Received May 13, 2019, accepted May 28, 2019, date of publication May 30, 2019, date of current version June 11, 2019.

*Digital Object Identifier 10.1109/ACCESS.2019.2919979*

# Influence of Time Delay on Bifurcation in Fractional Order BAM Neural Networks With Four Delays

**CHANGJIN XU[1], MAOXIN LIAO[2], PEILUAN LI[3], AND JINLING YAN[3]**

[1]Guizhou Key Laboratory of Economics System Simulation, Guizhou University of Finance and Economics, Guiyang 550004, China
[2]School of Mathematics and Physics, University of South China, Hengyang 421001, China
[3]School of Mathematics and Statistics, Henan University of Science and Technology, Luoyang 471023, China

Corresponding author: Changjin Xu (xcj403@126.com)

This work was supported in part by the National Natural Science Foundation of China under Grant 61673008, in part by the Project of High-level Innovative Talents of Guizhou Province under Grant [2016]5651, in part by the Major Research Project of the Innovation Group of the Education Department of Guizhou Province under Grant [2017]039, in part by the Project of Key Laboratory of Guizhou Province with Financial and Physical Features under Grant [2017]004, in part by the Foundation of Science and Technology of Guizhou Province under Grant [2018]1025, and in part by the Innovative Exploration Project of Guizhou University of Finance and Economics under Grant [2017]5736-015.

**ABSTRACT** Over the past several decades, numerous scholars have studied the stability and Hopf bifurcation problem of integer-order delayed neural networks. However, the fruits about the stability and Hopf bifurcation for fractional-order delayed neural networks are very scarce. In this paper, we will consider the stability and the existence of Hopf bifurcation of fractional-order bidirectional associative memory (BAM) neural networks with four delays. A set of sufficient criteria to ensure the stability and the existence of Hopf bifurcation for the fractional-order BAM neural networks with four delays are established by choosing the sum of two different delays as a bifurcation parameter. This paper manifests that the delay has an important influence on the stability and Hopf bifurcation of involved networks. An example is displayed to test the rationality of the derived theoretical findings. The derived results of this paper are new and play a key role in optimizing networks and improving human life.

**INDEX TERMS** BAM neural networks, stability, Hopf bifurcation, fractional order, delay.

## I. INTRODUCTION

At present, neural networks have attracted a great deal of attention from various areas due to their promising application in signal and image processing, optimization solvers, intelligent control, quadratic optimization, automatic control and so on [1]. Since the classical work of Marcus and Westervelt [2], great progress on the research on neural networks has been made during the last few decades. The study on the dynamical behavior of neural networks plays an important role in designing neural networks and serving human beings.

Since the neuron amplifiers and the communication time between two neurons have the finite switching speed, so it is suitable to introduce the time delay into neural networks. The investigation shows that time delay can affect the dynamics of neural networks. Based on this point, the impact of time delay on the dynamical behaviors of delayed neural networks has become a hot topic and focus issue in mathematical fields and various engineering disciplines. In recent years, numerous excellent and interesting results on various dynamics on delayed neural networks are springing up. For example, Li *et al.* [3] investigated the periodicity and stability of impulsive neural networks, Li and Yang [4], Li and Li [5], and Aouiti *et al.* [6] considered the almost automorphic solution, almost periodic solution, pseudo almost periodic solution of neural networks, Xu and Li [7], [8] pointed out that leakage delay and proportional delay has important effect on the global exponential convergence of neural networks, Long [9] established new sufficient conditions to ensure the existence and global exponentially stability of involved neural networks. For more detailed works, we refer the readers to [10]–[29].

The fractional calculus is generalization of ordinary differentiation and integration to arbitrary non-integer order [27], [30], [57]–[60]. No progress about the fractional

The associate editor coordinating the review of this manuscript and approving it for publication was Rathinasamy Sakthivel.

calculus has been made for a long time in virtue of the lack of actual background and theoretical basis. Until recently, a lot of researchers find that fractional-order differential equations are important instruments which can be used to model various phenomena in many areas and describe memory, hereditary nature of all sorts of materials and processes. Thus it is more reasonable to depict the objective problems by fractional-order derivatives than the classical integer-order ones. The research shows that fractional-order differential equations have widely applied in numerous fields such as material sciences, fluid mechanics, biology, medicine, etc. [31]. Based on the idea, the incorporation of memory element into neural networks will display great theoretical value and practical significance. Nowadays, the study on fractional-order neural networks has become a hot issue with great development prospect. Many outstanding results on this aspect have been available. For instance, Ding *et al.* [30] studied the global Mittag-Leffler synchronization for fractional-order neural networks, Yang *et al.* [31] obtained the sufficient conditions to ensure the finite-time stability of fractional-order delayed neural networks, Chen *et al.* [32] discussed the global Mittag-Leffler stability and synchronization issue of memristor-based fractional-order neural networks. For more related works, we refer the readers to [33]–[37], [61]–[63].

