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An Optimal Imperfect Maintenance Policy for a Partially Observed System With Obvious Failures and Limited Repairs

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ABSTRACT In this paper, we investigate an imperfect maintenance optimization problem for a multi-state, Markovian deterioration system with obvious failures under repair restriction based on those non-periodically collected sensor information. Our aim is to adaptively schedule observations and other maintenance actions with taking imperfect maintenance effect into consideration. Different from most existing works, imperfect maintenance here means that repair action can not only restore the system to a less deteriorated level instead of the good-as-new state but also accelerate the deterioration process so that the system can be repaired only a limited number of times before it must be replaced with a new one. Assuming that the system's deterioration state evolves as a discrete-time Markov chain with a finite state space, and then choosing the information state together with the number of completed repair times as state variable, we formulate the problem as a Markov decision process over an infinite time horizon. In order to increase the computational efficiency, several key structural properties are developed by minimizing the long-run average cost per unit time. Then, special algorithms are proposed to find the optimal maintenance policies. Finally, a numerical example is given to illustrate the effectiveness of the proposed algorithms.

INDEX TERMS Condition-based maintenance, deteriorating system, imperfect maintenance, Markov decision process, repair restriction.

I. INTRODUCTION

Most complex engineering systems are always subject to deterioration due to their age and everyday operations, and eventually fail unless some intervention is taken. As a useful approach, maintenance has been introduced to keep the system reliability above a satisfactory level. Maintenance optimization mainly focuses on finding an optimal maintenance policy to make the balance between the costs and benefits of performing the maintenance actions on the complex system subject to performance degradation. In the past decades, maintenance policies for repairable systems have been extensively studied, e.g. [1]–[6] and the references therein.

As an efficient maintenance approach, condition-based maintenance (CBM) utilizes the information gathered through condition monitoring to recommend appropriate maintenance actions, and has captured more and more people's attention. Jardin *et al.* [3] summarized those work about CBM appearing before 2006. However, research in this area still grows rapidly. A large amount of papers associated with CBM have appeared during the past several years [7]–[10].

In the area of CBM, several important aspects should be carefully examined. *One is the cost associated with collecting the condition information used for decision making.* Most works on CBM consider that information can be periodically obtained based on the assumption that the sampling cost can be neglected, e.g. [7]–[9], [11], [12]. However, in many real applications, the deterioration state is very expensive or hard to collect. Hence, it is necessary to determine when to collect the condition information and how to utilize that information to make the optimal maintenance policy together. *Another important aspect is whether the obtained information is*

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perfect or not. Perfect information can exactly reveal the true state of the system, while the imperfect information is only stochastically related to the underlying state. Therefore, it is very interesting to study the problem of joint optimization of sampling and maintenance with imperfect information. As early as in 1968, Eckles [13] studied partially observable systems with imperfect information under a very general setting. Other classical works include [14], [15]. However, few structural properties have been reported in these papers. Fortunately, Maillart [16], [17] considered maintenance systems with perfect and imperfect information and developed structural properties for perfect information which is then used to motivate heuristic policies for the information case. Recently, Kim and Makis [18] studied the similar problem with the main difference in that the system state process was modelled in continuous time. They proved that the optimal maintenance policy could be represented as a control chart with three critical thresholds. *In addition, maintenance effect is also an important aspect which should be carefully considered in CBM*. Maintenance effect has been considered in many papers, e.g. [19]–[21]. Imperfect maintenance here means that repair action can only restore the system to a less deteriorated level instead of the good-as-new state (Imperfect maintenance 1, IM1), or that the system can be repaired only a limited number of times before they must be replaced with a new one due to physical, safety, technological and/or economical restrictions (Imperfect maintenance 2, IM2). The former case can be easily understood and has been studied by many researchers. While the latter case means that a system subject to IM2 becomes more prone to deterioration as the number of performed repair actions increases [22]–[25]. And such case can also be found in many engineering and service applications [11]. For example, aircraft engine turbine blades always suffer from degradation of fatigue strength for the reason that they operate at high temperatures and experience centrifugal stresses. They can be reworked only a limited number of times followed by a replacement action to ensure flight safety. As such, Kurt *et al.* [11] studied the maintenance problem for a Markovian deteriorating system based on IM2. They formulated the problem as an infinite-horizon Markov decision process, and derived key structural properties of the optimal cost function to develop a more efficient computational algorithm. But the system considered in [11] is periodically inspected and completely observable.

Therefore, in this paper we try to combine the first and last aspects together into one maintenance model. Specially, we study the imperfect maintenance decision problem for a multi-state system with non-periodical perfect observations under the constraint that only a limited number of imperfect preventive repair actions can be performed between two successive preventive replacements. That is, both IM1 and IM2 are considered in this paper. As far as we know, there exist few works on the above-mentioned problem. Fan *et al.* [25] considered the similar problem of optimally maintaining a multi-state system only based on IM2, and no observation was used to make the maintenance decision. Chen *et al.* [28] improved the work in [25] through taking IM1 into consideration. However, IM1 was only considered in maintenance modelling. The structural properties were obtained still based on perfect maintenance. Hence, our work can be viewed as an extension of the works of [25], [28].

The remainder is organized as follows. The next section formulates the problem and constructs the optimality equations. In Section 3 we analyze the structural properties of the optimal value function based on which an algorithm to determine the optimal maintenance policies is developed in Section 4. A numerical example is then given to show the effectiveness of the proposed algorithm. The last section draws up our conclusions and shows the future work.

II. MODEL DESCRIPTION AND FORMULATION

In this paper, a repairable system subject to continuous deterioration in time is considered. In order to keep the system health, maintenance decisions are made at discrete equidistant times $t_n = n\Delta$, where $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and Δ is the decision interval. The system deterioration at t_n is characterized by a random variable $X_n = X(t_n)$ whose value should be chosen from the set $\mathcal{S} = \{1, \ldots, m+1\}$ with a natural order on its elements. That is, the numbers are arranged in order of increasing deterioration level. Level 1 denotes the best health condition meaning that the system is as good as new, m denotes the most deteriorated condition, and $m+1$ shows that the system has failed. Furthermore, $\{X_n, n \in \mathbb{N}_0\}$ is assumed to be a discrete time Markov chain (DTMC) on the state space $\mathcal{S}$. Although X_n can describe the system deterioration condition, its realization, i.e. the system true deterioration state, can't always be obtained at t_n except when the observation action is performed on the system. Therefore, information state $\boldsymbol{\pi} = [\pi_1, \pi_2, \cdots, \pi_{m+1}] \in \Omega$ is adopted to describe the belief over the actual deteriorated condition, where $\Omega = \{\boldsymbol{\pi} : \sum_{i=1}^{m+1} \pi_i = 1\}$ and π_i represents the probability that the system is now in the deterioration level i, $i = 1, \ldots, m+1$. Since the system failure is obvious, there exist $\pi_{m+1} = 0$ if the system is still working, otherwise, $\pi_{m+1} = 1$. As mentioned in the previous section, the system in this paper is subject to IM2, that is, it can only be repaired for a limited number of times, which means that each repair action will affect the deterioration process, thus it is reasonable to assume that the transition probability matrix is influnced by completed repair times. Furthermore, let $\mathbf{P}_k = [p_{ij}^k]_{(m+1)\times(m+1)}$ denote transition probability matrix when the system has been repaired k times, where p_{ij}^k is one-step transition probability from deterioration state i to j, $p_{ij}^k = P\{X_{n+1} = j | X_n = i, N(t_n) = k\}$. Here, $N(t)$ is the number of performed repairs up to time t and satisfies $0 \leq N(t) \leq K$, where K is the maximal number of available repairs. For simplification, let $\mathcal{K} = \{0, 1, \ldots, K\}$ and $\mathcal{K}' = \mathcal{K} \backslash \{K\}$.

At the beginning of each decision epoch, if $N(t) \in \mathcal{K}'$, there are three available maintenance actions: no action (NA), taking an observation (OB), and preventive maintenance (PM). However, the system cannot be repaired any more after the Kth performed repair action, the available maintenance actions for $N(t) = K$ become NA, OB, and preventive replacement (PR). It should be noted that the OB action sometimes is so complicated that it cannot reveal the true state immediately. Hence, in this paper we make an assumption that only one action can be chosen at the beginning of each decision epoch. In summary, given that the current system information state is $\boldsymbol{\pi}$ and the system has been repaired k times, the action space $\mathcal{A}(\boldsymbol{\pi}, k)$ is

$$\mathcal{A}(\boldsymbol{\pi}, k) = \begin{cases} \{NA, OB, PM\}, & \text{for } k \in \mathcal{K}'; \\ \{NA, OB, PR\}, & \text{for } k = K. \end{cases}$$

The four actions are explained in detail as follows.

- NA: let the system continue to operate without any intervention. If this action is adopted, the system will still be operable after the next transition with probability $R_k(\boldsymbol{\pi}) = 1 - \sum_{i=1}^{m} \pi_i p_{i,m+1}^k$, and the current information state $\boldsymbol{\pi}$ will transit to state $\boldsymbol{\pi}'(\boldsymbol{\pi}, k)$ at the begining of the next decision period. Particularly, the new information state $\boldsymbol{\pi}'(\boldsymbol{\pi}, k)$ can be obtained through

$$\pi_j'(\boldsymbol{\pi}, k) = \begin{cases} \dfrac{\sum_{i=1}^{m} \pi_i p_{ij}^k}{R_k(\boldsymbol{\pi})}, & j = 1, 2, \ldots, m; \\ 0, & j = m+1. \end{cases} \quad (1)$$

  But, if the system fails with probability $1 - R_k(\boldsymbol{\pi})$, it will be replaced with a new one at cost c_{fr}, and then the new information state is $\boldsymbol{e}_{m+1}$, where $\boldsymbol{e}_i = [0, \ldots, 1, \ldots, 0]$ is an $(m+1) \times 1$ dimensional row vector with a 1 in the i-th position, and 0 elsewhere.
- PM: preventively maintain the system at cost c_{pm}. PM action is imperfect in the sense that it can only restore the system to a less deteriorated level $j, j \leq i$ with probability q_{ij} if the current deterioration level is i, where $\sum_{j=1}^{i} q_{ij} = 1$, $0 \leq q_{ij} \leq 1$, $q_{il} = 0, i < l \leq m+1$, $i \in \mathcal{S}' = \mathcal{S} \backslash \{m+1\}$. Since the failure is obvious and each failure replacement will restore the system to the level 1, we define $q_{(m+1)1} = 1$ and $q_{(m+1)j} = 0, j \in \mathcal{S}'$. The matrix $\mathbf{Q} = [q_{ij}]_{(m+1)\times(m+1)}$ is called maintenance effect matrix in this paper. Thus, after PM, the current system deterioration state becomes $j (j = 1, \ldots, m)$ with probability $\sum_{i=j}^{m} \pi_i q_{ij}$ based on the assumption that the PM action is performed instantaneously, and then the information state at the current time becomes $\left[\pi_1'(\boldsymbol{\pi}, k), \ldots, \pi_m'(\boldsymbol{\pi}, k), 0\right]$, where $\pi_j'(\boldsymbol{\pi}, k) = \sum_{i=j}^{m} \pi_i q_{ij}, j = 1, 2, \ldots, m$. As mentioned previously, the PM action can be considered only when $k \in \mathcal{K}'$.
- PR: restore the system deterioration condition to the best level 1 with negligible time at cost $c_{pr} (c_{pm} < c_{pr} < c_{fr} < \infty)$. After PR, the information state at the current time becomes $\boldsymbol{e}_1$. This action can be available only when $k = K$.
- OB: evaluate the exact deterioration level at cost $c_o (c_o + c_{pr} < c_{fr})$. After OB, the true deterioration state can be obtained, and the information state becomes $\boldsymbol{e}_i$.

Considering that OB actions are performed non-periodically, we can formulate the maintenance problem as a partially observed Markov decision process (POMDP) model. However, through introducing an ordered pair $(\boldsymbol{\pi}, k) \in \Omega \times \mathcal{K}$ as the state of the decision process, the problem can then be reformulated as a Markov decision process (MDP) model. At first, we consider the maintenance problem over a finite time horizon. Let $V_n(\boldsymbol{\pi}, k)$ denote the expected total cost over the remaining n decision periods when the system has been repaired k times and the current information state is $\boldsymbol{\pi} \in \Omega$. In this paper, we consider maintenance decisions are made frequently, hence the discount rate is approximately equal to 1. Therefore, the optimality equation can be written as

$$V_n(\boldsymbol{\pi}, k) = \min\{NA_n(\boldsymbol{\pi}, k), PX_n(\boldsymbol{\pi}, k), OB_n(\boldsymbol{\pi}, k)\} \quad (2)$$

where

$$NA_n(\boldsymbol{\pi}, k) = (c_{fr} + V_{n-1}(\boldsymbol{e}_1, 0))(1 - R_k(\boldsymbol{\pi})) + V_{n-1}(\boldsymbol{\pi}'(\boldsymbol{\pi}, k), k) R_k(\boldsymbol{\pi}), \quad k \in \mathcal{K}, \quad (3)$$

$$OB_n(\boldsymbol{\pi}, k) = c_o + \sum_{i=1}^{m} \pi_i V_n(\boldsymbol{e}_i, k), \quad k \in \mathcal{K}, \quad (4)$$

$$PX_n(\boldsymbol{\pi}, k) = \begin{cases} PM_n(\boldsymbol{\pi}, k), & k \in \mathcal{K}', \\ PR_n(\boldsymbol{\pi}, k), & k = K, \end{cases} \quad (5)$$

$$PM_n(\boldsymbol{\pi}, k) = c_{pm} + \sum_{i=1}^{m} \pi_i \sum_{j=1}^{i} q_{ij} V_n(\boldsymbol{e}_j, k+1), \quad (6)$$

$$PR_n(\boldsymbol{\pi}, K) = c_{pr} + V_n(\boldsymbol{e}_1, 0). \quad (7)$$

Eq. (3) represents the expected cost incurred by NA. The first term in right hand side (r.h.s.) of Eq. (3) is the cost incurred when a failure happens with probability $1 - R_k(\boldsymbol{\pi})$. Since the failure replacement action makes the state of MDP revert to $(\boldsymbol{e}_1, 0)$, the failure associated cost is calculated through adding failure replacement cost c_{fr} to the cost-to-go during the remaining $n-1$ periods $V_{n-1}(\boldsymbol{e}_1, 0)$. While the second term in r.h.s. of Eq. (3) is the cost incurred when the system is still operable with probability $R_k(\boldsymbol{\pi})$ at the begining of the next decision period. In this case, the information state is updated to $\boldsymbol{\pi}'(\boldsymbol{\pi}, k)$, and the number of completed repairs is still equal to k. Therefore, the new state of the MDP becomes $(\boldsymbol{\pi}'(\boldsymbol{\pi}, k), k)$, and the cost-to-go during the remaining $n-1$ periods is $V_{n-1}(\boldsymbol{\pi}'(\boldsymbol{\pi}, k), k)$. Eq. (4) shows that OB will cause observation cost c_o as well as the cost-to-go associated with starting in the state revealed through perfect observation. Eq. (6) calculates the expected cost incurred by PM. The first term in r.h.s. of Eq. (6) is the instantaneous cost c_{pm}, while the second term is the cost-to-go during the remaining n periods. As previously mentioned, the information state after PM becomes $\boldsymbol{e}_j$ with probability $\pi_i q_{ij}$, $i = 1, \ldots, m, j = 1, \ldots, i$, and the number of performed PM action increases by 1, thus the updated state of the MDP becomes $(\boldsymbol{e}_j, k+1)$ with probability $\pi_i q_{ij}$, based on which the second term in r.h.s. of Eq. (6) can be obtained. Finally, Eq. (7) reflects the fact that PR incurs cost c_{pr} plus the cost-to-go associated with starting in the state $(\boldsymbol{e}_1, 0)$.

