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Anti-Synchronization and Synchronization of Coupled Chaotic System With Ring Connection and Stochastic Perturbations

CHAOLONG ZHANG[1], **FEIQI DENG**[2], **WENFENG ZHANG**[3], **TING HOU**[4], **AND ZHIWEI YANG**[1]

[1]College of Computational Science, Zhongkai University of Agriculture and Engineering, Guangzhou 510225, China
[2]Systems Engineering Institute, South China University of Technology, Guangzhou 510640, China
[3]Institute of Rural Development, Zhongkai University of Agriculture and Engineering, Guangzhou 510225, China
[4]College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590, China

Corresponding authors: Feiqi Deng (aufqdeng@scut.edu.cn) and Wenfeng Zhang (zhwf999@163.com)

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ABSTRACT This paper focuses on anti-synchronization and synchronization of a stochastic multi-coupled chaotic system with ring connection and control schemes. First, the system is simplified by means of the formula deformation technique and the controller. Based on the Lyapunov method and stochastic differential theory, some sufficient conditions are obtained by some control methods. At last, the Chen system, Lorenz system, and Lü system with stochastic perturbations are used to verify the correctness of the conclusions.

INDEX TERMS Anti-synchronization, synchronization, stochastic, coupled chaotic system, ring connection.

I. INTRODUCTION

Chaotic systems exist widely in nature and human society,such as Lorenz system, Chen system, Lü system and Hyperchaotic Chua Systems and so on.It is an interesting and challenging issue to make more chaotic systems achieve synchronization and anti-synchronization.

Up to now, many methods for the synchronization and anti-synchronization of chaotic systems have been investigated mainly including the adaptive control [1]–[6], nonlinear control [1], [7], sliding mode control [8], active control [9], [10], impulsive control [11], intermittent adjustment feedback control [12], [13], nonlinear parametric variation [14], nonlinear delay control [15]–[17], etc.

In [18], the two systems can be obtained synchronization and anti-synchronization by the coupling method. In [19], [20], the authors investigate the complete synchronization and anti-synchronization behavior in an array of coupled chaotic systems with ring connection.

Example 1: The systems: $\begin{pmatrix} x'_{i1} \\ x'_{i2} \end{pmatrix} = A_i \begin{pmatrix} x_{i1} \\ x_{i2} \end{pmatrix}, \quad i = 1, 2.$ Let $A_i = \begin{pmatrix} -2 & 2 \\ -2 & 1 \end{pmatrix}$, the eigenvalues of A_i are $-\frac{1}{2} \pm \frac{\sqrt{7}}{2} i$. We know the solution of the vertex systems is globally asymptotically stable.

Consider the following linearly coupled system

$$\begin{pmatrix} x'_{11} \\ x'_{12} \end{pmatrix} = A_1 \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} + \begin{pmatrix} 2x_{21} - x_{11} \\ 0 \end{pmatrix},$$
$$\begin{pmatrix} x'_{21} \\ x'_{22} \end{pmatrix} = A_2 \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix} + \begin{pmatrix} 2x_{11} - x_{21} \\ 0 \end{pmatrix},$$

whose coefficient matrix

$$A = \begin{pmatrix} -3 & 2 & 2 & 0 \\ -2 & 1 & 0 & 0 \\ 2 & 0 & -3 & 2 \\ 0 & 0 & -2 & 1 \end{pmatrix}.$$

has a positive eigenvalue 0.2361, and thus the zero solution of the coupled system is unstable.

However, a real system is usually affected by external perturbations which in many cases are of great uncertainty and hence may be treated as random. Noise is unavoidable and should be taken into consideration in modeling. Noise disturbance is a major source of instability and may lead to poor performances in networks.

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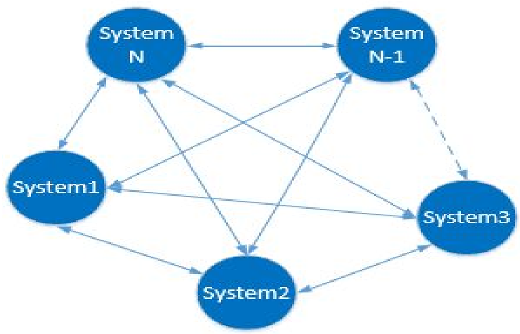

