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Stochastic Finite-Time Output Feedback Stabilization of Feedforward Nonlinear Systems

XINGHUI ZHANG, ANCAI ZHANG, AND GUOCHEN PANG
School of Automation and Electrical Engineering, Linyi University, Linyi 276000, China

Corresponding author: Xinghui Zhang (lyzhangxinghui@163.com)

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ABSTRACT The finite-time stabilization for stochastic high-order nonlinear systems is considered in this paper using output feedback. The power orders in the system nonlinearities are relaxed to be rational number greater than one. A new constructive output-feedback, finite-time controller is designed based on a finite-time observer together with the homogeneous domination. The stochastic finite-time stability analysis is given rigorously for the closed-loop stochastic system.

INDEX TERMS Stochastic, feedforward systems, finite-time, output feedback control.

I. INTRODUCTION

Stabilization of feedforward systems is of practical importance [1], [2]. There are many deterministic results on feedforward systems, see [3]–[12] and the references therein. In recent years, some stabilization results have been achieved for stochastic nonlinear feedforward systems

$$\begin{aligned} d\eta_i &= (\eta_{i+1}^{p_i} + f_i(\eta_{i+2}, \cdots, \eta_n, v))dt \\ &\quad + g_i^\top(\eta_{i+2}, \cdots, \eta_n, v)d\omega, \ i = 1, \cdots, n-2, \\ d\eta_{n-1} &= (\eta_n^{p_{n-1}} + f_{n-1}(v))dt + g_{n-1}^\top(v)d\omega, \ \eta(0) = \eta_0, \\ d\eta_n &= v^{p_n}dt, \\ y &= \eta_1, \end{aligned} \tag{1}$$

where $n \geq 2$, and $\eta = (\eta_1, \cdots, \eta_n)^\top \in R^n, v \in R, y = \eta_1$ and $\eta_0 \in R$ are the system state, input, output and initial value. ω is an m-dimensional standard Wiener process defined on the complete probability space $(\Omega, \mathcal{F}, P)$. For $i = 1, \cdots, n$, $p_i \in R_{odd}^{\geq 1} \triangleq \{\frac{p}{q} \in R^+$: p and q are odd integers, $p \geq q\}$, system (1) is called as high-order system if one $p_i > 1$, $f_i : R^{n-i-1} \times R \to R$ and $g_i : R^{n-i-1} \times R \to R^m$ are continuously differential with $f_i(0, 0) = 0$, $g_i(0, 0) = 0$.

For instance, [13] studied the stochastic stabilization of nonlinear systems in feedforward form with noisy outputs. [14] constructed a state feedback stabilized controller for stochastic high-order nonlinear feedforward systems. Reference [15] considered the global stabilization of feedforward systems with time-delay. However, all of these results only consider the asymptotic stabilization.

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Recently, more attention have been paid on the finite-time stabilization of deterministic feedforward systems [9], stochastic lower-triangular systems [17], [18], switched stochastic nonlinear systems [19], [20]. For stochastic nonlinear feedforward systems, the finite-time stabilization is considered in [21] but where only a finite-time state feedback stabilized controller is designed.

In this paper, we will design a finite-time output feedback controller for stochastic high-order nonlinear feedforward systems (1). Based on stochastic Lyapunov theorem on finite-time stability in probability established in [22], by using the homogeneous domination method, the adding one power integrator and sign function method, constructing a $\mathcal{C}^2$ Lyapunov function and verifying the existence and uniqueness of solution, a continuous output feedback controller is designed to guarantee the closed-loop system finite-time stable in probability. The effectiveness of control method is showed by a simulation example.

II. PRELIMINARIES

The following notations are to be used in this paper.

Notations: R^n denotes the n-dimensional Euclidean space, R^+ stands for the set of all nonnegative real numbers. For a vector or matrix X, $X^\top$ denotes its transpose, $\mathrm{Tr}\{X\}$ denotes its trace when X is square. $\|X\|$ denotes Euclidean norm $(\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$ and Frobenius norm $(\mathrm{Tr}\{X^\top X\})^{\frac{1}{2}}$ for vector X and matrix X, respectively. A function $f : R^n \to R$ is $\mathcal{C}^i$ if it is i-times differential. $\mathcal{K}$ denotes the set of all functions: $R^+ \to R^+$ that are continuous, strictly increasing and vanishing at zero, $\mathcal{K}_\infty$ denotes the set of all functions that are of class $\mathcal{K}$ and unbounded. $\mathrm{sgn}(x)$ is the sign function defined as: $\mathrm{sgn}(x) = 1$ if $x > 0$, $\mathrm{sgn}(x) = 0$ if $x = 0$, and $\mathrm{sgn}(x) = -1$ if $x < 0$.

For system

$$\mathrm{d}x = f(x)\mathrm{d}t + g^\top(x)\mathrm{d}\omega, \quad x(0) = x_0 \in R^n, \tag{2}$$

where $f(x)$, $g(x)$ are continuous in x, $f(0) = 0$, $g(0) = 0$, x_0 is the initial value, we have the following three lemmas.

Lemma 1 [23]: Suppose continuous functions $f(x)$ and $g(x)$ satisfy $\|f(x)\|^2 + \|g(x)\|^2 \le K(1 + x^2)$ for a constant $K > 0$, then for any $x_0 \in R^n$, system (2) has a solution.

Lemma 2 [17]: If one nonnegative, radially unbounded and $\mathcal{C}^2$ function $V(x)$ satisfies $\mathcal{L}V(x)|_{(2)} \le 0$ for all $x \in R^n$, then system (2) has a solution for any $x_0 \in R^n$.

Lemma 3 [22]: Suppose system (2) admits a unique solution, if there is a $\mathcal{C}^2$ function $V : R^n \to R^+$, $\mathcal{K}_\infty$ functions γ_1 and γ_2, real numbers $\lambda > 0$ and $0 < \gamma < 1$ such that for all $t > 0$ and $x \in R^n$, $\gamma_1(\|x\|) \le V(x) \le \gamma_2(\|x\|)$, $\mathcal{L}V(x) \le -\lambda V^\gamma(x)$, then the trivial solution of system (2) is finite-time stable in probability.

Lemma 4 [25]: For $x, y \in R$, if $p \ge 1 \in R$, then

$$|x + y|^p \le 2^{p-1}|x^p + y^p|, \quad (|x| + |y|)^{\frac{1}{p}} \le |x|^{\frac{1}{p}} + |y|^{\frac{1}{p}}. \tag{3}$$

If $p \in R^{\ge 1}_{odd}$, then

$$|x^p - y^p| \le p(1 + 2^{p-3})|x - y|((x - y)^{p-1} + y^{p-1}), \tag{4}$$

$$|x - y|^p \le 2^{p-1}|x^p - y^p|. \tag{5}$$

Lemma 5: Let real numbers $p \ge 1$ and $q \ge 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$, then for any $x, y \in R$ and any given positive number $\gamma > 0$, $xy \le \gamma|x|^p + \frac{1}{q}(p\gamma)^{-\frac{q}{p}}|y|^q$.