Because both failure replacement and preventive replacement can renew the system, it is possible that the model is unichain [17]. Hence, for each $(\boldsymbol{\pi}, k) \in \Omega \times \mathcal{K}$, according to the theory of MDP [29], there exists the following equation

$$\lim_{n\to\infty} V_n(\boldsymbol{\pi}, k) = ng + b(\boldsymbol{\pi}, k) \tag{8}$$

where g is the minimum expected cost per unit time, and $b(\boldsymbol{\pi}, k)$ is the relative cost of starting in state $(\boldsymbol{\pi}, k)$ under the optimal policy. Thus, making n in Eq. (2) tend to infinity, and then using Eq. (8), we can have

$$b(\boldsymbol{\pi}, k) = \min\{b_{NA}(\boldsymbol{\pi}, k), b_{PX}(\boldsymbol{\pi}, k), b_{OB}(\boldsymbol{\pi}, k)\} \tag{9}$$

where

$$\begin{aligned} b_{NA}(\boldsymbol{\pi}, k) =& (c_{fr} + b(\boldsymbol{e}_1, 0))(1 - R_k(\boldsymbol{\pi})) \\ &+ b(\boldsymbol{\pi}'(\boldsymbol{\pi}, k), k)R_k(\boldsymbol{\pi}) - g, \quad k \in \mathcal{K}, \end{aligned} \tag{10}$$

$$b_{OB}(\boldsymbol{\pi}, k) = c_o + \sum_{i=1}^{m} \pi_i b(\boldsymbol{e}_i, k), \quad k \in \mathcal{K}, \tag{11}$$

$$b_{PX}(\boldsymbol{\pi}, k) = \begin{cases} b_{PM}(\boldsymbol{\pi}, k), & k \in \mathcal{K}', \\ b_{PR}(\boldsymbol{\pi}, k), & k = K. \end{cases} \tag{12}$$

$$b_{PM}(\boldsymbol{\pi}, k) = c_{pm} + \sum_{i=1}^{m} \pi_i \sum_{j=1}^{i} q_{ij} b(\boldsymbol{e}_j, k+1), \tag{13}$$

$$b_{PR}(\boldsymbol{\pi}, K) = c_{pr} + b(\boldsymbol{e}_1, 0). \tag{14}$$

After constructing the above optimality equation, the remaining work is to solve it. If the state space is finite, traditional policy or value iteration algorithm can be adopted to obtain the optimal maintenance policy [29]. However, in this paper, belief state space Ω is continuous and infinite. In order to overcome the difficulty, the concept of sample path is introduced at first. Suppose that the system initial belief state is $\boldsymbol{\pi}$ and then goes without any intervention, all the information states it would occupy will form a sequence of information states over time which is called a sample path emating from $\boldsymbol{\pi}$. Assuming that the system has been repaired k times, let $\Omega_{\boldsymbol{\pi}}^k = \left\{\boldsymbol{\pi}, \boldsymbol{\pi}_k^2, \ldots, \boldsymbol{\pi}_k^l, \ldots,\right\}$ denote the sample path emanating from $\boldsymbol{\pi}$ where $\boldsymbol{\pi}_k^l = \boldsymbol{\pi}'(\boldsymbol{\pi}_k^{l-1}, k), l \geq 2$, and $\boldsymbol{\pi}_k^1 = \boldsymbol{\pi}$. In addition, if the Markov chain is acyclic, the sample path converges to the absorbing state which is defined as $\boldsymbol{\pi}_k^{L_k}$, where $L_k = \min\left\{l;\ \|\boldsymbol{\pi}_k^{l+1} - \boldsymbol{\pi}_k^l\| \leq \varepsilon\right\}$ with any $\varepsilon > 0$ [17]. In this paper, both perfect OB and imperfect PM only can make the process restarting from state $\boldsymbol{e}_i, i = 1, \ldots, m$. Furthermore, we assume the deterioration state at time 0 is known. That is, initial information state is one of the states $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_m$. Hence, we can confine the belief state space to $\Omega'(\Psi) = \bigcup_{\pi\in\Psi,k\in\mathcal{K}} \Omega_{\boldsymbol{\pi}}^k$ which is finite, where $\Psi = \{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_m\}$. After approximating Ω as $\Omega'(\Psi)$, traditional policy or value iteration algorithm can be directly applied for solving the optimality equation. However, in order to develop a more efficient and effective algorithm, it is still meaningful to derive some key structural properties which is discussed in the next section.

III. STRUCTURAL RESULTS

In this section, we establish several structural properties of the optimal value function to reduce the computation time cost in solving Eq. (9). At first, some useful definitions are introduced. After that, several preliminary results are given to provide the basis for deriving useful structural properties of the optimal value function.

A. PRELIMINARY RESULTS AND MAIN ASSUMPTIONS

Although the following definitions are widely used in POMDP related literature, e.g. [17], [26]–[28], [30], we still introduce them here to keep the integrity of the paper. At first, we give two definitions about how to compare two information states.

Definition 1: The information state $\boldsymbol{\pi}$ is stochastically less than the information state $\hat{\boldsymbol{\pi}}$, denoted as $\boldsymbol{\pi} \prec_{st} \hat{\boldsymbol{\pi}}$, if and only if $\sum_{i\geq l} \pi_i \leq \sum_{i\geq l} \hat{\pi}_i$ for all l.

Definition 2: The information state $\boldsymbol{\pi}$ is less in likelihood than the information state $\hat{\boldsymbol{\pi}}$, denoted as $\boldsymbol{\pi} \prec_{lr} \hat{\boldsymbol{\pi}}$, if $\pi_i\hat{\pi}_j \geq \pi_j\hat{\pi}_i$ for all $j \geq i$.

Both of the above two definitions mean that if one information state is stochastically less than another one, then the system in the smaller information state is less deteriorated. But, the condition of Definition 2 is more stringent than that of Definition 1. Next, two definitions on the probability transition matrix $\mathbf{P}$ are also provided.

Definition 3: A probability transition matrix $\mathbf{P}$ has an Increasing Failure Rate (IFR) if $\sum_{j\geq l} p_{i,j} \leq \sum_{j\geq l} p_{i',j}$ for $i \leq i'$ and $\forall l$.

Definition 4: A probability transition matrix $\mathbf{P}$ is Totally Positive of order 2 (TP2) if $p_{ij}p_{i'j'} \geq p_{i'j}p_{ij'}, \forall i' \geq i, j' \geq j$.

The condition of definition 3 implies that the system is more prone to deteriorate further and fail when the system is now in the stochastically larger information state. Definition 4 implies the same meaning, but in a stronger sense, which is illustrated in the following proposition.

Proposition 1 [30]: (a) If $\boldsymbol{\pi} \prec_{lr} \hat{\boldsymbol{\pi}}$, then $\boldsymbol{\pi} \prec_{st} \hat{\boldsymbol{\pi}}$. (b) If $\mathbf{P}$ is TP2, $\mathbf{P}$ is IFR.

To proceed the discussion, we make the following three assumptions about the probability transition matrix $\mathbf{P}_k$ based on the above definitions.

Assumption 1: $\mathbf{P}_k$ is TP2. That is, $\boldsymbol{p}_{i,:}^k \prec_{lr} \boldsymbol{p}_{j,:}^k$ for $i \leq j$, where $\boldsymbol{p}_{i,:}^k$ denotes the ith row vector of $\mathbf{P}_k$ with $k \in \mathcal{K}$.

Assumption 2: If $k_1 \leq k_2$, then $p_{ij}^{k_1} \geq p_{ij}^{k_2}$ for $j \leq i$, while for $j > i$, $p_{ij}^{k_1} \leq p_{ij}^{k_2}$, and furthermore $\boldsymbol{p}_{i,:}^{k_1} \prec_{lr} \boldsymbol{p}_{i,:}^{k_2}$, where $i \in \mathcal{S}$ and $k_1, k_2 \in \mathcal{K}$.

Assumption 3: $\boldsymbol{q}_{i_1,:} \prec_{st} \boldsymbol{q}_{i_2,:}$ holds for $i_1 \leq i_2$, where $\boldsymbol{q}_{i_1,:}$ and $\boldsymbol{q}_{i_2,:}$ denote the i_1th row vector and the i_2th row vector of $\mathbf{Q}$ respectively, $i_1, i_2 \in \mathcal{S}'$.

Assumption 1 shows that the system in larger deterioration level j is more prone to deteriorate further than the one in smaller deterioration level i ($i \leq j$) when the they have the same cumulative completed repairs. Like Assumption 1, Assumption 2 implies that, given two systems in the same deterioration level, the system with more completed repair times is more likely to get worse than the other one with less completed repair times. Assumption 3 relates the maintenance effect with the system's deterioration level. After maintenance action, the system at smaller deterioration level i_1 is more likely to become less deteriorated than the other. That is, the larger the system deterioration level is, the harder people can repair it.

Before providing our results, we introduce another well-known result in the following proposition.

Proposition 2 [31]: For any column vector $\boldsymbol{v}$ *such that* $v_i \leq v_{i+1}$, $\forall i$, *if* $\boldsymbol{\pi} \prec_{st} \hat{\boldsymbol{\pi}}$, *then* $\boldsymbol{\pi v} \leq \hat{\boldsymbol{\pi}} v$.

According to the above Proposition 2, we can obtain the results about the reliability of the system.

Proposition 3: (a) Suppose Assumption 1 is satisfied. If $\boldsymbol{\pi} \prec_{st} \hat{\boldsymbol{\pi}}$, *then* $R_k(\boldsymbol{\pi}) \geq R_k(\hat{\boldsymbol{\pi}})$ *for all* $k \in \mathcal{K}$. *(b) If Assumption 2 is satisfied, then* $R_k(\boldsymbol{\pi})$ *is non-increasing in* $k \in \mathcal{K}$ *for* $\forall \boldsymbol{\pi} \in \Omega$.

Proof: The proof of part (a) is similar to that in [17]. We focus on the proof of part (b). According to Assumption 2, we have $p^{k_1}_{i,m+1} \leq p^{k_2}_{i,m+1}$ for $k_1 < k_2$ and $i = 1, 2, \ldots, m+1$. Then,

$$R_{k_1}(\boldsymbol{\pi}) = 1 - \sum_{i=1}^{m} \pi_i p^{k_1}_{i,m+1} \geq 1 - \sum_{i=1}^{m} \pi_i p^{k_2}_{i,m+1} = R_{k_2}(\boldsymbol{\pi}),$$

which implies that $R_k(\boldsymbol{\pi})$ is non-increasing in $k \in \mathcal{K}$. ■

Proposition 3 is consistent with our intuition in that a less deteriorated system has a higher reliability and the system repaired more times has the lower ability to perform the required functions.

Proposition 4: (a) If Assumption 1 is satisfied, then for $\boldsymbol{\pi} \prec_{lr} \hat{\boldsymbol{\pi}}$, $\boldsymbol{\pi}'(\boldsymbol{\pi}, k) \prec_{lr} \boldsymbol{\pi}'(\hat{\boldsymbol{\pi}}, k)$ *for all* $k \in \mathcal{K}$. *(b) If Assumptions 1 and 2 are both satisfied, then* $\boldsymbol{\pi}'(\boldsymbol{\pi}, k_1) \prec_{lr} \boldsymbol{\pi}'(\boldsymbol{\pi}, k_2)$ *for* $k_1 \leq k_2$ *and* $\boldsymbol{\pi} \in \Omega$.

Proof: The proof of the case when k is fixed is the same as that of Proposition 4 in [17] and omitted here. More attention is paid on proving part (b). For $k_1 \leq k_2$ and $l \geq i$, according to Assumptions 1 and 2, we have $\boldsymbol{p}^{k_1}_{i,:} \prec_{lr} \boldsymbol{p}^{k_2}_{i,:} \prec_{lr} \boldsymbol{p}^{k_2}_{l,:}$, which implies that $p^{k_1}_{ij} p^{k_2}_{lj'} \geq p^{k_1}_{ij'} p^{k_2}_{lj}$ for $j' \geq j$.