FIGURE 1. **Chaotic coupled systems with ring connection for Eq. (1).**

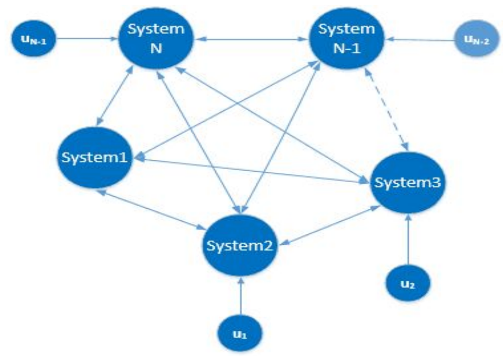

FIGURE 2. **Chaotic coupled systems with ring connection and the controllers for Eq. (2).**

Example 2: For the system

$$\dot{x}(t) = ax(t) \quad (a > 0),$$

it is easy to know the equation is a unstable system. Let us look at the scalar linear *Itô* equation

$$dx(t) = ax(t)dt + \sum_{i=1}^{m} b_i x(t) dB_i(t).$$

If $a - \frac{1}{2}\sum_{i=1}^{m} b_i^2 < 0$, the stochastically perturbed system can be stable in Ref. [21].

From the above two examples and some backgrounds, we consider the chaotic coupled models with ring connection and stochastic perturbations in the following:

$$\dot{x}(t) = ax(t) \quad (a > 0),$$

$$\begin{cases} dx_1 = \big(f_1(x_1) + \sum_{h=1}^{N} B_{1h}(x_h - x_1)\big)dt + Cx_1 d\omega(t), \\ dx_2 = \big(f_2(x_2) + \sum_{h=1}^{N} B_{2h}(x_h - x_2)\big)dt + Cx_2 d\omega(t), \\ \cdots\cdots \\ dx_N = \big(f_N(x_N) + \sum_{h=1}^{N} B_{Nh}(x_h - x_N)\big)dt + Cx_N d\omega(t), \end{cases} \tag{1}$$

where x_i is the state vectors,and $x_i = (x_{i1}, x_{i2}, \cdots, x_{in})^T$; $B_{ih} = diag(b_{ih_1}, b_{ih_2}, \cdots, b_{ih_n})$ $(i = 1, 2, \cdots, N, h = 1, 2, \cdots, N)$ is n dimensional coupled diagonal matrix, and $b_{ih_k} \geq 0$; $C = diag(c_1, c_2, \cdots, c_n)$ is $R^n \times R^n$ constant matrix; $\omega(t)$ is n dimension Brownian motion.

This paper is organized as follows. In section 2, we introduce some necessary notations which will be used later. In section 3, some sufficient conditions are obtained for anti-synchronization and synchronization of chaotic coupled system with stochastic perturbations. In Section 4, an example and it's simulations are given to show the effectiveness of the obtained results.

II. PRELIMINARIES

Assumed that $A_i \neq A_j, \psi_i(\cdot) \neq \psi_j(\cdot), (i \neq j), \psi_i(x_i) = f_i(x_i) - A_i x_i, i, j = 1, 2, \cdots, N$. A_i is $R^n \times R^n$ constant matrix. Substitute the controllers $u_1, u_2, \cdots, u_{N-1}$ into the Equ.(1), then the Equ.(1) can be described as follows:

$$\begin{cases} dx_1 = \big\{A_1 x_1 + \psi_1(x_1) + \sum_{h=1}^{N} B_{1h}(x_h - x_1)\big\}dt + Cx_1 d\omega(t), \\ dx_2 = \big\{A_2 x_2 + \psi_2(x_2) + \sum_{h=1}^{N} B_{2h}(x_h - x_2) + u_1\big\}dt \\ \qquad + Cx_2 d\omega(t), \\ \cdots\cdots \\ dx_N = \big\{A_N x_N + \psi_N(x_N) + \sum_{h=1}^{N} B_{Nh}(x_h - x_N) + u_{N-1}\big\}dt \\ \qquad + Cx_N d\omega(t), \end{cases} \tag{2}$$

Let the error be

$$e_i(t) = x_{i+1}(t) + x_i(t),$$

$$F_1 = \begin{pmatrix} M_1 & M_2 & M_3 & M_4 & \cdots & M_{N-2} & B_{1N} \\ B_{21} & A_3 - B_{32} & 0 & 0 & \cdots & 0 & 0 \\ 0 & B_{32} & A_4 - B_{43} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & B_{(N-1)(N-2)} & M_N \end{pmatrix}$$

and F_1, as shown at the bottom of the previous page, where $M_1 = A_2 - (-1)^{N-1}B_{1N} - B_{21}, M_j = -(-1)^{N-j}B_{1N}, (j = 2, 3, \cdots, N-1), M_N = A_N - B_{N(N-1)}$, F_2, as shown at the top of the next page, and

$$H = \begin{pmatrix} C & 0 & 0 & \cdots & 0 & 0 \\ 0 & C & 0 & \cdots & 0 & 0 \\ 0 & 0 & C & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & C \end{pmatrix},$$

Then the error dynamic system can be described in the following:

$$de = (F_1 e + F_2)dt + Hed\omega(t) \tag{3}$$

where $e = (e_1, e_2, \cdots, e_{N-1})^T$.