Lemma 6 [26]: For any real numbers $p = \frac{b_1}{b_0} \in R^{\ge 1}_{odd}$ with $b_1 \ge b_0 \ge 1$ and $a \ge 1$, there hold

$$|x^p - y^p| \le 2^{1-\frac{1}{b_0}}|\mathrm{sgn}(x)|x|^{b_1} - \mathrm{sgn}(y)|y|^{b_1}|^{\frac{1}{b_0}}, \tag{6}$$

$$\left|\mathrm{sgn}(x)|x|^{\frac{1}{a}} - \mathrm{sgn}(y)|y|^{\frac{1}{a}}\right| \le 2^{1-\frac{1}{a}}|x - y|^{\frac{1}{a}}. \tag{7}$$

Lemma 7 [27]: If $r \ge 1$ is any given rational number, then there are two constants $c_0(r) > 0$ and $1 \ge d(r) > 0$ both dependent on r such that for any $x, y \in R$,

$$|\mathrm{sgn}(x)|x|^a - \mathrm{sgn}(y)|y|^a| \le c_0|x - y|^d(|x - y|^{a-d} + |y|^{a-d}). \tag{8}$$

III. DESIGN OF FINITE-TIME OUTPUT FEEDBACK CONTROLLER

A. PROBLEM FORMULATION

To obtain the stochastic finite-time output feedback stability of system (1), in this paper, the nonlinear functions are assumed to satisfy the following conditions:

Assumption 1: There exist real numbers $M_1 > 0$ and $M_2 > 0$, and a rational number $\tau \in (-\frac{1}{2(1+\sum_{l=2}^{n} p_1 \cdots p_{l-1})}, 0)$ such that for $i = 1, \cdots, n-2$,

$$|f_i(\eta_{i+2}, \cdots, \eta_n, v)| \le M_1\Big(\sum_{j=i+2}^{n} |\eta_j|^{\frac{r_i+\tau}{r_j}} + |v|^{\frac{r_i+\tau}{r_{n+1}}}\Big),$$

$$\|g_i(\eta_{i+2}, \cdots, \eta_n, v)\| \le M_2\Big(\sum_{j=i+2}^{n} |\eta_j|^{\frac{2r_i+\tau}{2r_j}} + |v|^{\frac{2r_i+\tau}{2r_{n+1}}}\Big),$$

$$|f_{n-1}(v)| \le M_1|v|^{\frac{r_{n-1}+\tau}{r_{n+1}}}, \quad \|g_{n-1}(v)\| \le M_2|v|^{\frac{2r_{n-1}+\tau}{2r_{n+1}}}, \tag{9}$$

where

$$r_1 = \frac{1}{2}, \quad r_{i+1} = \frac{r_i + \tau}{p_i}, \ i = 1, \cdots, n. \tag{10}$$

Remark 1: Assumption 1 holds for system (1) with $n \ge 2$. It's worth noting that, when $n = 2$, system (1) reduces to

$$\begin{aligned} \mathrm{d}\eta_1 &= (\eta_2^{p_1} + f_1(v))\mathrm{d}t + g_1^\top(v)\mathrm{d}\omega, \\ \mathrm{d}\eta_2 &= v^{p_2}\mathrm{d}t, \\ y &= \eta_1. \end{aligned} \tag{11}$$

For this case, system (11) just needs to satisfy the condition $M_1|v|^{\frac{r_{n-1}+\tau}{r_{n+1}}}$, $\|g_{n-1}(v)\| \le M_2|v|^{\frac{2r_{n-1}+\tau}{2r_{n+1}}}$ with $n = 2$ in Assumption 1.

Remark 2: For the finite-time stabilization of stochastic nonlinear systems by output feedback, several latest results such as [29]–[31] have been achieved. However, all of them consider the systems in feedback form. To our knowledge, there is no stochastic output feedback finite-time stabilization result on feedforward systems, so Assumption 1 is the weakest until now. To ensure that Assumption 1 holds, one must find a constant $\tau \in (-\frac{1}{2(1+\sum_{l=2}^{n} p_1 \cdots p_{l-1})}, 0)$ such that f_i and g_i satisfy (9). For system $\mathrm{d}\eta_1 = \eta_2^{\frac{5}{3}}\mathrm{d}t + \eta_3^a\mathrm{d}\omega$, $\mathrm{d}\eta_2 = \eta_3^3\mathrm{d}t$, $\mathrm{d}\eta_3 = v\mathrm{d}t$, where a takes any real constant inside $(5, \frac{43}{6})$, by choosing $\tau = \frac{3a-15}{15-16a}$, Assumption 1 holds due to $g_1 \le |\eta_3|^a = |\eta_3|^{\frac{2r_1+\tau}{2r_3}}$.

There are a variety of nonlinearities satisfying Assumption 1. For example, if there is τ such that $\tau + r_i = a_{i+2}r_{i+2} + \cdots + a_n r_n$, then $f_i = \eta_{i+2}^{a_{i+2}} \cdots \eta_n^{a_n}$ satisfies Assumption 1. It's worthy to note that some nonlinearities such as $\sin\eta$, $\ln(1 + |\eta|)$ can be bounded by $|\eta|$. □

B. OUTPUT FEEDBACK CONTROLLER OF SYSTEM (1)

The design of output feedback controller of system (1) consists of three parts. In Part I, the output feedback controller of the nominal system for system (1) based on a observer is designed. Part II determines the observer gains $\ell_1, \cdots, \ell_{n-1}$. In Part III, a dynamic gain Γ is introduced into system (1) by the transformation, the output feedback controller of system (1) with the dynamic gain is obtained.

Part I. Output feedback controller of nominal system

In this subsection, we design a homogeneous output feedback controller for the nominal system:

$$\begin{aligned} \mathrm{d}x_i &= x_{i+1}^{p_i}\mathrm{d}t, \quad i = 1, \cdots, n-1, \\ \mathrm{d}x_n &= u^{p_n}\mathrm{d}t, \\ y &= x_1. \end{aligned} \tag{12}$$

Denote $\mathrm{sgn}(x)|x|^a \triangleq [x]^a$ for any $a \in R^+, x \in R$.

Step 1: Choose $V_1(x_1) = m_1 \int_0^{x_1} [s]^{\frac{4-r_2p_1}{r_1}} ds$ with $m_1 > 0$ being a constant, and define $z_1 = [x_1]^{\frac{1}{r_1}}$. Obviously, $V_1(x_1)$ is $\mathcal{C}^2$ since $4 - r_2p_1 \geq 1$. The infinitesimal generator[1] of V_1 along (12) satisfies $\mathcal{L}V_1(x_1)|_{(12)} = m_1[z_1]^{4-r_2p_1}(x_2^{p_1} - \alpha_1^{p_1}) + m_1[z_1]^{4-r_2p_1}\alpha_1^{p_1}$. Choosing $\alpha_1 = -\beta_1^{r_2}[z_1]^{r_2}$ with $\beta_1 = (\frac{a_{1,1}}{m_1})^{\frac{1}{r_2p_1}}$ and $a_{1,1}$ being a positive constant to be designed, one has $\mathcal{L}V_1(x_1)|_{(12)} \leq -a_{1,1}z_1^4 + m_1[z_1]^{4-r_2p_1}(x_2^{p_1} - \alpha_1^{p_1})$.

Step k $(k = 2, \cdots, n)$: We demonstrate this step by Proposition 1 whose proof is placed in the Appendix.