Hence,

$$\begin{aligned} 0 &\leq \sum_{i=1}^{m} \sum_{l=1}^{m} \pi_i \pi_l (p^{k_1}_{ij} p^{k_2}_{lj'} - p^{k_1}_{ij'} p^{k_2}_{lj}) \\ &= \sum_{i=1}^{m} \sum_{l=1}^{m} \pi_i \pi_l p^{k_1}_{ij} p^{k_2}_{lj'} - \sum_{i=1}^{m} \sum_{l=1}^{m} \pi_i \pi_l p^{k_1}_{ij'} p^{k_2}_{lj} \\ &= \sum_{i=1}^{m} \pi_i p^{k_1}_{ij} \sum_{l=1}^{m} \pi_l p^{k_2}_{lj'} - \sum_{i=1}^{m} \pi_i p^{k_1}_{ij'} \sum_{l=1}^{m} \pi_l p^{k_2}_{lj}. \end{aligned}$$

Since the failure is obvious, the reliability of the system in service is not equal to 0, i.e. $R_{k_1}(\boldsymbol{\pi}) \neq 0$ and $R_{k_2}(\boldsymbol{\pi}) \neq 0$. Then we have

$$\begin{aligned} 0 &\leq \frac{\sum_{i=1}^{m} \pi_i p^{k_1}_{ij} \sum_{l=1}^{m} \pi_l p^{k_2}_{lj'}}{R_{k_1}(\boldsymbol{\pi}) R_{k_2}(\boldsymbol{\pi})} - \frac{\sum_{i=1}^{m} \pi_i p^{k_1}_{ij'} \sum_{l=1}^{m} \pi_l p^{k_2}_{lj}}{R_{k_1}(\boldsymbol{\pi}) R_{k_2}(\boldsymbol{\pi})} \\ &= \pi'_j(\boldsymbol{\pi}, k_1)\pi'_{j'}(\boldsymbol{\pi}, k_2) - \pi'_{j'}(\boldsymbol{\pi}, k_1)\pi'_j(\boldsymbol{\pi}, k_2), \end{aligned}$$

which means that $\boldsymbol{\pi}'(\boldsymbol{\pi}, k_1) \prec_{lr} \boldsymbol{\pi}'(\boldsymbol{\pi}, k_2)$ for $k_1 \leq k_2$ and $\boldsymbol{\pi} \in \Omega$. ■

This proposition shows that the $\prec_{lr}$-ordered information states retain their order after the state transition.

According to Proposition 4, a sample path emanating from any state has the following property for $\boldsymbol{\pi} \prec_{lr} \boldsymbol{\pi}'(\boldsymbol{\pi}, k)$.

Proposition 5: Suppose Assumptions 1 and 2 are both satisfied. If $\boldsymbol{\pi} \prec_{lr} \boldsymbol{\pi}'(\boldsymbol{\pi}, k)$ *for* $\forall \boldsymbol{\pi} \in \Omega$, *then* $\boldsymbol{\pi} = \boldsymbol{\pi}^1_k \prec_{lr} \cdots \prec_{lr} \boldsymbol{\pi}^{L_k}_k$ *holds for the sample path starting from the information state* $\boldsymbol{\pi} \in \Omega$.

The proof of this proposition can be easily obtained according to part (a) of Proposition 4 and is omitted here.

Lemma 1: Suppose Assumptions 1 and 3 are satisfied. (a) $b(\boldsymbol{\pi}, K)$ *is nondecreasing in* $\prec_{lr}$. *(b) If* $b(\boldsymbol{e}_m, k) \leq c_{fr} + b(\boldsymbol{e}_1, 0)$, *then* $b(\boldsymbol{\pi}, k)$ *is nondecreasing in* $\prec_{lr}$ *for any fixed* $k \in \mathcal{K}'$.

Proof: In the following, the proofs of part (a) and part (b) are provided together. From Eq. (8), we know that $b(\boldsymbol{\pi}, k)$ can be obtained by taking the limits of $V_n(\boldsymbol{\pi}, k)$. Hence, it is sufficient to show that $V_n(\boldsymbol{\pi}, k)$ is nondecreasing in $\prec_{lr}$ for all $n = 1, 2, \ldots$ and $k \in \mathcal{K}$. Furthermore, since the minimum of nondecreasing functions is still nodecreasing, it suffices to show that $NA_n(\boldsymbol{\pi}, k)$, $PX_n(\boldsymbol{\pi}, k)$ and $OB_n(\boldsymbol{\pi}, k)$ are nondecreasing in $\prec_{lr}$ for all $n = 1, 2, \ldots$ and $k \in \mathcal{K}$, which is achieved through the induction method. Without loss of generality, we firstly suppose that $V_0(\boldsymbol{\pi}, k) = 0$ for all $\boldsymbol{\pi} \in \Omega$, $k \in \mathcal{K}$. Next, as the induction hypothesis, assume that $V_{n-1}(\boldsymbol{\pi}, k)$ is nondecreasing in $\prec_{lr}$ for all fixed $k \in \mathcal{K}$. Our aim is to show $V_n(\boldsymbol{\pi}, k)$ is also nondecreasing in $\prec_{lr}$. Firstly, the monotonicity of $NA_n(\boldsymbol{\pi}, k)$ in $\boldsymbol{\pi}$ is discussed. Suppose $\boldsymbol{\pi}_1 \prec_{lr} \boldsymbol{\pi}_2$. If $k = K$, then according to Proposition 3 and Proposition 4, there exists

$$\begin{aligned} NA_n(\boldsymbol{\pi}_1, k) &= (c_{fr} + V_{n-1}(\boldsymbol{e}_1, 0))(1 - R_k(\boldsymbol{\pi}_1)) \\ &\quad + V_{n-1}(\boldsymbol{\pi}'(\boldsymbol{\pi}_1, k), k) R_k(\boldsymbol{\pi}_1) \\ &\leq (c_{fr} + V_{n-1}(\boldsymbol{e}_1, 0))(1 - R_k(\boldsymbol{\pi}_1)) \\ &\quad + V_{n-1}(\boldsymbol{\pi}'(\boldsymbol{\pi}_2, k), k) R_k(\boldsymbol{\pi}_1) \qquad (15) \\ &= (c_{fr} + V_{n-1}(\boldsymbol{e}_1, 0)) + (V_{n-1}(\boldsymbol{\pi}'(\boldsymbol{\pi}_2, k), k) \\ &\quad - c_{fr} - V_{n-1}(\boldsymbol{e}_1, 0)) R_k(\boldsymbol{\pi}_1) \\ &\leq (c_{fr} + V_{n-1}(\boldsymbol{e}_1, 0)) + (V_{n-1}(\boldsymbol{\pi}'(\boldsymbol{\pi}_2, k), k) \\ &\quad - c_{fr} - V_{n-1}(\boldsymbol{e}_1, 0)) R_k(\boldsymbol{\pi}_2) \\ &= (c_{fr} + V_{n-1}(\boldsymbol{e}_1, 0))(1 - R_k(\boldsymbol{\pi}_2)) \\ &\quad + V_{n-1}(\boldsymbol{\pi}'(\boldsymbol{\pi}_2, k), k) R_k(\boldsymbol{\pi}_2) \\ &= NA_n(\boldsymbol{\pi}_2, k), \qquad (16) \end{aligned}$$

which implies that $NA_n(\boldsymbol{\pi}, K)$ is nondecreasing in $\boldsymbol{\pi}$. Inequality (15) follows from the induction hypothesis and Proposition 4, and holds for all $k \in \mathcal{K}$. Inequality (16) follows from Proposition 3 and the fact that $V_{n-1}(\boldsymbol{\pi}'(\boldsymbol{\pi}_2, K), K) \leq c_{pr} + V_{n-1}(\boldsymbol{e}_1, 0) < c_{fr} + V_{n-1}(\boldsymbol{e}_1, 0)$. However, for $k \in \mathcal{K}'$, according to the induction hypothesis, $V_{n-1}(\boldsymbol{\pi}'(\boldsymbol{\pi}_2, k), k) \leq V_{n-1}(\boldsymbol{e}_m, k)$. Hence, if $V_{n-1}(\boldsymbol{e}_m, k) \leq c_{fr} + V_{n-1}(\boldsymbol{e}_1, 0)$, then we have $V_{n-1}(\boldsymbol{\pi}'(\boldsymbol{\pi}_2, k), k) \leq c_{fr} + V_{n-1}(\boldsymbol{e}_1, 0)$ which makes inequality (16) holds. Thus, $NA_n(\boldsymbol{\pi}, k)$ is nondecreasing in $\boldsymbol{\pi}$ for any fixed $k \in \mathcal{K}'$ if $V_{n-1}(\boldsymbol{e}_m, k) \leq c_{fr} + V_{n-1}(\boldsymbol{e}_1, 0)$. Secondly, we examine the monotonicity of $PX_n(\boldsymbol{\pi}, k)$. Obviously, $PR_n(\boldsymbol{\pi}, K)$ is constant and nondecreasing in $\boldsymbol{\pi}$. Hence, we only needs to study the monotonicity of $PM_n(\boldsymbol{\pi}, k), k \in \mathcal{K}'$. To begin with, we study the monotonicity of $V_n(\boldsymbol{e}_i, K)$ in i. According to Eq. (4), $OB_n(\boldsymbol{e}_i, k) = c_o + V_n(\boldsymbol{e}_i, k) > V_n(\boldsymbol{e}_i, k)$ which implies that $V_n(\boldsymbol{e}_i, k) = \min\{NA_n(\boldsymbol{e}_i, k), PX_n(\boldsymbol{e}_i, k)\}$. Since we have already established that $NA_n(\boldsymbol{\pi}, K)$ is nondecreasing in $\boldsymbol{\pi}$ and $PR_n(\boldsymbol{\pi}, K)$ is constant, it is clear that $V_n(\boldsymbol{e}_i, K)$ is nondecreasing in i. Then, according to the Proposition 2 and Assumption 3, we can conclude that $\sum_{j=1}^{m} q_{i_1 j} V_n(\boldsymbol{e}_j, K) \leq \sum_{j=1}^{m} q_{i_2 j} V_n(\boldsymbol{e}_j, K)$ holds for $i_1 \leq i_2$, which further implies that $PM_n(\boldsymbol{\pi}_1, K-1) \leq PM_n(\boldsymbol{\pi}_2, K-1)$ for $\boldsymbol{\pi}_1 \prec_{lr} \boldsymbol{\pi}_2$. Thus, the value function $V_n(\boldsymbol{e}_i, K-1)$ is nondecreasing in i if $V_{n-1}(\boldsymbol{e}_m, K-1) \leq c_{fr} + V_{n-1}(\boldsymbol{e}_1, 0)$. Analogically, we can successively obtain that $PM_n(\boldsymbol{\pi}, k)$ and $V_n(\boldsymbol{e}_i, k), k = K-2, \ldots, 1$ are all nondecreasing in $\prec_{lr}$ if $V_{n-1}(\boldsymbol{e}_m, k) \leq c_{fr} + V_{n-1}(\boldsymbol{e}_1, 0)$. From the above analysis, it can be concluded that $PR_n(\boldsymbol{\pi}, K)$ is nondecreasing in $\boldsymbol{\pi}$ and $PM_n(\boldsymbol{\pi}, k)$ is nondecreasing in $\boldsymbol{\pi}$ if $V_{n-1}(\boldsymbol{e}_m, k) \leq c_{fr} + V_{n-1}(\boldsymbol{e}_1, 0)$ for any fixed $k \in \mathcal{K}'$. Furthermore, once given that $V_n(\boldsymbol{e}_i, k)$ is nondecreasing in i, then according to Proposition 2, there exists $OB_n(\boldsymbol{\pi}_1, k) = c_o + \sum_{i=1}^{m} \pi_{1,i} V_n(\boldsymbol{e}_i, k) \leq c_o + \sum_{i=1}^{m} \pi_{2,i} V_n(\boldsymbol{e}_i, k) = OB_n(\boldsymbol{\pi}_2, k)$ which shows that $OB_n(\boldsymbol{\pi}, k)$ is nondecreasing in $\prec_{lr}$. As a result, $V_n(\boldsymbol{\pi}, K)$ is nondecreasing in $\prec_{lr}$ for all n by induction, and the claim (a) holds, while for $k \in \mathcal{K}'$, if $V_{n-1}(\boldsymbol{e}_m, k) \leq c_{fr} + V_{n-1}(\boldsymbol{e}_1, 0)$ holds for all n, $V_n(\boldsymbol{\pi}, K)$ is nondecreasing in $\prec_{lr}$ for all n by induction, which implies claim (b) holds. ■

Lemma 1 shows that the optimal expected total cost does not decrease when the system with the same repair times gets more deteriorated.

B. ACTION BOUNDARY EXPRESSIONS

Based on the above conclusions, we further study the closed expressions for the optimal preventive repair region and other structural properties. Let $a^*(\boldsymbol{\pi}, k)$ denote the optimal maintenance action at the state $(\boldsymbol{\pi}, k)$.

