the synchronization error for cluster 2 under controllers (37) and (56), respectively.

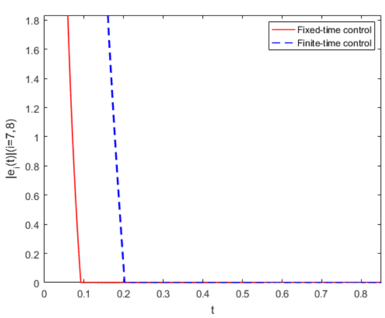

FIGURE 17. Trajectories of the synchronization error for cluster 3 under controllers (37) and (56), respectively.

fixed-time cluster synchronization ($T_f \leq 0.3448$) is faster than the rate of finite-time cluster synchronization ($T_s \leq 1.7052$). Therefore, compare with finite-time control

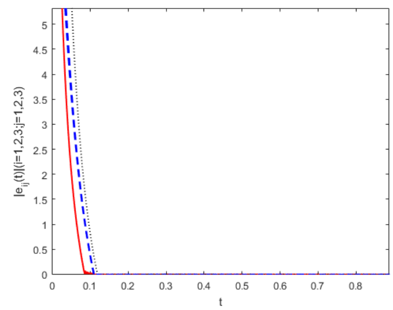

FIGURE 18. Trajectories of the synchronization error for cluster 1 under the aperiodically adaptive switching control (59).

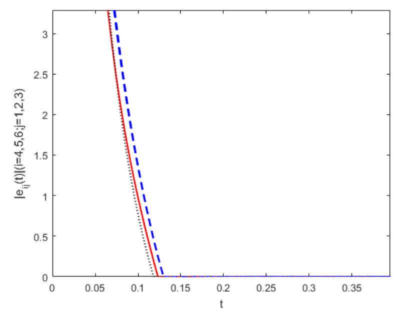

FIGURE 19. Trajectories of the synchronization error for cluster 2 under the aperiodically adaptive switching control (59).

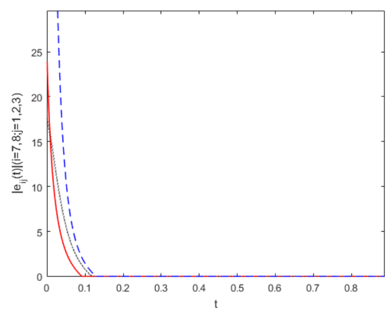

FIGURE 20. Trajectories of the synchronization error for cluster 3 under the aperiodically adaptive switching control (59).

strategy, fixed-time control scheme shows more effectiveness and superiority.

As we all know, the aperiodically switching controller is more comprehensive and practical than the periodically switching controller in the authentic application. Therefore, the fixed-time cluster synchronization by the aperiodically adaptive switching controllers will be given and time intervals of intermittent controllers as follows:

$$[0, 3] \cup [3.2, 6.4] \cup [6.5, 9.6] \cup [9.8, 12.8] \cup [13, 16]$$
$$[16.2, 19.2] \cup [19.5, 22.6] \cup [22.8, 25.8] \cup \cdots .$$

By the computing, we acquire $\vartheta = 0.5$ and the settling time satisfies $T_f^* \leq 0.5517$. From Theorem 2, the community network (1) can achieve cluster synchronization in fixed time under the aperiodically adaptive switching control (59) as shown in Figs. 18-20.

V. CONCLUSION

In this paper, we have studied the fixed-time cluster synchronization problem for a class of directed community networks with discontinuous nodes. By designing periodically switching control or aperiodically adaptive switching control, and utilizing the theory of differential inclusions, some fixed-time cluster synchronization criteria have been derived in which the settling time function is bounded for any initial values. Some remarks and numerical simulations have been used to demonstrate the effectiveness of the obtained results. In fact, time delays are widely existed in community networks. The integrating of internal delay and coupling delay into community networks make them more realistic by allowing to describe the effects of the finite speeds and spreading as well as traffic congestion for nodes' behaviors and transmission exchange of information between nodes, respectively. Therefore, this is an interesting problem and will become our future investigative direction.

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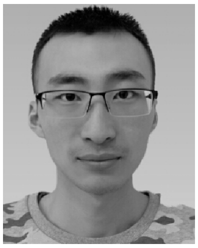

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