Let

$$\begin{cases} u_1 = v_1 - \Big[\Big((-1-(-1)^N)B_{1N} + 2B_{21} - (A_2 - A_1)\Big)x_1 \\ \qquad + \psi_2(x_2) + \psi_1(x_1) + \sum_{h=1}^{N-1} B_{1h}(x_h - x_1) \\ \qquad + \sum_{h=2}^{N} B_{2h}(x_h - x_2)\Big], \\ u_2 = v_2 - \Big[\Big(-2(B_{21} - B_{32}) + (A_2 - A_3)\Big)x_2 \\ \qquad + \psi_3(x_3) + \psi_2(x_2) + u_1 + \sum_{h=2, h\neq 1}^{N} B_{2h}(x_h - x_2) \\ \qquad + \sum_{h=1, h\neq 2}^{N} B_{3h}(x_h - x_3)\Big], \\ \qquad \cdots\cdots \\ u_{N-1} = v_{N-1} - \Big[\Big(-2(B_{(N-1)(N-2)} - B_{N(N-1)}) \\ \qquad + (A_{N-1} - A_N)\Big)x_{N-1} + \psi_N(x_N) + \psi_{N-1}(x_{N-1}) \\ \qquad + u_{N-2} + \sum_{h=1, h\neq N-2}^{N} B_{(N-1)h}(x_h - x_{N-1}) \\ \qquad + \sum_{h=1, h\neq N-1}^{N} B_{Nh}(x_h - x_{N-1})\Big]. \end{cases}$$

So, the error system (3) can be rewritten as follows:

$$de = (F_1 e + v)dt + Hed\omega(t), \tag{4}$$

where $v = (v_1, v_2, \cdots, v_{N-1})^T$.

Definition 1: The chaotic coupled system (4) is anti-synchronization under the controller $u_i(t)(i = 1, 2, 3, \cdots, N-1)$, if the trivial solution of the error system (4) is asymptotically stable,i.e.

$$\lim_{t\to\infty} \mathbb{E}\|e_i(t)\| = \lim_{t\to\infty} \mathbb{E}\|x_i(t) + x_{i+1}(t)\| = 0, \quad i = 1, 2, \cdots, N-1.$$

Definition 2: The chaotic coupled system (4) is complete synchronization under the controller $u_i(t)$, $(i = 1, 2, 3, \cdots, N-1)$, if the trivial solution of the error system $e_l(t)$ and $e_k(t)$ is asymptotically stable, i.e., if N is odd,

$$\lim_{t\to\infty} \|e_l(t)\| = \lim_{t\to\infty} \|x_{l+2}(t) - x_l(t)\| = 0, \quad (l = 1, 3, 5, \cdots, N-2);$$
$$\lim_{t\to\infty} \|e_k(t)\| = \lim_{t\to\infty} \|x_{k+2}(t) - x_k(t)\| = 0; \quad (k = 2, 4, 6, \cdots, N-3).$$

And, if the trivial solution of the error system $\tilde{e}_l(t)$ and $\tilde{e}_k(t)$ is asymptotically stable, i.e.,if N is even,

$$\lim_{t\to\infty} \|\tilde{e}_l(t)\| = \lim_{t\to\infty} \|x_{l+2}(t) - x_l(t)\| = 0, \quad (l = 1, 3, 5, \cdots, N-3);$$
$$\lim_{t\to\infty} \|\tilde{e}_k(t)\| = \lim_{t\to\infty} \|x_{k+2}(t) - x_k(t)\| = 0, \quad (k = 2, 4, 6, \cdots, N-2).$$

For the stochastic system:

$$dx(t) = f(x, t)dt + g(x, t)dB(t).$$

Definition 3: For each $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}^+; \mathbb{R}^+)$, we define an operator $\mathcal{L}V$ from $\mathbb{R}^n \times \mathbb{R}^+$ to $\mathbb{R}$ by $\mathcal{L}V(x, t) = V_t(x, t) + V_x(x, t)f(x, t) + \frac{1}{2}\text{trace}[g^T(x, t)V_{xx}(x, t)g(x, t)]$, where $V_x(x, t) = (\frac{\partial V(x,t)}{\partial x_1}, \frac{\partial V(x,t)}{\partial x_2}, \cdots, \frac{\partial V(x,t)}{\partial x_n}), V_{xx}(x, t) = (\frac{\partial^2 V(x,t))}{\partial x_i \partial x_j})_{n\times n}$.