Lemma 2: Define the bias value vector $\boldsymbol{b}_k = [b(\boldsymbol{e}_1, k), \ldots, b(\boldsymbol{e}_{m+1}, k)]'$ *for* $k \in \mathcal{K}$, $\Omega^k_{NA \leq PM} = \{\boldsymbol{\pi}; (\boldsymbol{\pi}'\mathbf{Q}\boldsymbol{b}_{k+1} - c_{fr} + c_{pm} - b(\boldsymbol{e}_1, 0))R_k(\boldsymbol{\pi}) \leq \boldsymbol{\pi}\mathbf{Q}\boldsymbol{b}_{k+1} - c_{fr} + c_{pm} - b(\boldsymbol{e}_1, 0) + g\}$ *for* $k \in \mathcal{K}'$. *(a) If* $\boldsymbol{\pi} \in \Omega^k_{NA \leq PM}$, *then* $a^*(\boldsymbol{\pi}, k) \neq PM$. *(b) Suppose Assumptions 1 and 3 are satisfied. If* $R_K(\boldsymbol{\pi}) \geq 1 - g/(c_{fr} - c_{pr})$, $a^*(\boldsymbol{\pi}, K) \neq PR$; *otherwise,* $a^*(\boldsymbol{\pi}, K) \neq NA$ *for* $\boldsymbol{\pi} \prec_{lr} \boldsymbol{\pi}'(\boldsymbol{\pi}, k)$.

Proof: Firstly, we compare $b_{NA}(\boldsymbol{\pi}, k)$ to $b_{PX}(\boldsymbol{\pi}, k)$ as follows. If $k \in \mathcal{K}'$, there exists

$$
\begin{aligned}
&b_{NA}(\boldsymbol{\pi}, k) - b_{PM}(\boldsymbol{\pi}, k)\\
&\quad= (c_{fr} + b(\boldsymbol{e}_1, 0))(1 - R_k(\boldsymbol{\pi})) + b(\boldsymbol{\pi}'(\boldsymbol{\pi}, k), k)R_k(\boldsymbol{\pi})\\
&\qquad - g - c_{pm} - \sum_{i=1}^{m} \pi_i \sum_{j=1}^{i} q_{ij} b(\boldsymbol{e}_j, k+1)\\
&\quad= \big(c_{fr} + b(\boldsymbol{e}_1, 0) - \boldsymbol{\pi}\mathbf{Q}\boldsymbol{b}_{k+1}\big)(1 - R_k(\boldsymbol{\pi}))\\
&\qquad - g + (b(\boldsymbol{\pi}'(\boldsymbol{\pi}, k), k) - c_{pm} - \boldsymbol{\pi}\mathbf{Q}\boldsymbol{b}_{k+1})R_k(\boldsymbol{\pi})\\
&\quad= \big(c_{fr} - c_{pm} + b(\boldsymbol{e}_1, 0) - \boldsymbol{\pi}\mathbf{Q}\boldsymbol{b}_{k+1}\big)(1 - R_k(\boldsymbol{\pi}))\\
&\qquad - g + (b(\boldsymbol{\pi}'(\boldsymbol{\pi}, k), k) - c_{pm} - \boldsymbol{\pi}'\mathbf{Q}\boldsymbol{b}_{k+1})R_k(\boldsymbol{\pi})\\
&\qquad + (\boldsymbol{\pi}'\mathbf{Q}\boldsymbol{b}_{k+1} - \boldsymbol{\pi}\mathbf{Q}\boldsymbol{b}_{k+1})R_k(\boldsymbol{\pi})\\
&\quad= \big(c_{fr} - c_{pm} + b(\boldsymbol{e}_1, 0)\big)(1 - R_k(\boldsymbol{\pi}))\\
&\qquad - \boldsymbol{\pi}\mathbf{Q}\boldsymbol{b}_{k+1}(1 - R_k(\boldsymbol{\pi})) - g\\
&\qquad + (b(\boldsymbol{\pi}'(\boldsymbol{\pi}, k), k) - c_{pm} - \boldsymbol{\pi}'\mathbf{Q}\boldsymbol{b}_{k+1})R_k(\boldsymbol{\pi})\\
&\qquad + (\boldsymbol{\pi}'\mathbf{Q}\boldsymbol{b}_{k+1} - \boldsymbol{\pi}\mathbf{Q}\boldsymbol{b}_{k+1})R_k(\boldsymbol{\pi})\\
&\quad= \big(c_{fr} - c_{pm} + b(\boldsymbol{e}_1, 0)\big)(1 - R_k(\boldsymbol{\pi})) - \boldsymbol{\pi}\mathbf{Q}\boldsymbol{b}_{k+1}\\
&\qquad + \boldsymbol{\pi}'\mathbf{Q}\boldsymbol{b}_{k+1}R_k(\boldsymbol{\pi}) - g\\
&\qquad + (b(\boldsymbol{\pi}'(\boldsymbol{\pi}, k), k) - c_{pm} - \boldsymbol{\pi}'\mathbf{Q}\boldsymbol{b}_{k+1})R_k(\boldsymbol{\pi})\\
&\quad= \big(c_{fr} - c_{pm} + b(\boldsymbol{e}_1, 0) - \boldsymbol{\pi}\mathbf{Q}\boldsymbol{b}_{k+1}\big) - g\\
&\qquad + R_k(\boldsymbol{\pi})\big(\boldsymbol{\pi}'\mathbf{Q}\boldsymbol{b}_{k+1} - c_{fr} + c_{pm} - b(\boldsymbol{e}_1, 0)\big)\\
&\qquad + (b(\boldsymbol{\pi}'(\boldsymbol{\pi}, k), k) - c_{pm} - \boldsymbol{\pi}'\mathbf{Q}\boldsymbol{b}_{k+1})R_k(\boldsymbol{\pi})
\end{aligned}
$$

From Eq. (9), we conclude that $b(\boldsymbol{\pi}'(\boldsymbol{\pi}, k), k) \leq c_{pm} + \boldsymbol{\pi}'\mathbf{Q}\boldsymbol{b}_{k+1}$. Thus, $b_{NA}(\boldsymbol{\pi}, k) \leq b_{PM}(\boldsymbol{\pi}, k)$ if $(c_{fr} - c_{pm} + b(\boldsymbol{e}_1, 0) - \boldsymbol{\pi}\mathbf{Q}\boldsymbol{b}_{k+1}) - g + R_k(\boldsymbol{\pi})(\boldsymbol{\pi}'\mathbf{Q}\boldsymbol{b}_{k+1} - c_{fr} + c_{pm} - b(\boldsymbol{e}_1, 0)) \leq 0$, i.e. doing nothing is preferred to preventive maintenance action.

For the special case that $k = K$, there exists

$$
\begin{aligned}
&b_{NA}(\boldsymbol{\pi}, K) - b_{PR}(\boldsymbol{\pi}, K)\\
&\quad= (c_{fr} + b(\boldsymbol{e}_1, 0))(1 - R_K(\boldsymbol{\pi}))\\
&\qquad + b(\boldsymbol{\pi}'(\boldsymbol{\pi}, K), K)R_K(\boldsymbol{\pi}) - g - c_{pr} - b(\boldsymbol{e}_1, 0)\\
&\quad= \big(c_{fr} - c_{pr}\big)(1 - R_K(\boldsymbol{\pi})) - g + (b(\boldsymbol{\pi}'(\boldsymbol{\pi}, K), K)\\
&\qquad - c_{pr} - b(\boldsymbol{e}_1, 0))R_K(\boldsymbol{\pi})
\end{aligned}
$$

Clearly, $b(\boldsymbol{\pi}'(\boldsymbol{\pi}, K), K) < c_{pr} + b(\boldsymbol{e}_1, 0)$. So if $(c_{fr} - c_{pr})(1 - R_K(\boldsymbol{\pi})) - g \leq 0$, or equivalently, $R_K(\boldsymbol{\pi}) \geq 1 - g/(c_{fr} - c_{pr})$, then $b_{NA}(\boldsymbol{\pi}, K) < b_{PR}(\boldsymbol{\pi}, K)$. For the situation that $R_K(\boldsymbol{\pi}) < 1 - g/(c_{fr} - c_{pr})$, suppose $a^*(\boldsymbol{\pi}, K) = NA$, then we have

$$
\begin{aligned}
&b(\boldsymbol{\pi}'(\boldsymbol{\pi}, K), K) - b(\boldsymbol{\pi}, K)\\
&\quad= b(\boldsymbol{\pi}'(\boldsymbol{\pi}, K), K) - (c_{fr} + b(\boldsymbol{e}_1, 0))(1 - R_K(\boldsymbol{\pi}))\\
&\qquad - b(\boldsymbol{\pi}'(\boldsymbol{\pi}, K), K)R_K(\boldsymbol{\pi}) + g\\
&\quad= (b(\boldsymbol{\pi}'(\boldsymbol{\pi}, K), K) - c_{pr} - b(\boldsymbol{e}_1, 0))(1 - R_K(\boldsymbol{\pi}))\\
&\qquad - (c_{fr} - c_{pr})(1 - R_K(\boldsymbol{\pi})) + g\\
&\quad< 0
\end{aligned}
$$

which contradicts the fact that $b(\boldsymbol{\pi}'(\boldsymbol{\pi}, K), K) \geq b(\boldsymbol{\pi}, K)$ for $\boldsymbol{\pi} \prec_{lr} \boldsymbol{\pi}'(\boldsymbol{\pi}, K)$. Therefore, the optimal action cannot be NA when $R_K(\boldsymbol{\pi}) \geq 1 - g/(c_{fr} - c_{pr})$ and $\boldsymbol{\pi} \prec_{lr} \boldsymbol{\pi}'(\boldsymbol{\pi}, K)$. ■

Corollary 1: Suppose Assumptions 1 and 3 are satisfied, and $c_{pm} + b(\boldsymbol{e}_m, k) \leq c_{fr} + b(\boldsymbol{e}_1, 0)$ *for any fixed* $k \in \mathcal{K}'$. *If* $R_k(\boldsymbol{\pi}) \geq (c_{fr} + b(\boldsymbol{e}_1, 0) - c_{pm} - \boldsymbol{\pi}\mathbf{Q}\boldsymbol{b}_{k+1} - g)/(c_{fr} + b(\boldsymbol{e}_1, 0) - c_{pm} - \boldsymbol{\pi}'\mathbf{Q}\boldsymbol{b}_{k+1})$ *then* $a^*(\boldsymbol{\pi}, k) \neq PM$; *otherwise, if* $R_k(\boldsymbol{\pi}) \leq 1 - g/(c_{fr} + b(\boldsymbol{e}_1, 0) - c_{pm} - \boldsymbol{\pi}'\mathbf{Q}\boldsymbol{b}_{k+1})$, *then* $a^*(\boldsymbol{\pi}, k) \neq NA$ *for* $\boldsymbol{\pi} \prec_{lr} \boldsymbol{\pi}'(\boldsymbol{\pi}, k)$.

Proof: Based on the assumption of this corollary, we have $b(\boldsymbol{e}_m, k) < c_{pm} + b(\boldsymbol{e}_m, k) \leq c_{fr} + b(\boldsymbol{e}_1, 0)$, then according to Lemma 1, there exists $c_{pm} + \boldsymbol{\pi}'\mathbf{Q}\boldsymbol{b}_{k+1} \leq c_{pm} + \boldsymbol{e}_m\mathbf{Q}\boldsymbol{b}_{k+1} = c_{pm} + \sum_{j=1}^{m} q_{mj} b(\boldsymbol{e}_j, k+1) \leq c_{pm} + b(\boldsymbol{e}_m, k+1) \leq c_{fr} + b(\boldsymbol{e}_1, 0)$. Therefore, the former part of this corollary can be obtained directly according to Lemma 2, while for the case that $R_k(\boldsymbol{\pi}) \leq 1 - g/(c_{fr} + b(\boldsymbol{e}_1, 0) - c_{pm} - \boldsymbol{\pi}'\mathbf{Q}\boldsymbol{b}_{k+1})$, suppose $a^*(\boldsymbol{\pi}, k) = NA$, then we have

$$\begin{aligned} & b(\boldsymbol{\pi}'(\boldsymbol{\pi}, k), k) - b(\boldsymbol{\pi}, k) \\ &= b(\boldsymbol{\pi}'(\boldsymbol{\pi}, k), k) - (c_{fr} + b(\boldsymbol{e}_1, 0))(1 - R_k(\boldsymbol{\pi})) \\ &\quad - b(\boldsymbol{\pi}'(\boldsymbol{\pi}, k), k)R_k(\boldsymbol{\pi}) + g \\ &= (b(\boldsymbol{\pi}'(\boldsymbol{\pi}, k), k) - c_{pm} - \boldsymbol{\pi}'\mathbf{Q}\boldsymbol{b}_{k+1})(1 - R_k(\boldsymbol{\pi})) \\ &\quad - (c_{fr} + b(\boldsymbol{e}_1, 0) - c_{pm} - \boldsymbol{\pi}'\mathbf{Q}\boldsymbol{b}_{k+1})(1 - R_k(\boldsymbol{\pi})) + g \\ &< 0 \end{aligned}$$

which contradicts the fact that $b(\boldsymbol{\pi}'(\boldsymbol{\pi}, k), k) \geq b(\boldsymbol{\pi}, k)$ for $\boldsymbol{\pi} \prec_{lr} \boldsymbol{\pi}'(\boldsymbol{\pi}, k)$. Therefore, the optimal action cannot be NA when $R_k(\boldsymbol{\pi}) \leq 1 - g/(c_{fr} + b(\boldsymbol{e}_1, 0) - c_{pm} - \boldsymbol{\pi}'\mathbf{Q}\boldsymbol{b}_{k+1})$. ■

Lemma 3: (a) Let $\Omega^k_{OB \leq PM} = \{\boldsymbol{\pi}; \boldsymbol{\pi}(\boldsymbol{b}_k - \mathbf{Q}\boldsymbol{b}_{k+1}) \leq c_{pm} - c_o\}$, $k \in \mathcal{K}'$. *If* $\boldsymbol{\pi} \in \Omega^k_{OB \leq PM}$, *then* $a * (\boldsymbol{\pi}, k) \neq PM$; *otherwise,* $a * (\boldsymbol{\pi}, k) \neq OB$. *(b) If* $\boldsymbol{\pi}\boldsymbol{b}_K < c_{pr} + b(\boldsymbol{e}_1, 0) - c_o$, *then* $a * (\boldsymbol{\pi}, K) \neq PR$; *otherwise,* $a * (\boldsymbol{\pi}, K) \neq OB$.