Proposition 1: Suppose that there is a $\mathcal{C}^2$, positive definite and radially unbounded function $V_{k-1}(\bar{x}_{k-1})$, and virtual controllers $\alpha_0, \cdots, \alpha_{k-1}$ defined by $\alpha_0 = 0$, $\alpha_{j-1} = -\beta_{j-1}^{r_j}[z_{j-1}]^{r_j}$, $z_{j-1} = [x_{j-1}]^{\frac{1}{r_{j-1}}} - [\alpha_{j-2}]^{\frac{1}{r_{j-1}}}$, $j = 1, \cdots, k$, such that

$$\begin{aligned}
&\mathcal{L}V_{k-1}(\bar{x}_{k-1})|_{(12)} \\
&\leq -\sum_{j=1}^{k-3}\Big(a_{j,j} - \tilde{a}_{j+1,j} - \bar{a}_{j+1,j} - \sum_{l=j+2}^{k-1}\bar{a}_{l,j}\Big)z_j^4 \\
&\quad - (a_{k-2,k-2} - \tilde{a}_{k-1,k-2} - \bar{a}_{k-1,k-2})z_{k-2}^4 \\
&\quad - a_{k-1,k-1}z_{k-1}^4 + m_{k-1}[z_{k-1}]^{4-r_kp_{k-1}}(x_k^{p_{k-1}} - \alpha_{k-1}^{p_{k-1}}),
\end{aligned} \tag{13}$$

then by defining $z_k = [x_k]^{\frac{1}{r_k}} - [\alpha_{k-1}]^{\frac{1}{r_k}}$, there is a virtual controller $\alpha_k = -\beta_k^{r_{k+1}}[z_k]^{r_{k+1}}$ such that the kth Lyapunov function

$$\begin{aligned}
V_k(\bar{x}_k) &= V_{k-1}(\bar{x}_{k-1}) + m_kW_k(\bar{x}_k), \\
W_k(\bar{x}_k) &= \int_{\alpha_{k-1}}^{x_k}\left[[s]^{\frac{1}{r_k}} - [\alpha_{k-1}]^{\frac{1}{r_k}}\right]^{4-r_{k+1}p_k} ds
\end{aligned} \tag{14}$$

satisfies

$$\begin{aligned}
\mathcal{L}V_k(\bar{x}_k)|_{(12)} &\leq -\sum_{j=1}^{k-2}\Big(a_{j,j} - \tilde{a}_{j+1,j} - \bar{a}_{j+1,j} - \sum_{l=j+2}^{k}\bar{a}_{l,j}\Big)z_j^4 \\
&\quad - (a_{k-1,k-1} - \tilde{a}_{k,k-1} - \bar{a}_{k,k-1})z_{k-1}^4 \\
&\quad - a_{k,k}z_k^4 + m_k[z_k]^{4-r_{k+1}p_k}(x_{k+1}^{p_k} - \alpha_k^{p_k}),
\end{aligned} \tag{15}$$

where m_{k-1}, m_k, $\tilde{a}_{j+1,j}$, $\bar{a}_{j+1,j}$, $\bar{a}_{l,j}$, $\tilde{a}_{k,k-1}$, $\bar{a}_{k,k-1}$, $j = 1, \cdots, k-2$, $l = j+2, \cdots, k$, are positive constants, $a_{1,1}, \cdots, a_{k-1,k-1}, a_{k,k}$ are positive constants to be designed, $\beta_1, \cdots, \beta_k$ are positive constants dependent on $a_{1,1}, \cdots, a_{k-1,k-1}, a_{k,k}$.

At step n, choose $V_n(x) = \sum_{k=1}^n m_kW_k(\bar{x}_k)$ with $m_1, \cdots, m_n$ being positive constants and design

$$\begin{aligned}
u^* &= -\beta_n^{r_{n+1}}[z_n]^{r_{n+1}}, \\
z_j &= [x_j]^{\frac{1}{r_j}} - [\alpha_{j-1}]^{\frac{1}{r_j}}, \\
\alpha_0 &= 0, \quad \alpha_{j-1} = -\beta_{j-1}^{r_j}[z_{j-1}]^{r_j}, \ j = 1, \cdots, n,
\end{aligned} \tag{16}$$

[1]For $\mathcal{C}^2$ function $V(x)$, the infinitesimal generator of $V(x)$ along system (2) is defined as $\mathcal{L}V(x) = (\frac{\partial V(x)}{\partial x})^\top f(x) + \frac{1}{2}\mathrm{Tr}\{g(x)\frac{\partial^2 V(x)}{\partial x^2}g^\top(x)\}$, where $\frac{1}{2}\mathrm{Tr}\{g(x)\frac{\partial^2 V(x)}{\partial x^2}g^\top(x)\}$ is called the Hessian term of $\mathcal{L}$.

then there are positive constants $\beta_1, \cdots, \beta_n$ such that

$$\begin{aligned}
\mathcal{L}V_n(x)|_{(12)} &\leq -\sum_{j=1}^{n-2}\Big(a_{j,j} - \tilde{a}_{j+1,j} - \bar{a}_{j+1,j} - \sum_{l=j+2}^{n}\bar{a}_{l,j}\Big)z_j^4 \\
&\quad - (a_{n-1,n-1} - \tilde{a}_{n,n-1} - \bar{a}_{n,n-1})z_{n-1}^4 \\
&\quad - a_{n,n}z_n^4 + m_n[z_n]^{4-r_{n+1}p_n}(u^{p_n} - u^{*p_n}),
\end{aligned} \tag{17}$$

where $f_n = 0$, $g_n = 0$, $\tilde{a}_{j+1,j}, \bar{a}_{j+1,j}, \bar{a}_{l,j}, \tilde{a}_{n,n-1}, \bar{a}_{n,n-1}$, $j = 1, \cdots, n-2$, $l = j+2, \cdots, n$, are positive constants, $a_{1,1}, \cdots, a_{n,n}$ are positive constants to be designed.

Introduce a homogeneous reduced-order observer:

$$d\hat{\xi}_k = -\ell_{k-1}\hat{x}_k^{p_{k-1}}dt, \quad \hat{x}_k = [\hat{\xi}_k + \ell_{k-1}\hat{x}_{k-1}]^{\frac{r_k}{r_{k-1}}}, \tag{18}$$

where $k = 2, \cdots, n$. Using the certainty equivalence principle together with (16), one obtains the implementable output feedback controller for system (12):

$$u = \hat{\alpha}_n = -\beta_n^{r_{n+1}}\left[[\hat{x}_n]^{\frac{1}{r_n}} + \sum_{i=1}^{n-1}\Big(\prod_{j=i}^{n-1}\beta_j\Big)[\hat{x}_i]^{\frac{1}{r_i}}\right]^{r_{n+1}}. \tag{19}$$

Part II. Determination of observer gains $\ell_1, \cdots, \ell_{n-1}$

For $k = 2, \cdots, n$, define $e_k = [x_k^{p_{k-1}} - \hat{x}_k^{p_{k-1}}]^{\frac{1}{r_kp_{k-1}}}$ and choose

$$\begin{aligned}
V = V_n + \sum_{k=2}^n \int_{[\hat{\xi}_k+\ell_{k-1}x_{k-1}]^{\frac{4-r_kp_{k-1}}{r_{k-1}}}}^{[x_k]^{\frac{4-r_kp_{k-1}}{r_k}}} \Big([s]^{\frac{r_{k-1}}{4-r_kp_{k-1}}} \\
- (\hat{\xi}_k + \ell_{k-1}x_{k-1})\Big)ds \triangleq \sum_{k=2}^n U_k + V_n.
\end{aligned} \tag{20}$$

Obviously, $\frac{4-r_kp_{k-1}}{r_k} \geq \frac{4-r_kp_{k-1}}{r_{k-1}} \geq 1$ imply that $\sum_{k=2}^n U_k$ is $\mathcal{C}^1$. Using (12), (18), (20) yields

$$\begin{aligned}
&\mathcal{L}U_k \\
&= \frac{4-r_kp_{k-1}}{r_k}\Big([x_k]^{\frac{r_{k-1}}{r_k}} - (\hat{\xi}_k + \ell_{k-1}x_{k-1})\Big)|x_k|^{\frac{4-r_kp_{k-1}-r_k}{r_k}} \\
&\quad \cdot x_{k+1}^{p_k} - \ell_{k-1}[e_k]^{r_kp_{k-1}}\Big([x_k]^{\frac{4-r_kp_{k-1}}{r_k}} - [\hat{x}_k]^{\frac{4-r_kp_{k-1}}{r_k}}\Big) \\
&\quad - \ell_{k-1}[e_k]^{r_kp_{k-1}}\Big([\hat{x}_k]^{\frac{4-r_kp_{k-1}}{r_k}} - [\hat{\xi}_k + \ell_{k-1}x_{k-1}]^{\frac{4-r_kp_{k-1}}{r_{k-1}}}\Big),
\end{aligned} \tag{21}$$

where $x_{n+1} = u$.