III. CONCLUSION

Theorem 1: Assumed that the following condition holds,

$$e^T v \le -e^T F_1 e - \frac{1}{2}\text{trace}\big[e^T H^T He\big],$$

then (4) is asymptotically stable,i.e.,the chaotic systems (2) is anti-synchronization under the control schemes.

Proof: Consider the Lyapunov function as follows:

$$V = \frac{1}{2}e^T e,$$

then $\mathcal{L}V$ can be computed by trajectory of (4),

$$\mathcal{L}V = e^T(F_1 e + v) + \frac{1}{2}\text{trace}\big[e^T H^T He\big],$$

by the condition of Theorem 1, we know $\mathcal{L}V < 0$.

Therefore, (4) is asymptotically stable,i.e.,the chaotic systems (2) is anti-synchronization under the control schemes.

Let

$$v = Ke, \quad \frac{1}{2}\text{trace}\big[e^T H^T He\big] \le \lambda e^T e, \quad \lambda > 0,$$

then the following conclusions are established.

Theorem 2: Assumed that the following condition holds, and

$$F_1 + K + \lambda I \le 0,$$

then (4) is asymptotically stable,i.e.,the coupled chaotic systems (2)is anti-synchronization under the control schemes.

Proof: Consider the Lyapunov function as follows:

$$V = \frac{1}{2}e^T e,$$

$$F_2 = \begin{pmatrix} \Big[\Big((-1-(-1)^N)B_{1N} + 2B_{21} - (A_2 - A_1)\Big)x_1 \\ \quad +\psi_2(x_2) + \psi_1(x_1) + u_1 \\ \quad + \sum_{h=1}^{N-1} B_{1h}(x_h - x_1) + \sum_{h=2}^{N} B_{2h}(x_h - x_2)\Big], \\ \Big[\big(-2(B_{21} - B_{32}) + (A_2 - A_3)\big)x_2 \\ \quad +\psi_3(x_3) + \psi_2(x_2) + u_2 + u_1 \\ \quad + \sum_{h=2,h\neq 1}^{N} B_{2h}(x_h - x_2) + \sum_{h=1,h\neq 2}^{N} B_{3h}(x_h - x_3)\Big], \\ \Big[\big(-2(B_{32} - B_{43}) + (A_3 - A_4)\big)x_3 \\ \quad +\psi_4(x_4) + \psi_3(x_3) + u_3 + u_2 \\ \quad + \sum_{h=1,h\neq 2}^{N} B_{3h}(x_h - x_3) + \sum_{h=1,h\neq 3}^{N} B_{4h}(x_h - x_4)\Big], \\ \Big[\big(-2(B_{43} - B_{54}) + (A_4 - A_5)\big)x_4 + \psi_5(x_5) + \psi_4(x_4) + u_4 + u_3 \\ \quad + \sum_{h=1,h\neq 3}^{N} B_{4h}(x_h - x_4) + \sum_{h=1,h\neq 4}^{N} B_{5h}(x_h - x_5)\Big], \\ \vdots \\ \Big[\big(-2(B_{(N-2)(N-3)} - B_{(N-1)(N-2)}) + (A_{N-2} - A_{N-1})\big)x_{N-2} \\ \quad +\psi_{N-1}(x_{N-1}) + \psi_{N-2}(x_{N-2}) + u_{N-2} + u_{N-3} \\ \quad + \sum_{h=1,h\neq N-3}^{N} B_{(N-2)h}(x_h - x_{N-2}) \\ \quad + \sum_{h=1,h\neq N-2}^{N} B_{(N-1)h}(x_h - x_{N-1})\Big], \\ \Big[\big(-2(B_{(N-1)(N-2)} - B_{N(N-1)}) + (A_{N-1} - A_N)\big)x_{N-1} \\ \quad +\psi_N(x_N) + \psi_{N-1}(x_{N-1}) + u_{N-1} + u_{N-2} \\ \quad + \sum_{h=1,h\neq N-2}^{N} B_{(N-1)h}(x_h - x_{N-1}) \\ \quad + \sum_{h=1,h\neq N-1}^{N} B_{Nh}(x_h - x_{N-1})\Big], \end{pmatrix}$$

then $\mathcal{L}V$ can be computed by trajectory of (4), $\mathcal{L}V = e^T(F_1e + v) + \frac{1}{2}\text{trace}\big[e^TH^THe\big] = e^T(F_1 + K + \lambda I)e \leq 0$, by the condition of Theorem 2, we know $\mathcal{L}V < 0$.

Therefore, it is asymptotically stable for Equ.(4),i.e.,the coupled chaotic systems (2) is anti-synchronization under the control schemes.