Proof: We first compae $b_{OB}(\boldsymbol{\pi}, k)$ with $b_{PX}(\boldsymbol{\pi}, k)$ for each $k \in \mathcal{K}'$ when the state is $(\boldsymbol{\pi}, k)$.

$$\begin{aligned} b_{OB}(\boldsymbol{\pi}, k) - b_{PM}(\boldsymbol{\pi}, k) &= c_o + \sum_{i=1}^{m} \pi_i b(\boldsymbol{e}_i, k) - c_{pm} \\ &\quad - \sum_{i=1}^{m} \pi_i \sum_{j=1}^{i} q_{ij} b(\boldsymbol{e}_j, k+1) \\ &= c_o - c_{pm} + \boldsymbol{\pi}\boldsymbol{b}_k - \boldsymbol{\pi}\mathbf{Q}\boldsymbol{b}_{k+1} \\ &= c_o - c_{pm} + \boldsymbol{\pi}(\boldsymbol{b}_k - \mathbf{Q}\boldsymbol{b}_{k+1}) \end{aligned}$$

Hence, if $\boldsymbol{\pi}(\boldsymbol{b}_k - \mathbf{Q}\boldsymbol{b}_{k+1}) \leq c_{pm} - c_o$, then $b_{OB}(\boldsymbol{\pi}, k) \leq b_{PM}(\boldsymbol{\pi}, k)$, otherwise $b_{OB}(\boldsymbol{\pi}, k) > b_{PM}(\boldsymbol{\pi}, k)$.

Then, compare $b_{OB}(\boldsymbol{\pi}, K)$ with $b_{PR}(\boldsymbol{\pi}, K)$ for the state $(\boldsymbol{\pi}, K)$. Clearly, $b_{OB}(\boldsymbol{\pi}, K) - b_{PR}(\boldsymbol{\pi}, K) = c_o + \sum_{i=1}^{m} \pi_i b(\boldsymbol{e}_i, K) - c_{pr} - b(\boldsymbol{e}_1, 0) = \boldsymbol{\pi}\boldsymbol{b}_K + c_o - c_{pr} - b(\boldsymbol{e}_1, 0)$. Thus, if $\boldsymbol{\pi}\boldsymbol{b}_K < c_{pr} + b(\boldsymbol{e}_1, 0) - c_o$, then OB is preferred to PR, otherwise, PR is preferred to OB. ■

According to Lemmas 2 and 3, the following corollary can be obtained to specify the sufficient conditions for NA and OB to be optimal respectively.

Corollary 2: Suppose Assumptions 1 and 3 are satisfied. (a) If $\boldsymbol{\pi} \in (\Omega^k_{NA \leq PM} \cap \bar{\Omega}^k_{OB \leq PM})$, $k \in \mathcal{K}'$, *or if* $R_K(\boldsymbol{\pi}) \geq 1 - g/(c_{fr} - c_{pr})$ *and* $\boldsymbol{\pi}\boldsymbol{b}_K \geq c_{pr} + b(\boldsymbol{e}_1, 0) - c_o$ *are both satisfied, then* $a^*(\boldsymbol{\pi}, k) = NA$. *(b) If* $R_K(\boldsymbol{\pi}) < 1 - g/(c_{fr} - c_{pr})$ *and* $\boldsymbol{\pi}\boldsymbol{b}_K < c_{pr} + b(\boldsymbol{e}_1, 0) - c_o$, $a^*(\boldsymbol{\pi}, K) = OB$ *for* $\boldsymbol{\pi} \prec_{lr} \boldsymbol{\pi}'$.

Furthermore, for the case $c_{pm} + b(\boldsymbol{e}_m, k) \leq c_{fr} + b(\boldsymbol{e}_1, 0)$ for any fixed $k \in \mathcal{K}'$, the following corollary is provided to specify the sufficient condition for PM to be optimal according to Corollary 1 and Lemma 3.

Corollary 3: Suppose Assumptions 1 and 3 are satisfied, and $c_{pm} + b(\boldsymbol{e}_m, k) \leq c_{fr} + b(\boldsymbol{e}_1, 0)$ *for any fixed* $k \in \mathcal{K}'$. *If* $\boldsymbol{\pi} \in \bar{\Omega}^k_{OB \leq PM}$ *and* $R_k(\boldsymbol{\pi}) \leq 1 - g/(c_{fr} + b(\boldsymbol{e}_1, 0) - c_{pm} - \boldsymbol{\pi}'\mathbf{Q}\boldsymbol{b}_{k+1})$, *then* $a^*(\boldsymbol{\pi}, k) = PM$ *for* $\boldsymbol{\pi} \prec_{lr} \boldsymbol{\pi}'(\boldsymbol{\pi}, k)$.

Particularly, for the special case $k = K$, we can get several beautiful structural properties. According to Lemmas 2 and 3, we can conclude that if $R_K(\boldsymbol{\pi}) < 1 - g/(c_{fr} - c_{pr})$, and $\boldsymbol{\pi}\boldsymbol{b}_K \geq c_{pr} + b(\boldsymbol{e}_1, 0) - c_o$ for $\boldsymbol{\pi} \prec_{lr} \boldsymbol{\pi}'(\boldsymbol{\pi}, K)$, the optimal policy is PR. Furthermore, from the facts that $b_{OB}(\boldsymbol{\pi}, K)$ is nondecreasing in $\prec_{lr}$ ordering, and $b_{PR}(\boldsymbol{\pi}, K)$ is constant, the control limit for PR can be derived in closed form. The following Theorem 1 summarizes the sufficient and necessary condition of the existence of the control limit for PR.

Theorem 1: Suppose Assumptions 1 and 3 are satisfied. (a) For $\boldsymbol{\pi} \prec_{lr} \boldsymbol{\pi}'(\boldsymbol{\pi}, K)$, *the region where the optimal policy is PR is defined by* $\Omega^K_{PR} = \{\boldsymbol{\pi}; R_K(\boldsymbol{\pi}) < 1 - g/(c_{fr} - c_{pr}), \boldsymbol{\pi}\boldsymbol{b}_K \geq c_{pr} + b(\boldsymbol{e}_1, 0) - c_o\}$, *whereas PR cannot be optimal for* $\notin \Omega^K_{PR}$. *(b) Furthermore, if* $a^*(\boldsymbol{\pi}, K) = PR$, $a^*(\hat{\boldsymbol{\pi}}, K) = PR$ *for* $\boldsymbol{\pi} \prec_{lr} \hat{\boldsymbol{\pi}}$.

The Theorem 1 points out that there exists a control limit for PR when $k = K$ due to $b_{PR}(\boldsymbol{\pi}, K)$ is constant. However, for the case $k \in \mathcal{K}'$, the similar structural properties cannot be obtained for the reason that $b_{PM}(\boldsymbol{\pi}, k)$ varies along with $\boldsymbol{\pi}$.

Finally, let us compare $b_{NA}(\boldsymbol{\pi}, k)$ with $b_{OB}(\boldsymbol{\pi}, k)$ to obtain the sufficient condisions under which NA is preferred to OB.

Lemma 4: Let $\Omega^k_{NA \leq OB} = \{\boldsymbol{\pi} : (c_o + \boldsymbol{\pi}'\boldsymbol{b}_k - c_{fr} - b(\boldsymbol{e}_1, 0))R_k(\boldsymbol{\pi}) \leq g + c_o + \boldsymbol{\pi}\boldsymbol{b}_k - c_{fr} - b(\boldsymbol{e}_1, 0)\}$ *for any fixed* $k \in \mathcal{K}'$. *(a) If* $\boldsymbol{\pi} \in \Omega^k_{NA \leq OB}$, *then* $a^*(\boldsymbol{\pi}, k) \neq OB$ *for* $k \in \mathcal{K}'$. *(b) If* $R_K(\boldsymbol{\pi}) \geq (c_{fr} + b(\boldsymbol{e}_1, 0) - c_o - \boldsymbol{\pi}\boldsymbol{b}_K - g)/(c_{fr} + b(\boldsymbol{e}_1, 0) - c_o - \boldsymbol{\pi}'\boldsymbol{b}_K)$, *then* $a^*(\boldsymbol{\pi}, K) \neq OB$.

Proof:

$$\begin{aligned} & b_{NA}(\boldsymbol{\pi}, k) - b_{OB}(\boldsymbol{\pi}, k) \\ &= (c_{fr} + b(\boldsymbol{e}_1, 0))(1 - R_k(\boldsymbol{\pi})) + b(\boldsymbol{\pi}'(\boldsymbol{\pi}, k), k)R_k(\boldsymbol{\pi}) \\ &\quad - g - c_o - \boldsymbol{\pi}\boldsymbol{b}_k \\ &= (c_{fr} + b(\boldsymbol{e}_1, 0) - c_o - \boldsymbol{\pi}\boldsymbol{b}_k)(1 - R_k(\boldsymbol{\pi})) - g \\ &\quad + (b(\boldsymbol{\pi}'(\boldsymbol{\pi}, k), k) - c_o - \boldsymbol{\pi}\boldsymbol{b}_k)R_k(\boldsymbol{\pi}) \\ &= (c_{fr} + b(\boldsymbol{e}_1, 0) - c_o - \boldsymbol{\pi}\boldsymbol{b}_k)(1 - R_k(\boldsymbol{\pi})) - g \\ &\quad + (b(\boldsymbol{\pi}'(\boldsymbol{\pi}, k), k) - c_o - \boldsymbol{\pi}'\boldsymbol{b}_k + \boldsymbol{\pi}'\boldsymbol{b}_k - \boldsymbol{\pi}\boldsymbol{b}_k)R_k(\boldsymbol{\pi}) \\ &= (c_{fr} + b(\boldsymbol{e}_1, 0) - c_o - \boldsymbol{\pi}\boldsymbol{b}_k)(1 - R_k(\boldsymbol{\pi})) - g \\ &\quad + (b(\boldsymbol{\pi}'(\boldsymbol{\pi}, k), k) - c_o - \boldsymbol{\pi}'\boldsymbol{b}_k)R_k(\boldsymbol{\pi}) \\ &\quad + (\boldsymbol{\pi}'\boldsymbol{b}_k - \boldsymbol{\pi}\boldsymbol{b}_k)R_k(\boldsymbol{\pi}) \end{aligned}$$

Since $b(\boldsymbol{\pi}'(\boldsymbol{\pi}, k), k) < c_o + \boldsymbol{\pi}'\boldsymbol{b}_k$, if $(c_{fr} + b(\boldsymbol{e}_1, 0) - c_o - \boldsymbol{\pi}\boldsymbol{b}_k)(1 - R_k(\boldsymbol{\pi})) - g + (\boldsymbol{\pi}'\boldsymbol{b}_k - \boldsymbol{\pi}\boldsymbol{b}_k)R_k(\boldsymbol{\pi}) \leq 0$, or equivalently $\boldsymbol{\pi} \in \Omega^k_{NA \leq OB} = \{\boldsymbol{\pi} : (c_o + \boldsymbol{\pi}'\boldsymbol{b}_k - c_{fr} - b(\boldsymbol{e}_1, 0))R_k(\boldsymbol{\pi}) \leq g + c_o + \boldsymbol{\pi}\boldsymbol{b}_k - c_{fr} - b(\boldsymbol{e}_1, 0)\}$, then $b_{NA}(\boldsymbol{\pi}, k) \leq b_{OB}(\boldsymbol{\pi}, k)$. In detail, if $c_{fr} + b(\boldsymbol{e}_1, 0) - c_o < \boldsymbol{\pi}'\boldsymbol{b}_k$, then $b_{NA}(\boldsymbol{\pi}, k) < b_{OB}(\boldsymbol{\pi}, k)$ satisfies under the condition that $R_k(\boldsymbol{\pi}) < 1 - (\boldsymbol{\pi}'\boldsymbol{b}_k - \boldsymbol{\pi}\boldsymbol{b}_k - g)/(c_o + \boldsymbol{\pi}'\boldsymbol{b}_k - c_{fr} - b(\boldsymbol{e}_1, 0))$. On the other side, if $c_{fr} + b(\boldsymbol{e}_1, 0) - c_o > \boldsymbol{\pi}'\boldsymbol{b}_k$, then $b_{NA}(\boldsymbol{\pi}, k) \leq b_{OB}(\boldsymbol{\pi}, k)$ satisfies under the condition that $R_k(\boldsymbol{\pi}) \geq 1 - (\boldsymbol{\pi}'\boldsymbol{b}_k - \boldsymbol{\pi}\boldsymbol{b}_k - g)/(c_o + \boldsymbol{\pi}'\boldsymbol{b}_k - c_{fr} - b(\boldsymbol{e}_1, 0))$.

For the case $k = K$, there exists $b(\boldsymbol{e}_i, k) \leq c_{pr} + b(\boldsymbol{e}_1, 0)$. Thus, $c_o + \boldsymbol{\pi}'\boldsymbol{b}_k \leq c_o + c_{pr} + b(\boldsymbol{e}_1, 0) < c_{fr} + b(\boldsymbol{e}_1, 0)$ based on the assumption that $c_o + c_{pr} < c_{fr}$. Therefore, part (b) can be obtained based on the proof of part (a). ■

The following corollary specifies the sufficient condition under which NA is optimal; its proof follows directly from Lemma 3 and Lemma 4, and is omitted here.

Corollary 4: Suppose Assumptions 1-3 are all satisfied. If $\boldsymbol{\pi} \in (\Omega^k_{NA \leq OB} \cap \Omega^k_{OB \leq PM})$, $k \in \mathcal{K}'$, *or if* $R_K(\boldsymbol{\pi}) \geq \max\{1 - g/(c_{fr} - c_{pr}), (c_{fr} + b(\boldsymbol{e}_1, 0) - c_o - \boldsymbol{\pi}\boldsymbol{b}_K - g)/(c_{fr} + b(\boldsymbol{e}_1, 0) - c_o - \boldsymbol{\pi}'\boldsymbol{b}_K)\}$, *then* $a^*(\boldsymbol{\pi}, K) = NA$.