We can easily verify the following Proposition 2 directly by Lemmas 3,6, we omit the proof here.

Proposition 2: For $k = 2, \cdots, n$, there holds $-\ell_{k-1}[e_k]^{r_kp_{k-1}}([x_k]^{\frac{4-r_kp_{k-1}}{r_k}} - [\hat{x}_k]^{\frac{4-r_kp_{k-1}}{r_k}}) \leq -\ell_{k-1} 2^{1-\frac{4-r_kp_{k-1}}{r_kp_{k-1}}} e_k^4$.

To further estimate (21), we give the following Proposition 3-Proposition 5 whose proofs are in the Appendix.

Proposition 3: For $k = 2, \cdots, n-1$, there holds $\frac{4-r_kp_{k-1}}{r_k}([x_k]^{\frac{r_{k-1}}{r_k}} - (\hat{\xi}_k + \ell_{k-1}x_{k-1}))|x_k|^{\frac{4-r_kp_{k-1}-r_k}{r_k}} x_{k+1}^{p_k} \leq b_1\sum_{j=k-1}^{k+1} z_j^4 + c_{1,k}e_k^4 + h_{1,k}(\ell_{k-1})e_{k-1}^4$, where b_1 and $c_{1,k}$ are positive constants, $h_{1,k}$ is a positive constant related to ℓ_{k-1}.

Proposition 4: For $k = 3, \cdots, n$, there holds $-\ell_{k-1}[e_k]^{r_k p_{k-1}}([\hat{x}_k]^{\frac{4-r_k p_{k-1}}{r_k}} - [\hat{\xi}_k + \ell_{k-1}x_{k-1}]^{\frac{4-r_k p_{k-1}}{r_{k-1}}}) \leq b_2 \sum_{j=k-1}^{k} z_j^4 + c_{2,k}e_k^4 + h_{2,k}(\ell_{k-1})e_{k-1}^4$, where b_2 and $c_{2,k}$ are positive constants, $h_{2,k}$ is a positive constant related to ℓ_{k-1}.

Proposition 5: There holds $\frac{4-r_n p_{n-1}}{r_n}([x_n]^{\frac{r_{n-1}}{r_n}} - (\hat{\xi}_n + \ell_{n-1}x_{n-1}))|x_n|^{\frac{4-r_n p_{n-1}-r_n}{r_n}} u^{p_n} \leq b_3 \sum_{j=1}^{n} z_j^4 + c_3 \sum_{k=2}^{n} e_k^4 + h_3(\ell_{n-1})e_{n-1}^4$, where b_3 and c_3 are positive constants, h_3 is a positive constant related to ℓ_{n-1}.

We estimate $m_n[z_n]^{4-r_{n+1}p_n}(u^{p_n} - u^{*p_n})$ in (17) by the following proposition whose proof is in the Appendix.

Proposition 6: $m_n[z_n]^{4-r_{n+1}p_n}(u^{p_n} - u^{*p_n}) \leq b_4 \sum_{k=1}^{n} z_k^4 + c_4 \sum_{k=2}^{n} e_k^4$.

From (17), (20), (21) and Propositions 2-6, it follows that

$$\begin{aligned}
&\mathcal{L}V|_{(12),(18),(19)} \\
&\leq -\sum_{j=1}^{n-2}\Big(a_{j,j} - \tilde{a}_{j+1,j} - \bar{a}_{j+1,j} - \sum_{l=j+2}^{n} \bar{a}_{l,j} - 2b_1 \\
&\quad - 2b_2 - b_3 - b_4\Big)z_j^4 - (a_{n-1,n-1} - \tilde{a}_{n,n-1} - \bar{a}_{n,n-1} \\
&\quad - 2b_1 - 2b_2 - b_3 - b_4)z_{n-1}^4 - (a_{n,n} - 2b_1 - 2b_2 - b_3 \\
&\quad - b_4)z_n^4 - \Big(2^{1-\frac{4-r_2p_1}{r_2p_1}}\ell_1 - c_{1,2} - h_{1,3}(\ell_2) - h_{2,3}(\ell_2) \\
&\quad - c_3 - c_4\Big)e_2^4 - \sum_{k=3}^{n-2}\Big(2^{1-\frac{4-r_kp_{k-1}}{r_kp_{k-1}}}\ell_{k-1} - c_{1,k} - c_{2,k} \\
&\quad - h_{1,k+1}(\ell_k) - h_{2,k+1}(\ell_k) - c_3 - c_4\Big)e_k^4 \\
&\quad - \Big(2^{1-\frac{4-r_{n-1}p_{n-2}}{r_{n-1}p_{n-2}}}\ell_{n-2} - c_{1,n-1} - c_{2,n-1} - h_{2,n}(\ell_{n-1}) \\
&\quad - c_3 - h_3(\ell_{n-1}) - c_4\Big)e_{n-1}^4 - \Big(2^{1-\frac{4-r_np_{n-1}}{r_np_{n-1}}}\ell_{n-1} \\
&\quad - c_{2,n} - c_3 - c_4\Big)e_n^4.
\end{aligned}$$

By designing constants $a_{1,1}, \cdots, a_{n,n}, \ell_1, \cdots, \ell_{n-1}$ as

$$\begin{aligned}
a_{j,j} &\geq 1 + \tilde{a}_{j+1,j} + \bar{a}_{j+1,j} + \sum_{l=j+2}^{n} \bar{a}_{l,j} + 2b_1 + 2b_2 \\
&\quad + b_3 + b_4, \quad j = 1, \cdots, n-1, \\
a_{n,n} &\geq 1 + 2b_1 + 2b_2 + b_3 + b_4, \\
\ell_{n-1} &\geq 2^{\frac{4-r_np_{n-1}}{r_np_{n-1}}-1}(1 + c_{2,n} + c_3 + c_4), \\
\ell_{n-2} &\geq 2^{\frac{4-r_{n-1}p_{n-2}}{r_{n-1}p_{n-2}}-1}(1 + c_{1,n-1} + c_{2,n-1} + c_4 \\
&\quad + c_3 + h_3(\ell_{n-1}) + h_{2,n}(\ell_{n-1})), \\
\ell_{k-1} &\geq 2^{\frac{4-r_kp_{k-1}}{r_kp_{k-1}}-1}(1 + c_{1,k} + c_{2,k} + h_{1,k+1}(\ell_k) \\
&\quad + h_{2,k+1}(\ell_k) + c_3 + c_4), \quad k = n-2, \cdots, 3, \\
\ell_1 &\geq 2^{\frac{4-r_2p_1}{r_2p_1}-1}(1+c_{1,2}+h_{1,3}(\ell_2)+h_{2,3}(\ell_2)+c_3+c_4),
\end{aligned} \tag{22}$$

one obtains

$$\mathcal{L}V|_{(12),(18),(19)} \leq -\sum_{k=1}^{n} z_k^4 - \sum_{k=2}^{n} e_k^4. \tag{23}$$

Denote $\mathcal{X} = (x_1, \cdots, x_n, \hat{\xi}_2, \cdots, \hat{\xi}_n)^\top$, the closed-loop system (12), (18), (19) can be rewritten as

$$\begin{aligned}
\mathrm{d}\mathcal{X} &= \Phi(\mathcal{X})\mathrm{d}t \\
&= (x_2^{p_1}, \cdots, u^{p_n}, -\ell_1\hat{x}_2^{p_1}, \cdots, -\ell_{n-1}\hat{x}_n^{p_{n-1}})^\top \mathrm{d}t.
\end{aligned} \tag{24}$$

Introducing the dilation weight[2] $\Delta = (r_1, \cdots, r_n, r_1, \cdots, r_{n-1})$ for $\mathcal{X}$, we have the following proposition whose proof is in the Appendix.