Let's consider the complete synchronization behavior for such chaotic systems under the anti-synchronization controllers. According to the number of the systems, two cases are discussed.

Case I: If the number of chaotic systems $N(N \geq 3)$ is odd, let

$$\begin{aligned} e_l(t) &= x_{l+2}(t) - x_l(t), \quad l = 1, 3, 5, \cdots, N-2; \\ e_k &= x_{k+2}(t) - x_k(t), \quad k = 2, 4, 6, \cdots, N-3. \end{aligned} \tag{5}$$

Theorem 3: Assumed that (3) is asymptotically stable, for the errors (5), then

$$\begin{aligned} \lim_{t\to\infty} \|e_l(t)\| &= \lim_{t\to\infty} \|x_{l+2}(t) - x_l(t)\| = 0, \\ &\quad (l = 1, 3, 5, \cdots, N-2); \\ \lim_{t\to\infty} \|e_k(t)\| &= \lim_{t\to\infty} \|x_{k+2}(t) - x_k(t)\| = 0; \\ &\quad (k = 2, 4, 6, \cdots, N-3). \end{aligned}$$

i.e., for the chaotic coupled system (2), the x_j and $x_{j+2}(j = l, k)$ is the complete synchronization under the controller $u_i(t)$.

Proof: By the conclusion of Theorems 1 and 2, we have

$$\begin{aligned} \lim_{t\to\infty} \|e_i(t)\| &= \lim_{t\to\infty} \|x_{i+1}(t) + x_i(t)\| = 0, \\ &\quad (i = 1, 2, 3, \cdots, N-1). \end{aligned} \tag{6}$$

In view of (5), we have

$$e_l(t) = x_{l+2}(t) - x_l(t) = x_{l+2}(t) + x_{l+1}(t) - (x_{l+1}(t) + x_l(t));$$
$$e_k = x_{k+2}(t) - x_k(t) = x_{k+2}(t) + x_{k+1}(t) - (x_{k+1}(t) + .x_k(t)).$$

By (6),

$$\begin{aligned}\lim_{t\to\infty} \|e_l(t)\| &= \lim_{t\to\infty} \|x_{l+2}(t) - x_l(t)\| \\ &\le \lim_{t\to\infty} \|x_{l+2}(t) + x_{l+1}(t)\| \\ &\quad + \lim_{t\to\infty} \|x_{l+1}(t) + x_l(t)\| \\ &= 0, \quad (l = 1, 3, 5, \cdots, N-2),\end{aligned}$$

and

$$\begin{aligned}\lim_{t\to\infty} \|e_k(t)\| &= \lim_{t\to\infty} \|x_{k+2}(t) - x_k(t)\| \\ &\le \lim_{t\to\infty} \|x_{k+2}(t) + x_{k+1}(t)\| \\ &\quad + \lim_{t\to\infty} \|x_{k+1}(t) + x_k(t)\| \\ &= 0, \quad (l = 2, 4, 6, \cdots, N-3).\end{aligned}$$

Therefore, the errors e_l and e_k converge to 0, i.e.,the x_{j+2} and x_j $(j = l, k)$ is complete synchronization.

Case II: If the number of chaotic systems $N(N \ge 3)$ is even, let

$$\begin{aligned}\tilde{e}_l(t) &= x_{l+2}(t) - x_l(t), \quad l = 1, 3, 5, \cdots, N-3; \\ \tilde{e}_k &= x_{k+2}(t) - x_k(t), \quad k = 2, 4, 6, \cdots, N-2. \end{aligned} \tag{7}$$

Theorem 4: Assumed that (3) is asymptotically stable, for the errors (7), then

$$\begin{aligned}\lim_{t\to\infty} \|\tilde{e}_l(t)\| &= \lim_{t\to\infty} \|x_{l+2}(t) - x_l(t)\| = 0, \\ &\quad (l = 1, 3, 5, \cdots, N-3); \\ \lim_{t\to\infty} \|\tilde{e}_k(t)\| &= \lim_{t\to\infty} \|x_{k+2}(t) - x_k(t)\| = 0, \\ &\quad (k = 2, 4, 6, \cdots, N-2).\end{aligned}$$

i.e., for the chaotic system (2), the x_j and $x_{j+2}(j = l, k)$ is the complete synchronization under the controller $u_i(t)$.

Remark : The proof of Theorem 4 is similar to Theorem 3. Therefore it's proof is omitted here.

IV. APPLICATION

Here we give many different chaotic systems to verify those results, such as Lorenz System, Chen System, Lü System.