From the above-mentioned analysis, we can find that there are many good properties for the special case $k = K$. The detailed results are discussed in the following.

According to Lemma 3, it can be easily obtained that the action space for any order pair $(\boldsymbol{\pi}, K) \in \Omega \times K$ is represented as

$$\mathcal{A}(\boldsymbol{\pi}, K) = \begin{cases} \{NA, OB\}, & \boldsymbol{\pi}\boldsymbol{b}_K < c_{pr} + b(\boldsymbol{e}_1, 0) - c_o; \\ \{NA, PR\}, & \boldsymbol{\pi}\boldsymbol{b}_K \geq c_{pr} + b(\boldsymbol{e}_1, 0) - c_o. \end{cases}$$

Thus, for the fixed repair number K, the region $\{ \boldsymbol{\pi} : \boldsymbol{\pi}\boldsymbol{b}_K \geq c_{pr} - c_o + b(\boldsymbol{e}_1, 0) \}$ can be divided into two parts in terms of the threshold $1 - g/(c_{fr} - c_{pr})$ according to Lemma 2 and Theorem 1. One is Ω_{PR}^K, the other is doing nothing region. However, for the region $\{ \boldsymbol{\pi} : \boldsymbol{\pi}\boldsymbol{b}_K < c_{pr} - c_o + b(\boldsymbol{e}_1, 0) \}$, if $1 - g/(c_{fr} - c_{pr}) \geq (c_{fr} + b(\boldsymbol{e}_1, 0) - c_o - \boldsymbol{\pi}\boldsymbol{b}_K - g)/(c_{fr} + b(\boldsymbol{e}_1, 0) - c_o - \boldsymbol{\pi}'\boldsymbol{b}_K)$, it can also be be divided into two parts in terms of the threshold $1 - g/(c_{fr} - c_{pr})$ according to Corollary 2 and Corollary 4; otherwise, the bound distinguishing NA from OB cannot be determined exactly.

C. THE MONOTONIC AT-MOST-FOUR-REGION POLICY FOR $k = K$

To speed the computation time, the AM4R policy has been studied in several previous studies, eg. [17], [32], [33]. A monotonic AM4R structure means that the state space can be divided at most four regions, and the action corresponding to each region occurs with a certain order. In this paper, the similar results can also be established for the special case $k = K$. Specially, the monotonic AM4R policy for our maintenance model means that along any straight line of $\prec_{lr}$-ordered information states $\boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \ldots$ there are at most three numbers $0 \leq n_1^* \leq n_2^* \leq n_3^*$ to divide the optimal region as follows.

$$a^*(\boldsymbol{\pi}_n, K) = \begin{cases} NA, & n < n_1^* \text{ or } n_2^* < n \leq n_3^*, \\ OB, & n_1^* \leq n \leq n_2^*, \\ PR, & n > n_3^*. \end{cases} \tag{17}$$

Furthermore, as n increases, the action corresponding to each region occurs with the order NA, OB, NA, PR.

In order to obtain this AM4R structure for the special case $k = K$, the following lemma is provided firstly.

Lemma 5: $b(\boldsymbol{\pi}, k)$ given by Equation (9) is piecewise-linear concave for any fixed $k \in \mathcal{K}$.

Proof: According to Equation (8), it can be found that $b(\boldsymbol{\pi}, k)$ is piecewise-linear concave if $V_n(\boldsymbol{\pi}, k)$ is piecewise-linear concave for all n. Obviously, $OB_n(\boldsymbol{\pi}, k)$ and $PX_n(\boldsymbol{\pi}, k)$ are both linear concave functions of $\boldsymbol{\pi}$ for any fixed $k \in \mathcal{K}$. Considering the fact that the minimum of piecewise-linear concave functions is still piecewise-linear concave, we only need to show $NA_n(\boldsymbol{\pi}, k)$ is piecewise-linear concave. To achieve this aim, induction technique is adopted here. Suppose $V_0(\boldsymbol{\pi}, k) = 0$ for $\forall \boldsymbol{\pi}$ without loss of generality. $NA_1(\boldsymbol{\pi}, k) = c_{fr}(1 - R_k(\boldsymbol{\pi}))$ is linear in $\boldsymbol{\pi}$. Now, suppose that $NA_n(\boldsymbol{\pi}, k)$ is piecewise-linear concave, which implies that $V_n(\boldsymbol{\pi}, k) = \min\{\boldsymbol{\pi} \cdot \boldsymbol{u}_n^T; \boldsymbol{u}_n \in \boldsymbol{U}_n\}$ where $\boldsymbol{u}$ is a $1 \times (m+1)$ dimensional row vector. After that, the piecewise-linear concavity of $NA_{n+1}(\boldsymbol{\pi}, k)$ is examined. Clearly, the first term of $NA_{n+1}(\boldsymbol{\pi}, k)$, that is, $(c_{fr} + V_n(\boldsymbol{\pi}, k))(1 - R_k(\boldsymbol{\pi}))$ is linear in $\boldsymbol{\pi}$. Therefore, we only need to consider the second term which is $V_n(\boldsymbol{\pi}'(\boldsymbol{\pi}, k), k)R_k(\boldsymbol{\pi})$. There exists

$$\begin{aligned} &V_n(\boldsymbol{\pi}'(\boldsymbol{\pi}, k), k)R_k(\boldsymbol{\pi}) \\ &= \min\{\boldsymbol{\pi}'(\boldsymbol{\pi}, k), k) \cdot \boldsymbol{u}_n^T; \boldsymbol{u}_n \in \boldsymbol{U}_n\}R_k(\boldsymbol{\pi}) \\ &= \min\left\{\left[\frac{\sum_{i=1}^m \pi_i p_{i1}^k}{R_k(\boldsymbol{\pi})}, \ldots, \frac{\sum_{i=1}^m \pi_i p_{im}^k}{R_k(\boldsymbol{\pi})}, 0\right] \cdot \boldsymbol{u}_n^T\right\} R_k(\boldsymbol{\pi}) \\ &= \min\left\{\left[\sum_{i=1}^m \pi_i p_{i1}^k, \sum_{i=1}^m \pi_i p_{i2}^k, \ldots, \sum_{i=1}^m \pi_i p_{im}^k, 0\right] \cdot \boldsymbol{u}_n^T\right\} \\ &= \min\{\boldsymbol{\pi} \cdot \boldsymbol{u}_{n+1}^T; \boldsymbol{u}_{n+1} \in \boldsymbol{U}_{n+1}\} \end{aligned} \tag{18}$$

which is piecewise-linear concave. Thus, $NA_{n+1}(\boldsymbol{\pi}, k)$ is also piecewise-linear concave. As a result, $V_{n+1}(\boldsymbol{\pi}, k)$ is piecewise-linear concave and the claims holds for all n by induction. ■

According to the above Lemma 5, the following Theorem 2 can be obtained to show that the optimal maintenance policy for the case $k = K$ in our problem is characterized by the monotonic AM4R structure along a $\prec_{lr}$-increasing line.

Theorem 2: If $\mathbf{P}_K$ is TP2, the optimal policy for $k = K$ has the monotonic AM4R structure along any straight line of information states in $\prec_{lr}$-increasing order.

Proof: Let Ω_{NA}^k, Ω_{OB}^k and Ω_{PX}^k denote the set of information states corresponding to $a^*(\boldsymbol{\pi}, k) = NA$, $a^*(\boldsymbol{\pi}, k) = OB$ and $a^*(\boldsymbol{\pi}, k) = PX$, respectively. Since both $b_{OB}(\boldsymbol{\pi}, k)$ and $b_{PX}(\boldsymbol{\pi}, k)$ are hyperplanes, Ω_{PX}^k and Ω_{OB}^k are convex subsets of Ω according to Lemma 5 in this paper and Lemma 1 in [15]. While, $b_{NA}(\boldsymbol{\pi}, k)$ is not a hyperplane but a piecewise-linear function in $\boldsymbol{\pi}$, hence Ω_{NA}^k is not a convex subset. Furthermore, for $k = K$, once an information state $\boldsymbol{\pi}$ which increases along a $\prec_{lr}$-increasing line enters into Ω_{PR}^K, it will not leave Ω_{PR}^K due to the fact that $b_{PR}(\boldsymbol{\pi}, K)$ is constant and $b(\boldsymbol{\pi}, K)$ is nondecreasing in $\prec_{lr}$-increasing order. Hence, there are at most two NA regions, and then the action changes in the order $NA \rightarrow OB \rightarrow NA \rightarrow PR$ along any $\prec_{lr}$-ordered straight line of information states, which implies the claim holds. ■

Unfortunately, we cannot obtain the monotonic AM4R for any fixed $k \in \mathcal{K}'$ due to that $b_{PM}(\boldsymbol{\pi}, k)$ is not constant. In this situtaion, only the conclusion that Ω_{OB}^k and Ω_{PM}^k are both convex subsets of Ω can be achieved, which implies that either OB or PM action appears at most one times.

IV. ALGORITHMS

In this section, we use the structural properties obtained in the previous section to develop a more efficient and computationally tractable algorithm to determine the optimal maintenance action for each state $(\boldsymbol{\pi}, k) \in \Omega \times \mathcal{K}$.

Parameters such as the cost c_{fr}, c_{pr}, c_o, the transition probability matrix $\mathbf{P}_k$ and the maximal number of repair actions K are specified at first. From the analysis of the structural properties of the optimal policy, we should calculate the biases

$b(\boldsymbol{e}_i, k), i \in \mathcal{S}, k \in \mathcal{K}$. As far as we know, policy iteration and value iteration algorithms are widely used algorithms to solve the Markov decision problems [29]. Hence, we first use value iteration to obtain $b(\boldsymbol{e}_i, k)(i \in \mathcal{S}, k \in \mathcal{K})$ based on the sample path emanating from $\boldsymbol{e}_i$, $i \in \mathcal{S}'$. The details of the value iteration algorithm can be found in [29] and omitted here. Then for each state $(\boldsymbol{\pi}, k) \in \Omega \times \mathcal{K}$, the following algorithms are developed based on the structural properties provided in the previous section to obtain the optimal policy.

Algorithm 1 $k \in \mathcal{K}'$

1) *Suppose* $\boldsymbol{\pi} \in \Omega^k_{OB\leq PM}$. *If* $\boldsymbol{\pi} \in \Omega^k_{NA\leq OB}$, *then* $a^*(\boldsymbol{\pi}, k) = NA$; *otherwise, go to setp 4).*
2) *Suppose* $\boldsymbol{\pi} \notin \Omega^k_{OB\leq PM}$ *and* $c_{pm} + b(\boldsymbol{e}_m, k) \leq c_{fr} + b(\boldsymbol{e}_1, 0)$. *If* $R_k(\boldsymbol{\pi}) \leq 1 - g/(c_{fr} + b(\boldsymbol{e}_1, 0) - c_{pm} - \boldsymbol{\pi}'\mathbf{Q}\boldsymbol{b}_{k+1})$, *then* $a^*(\boldsymbol{\pi}, k) = PM$; *otherwise, if* $R_k(\boldsymbol{\pi}) \geq (c_{fr} + b(\boldsymbol{e}_1, 0) - c_{pm} - \boldsymbol{\pi}\mathbf{Q}\boldsymbol{b}_{k+1} - g)/(c_{fr} + b(\boldsymbol{e}_1, 0) - c_{pm} - \boldsymbol{\pi}'\mathbf{Q}\boldsymbol{b}_{k+1})$, *then* $a^*(\boldsymbol{\pi}, k) = NA$; *otherwise, go to setp 4).*
3) *Suppose* $\boldsymbol{\pi} \notin \Omega^k_{OB\leq PM}$ *and* $c_{pm} + b(\boldsymbol{e}_m, k) > c_{fr} + b(\boldsymbol{e}_1, 0)$. *If* $\boldsymbol{\pi} \in \Omega^k_{NA\leq PM}$, *then* $a^*(\boldsymbol{\pi}, k) = NA$; *otherwise, go to setp 4).*
4) *The following steps are used to decide the optimal policy for the remaining information states.*
   a) *Set* $l = 1$, $\boldsymbol{\pi}^1_k = \boldsymbol{\pi}$.
   b) *Compute* $\boldsymbol{\pi}^l_k = \boldsymbol{\pi}'(\boldsymbol{\pi}^{l-1}_k, k)$ *according to Eq. (1).*
   c) *If* $c_{pm} + b(\boldsymbol{e}_m, k) \leq c_{fr} + b(\boldsymbol{e}_1, 0)$ *and* $R_k(\boldsymbol{\pi}) \leq 1 - g/(c_{fr} + b(\boldsymbol{e}_1, 0) - c_{pm} - \boldsymbol{\pi}'\mathbf{Q}\boldsymbol{b}_{k+1})$, *or if* $c_{pm} + b(\boldsymbol{e}_m, k) > c_{fr} + b(\boldsymbol{e}_1, 0)$ *and* $\boldsymbol{\pi}^l_k \notin (\Omega^k_{NA\leq OB} \cup \Omega^k_{NA\leq PM})$, *then* $b(\boldsymbol{\pi}^l_k, k) = \min\{b_{OB}(\boldsymbol{\pi}^l_k, k), b_{PM}(\boldsymbol{\pi}^l_k, k)\}$ *which can be easily obtained through (11) and (13). Then apply the recursive set of (9) backward to get* $b(\boldsymbol{\pi}^{l-1}, k)$, $\ldots, b(\boldsymbol{\pi}, k)$. *Otherwise, go to setp d).*
   d) *If* $\|\boldsymbol{\pi}^{l+1}_k - \boldsymbol{\pi}^l_k\| \leq \varepsilon$, *then let* $\Pi_k(\boldsymbol{\pi}) = \pi^l_k$ *and replace both* $\boldsymbol{\pi}$ *and* $\boldsymbol{\pi}'(\boldsymbol{\pi}, k)$ *in Eq. (10) with* $\Pi_k(\boldsymbol{\pi})$ *to obtain* $b_{NA}(\Pi_k(\boldsymbol{\pi}), k)$. *Similarly, we can obtain* $b_{OB}(\Pi_k(\boldsymbol{\pi}), k)$ *and* $b_{PM}(\Pi_k(\boldsymbol{\pi}), k)$ *according to (11) and (13) respectively. Thus,* $b(\Pi_k(\boldsymbol{\pi}), k)$ *can be calculated through comparing the previous three terms. Finally, step backwards recursively along the sample path to get* $b(\boldsymbol{\pi}^{L_k-1}, k), \ldots, b(\boldsymbol{\pi}, k)$ *and the optimal maintenance action for the current state* $(\boldsymbol{\pi}, k)$. *Otherwise, set* $l = l + 1$ *and go back to the Step b).*