Proposition 7: There exist positive constants π_1, π_2 and π_3 such that $\pi_1\|\mathcal{X}\|_\Delta^{4-\tau} \leq V(\mathcal{X}) \leq \pi_2\|\mathcal{X}\|_\Delta^{4-\tau}$, $\mathcal{L}V|_{(19),(24)} \leq -\pi_3\|\mathcal{X}\|_\Delta^4$.

Part III. Output feedback controller of system (1)

Introduce the transformation

$$x_i = \Gamma^{-d_i}\eta_i, \quad u = \Gamma^{-d_{n+1}}v, \ i = 1, \cdots, n, \tag{25}$$

then system (1) can be converted into

$$\begin{aligned}
\mathrm{d}x_i &= (\Gamma x_{i+1}^{p_i} + \tilde{f}_i(x_{i+2}, \cdots, x_n, u))\mathrm{d}t \\
&\quad + \tilde{g}_i^\top(x_{i+2}, \cdots, x_n, u)\mathrm{d}\omega, \quad i = 1, \cdots, n-2, \\
\mathrm{d}x_{n-1} &= (\Gamma x_n^{p_{n-1}} + \tilde{f}_{n-1}(u))\mathrm{d}t + \tilde{g}_{n-1}^\top(u)\mathrm{d}\omega, \\
\mathrm{d}x_n &= \Gamma u^{p_n}\mathrm{d}t, \quad x(0) = x_0(\eta_0), \\
y &= x_1,
\end{aligned} \tag{26}$$

where $d_1 = 0$, $d_i = \frac{d_{i-1}+1}{p_{i-1}}$, $i = 2, \cdots, n$, $0 < \Gamma \leq 1$ is a constant to be determined, $\tilde{f}_i = \Gamma^{-d_i}f_i$ and $\tilde{g}_i = \Gamma^{-d_i}g_i$, $i = 1, \cdots, n-1$.

In view of (18), the observer of system (26) is constructed as:

$$\begin{aligned}
\mathrm{d}\hat{\xi}_k &= -\Gamma\ell_{k-1}\hat{x}_k^{p_{k-1}}\mathrm{d}t, \\
\hat{x}_k &= [\hat{\xi}_k + \ell_{k-1}\hat{x}_{k-1}]^{\frac{r_k}{r_{k-1}}}, \quad k = 2, \cdots, n,
\end{aligned} \tag{27}$$

where $\ell_1, \cdots, \ell_{n-1}$ are chosen in (22). Then, (19), (26) and (27) can be rewritten in the compact form:

$$\mathrm{d}\mathcal{X} = (\Gamma\Phi(\mathcal{X}) + F(\mathcal{X}))\mathrm{d}t + G^\top(\mathcal{X})\mathrm{d}\omega, \tag{28}$$

where $\Phi(\mathcal{X})$ is defined in (24), $F = (\tilde{f}_1, \cdots, 0, 0, \cdots, 0)^\top$, $G = (\tilde{g}_1, \cdots, 0, 0, \cdots, 0)$. Noting that Φ in (28) has the same structure as (24), by adopting the same Lyapunov function $V(\mathcal{X})$ as in the preceding subsection, one can conclude from Proposition 7 that

$$\mathcal{L}V|_{(19),(28)} \leq -\Gamma\pi_3\|\mathcal{X}\|_\Delta^4 + \frac{\partial V}{\partial \mathcal{X}}F + \frac{1}{2}\mathrm{Tr}\left\{G\frac{\partial^2 V}{\partial \mathcal{X}^2}G^\top\right\}. \tag{29}$$

We now estimate the last two terms on the right-hand side of (29) by the following proposition whose proof is in the Appendix.

[2]For a more precise definition of dilation and homogeneity, please refer to [28].

Proposition 8: There exists a positive constant γ_0 such that

$$\frac{\partial V(\mathcal{X})}{\partial \mathcal{X}}F(\mathcal{X})+\frac{1}{2}\mathrm{Tr}\left\{G(\mathcal{X})\frac{\partial^2 V(\mathcal{X})}{\partial \mathcal{X}^2}G^\top(\mathcal{X})\right\} \le \gamma_0\Gamma^{1+\nu}\|\mathcal{X}\|_\Delta^4. \quad (30)$$

Substituting Proposition 8 into (29), and choosing $\Gamma = \min\{(\frac{\pi_3}{2\gamma_0})^{\frac{1}{\nu}}, 1\}$, one gets

$$\mathcal{L}V(\mathcal{X})|_{(19),(28)} \le -\Gamma(\pi_3-\gamma_0\Gamma^\nu)\|\mathcal{X}\|_\Delta^4 \le -\frac{\pi_3\Gamma}{2}\|\mathcal{X}\|_\Delta^4. \quad (31)$$

Up to now, by (19), (25), (27), one obtains output feedback controller of system (1):

$$v=-\Gamma^{d_{n+1}}\beta_n^{r_{n+1}}\left[[\hat{x}_n]^{\frac{1}{r_n}}+\sum_{i=1}^{n-1}\left(\prod_{j=i}^{n-1}\beta_j\right)[\hat{x}_i]^{\frac{1}{r_i}}\right]^{r_{n+1}}, \quad (32)$$

where $\hat{x}_1=\eta_1$ and $\hat{x}_2,\cdots,\hat{x}_n$ are observed by (27).

IV. FINITE-TIME STABILITY

We state the main result in this paper.

Theorem 1: When Assumption 1 holds, the equilibrium $\eta=0$ of the closed-loop system (1) with (27), (32) is finite-time stable in probability.

Proof: With (31), Lemma 2, the closed-loop stochastic system (19) and (28) has a solution for any initial value $\mathcal{X}_0 \in R^n$. Since $\Gamma\Phi(\mathcal{X})+F(\mathcal{X})$ and $G^\top(\mathcal{X})$ are $\mathcal{C}^1$ on $R^n/\{0\}$, the closed-loop stochastic system obviously has a unique solution in forward time for any $\mathcal{X}_0 \in R^n/\{0\}$. When $\mathcal{X}_0=0$, the closed-loop stochastic system has a unique zero solution, that is, when $\mathcal{X}_0=0$, $P(\|\mathcal{X}(t;\mathcal{X}_0)\| \ge r)=0$ for all $r>0$ and $t \ge 0$, which can be verified by the following counter-evidence:

Suppose that there are $r_0>0$ and $t_0 \ge 0$ such that

$$P(\|\mathcal{X}(t_0;\mathcal{X}_0)\| \ge r_0) > \varepsilon_0, \quad (33)$$

where $1 \ge \varepsilon_0 \ge 0$. By Definition 2.1 and Theorem 2.2 in [32, Sec. 4.2], one concludes from (31) that the trivial solution of stochastic system (19) and (28) is stable in probability, that is, for every pair of $\varepsilon \in (0,1)$ and $r>0$, there exists a $\delta=\delta(\varepsilon,r)>0$ such that $P(\|\mathcal{X}(t;\mathcal{X}_0)\| < r$ for all $t \ge 0) \ge 1-\varepsilon$ whenever $\|\mathcal{X}_0\| \le \delta$. This conclusion implies that for above r_0 and ε_0, $P(\|\mathcal{X}(t;\mathcal{X}_0)\| < r_0$ for all $t \ge 0) \ge 1-\varepsilon_0$. This contradicts (33).