Let $N = 3$,

$$\begin{cases} dx_1 = \Big(A_1x_1 + \psi_1(x_1) + B_{12}(x_2 - x_1) \\ \qquad\qquad + B_{13}(x_3 - x_1)\Big)dt + Cx_1d\omega(t), \\ dx_2 = \Big(A_2x_2 + \psi_2(x_2) + B_{21}(x_1 - x_2) \\ \qquad\qquad + B_{23}(x_3 - x_2)\Big)dt + Cx_2d\omega(t), \\ dx_3 = \Big(A_3x_3 + \psi_3(x_3) + B_{31}(x_1 - x_3) \\ \qquad\qquad + B_{32}(x_2 - x_3)\Big)dt + Cx_3d\omega(t), \end{cases} \tag{8}$$

where $x_1 = (x_{11}, x_{12}, x_{13})^T$, $x_2 = (x_{21}, x_{22}, x_{23})^T$, $x_3 = (x_{31}, x_{32}, x_{33})^T$,

$$A_1 = \begin{pmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -\frac{8}{3} \end{pmatrix}, \quad \psi_1(x_1) = \begin{pmatrix} 0 \\ -x_{11}x_{13} \\ x_{11}x_{12} \end{pmatrix},$$
$$A_2 = \begin{pmatrix} -35 & 35 & 0 \\ -7 & 28 & 0 \\ 0 & 0 & -3 \end{pmatrix}, \quad \psi_2(x_2) = \begin{pmatrix} 0 \\ -x_{21}x_{23} \\ x_{21}x_{22} \end{pmatrix},$$
$$A_3 = \begin{pmatrix} -36 & 36 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & -3 \end{pmatrix}, \quad \psi_3(x_3) = \begin{pmatrix} 0 \\ -x_{31}x_{33} \\ x_{31}x_{32} \end{pmatrix},$$

$$B_{12} = diag(b_{12_1}, b_{12_2}, b_{12_3}), \quad B_{13} = diag(b_{13_1}, b_{13_2}, b_{13_3}),$$
$$B_{21} = diag(b_{21_1}, b_{21_2}, b_{21_3}), \quad B_{23} = diag(b_{23_1}, b_{23_2}, b_{23_3}),$$
$$B_{31} = diag(b_{31_1}, b_{31_2}, b_{31_3}), \quad B_{32} = diag(b_{32_1}, b_{32_2}, b_{32_3}),$$
$$C = diag(c_1, c_2, c_3).$$

Substitute the controllers $u_1 = (u_{11}, u_{12}, u_{13})^T, u_2 = (u_{21}, u_{22}, u_{23})^T$ into the Equ.(8), we can obtain

$$\begin{cases} dx_1 = \Big(A_1x_1 + \psi_1(x_1) + B_{12}(x_2 - x_1) \\ \qquad\qquad + B_{13}(x_3 - x_1)\Big)dt + Cx_1d\omega(t), \\ dx_2 = \Big(A_2x_2 + \psi_2(x_2) + B_{21}(x_1 - x_2) \\ \qquad\qquad + B_{23}(x_3 - x_2) + u_1\Big)dt + Cx_2d\omega(t), \\ dx_3 = \Big(A_3x_3 + \psi_3(x_3) + B_{31}(x_1 - x_3) \\ \qquad\qquad + B_{32}(x_2 - x_3) + u_2\Big)dt + Cx_3d\omega(t). \end{cases} \tag{9}$$

Remark: If $B_{ih} = 0(i = 1, 2, 3, h = 1, 2, 3)$, $C = 0$, then

$$\begin{cases} dx_1 = \Big(A_1x_1 + \psi_1(x_1)\Big)dt, \\ dx_2 = \Big(A_2x_2 + \psi_2(x_2)\Big)dt, \\ dx_3 = \Big(A_3x_3 + \psi_3(x_3)\Big)dt. \end{cases}$$

are respectively Lorenz system, Chen system, Lü system. It is well known that the three systems are all chaotic systems.

Let $e_i = x_i + x_{i+1}(i = 1, 2)$, then the error system can be described in the following:

$$de = (F_1e + F_2)dt + Hed\omega(t), \tag{10}$$

where

$$F_1 = \begin{pmatrix} K_1 & 35 & 0 & b_{13_1} & 0 & 0 \\ -7 & K_2 & 0 & 0 & b_{13_2} & 0 \\ 0 & 0 & K_3 & 0 & 0 & b_{13_3} \\ b_{21_1} & 0 & 0 & K_4 & 36 & 0 \\ 0 & b_{21_2} & 0 & 0 & K_5 & 0 \\ 0 & 0 & b_{21_3} & 0 & 0 & K_6 \end{pmatrix},$$