The above algorithm is constructed according to the action boundary expressions and can be used to find an optimal maintenance policy. Step 1), Step 2) and Step 3) are provided based on Corollary 2, Corollary 3 and Corollary 4 respectively. The optimal maintenance policy for those states which don't satisfy the conditions of Step 1)- Step 3) needs to be determined based on the sample path as Step a)-Step d) shows. Step c) means that there is no need to proceed until the sample path converges at a stationary state $\Pi_k(\boldsymbol{\pi})$ if $\boldsymbol{\pi}^l_k \notin (\Omega^k_{NA\leq OB} \cup \Omega^k_{NA\leq PM})$ for $c_{pm} + b(\boldsymbol{e}_m, k) > c_{fr} + b(\boldsymbol{e}_1, 0)$ or $R_k(\boldsymbol{\pi}) \leq 1 - g/(c_{fr} + b(\boldsymbol{e}_1, 0) - c_{pm} - \boldsymbol{\pi}'\mathbf{Q}\boldsymbol{b}_{k+1})$ for $c_{pm} + b(\boldsymbol{e}_m, k) \leq c_{fr} + b(\boldsymbol{e}_1, 0)$ satisfies. Clearly, the optimal maintenance action is not NA in this case. Hence, $b(\boldsymbol{\pi}^l_k, k)$ can be obtained through comparing $b_{OB}(\boldsymbol{\pi}^l_k, k)$ with $b_{PM}(\boldsymbol{\pi}^l_k, k)$. Then we can step backwards through (9) to get $b(\boldsymbol{\pi}, k)$ with its corresponding maintenance action. Otherwise, proceed to Step d). $b_{NA}(\Pi_k(\boldsymbol{\pi}), k)$ can then be calculated through replacing both $\boldsymbol{\pi}$ and $\boldsymbol{\pi}'(\boldsymbol{\pi}, k)$ in Eq. (10) with $\Pi_k(\boldsymbol{\pi})$ based on the fact that $b(\boldsymbol{e}_i, k), i = 1, \ldots, m, k \in \mathcal{K}'$ and the average cost g which have been obtained. In the same way, $b_{OB}(\Pi_k(\boldsymbol{\pi}), k)$ and $b_{PM}(\Pi_k(\boldsymbol{\pi}), k)$ can also be easily computed through (11) and (13) respectively. As a result, $b(\Pi_k(\boldsymbol{\pi}), k)$ can be determined. Finally, step backwards to $\boldsymbol{\pi}$ to determine the optimal policy.

Algorithm 2 $k = K$

1) *If* $a^*(\boldsymbol{\pi}, K) = PR$, *then* $a^*(\hat{\boldsymbol{\pi}}, K) = PR$ *for any* $\hat{\boldsymbol{\pi}}$ *subject to* $\boldsymbol{\pi} \prec_{lr} \hat{\boldsymbol{\pi}}$.
2) *Suppose* $R_K(\boldsymbol{\pi}) < 1 - g/(c_{fr} - c_{pr})$. *If* $\boldsymbol{\pi}\boldsymbol{b}_K < c_{pr} + b(\boldsymbol{e}_1, 0) - c_o$, *then* $a^*(\boldsymbol{\pi}, k) = OB$. *Otherwise,* $a^*(\boldsymbol{\pi}, k) = PR$.
3) *Suppose* $R_K(\boldsymbol{\pi}) \geq 1 - g/(c_{fr} - c_{pr})$. *If* $\boldsymbol{\pi}\boldsymbol{b}_K \geq c_{pr} + b(\boldsymbol{e}_1, 0) - c_o$, *or if* $R_K(\boldsymbol{\pi}) \geq (c_{fr} + b(\boldsymbol{e}_1, 0) - c_o - \boldsymbol{\pi}\boldsymbol{b}_K - g)/(c_{fr} + b(\boldsymbol{e}_1, 0) - c_o - \boldsymbol{\pi}'\boldsymbol{b}_K)$, *then* $a^*(\boldsymbol{\pi}, k) = NA$.
4) *Suppose* $1 - g/(c_{fr} - c_{pr}) \leq R_K(\boldsymbol{\pi}) \leq (c_{fr} + b(\boldsymbol{e}_1, 0) - c_o - \boldsymbol{\pi}\boldsymbol{b}_K - g)/(c_{fr} + b(\boldsymbol{e}_1, 0) - c_o - \boldsymbol{\pi}'\boldsymbol{b}_K)$ *and* $\boldsymbol{\pi}\boldsymbol{b}_K \geq c_{pr} + b(\boldsymbol{e}_1, 0) - c_o$. *Then the following steps are used to obtain the optimal maintenance policy instead of the pure recursive technique.*
   a) *Set* $l = 1$, $\boldsymbol{\pi}^1_K = \boldsymbol{\pi}$.
   b) *Calculate* $\boldsymbol{\pi}^l_K = \boldsymbol{\pi}'(\boldsymbol{\pi}^{l-1}_K, K)$ *according to (1).*
   c) *If* $R_K(\boldsymbol{\pi}^l_K) < 1 - g/(c_{fr} - c_{pr})$, *then* $b(\boldsymbol{\pi}^l_K, K) = \min\{b_{OB}(\boldsymbol{\pi}^l_K, K), b_{PR}(\boldsymbol{\pi}^l_K, K)\}$ *which can be easily obtained through (11) and (13). Then apply the recursive set of (9) backward to get* $b(\boldsymbol{\pi}^{l-1}, K)$, $\ldots, b(\boldsymbol{\pi}, K)$. *Otherwise, go to setp d).*
   d) *If* $\|\boldsymbol{\pi}^{l+1}_K - \boldsymbol{\pi}^l_K\| \leq \varepsilon$, *then let* $\Pi_K(\boldsymbol{\pi}) = \pi^l_K$ *and replace both* $\boldsymbol{\pi}$ *and* $\boldsymbol{\pi}'(\boldsymbol{\pi}, K)$ *in Eq. (10) with* $\Pi_K(\boldsymbol{\pi})$ *to obtain* $b_{NA}(\Pi_K(\boldsymbol{\pi}), K)$. *Then we obtain* $b(\Pi_K(\boldsymbol{\pi}), K)$ *through comparing* $b_{NA}(\Pi_K(\boldsymbol{\pi}), K)$ *and* $b_{OB}(\Pi_K(\boldsymbol{\pi}), K)$. *After that, step backwards recursively along the sample path through comparing* $b_{NA}(\boldsymbol{\pi}^l_K, K)$ *and* $b_{OB}(\boldsymbol{\pi}^l_K, K)$ *to get* $b(\boldsymbol{\pi}, K)$ *and the optimal maintenance action for the current state* $(\boldsymbol{\pi}, K)$. *Otherwise, set* $l = l + 1$ *and go back to the Step b).*

This algorithm is developed for the case $k = K$. In this situation, the recursive method can only be used for those states satisfying $1 - g/(c_{fr} - c_{pr}) \leq R_K(\boldsymbol{\pi}) \leq (c_{fr} + b(\boldsymbol{e}_1, 0) - c_o - \boldsymbol{\pi}\boldsymbol{b}_K - g)/(c_{fr} + b(\boldsymbol{e}_1, 0) - c_o - \boldsymbol{\pi}'\boldsymbol{b}_K)$ and $\boldsymbol{\pi}\boldsymbol{b}_K \geq c_{pr} + b(\boldsymbol{e}_1, 0) - c_o$. Even that, we also don't need to proceed until the sample path converges. Once the state which varies along the sample path satisfies $R_K(\boldsymbol{\pi}^l_K) < 1 - g/(c_{fr} - c_{pr})$, we can obtain $b(\boldsymbol{\pi}^l_K, K)$ through comparing $b_{OB}(\boldsymbol{\pi}^l_K, K)$ and $b_{PR}(\boldsymbol{\pi}^l_K, K)$ according to the part (b) of Lemma 2.

V. A NUMERICAL EXAMPLE

Since maintenance for the system with repair time limit is scarcely considered in the practical life, there exists few real data to generate the probability transition matrix. Thus, in this section, a numerical example is studied to illustrate the effectiveness of the optimal maintenance decision algorithms proposed in the previous section. Assume that the deterioration condition can be categorized 5 levels, i.e. $m = 4$, and the maximal number of available repairs K is 8. As done in [11], we introduce the functions $g(i) = 0.05 + 0.005(i-1)$ for $i \in \mathcal{S}$, $h(k) = 1 + 0.04k$ for $k \in \mathcal{K}$ and a stochastic matrix $\mathbf{D} = [d_{ij}]_{4\times4}, i, j \in \mathcal{S}'$ in order to define the probability transition matrix, where

$$\mathbf{D} = \begin{bmatrix} 0.895 & 0.1 & 0 & 0.005 \\ 0 & 0.9 & 0.01 & 0.09 \\ 0 & 0 & 0.9 & 0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{19}$$

Then we define the probability transition matrix $\mathbf{P}_k$ as follows:

$$p^k_{ij} = \begin{cases} g(i)h(k), & \text{for } i \in \mathcal{S}', j = m+1, k \in \mathcal{K}; \\ (1 - g(i)h(k))\, d_{ij}, & \text{for } i, j \in \mathcal{S}' \text{ and } k \in \mathcal{K}; \\ 1, & \text{for } i = j = m+1, k \in \mathcal{K}; \\ 0, & \text{otherwise.} \end{cases} \tag{20}$$

By choosing appropriate $\mathbf{D}$, we can guarantee the probability transition matrix $\mathbf{P}_k, k \in \mathcal{K}$ satisfies Assumptions 1-3. In addition, maintenance costs are given as follows: $c_o = 1$, $c_{pm} = 30$, $c_{pr} = 120$ and $c_{fr} = 500$.

Finally, according to the definition of the maintenance effect matrix and the Assumption 3, the matrix $\mathbf{Q}$ is chosen as follows:

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.95 & 0.05 & 0 & 0 & 0 \\ 0.90 & 0.075 & 0.025 & 0 & 0 \\ 0.8 & 0.1 & 0.05 & 0.05 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{21}$$

To begin with, value iteration algorithm is adopted to get the biases $b(\boldsymbol{e}_i, k), i \in \mathcal{S}, k \in \mathcal{K}$ and average cost g. In this example, the cost per unit time g is 28.4116. Fig. 1 shows the evolution of the bias value of $b(\boldsymbol{e}_i, k)$ as a function of $i \in \mathcal{S}'$ for a fixed $k \in \mathcal{K}$. From Fig. 1, we can find that the bias $b(\boldsymbol{e}_i, k)$ is increasing in i for a fixed k, which is in accordance with Lemma 1. Furthermore, it is obvious that $b(\boldsymbol{e}_4, 8) < b(\boldsymbol{e}_4, 6)$, which implies that the monotonity of $b(\boldsymbol{e}_i, k)$ in k for the fixed $\boldsymbol{e}_i$ cannot be obtained always.

Due to space limitation, only the optimal maintenance policies for $k = 0$ and 8 are provided in Figs. 2 and 3 respectively for the purpose of illustration. For clarification, we manually set $\pi_1 = 0$, which doesn't affect the correctness of the result, and obtain the optimal maintenance policies for each $k \in \mathcal{K}$ shown in Fig. 4. Through examining Figs. 2-4,

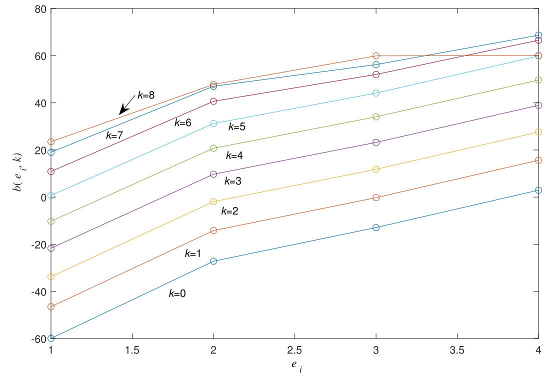

FIGURE 1. $b(\boldsymbol{e}_i, k)$ **versus** $k \in \mathcal{K}$.

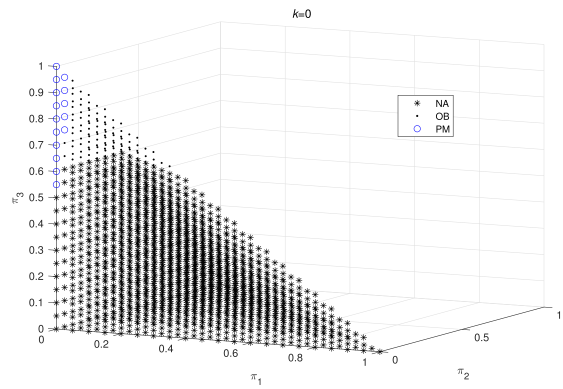

FIGURE 2. **Optimal maintenance decision rules for** $k = 0$.