According to the lemmas of weighted homogeneity in [28], there is a constant $\pi_0>0$ such that

$$\mathcal{L}V(\mathcal{X})|_{(19),(28)} \le -\frac{\pi_3}{2}\Gamma\|\mathcal{X}\|_\Delta^4 \le -\pi_0 V(\mathcal{X})^{\frac{4}{4-\tau}}. \quad (34)$$

Since the closed-loop system (19), (28) has a unique solution, then it follows from $0<\frac{4}{4-\tau}<1$, Lemma 3 and Proposition 7 that the origin of the closed-loop stochastic system is finite-time stable in probability. Since (25) is an equivalent transformation, one concludes that origin of closed-loop stochastic system (1), (27), (32) is finite-time stable in probability. □

V. A SIMULATION EXAMPLE

Without loss of generality, we consider a simple example:

$$\begin{aligned} d\eta_1 &= \left(\eta_2^{\frac{5}{3}}+0.2\ln\left(1+|v|^{\frac{25}{13}}\right)\right)dt+0.05v^2 d\omega,\\ d\eta_2 &= v dt,\\ y &= \eta_1. \end{aligned} \quad (35)$$

By choosing $\tau=-\frac{1}{27}\in(-\frac{1}{2(1+p_1)},0)=(-\frac{3}{16},0)$ and (10), one has $r_1=\frac{1}{2}$, $r_2=\frac{5}{18}$, $r_3=\frac{13}{54}$. Assumption 1 holds with $f_1=0.2\ln(1+|v|^{\frac{25}{13}}) \le |v|^{\frac{25}{13}}=|v|^{\frac{r_1+\tau}{r_3}}$, $g_1=0.05v^2 \le 0.05|v|^{\frac{2r_1+\tau}{2r_3}}$.

Following the design procedure as in Section III, one constructs the finite-time output feedback controller:

$$\begin{aligned} v &= -\Gamma^{\frac{8}{5}}\beta_2^{\frac{13}{54}}\left[[\hat{x}_2]^{\frac{18}{5}}+\beta_1[\eta_1]^2\right]^{\frac{13}{54}},\\ d\hat{\xi}_2 &= -\Gamma\ell_1\hat{x}_2^{\frac{5}{3}}dt, \quad \hat{x}_2=[\hat{\xi}_2+\ell_1\eta_1]^{\frac{5}{9}}, \end{aligned} \quad (36)$$

where $\beta_1>0$, $\beta_2>0$, $\ell_1>0$, $0<\Gamma\le 1$ are design parameters.

Choose $(\eta_1(0),\eta_2(0),\hat{\xi}_2(0))^\top=(0.04,-0.002,0.02)^\top$, $\Gamma=0.09$, $\ell_1=500$, $\beta_1=1$, $\beta_2=20$. Fig.1 shows the effectiveness of control method.

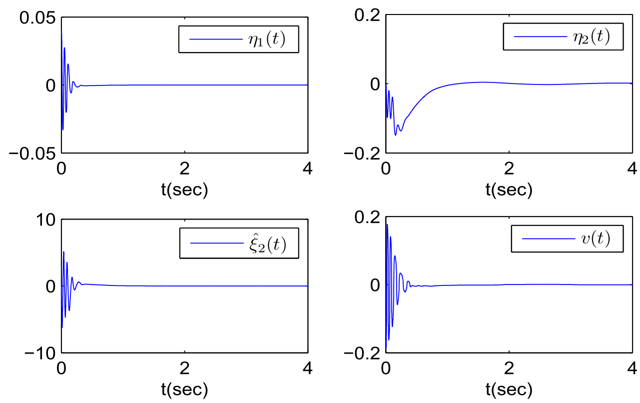

FIGURE 1. The responses of closed-loop system (35)-(36).

VI. A CONCLUDING REMARK

The finite-time output feedback stabilization in probability of stochastic high-order nonlinear feedforward autonomous systems is considered. In this paper, we initially list some lemmas which will deal with the nonlinearities with powers being any rational number and bigger than one, then based on the homogeneous domination and stochastic Lyapunov finite-time stability theory, the stochastic finite-time output feedback stabilizer is designed and analysed. A remaining problem is whether the finite-time stabilization can be established for stochastic nonlinear feedforward nonautonomous systems.

APPENDIX

Proof of Proposition 1: Firstly, $V_k(\bar{x}_k)$ is positive definite and radially unbounded, one can see the detailed proof in [27].

In view of $-\frac{1}{\sum_{l=1}^{n} p_1\cdots p_{l-1}} < \tau < 0$, one deduces that $\frac{1}{r_k} \geq 2$, $4-r_{k+1}p_k \geq 2$, $k=1,\cdots,n$. This implies that V_k is $\mathcal{C}^2$. Define $z_k = [x_k]^{\frac{1}{r_k}} - [\alpha_{k-1}]^{\frac{1}{r_k}}$, it follows from (13) that

$$\begin{aligned}&\mathcal{L}V_{k-1}(\bar{x}_{k-1})|_{(12)}\\ &\leq -\sum_{j=1}^{k-3}\left(a_{j,j}-\tilde{a}_{j+1,j}-\bar{a}_{j+1,j}-\sum_{l=j+2}^{k-1}\bar{a}_{l,j}\right)z_j^4\\ &\quad-(a_{k-2,k-2}-\tilde{a}_{k-1,k-2}-\bar{a}_{k-1,k-2})z_{k-2}^4-a_{k-1,k-1}z_{k-1}^4\\ &\quad+m_k[z_k]^{4-r_{k+1}p_k}(x_{k+1}^{p_k}-\alpha_k^{p_k})+m_k[z_k]^{4-r_{k+1}p_k}\alpha_k^{p_k}\\ &\quad+m_{k-1}[z_{k-1}]^{4-r_kp_{k-1}}(x_k^{p_{k-1}}-\alpha_{k-1}^{p_{k-1}})+m_k\sum_{j=1}^{k-1}\frac{\partial W_k}{\partial x_j}x_{j+1}^{p_j}.\end{aligned} \tag{37}$$

Lemmas 5,6 yield

$$\begin{aligned}m_{k-1}[z_{k-1}]^{4-r_kp_{k-1}}&(x_k^{p_{k-1}}-\alpha_{k-1}^{p_{k-1}})\\ &\leq \tilde{a}_{k,k-1}z_{k-1}^4+\lambda_{k,1}z_k^4,\end{aligned} \tag{38}$$

where $\tilde{a}_{k,k-1}$ is a positive constant, $\lambda_{k,1}$ is a positive constant dependent on $\tilde{a}_{k,k-1}$.