$K_1 = -35 - b_{13_1} - b_{21_1}, K_2 = 28 - b_{13_2} - b_{21_2}, K_3 = -3 - b_{13_3} - b_{21_3}, K_4 = -36 - b_{32_1}, K_5 = 20 - b_{32_2}, K_6 = -3 - b_{32_3}$.

$$F_2 = \begin{pmatrix} \Big[\Big(2B_{21} - (A_2 - A_1)\Big)x_1 + \psi_2(x_2) + \psi_1(x_1) + u_1 \\ +B_{12}(x_2 - x_1) + B_{23}(x_3 - x_2)\Big], \\ \Big[\big(-2(B_{21} - B_{32}) + (A_2 - A_3)\big)x_2 + \psi_3(x_3) \\ +\psi_2(x_2) + u_2 + u_1 + B_{23}(x_3 - x_2) \\ +B_{31}(x_1 - x_3)\Big] \end{pmatrix}$$

The controllers u_1 and u_2 are designed in the following:

$$\begin{cases} u_1 = v_1 - \Big[\Big(2B_{21} - (A_2 - A_1)\Big)x_1 + \psi_2(x_2) + \psi_1(x_1) \\ \qquad +B_{12}(x_2 - x_1) + B_{23}(x_3 - x_2)\Big], \\ u_2 = v_2 - \Big[\big(-2(B_{21} - B_{32}) + (A_2 - A_3)\big)x_2 + \psi_3(x_3) \\ \qquad +\psi_2(x_2) + u_1 + B_{23}(x_3 - x_2) + B_{31}(x_1 - x_3)\Big]. \end{cases}$$

Let v, as shown at the bottom of this page.

Therefore, the Equ.(10) can be rewritten as follows:

$$de = (F_1 e + v)dt + Hed\omega(t). \tag{11}$$

It is easy to compute that

$$F_1 + K + \lambda I = \begin{pmatrix} K_1+\lambda & 35 & 0 & b_{13_1} & 0 & 0 \\ -35 & K_2+\lambda & 0 & 0 & b_{13_2} & 0 \\ 0 & 0 & K_3+\lambda & 0 & 0 & b_{13_3} \\ -b_{13_1} & 0 & 0 & K_4+\lambda & 36 & 0 \\ 0 & -b_{13_2} & 0 & -36 & K_5+\lambda & 0 \\ 0 & 0 & -b_{13_3} & 0 & 0 & K_6+\lambda \end{pmatrix},$$

where

$$\lambda = \max\{2c_1^2, 2c_2^2, 2c_3^2\}.$$

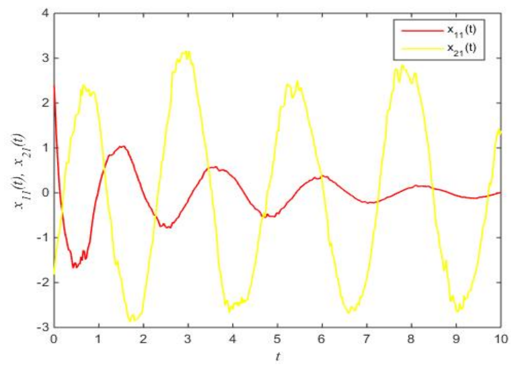

FIGURE 3. $x_{11}(t), x_{21}(t)$ **state trajectories.**

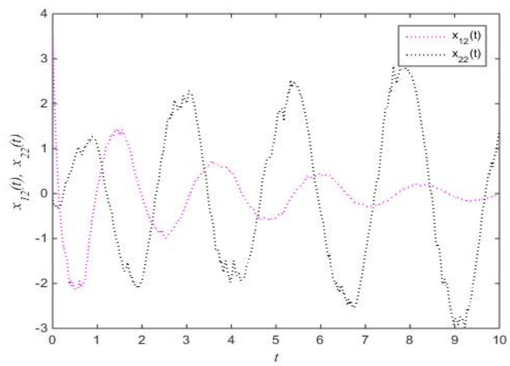

FIGURE 4. $x_{12}(t), x_{22}(t)$ **state trajectories.**

If those conditions $K_1 + \lambda = -35 - b_{13_1} - b_{21_1} + \lambda < 0$, $K_2 + \lambda = 28 - b_{13_2} - b_{21_2} + \lambda < 0$, $K_3 + \lambda = -3 - b_{13_3} - b_{21_3} + \lambda < 0$, $K_4 + \lambda = -36 - b_{32_1} + \lambda < 0$, $K_5 + \lambda = 20 - b_{32_2} + \lambda < 0$, $K_6 + \lambda = -3 - b_{32_3} + \lambda < 0$, hold. The conditions of the Theorem 2 are satisfied. Therefore, it is asymptotically stable for Equ.(11),i.e.,the chaotic systems (9)is anti-synchronization under the control schemes. The state trajectories of $x_{11}(t), x_{21}(t), x_{12}(t), x_{22}(t)$ and $x_{13}(t), x_{23}(t)$ are shown in Fig.3, Fig.4, Fig.5,respectively. The error state trajectories of $e(t)$ are shown in Fig.6, which obviously supports our theoretical result.