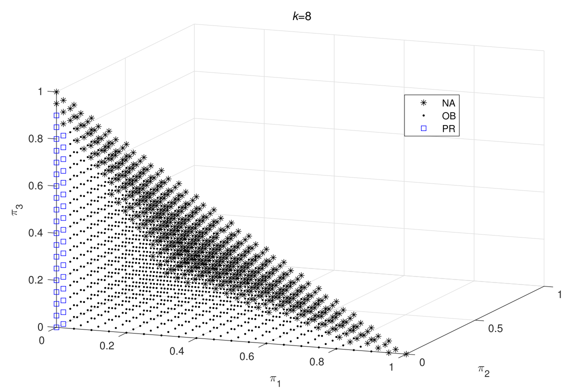

FIGURE 3. **Optimal maintenance decision rules for** $k = 0$.

we can find that if the repaired times $k (0 \le k \le 7)$ gest larger, the region corresponding to NA gets smaller, while the region corresponding to OB or PM gets larger. This is because the system subject to more repair times is more likely to get worse, and naturally needs more observation actions to reveal the true deterioration level for making proper maintenance actions. However, for the case $k = 8$, the region corresponding to NA looks larger than that for the case $k = 7$, which is due to the fact that the PR costs are larger than the PM costs.

Figs. 5 and 6 superimposed the action boundaries developed in Section III-B on the optimal maintenance decision rules for the system repaired $k = 6$ and K times respectively.

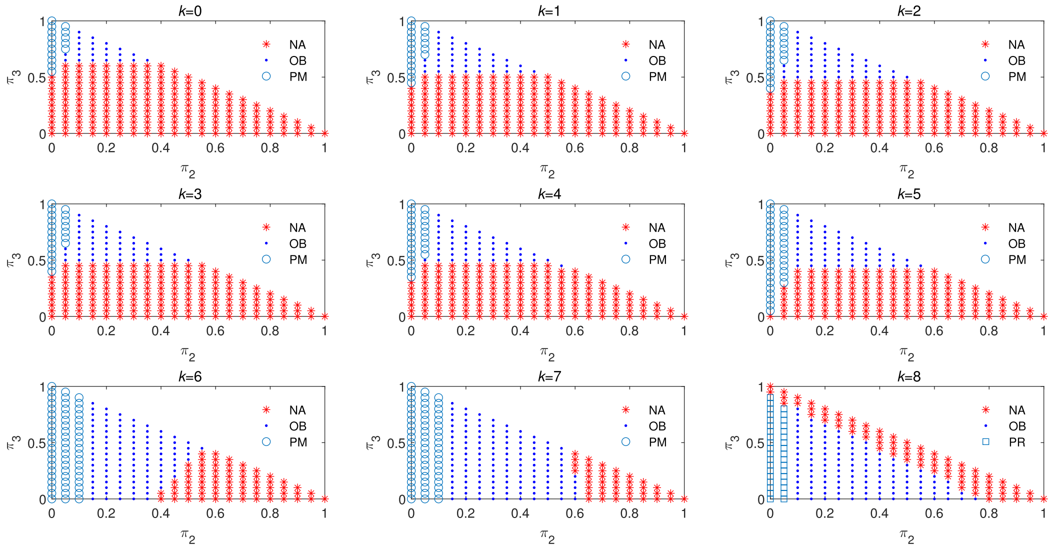

FIGURE 4. Optimal maintenance decision rules with taking maintenance effect into consideration.

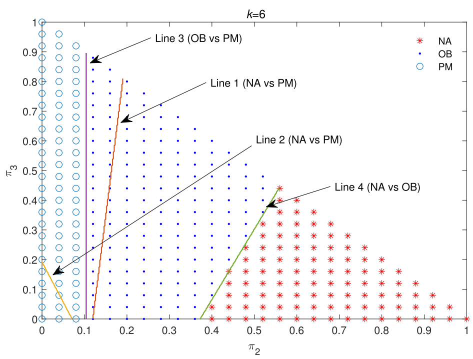

FIGURE 5. Action boundaries superimposed on the optimal policy when $k = 6$.

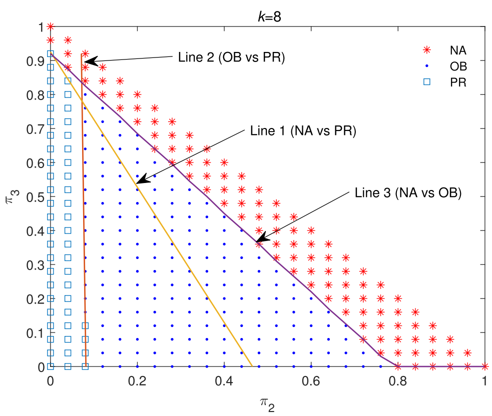

FIGURE 6. Action boundaries superimposed on the optimal policy when $k = 8$.

It should be noted that, we also only select two typical cases for illustration. For $k = 6$, since $c_{pm} + b(\boldsymbol{e}_4, 6) = 96.5403 < c_{fr} + b(\boldsymbol{e}_1, 0) = 440.0324$, two lines can be obtained according to Corollary 1 to depict the preference of NA to PM, or vice versa. The two lines are shown in in Fig. 5 as Line 1 and Line 2, where Line 1 defines the region where NA is preferred to PM with $R_k(\boldsymbol{\pi}) = (c_{fr} + b(\boldsymbol{e}_1, 0) - c_{pm} - \boldsymbol{\pi}\mathbf{Q}\boldsymbol{b}_{k+1} - g)/(c_{fr} + b(\boldsymbol{e}_1, 0) - c_{pm} - \boldsymbol{\pi}'\mathbf{Q}\boldsymbol{b}_{k+1})$, while Line 2 depicts the preference of PM to NA with $R_k(\boldsymbol{\pi}) = 1 - g/(c_{fr} + b(\boldsymbol{e}_1, 0) - c_{pm} - \boldsymbol{\pi}'\mathbf{Q}\boldsymbol{b}_{k+1})$. However, for $k = K$, only one line is needed to depict the preference of NA to PM, or vice versa. That line is shown as Line 1 in Fig. 6, and obtained with $R_K(\boldsymbol{\pi}) = 1 - g/(c_{fr} - c_{pr})$. Line 3 in Fig. 5 and Line 2 in Fig. 6 are obtained through comparison of b_{OB} and b_{PX} with $\boldsymbol{\pi}(\boldsymbol{b}_k - \mathbf{Q}\boldsymbol{b}_{k+1}) = c_{pm} - c_o$ and $\boldsymbol{\pi}\boldsymbol{b}_K = c_{pr} + b(\boldsymbol{e}_1, 0) - c_o$, respectively. Finally, Line 4 in Fig. 5 and Line 3 in Fig. 6 depict the preference of NA to OB with $(c_o + \boldsymbol{\pi}'\boldsymbol{b}_k - c_{fr} - b(\boldsymbol{e}_1, 0))R_k(\boldsymbol{\pi}) = g + c_o + \boldsymbol{\pi}\boldsymbol{b}_k - c_{fr} - b(\boldsymbol{e}_1, 0)$ and $R_K(\boldsymbol{\pi}) = (c_{fr} + b(\boldsymbol{e}_1, 0) - c_o - \boldsymbol{\pi}\boldsymbol{b}_K - g)/(c_{fr} + b(\boldsymbol{e}_1, 0) - c_o - \boldsymbol{\pi}'\boldsymbol{b}_K)$, respectively.

For the purpose of comparison, we study the optimal maintenance policies without taking the maintenance effect into consideration. In this case, the matrix $\mathbf{Q}$ is chosen as follows:

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{22}$$

The average cost g in this case is 27.9564 which is smaller than that in the case with taking maintenance effect into consideration. Similarly, we manually set $\pi_1 = 0$ and obtain the optimal maintenance decision rules without considering the maintenance effect shown in Fig. 7. It can be found that Fig. 7 is very similar to Fig. 4(a) in [17], which partially verifies the correctness of our results. Obviously, the optimal decision rules with or without maintenance effect are different. For example, if $k = 0$ and we further set $\pi_2 = 0$, the optimal maintenance action with maintenance effect changes from PM to NA when the system gets worse, but vice versa in the case without maintenance effect. This difference can be

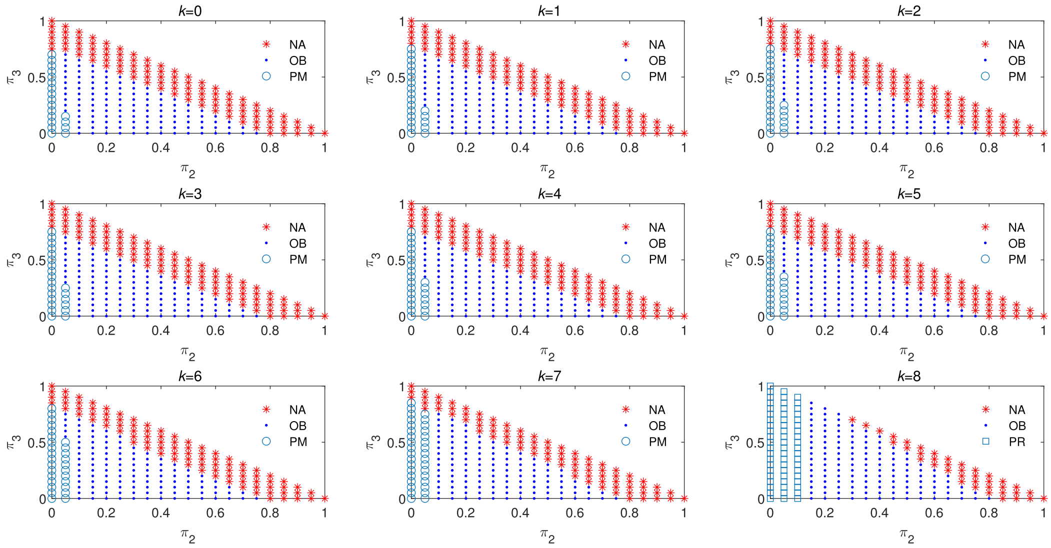

FIGURE 7. Optimal maintenance decision rules without taking maintenance effect into consideration.

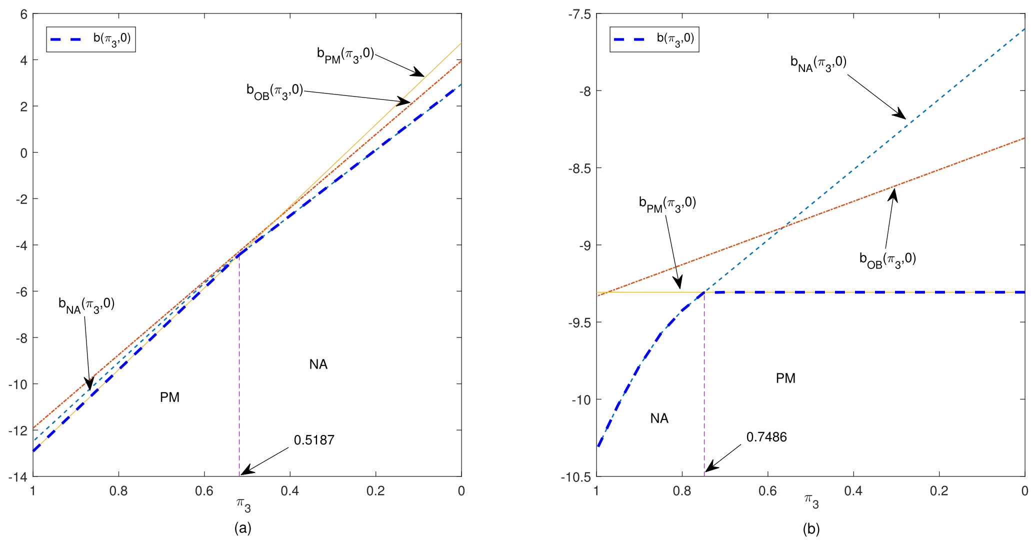

FIGURE 8. The bias value versus π_3 with or without maintenance effect for $k = 0$. (a) Bias value versus π_3 with maintenance effect. (b) Bias value versus π_3 without maintenance effect.

clearly illustrated in Fig. 8 which is a plot of $b_{NA}(\boldsymbol{\pi}, k)$, $b_{OB}(\boldsymbol{\pi}, k)$, $b_{PM}(\boldsymbol{\pi}, k)$ and $b(\boldsymbol{\pi}, k)$ as a function of the value of π_3 using $k = 0$, $\pi_1 = \pi_2 = 0$ with or without maintenance effect. The major reason for the difference is composed of two aspects. One is that we have set $\pi_1 = \pi_2 = 0$ which means the system suffers from more deteriorated condition. The other is that the PM cost is constant for the case without maintenance effect, while increasing in $\boldsymbol{\pi}$ for the case with maintenance effect. Fig. 8 also shows that $b(\pi_3, 0)$ with or without maintenance effect is a concave function which is consistent with Lemma 5.

VI. CONCLUSION AND THE FUTURE WORK

In this paper, we consider the problem of optimally maintaining a multi-state system under the constraint that only a limited number of imperfect repairs can be performed. Since the observation action is perfect but non-periodical, the problem can be modeled as a POMDP to adaptively schedule observation and preventive maintenance actions, which is different from the work [11]. After combining the information state together with the completed repair number as the state variable, the problem is reformulated as a Markov decision process. In order to increase the computational efficiency, several structural properties are developed, and then the related maintenance algorithms are developed. Finally, a numerical example is provided to validate the claims in our paper as well as the effectiveness of the proposed algorithms.

In the future, we will proceed the work from the following aspects. First, due to the widely use of sensors, focus on condition-based maintenance grows more rapidly in recent years. Furthermore, the collected sensor information is always imperfect. Therefore, it is necessary to introduce the effect of imperfect observation into the proposed maintenance model proposed in this paper. In addition, the sensitivity analysis of probability transition matrix should be included in the future work.

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