With the help of Lemmas 5,6,8 and the integral mean value theorem, one can deduce that

$$\begin{aligned}&m_k\sum_{j=1}^{k-1}\frac{\partial W_k}{\partial x_j}\cdot x_{j+1}^{p_j}\\ &\leq c\sum_{j=1}^{k-1}|z_k|^{3-\tau}\left(|z_j|^{1-r_j}+|z_{j-1}|^{1-r_j}\right)\left(|z_{j+1}|^{r_{j+1}p_j}+|z_j|^{r_{j+1}p_j}\right)\\ &\leq \sum_{j=1}^{k-1}\bar{a}_{k,j}z_j^4+\lambda_{k,2}z_k^4,\end{aligned} \tag{39}$$

where $\bar{a}_{k,1},\cdots,\bar{a}_{k,k-1}$ are positive constants, c, $\lambda_{k,2}$ are appropriate positive constants, $\lambda_{k,2}$ is dependent on $\bar{a}_{k,1},\cdots,\bar{a}_{k,k-1}$.

Defining $\lambda_k = \lambda_{k,1}+\lambda_{k,2}$, choosing $\beta_k = (\frac{a_{k,k}+\lambda_k}{m_k})^{\frac{1}{r_{k+1}p_k}}$ with $a_{k,k} > 0$ being a positive constant to be designed, and substituting (38), (39) into (37), one gets (15). □

Proof of Proposition 3: By (16), (18), Lemmas 4,5,7, one deduces that

$$\begin{aligned}&\frac{4-r_kp_{k-1}}{r_k}([x_k]^{\frac{r_{k-1}}{r_k}}-(\hat{\xi}_k+\ell_{k-1}x_{k-1}))x_{k+1}^{p_k}|x_k|^{\frac{4-r_kp_{k-1}-r_k}{r_k}}\\ &\leq \frac{4-r_kp_{k-1}}{r_k}(|[x_k]^{\frac{r_{k-1}}{r_k}}-[\hat{x}_k]^{\frac{r_{k-1}}{r_k}}|+\ell_{k-1}|x_{k-1}-\hat{x}_{k-1}|)\\ &\quad\cdot|x_k|^{\frac{4-r_kp_{k-1}-r_k}{r_k}}|x_{k+1}|^{p_k}\\ &\leq (c|e_k|^{d_{1,k}r_k}(|e_k|^{r_{k-1}-d_{1,k}r_k}+|z_k|^{r_{k-1}-d_{1,k}r_k}\\ &\quad+|z_{k-1}|^{r_{k-1}-d_{1,k}r_k})+\bar{h}_{1,k}(\ell_{k-1})|e_{k-1}|^{r_{k-1}})\\ &\quad\cdot(|z_k|^{4-r_kp_{k-1}-r_k}+|z_{k-1}|^{4-r_kp_{k-1}-r_k})\\ &\quad\cdot(|z_{k+1}|^{r_{k+1}p_k}+|z_k|^{r_{k+1}p_k})\\ &\leq b_1\sum_{j=k-1}^{k+1}z_j^4+c_{1,k}e_k^4+h_{1,k}(\ell_{k-1})e_{k-1}^4,\end{aligned} \tag{40}$$

where $\bar{c}_{1,k} = c_0(\frac{r_{k-1}}{r_k})$, $d_{1,k} = d(\frac{r_{k-1}}{r_k})$, $\bar{h}_{1,k}$ is a positive constant related to ℓ_{k-1}, and throughout this paper we use c to indicate a general constant. □

Proof of Proposition 4: From (16), (18), Lemmas 4,5,7, it follows that

$$\begin{aligned}&-\ell_{k-1}[e_k]^{r_kp_{k-1}}([\hat{x}_k]^{\frac{4-r_kp_{k-1}}{r_k}}-[\hat{\xi}_k+\ell_{k-1}x_{k-1}]^{\frac{4-r_kp_{k-1}}{r_{k-1}}})\\ &\leq \ell_{k-1}^{1+d_{2,k}}\bar{c}_{2,k}|e_k|^{r_kp_{k-1}}|x_{k-1}-\hat{x}_{k-1}|^{d_{2,k}}\\ &\quad\cdot(|\hat{x}_k|^{\frac{4-r_kp_{k-1}-d_{2,k}r_{k-1}}{r_k}}+|\ell_{k-1}(x_k^{p_{k-1}}-\hat{x}_k^{p_{k-1}})|^{\frac{4-r_kp_{k-1}}{r_{k-1}}-d_{2,k}})\\ &\leq b_2\sum_{j=k-1}^{k}z_j^4+c_{2,k}e_k^4+h_{2,k}(\ell_{k-1})e_{k-1}^4,\end{aligned} \tag{41}$$

where $\bar{c}_{2,k} = c_0(\frac{4-r_kp_{k-1}}{r_{k-1}})$, $d_{2,k} = d(\frac{4-r_kp_{k-1}}{r_{k-1}})$, $\bar{h}_{2,k}$ is a positive constant related to ℓ_{k-1}. □

Proof of Proposition 5: By (16), (18), (19), Lemmas 4,5,7, one deduces that

$$\begin{aligned}&\frac{4-r_np_{n-1}}{r_n}([x_n]^{\frac{r_{n-1}}{r_n}}-(\hat{\xi}_n+\ell_{n-1}x_{n-1}))|x_n|^{\frac{4-r_np_{n-1}-r_n}{r_n}}u^{p_n}\\ &\leq \bar{c}_3|e_n|^{d_{1,n}r_n}(|z_n|^{4-r_np_{n-1}-r_n}+|z_{n-1}|^{4-r_np_{n-1}-r_n})\\ &\quad\cdot(|e_n|^{r_{n-1}-d_{1,n}r_n}+|z_n|^{r_{n-1}-d_{1,n}r_n}+|z_{n-1}|^{r_{n-1}-d_{1,n}r_n})\\ &\quad\cdot\Big(\sum_{j=2}^{n}|e_j|^{r_{n+1}p_n}+\sum_{j=1}^{n}|z_j|^{r_{n+1}p_n}\Big)+\bar{h}_3(\ell_{n-1})|e_{n-1}|^{r_{n-1}}\\ &\quad\cdot(|z_n|^{4-r_np_{n-1}-r_n}+|z_{n-1}|^{4-r_np_{n-1}-r_n})\\ &\quad\cdot\Big(\sum_{j=2}^{n}|e_j|^{r_{n+1}p_n}+\sum_{j=1}^{n}|z_j|^{r_{n+1}p_n}\Big)\\ &\leq b_3\sum_{j=1}^{n}z_j^4+c_3\sum_{k=2}^{n}e_k^4+h_3(\ell_{n-1})e_{n-1}^4,\end{aligned}$$

where $\bar{c}_3$ is a positive constant, $\bar{h}_3$ is a positive constant related to ℓ_{n-1}. □

Proof of Proposition 6: With (16), (19) and Lemmas 4,5,7, one can deduce that

$$\begin{aligned}&m_n[z_n]^{4-r_{n+1}p_n}(u^{p_n}-u^{*p_n})\\ &\leq \tilde{c}_4|z_n|^{4-r_{n+1}p_n}\sum_{k=2}^{n}|x_k^{p_{k-1}}-\hat{x}_k^{p_{k-1}}|^{d_{3,k}}\\ &\quad\cdot(|x_k^{p_{k-1}}-\hat{x}_k^{p_{k-1}}|^{\frac{r_{n+1}p_n}{r_kp_{k-1}}-d_{3,k}}+|x_k|^{\frac{r_{n+1}p_n}{r_k}-d_{3,k}p_{k-1}})\\ &\leq \tilde{c}_4|z_n|^{4-r_{n+1}p_n}\sum_{k=2}^{n}|e_k|^{d_{3,k}r_kp_{k-1}}(|e_k|^{r_{n+1}p_n-d_{3,k}r_kp_{k-1}}\\ &\quad+|z_k|^{r_{n+1}p_n-d_{3,k}r_kp_{k-1}}+|z_{k-1}|^{r_{n+1}p_n-d_{3,k}r_kp_{k-1}})\\ &\leq b_4\sum_{k=1}^{n}z_k^4+c_4\sum_{k=2}^{n}e_k^4,\end{aligned}$$

where $\tilde{c}_4$, c_4 are positive constants, $d_{3,j} = d(\frac{r_{n+1}p_n}{r_jp_{j-1}})$. □