Let $e_i' = x_3 - x_1$, by the Theorem 3, for the chaotic system (9), the x_1 and x_3 is the complete synchronization under the controller $u_1(t)$ and $u_2(t)$.

$$v = Ke = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -28 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -b_{13_1} - b_{21_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & -b_{13_2} - b_{21_2} & 0 & -36 & 0 & 0 \\ 0 & 0 & -b_{13_3} - b_{21_3} & 0 & 0 & 0 \end{pmatrix} e$$

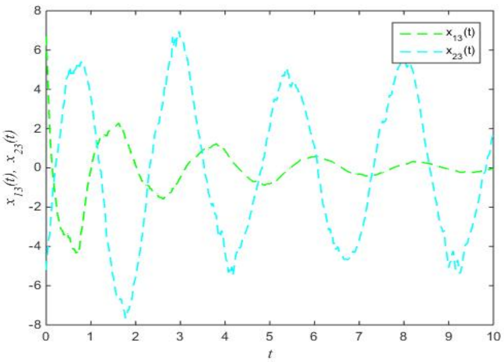

FIGURE 5. $x_{13}(t), x_{23}(t)$ **state trajectories.**

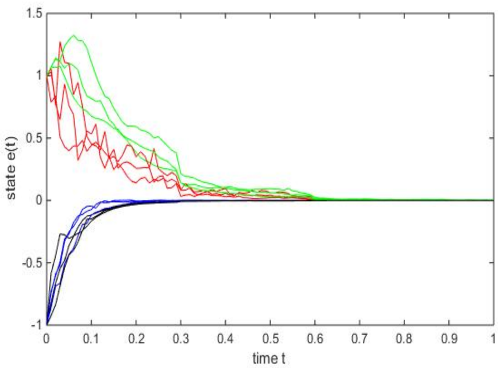

FIGURE 6. **Error state trajectories by controller.**

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CHAOLONG ZHANG was born in 1974. He received the master's degree in mathematics from South China Normal University, China, in 2004, and the Ph.D. degree in control theory and control engineering from the South China University of Technology, Guangzhou, in 2016. Since 2009, he has been an Associate Professor with the Zhongkai University of Agriculture and Engineering. His main research interests include stability and control of stochastic systems, synchronization of network neural, and stability and control of impulsive systems.

FEIQI DENG was born in 1962. He received the Ph.D. degree in control theory and control engineering from the South China University of Technology, Guangzhou, in 1997. Since 1999, he has been a Professor with the South China University of Technology and also the Director of the Systems Engineering Institute. His main research interests include stability, stabilization, and robust control theory of complex systems, including time-delay systems, non-linear systems, and stochastic systems. He is currently a member of the Technical Committee on Control Theory (TCCT) and the Chinese Association of Automation. He is currently serving as the Chair of the IEEE SMC Guangzhou Chapter, an Associate Editor-in-Chief of the *Journal of South China University of Technology*, and a member of the editorial boards of the following journals: the *Control Theory and Applications, All about Systems and Control*; the *Journal of Systems Engineering and Electronics*; and the *Journal of Systems Engineering*.

WENFENG ZHANG was born in 1968. He received the Ph.D. degree in control theory and control engineering from the South China University of Technology, Xi'an, in 2007. Since 2008, he has been a Professor with the Zhongkai University of Agriculture and Engineering. His main research interests include stochastic complex systems modeling and analysis, and fractional order stochastic systems and controls.

TING HOU received the M.S. degree from Shandong Normal University, Jinan, China, and the Ph.D. degree from the Shandong University of Science and Technology, Qingdao, in 2004 and 2010, respectively, where she is currently a Professor with the Shandong University of Science and Technology. Her research interests include stochastic stability and robust control.

ZHIWEI YANG received the B.Sc., M.Sc., and Ph.D. degrees from Sun Yat-sen University, China, in 2006, 2008, and 2018, respectively. He is currently an Associate Professor with the College of Computational Science, Zhongkai University of Agriculture and Engineering, China. His research interests include vehicular ad-hoc networks and distributed systems. He mainly focuses on information dissemination and dynamic models in vehicular ad-hoc networks, and has published related papers.