Proof of Proposition 7: Obviously V is homogeneous of degree $4-\tau$ and the term $-\sum_{k=1}^{n}z_k^4-\sum_{k=2}^{n}e_k^4 = -W$ on right-hand side of (23) is is homogeneous of degree 4, which is verified by

$$
\begin{aligned}
&V(\Delta_\varepsilon(\mathcal{X}))\\
&= \sum_{k=1}^{n}\int_{\varepsilon^{r_k}\alpha_{k-1}}^{\varepsilon^{r_k}x_k}[[s]^{\frac{1}{r_k}}-\varepsilon[\alpha_{k-1}]^{\frac{1}{r_k}}]^{4-r_{k+1}p_k}\mathrm{d}s\\
&\quad+\sum_{k=2}^{n}\int_{[\varepsilon^{r_{k-1}}\hat{\xi}_k+\ell_{k-1}\varepsilon^{r_{k-1}}x_{k-1}]^{\frac{4-r_kp_{k-1}}{r_{k-1}}}}^{[\varepsilon^{r_k}x_k]^{\frac{4-r_kp_{k-1}}{r_k}}}([\iota]^{\frac{r_{k-1}}{4-r_kp_{k-1}}}\\
&\quad-\varepsilon^{r_{k-1}}(\hat{\xi}_k+\ell_{k-1}x_{k-1}))\mathrm{d}\iota\\
&= \varepsilon^{4-\tau}V(\mathcal{X}),\\
&\quad-W(\Delta_\varepsilon(\mathcal{X}))\\
&= -\varepsilon^4\sum_{k=1}^{n}(\varepsilon[x_k]^{\frac{1}{r_k}}+\sum_{i=1}^{k-1}(\prod_{j=i}^{k-1}\beta_j)\varepsilon[x_i]^{\frac{1}{r_i}})^4\\
&\quad-\varepsilon^4\sum_{k=2}^{n}(x_k^{p_{k-1}}-[\hat{\xi}_k+\ell_{k-1}[\hat{\xi}_{k-1}+\ell_{k-2}[\cdots\\
&\quad+\ell_2[\hat{\xi}_2+\ell_1x_1]^{\frac{r_2}{r_1}}\cdots]^{\frac{r_{k-2}}{r_{k-3}}}]^{\frac{r_{k-1}}{r_{k-2}}}]^{\frac{r_kp_{k-1}}{r_{k-1}}})^{\frac{4}{r_kp_{k-1}}}\\
&= -\varepsilon^4W(\mathcal{X}).
\end{aligned}
$$

Noting that $V(\mathcal{X})$ and $W(\mathcal{X})$ are both positive definite and radially unbounded, by the lemmas of weighted homogeneity in [28], one can find positive constants π_1, π_2, and π_3 such that Proposition 7 holds. □

Proof of Proposition 8: By (16), (19), (25), Lemmas 4,9 and Assumption 1, $|\tilde{f}_i| \le \gamma_1\Gamma^{1+\nu}\|\mathcal{X}\|_\Delta^{r_i+\tau}$, $\|\tilde{g}_i\| \le \gamma_2\Gamma^{1+\nu}\|\mathcal{X}\|_\Delta^{\frac{2r_i+\tau}{2}}$, where $0<\nu=\min\limits_{\substack{i=1,\cdots,n-1,\\ j=i+2,\cdots,n+1}}\{\frac{d_j(r_i+\tau)}{r_j}-d_i-1\}$, γ_1, γ_2 are positive constants. Then

$$
\begin{aligned}
&\frac{\partial V(\mathcal{X})}{\partial\mathcal{X}}F(\mathcal{X})+\frac{1}{2}\mathrm{Tr}\left\{\partial G(\mathcal{X})\frac{\partial^2V(\mathcal{X})}{\partial\mathcal{X}^2}G^\top(\mathcal{X})\right\}\\
&=\sum_{j=1}^{n-1}\frac{\partial V}{\partial x_j}\tilde{f}_j+\frac{1}{2}\sum_{i,j=1}^{n-1}\mathrm{Tr}\{\tilde{g}_i\frac{\partial^2V}{\partial x_i\partial x_j}\tilde{g}_j^\top\}\\
&\le\sum_{j=1}^{n-1}\left|\frac{\partial V}{\partial x_j}\right|\cdot|\tilde{f}_j|+\bar{\gamma}_0\sum_{i,j=1}^{n-1}\left\|\frac{\partial^2V}{\partial x_i\partial x_j}\right\|\cdot\|\tilde{g}_i\|\cdot\|\tilde{g}_j^\top\|\\
&\le\bar{\gamma}_{01}\Gamma^{1+\nu}\sum_{j=1}^{n-1}\|\mathcal{X}\|_\Delta^{4-\tau-r_j}\cdot\|\mathcal{X}\|_\Delta^{r_j+\tau}\\
&\quad+\bar{\gamma}_{02}\Gamma^{1+\nu}\sum_{i,j=1}^{n-1}\|\mathcal{X}\|_\Delta^{4-\tau-r_i-r_j}\cdot\|\mathcal{X}\|_\Delta^{\frac{2r_i+\tau}{2}}\cdot\|\mathcal{X}\|_\Delta^{\frac{2r_j+\tau}{2}}\\
&\le\gamma_0\Gamma^{1+\nu}\|\mathcal{X}\|_\Delta^4,
\end{aligned}\tag{42}
$$

where $\bar{\gamma}_0$, $\bar{\gamma}_{01}$, $\bar{\gamma}_{02}$ and γ_0 are appropriate constants. □

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XINGHUI ZHANG received the M.S. and Ph.D. degrees in control science and engineering from Qufu Normal University, in 2013 and 2016, respectively. Since 2016, she has been with Linyi University, where she is currently a Professor with the School of Automation and Electrical Engineering. Her current research interests include stochastic nonlinear control and adaptive optimal control of complex systems.

ANCAI ZHANG received the B.S. degree in mathematics from Linyi University, Linyi, China, in 2005, and the M.S. degree in applied mathematics and the Ph.D. degree in control science and engineering from Central South University, Hunan, China, in 2008 and 2012, respectively. Since 2012, he has been with Linyi University, where he is currently a Professor with the School of Automation and Electrical Engineering. He carried out his study with the Tokyo University of Technology, Tokyo, Japan, with the support of the scholarship of Chinese Government from 2009 to 2011, and the Post-Doctoral Fellowship of Japan Society for the Promotion of Science from 2017 to 2018. His current research interests include underactuated mechanical systems, nonlinear systems, and robotics.

GUOCHEN PANG was born in Linyi, Shandong, China, in 1987. He received the M.S. degree in operational research and cybernetics from Qufu Normal University, Qufu, Shangdong, in 2012, and the Ph.D. degree in control theory and control engineering from Southeast University, China, in 2016. Since 2016, he has been a Lecturer with the School of Automation and Electrical Engineering, Linyi University. His research interests include fault detection, robust control, actuator saturation, and anti-disturbance control and its applications.