Hopf bifurcation is an important dynamical behavior of delayed differential equations. Hopf bifurcation phenomena of integer-order neural networks have been widely studied. A great deal of significant results have been reported (see [38]–[44]). However, the analysis methods on Hopf bifuration of inter-order systems can not be simply applied to focus on the Hopf bifurcation of fractional-order differential equations. Recently, some authors have analyzed Hopf bifurcation problems for fractional-order neural networks (see [45]–[53]).

Here we would like to point out that the majority of the papers mentioned above consider the Hopf bifurcation of neural networks with single delay. Up to now, there are rare papers that handle the Hopf bifurcation problem of neural networks with multiple delays. Due to the increase of the number of delay, the characteristic equation of involved neural networks will be more complicated than that of neural networks with single delay. Maybe we will face some new challenge.

Stimulated by the discussion above, it is necessary for us to investigate the Hopf bifurcation of neural networks with multiple delays. It is well known that the delayed BAM neural networks take the following form:

$$\begin{cases} \dot{u}_l(t) = -\alpha_l u_l(t) + \sum_{p=1}^{m} a_{pl} h_l(v_j(t-\sigma_{pl})) + U_l, \\ \dot{v}_p(t) = -\beta_p v_p(t) + \sum_{l=1}^{n} b_{lp} r_p(u_i(t-\varrho_{lp})) + V_p, \end{cases} \tag{1}$$

where $l = 1, 2, \cdots, n; p = 1, 2, \cdots, m$, $a_{pl}, b_{lp} (l = 1, 2, \cdots, n; p = 1, 2, \cdots, m)$ are the connection weights through neurons in two layers: the $U$-layer and $V$-layer; $\alpha_l$ and $\beta_p$ describe the stability of internal neuron processes on the $U$-layer and $V$-layer, respectively. On the $U$-layer, the neurons whose states are denoted by $u_l(t)$ receive the inputs $U_l$ and the inputs outputted by those neurons in the $J$-layer via activation functions $h_l$, while on the $V$-layer, the neurons whose associated states are denoted by $v_p(t)$ receive the inputs $V_p$ and the inputs outputted by those neurons in the $U$-layer via activation functions $r_p$ (see [49]). Although system (1) can be mathematically regarded as Hopfied-type neural networks with dimension $n+m$, it is really produces many nice properties due to the special structure of connection weights and has practical applications in storing paired patterns or memories. In details, one can see [50].

For model (1), Huang *et al.* [48] assumed that there are two neurons on the $U$-layer and one neuron on the $V$-layer and the time delay from the neuron $v_1$ on $V$-layer to neurons $u_1, u_2$ on $U$-layer are $\sigma_1$ and $\sigma_2$, respectively; The time delays from the neurons $u_1$ and $\sigma_2$ on $U$-layer to the neuron $v_1$ on $V$-layer are $\sigma_3$ and $\sigma_4$, respectively. Then he obtain the following simplified BAM neural network

$$\begin{cases} \dot{u}_1(t) = -\alpha_1 u_1(t) + \gamma_{11} h_1(v_1(t-\sigma_1)), \\ \dot{u}_2(t) = -\alpha_2 u_2(t) + \gamma_{12} h_2(v_1(t-\sigma_2)), \\ \dot{v}_1(t) = -\alpha_3 v_1(t) + \gamma_{13} h_3(u_1(t-\sigma_3)) \\ \qquad +\gamma_{23} h_3(u_2(t-\sigma_4)), \end{cases} \tag{2}$$

where $\alpha_i (i = 1, 2)$ and $\alpha_3$ describe the stability of internal neuron processes on the $U$-layer and $V$-layer, respectively, $\gamma_{11}, \gamma_{12}, \gamma_{13}, \gamma_{23}$ are the connection weights through neurons in two layers, $h_1, h_2$ and $h_3$ stand for different activation functions, $\sigma_i (i = 1, 2, 3, 4)$ is time delay. Taking the sum of the delays $\sigma = \sigma_1 + \sigma_3 = \sigma_2 + \sigma_4$ as a parameter, the author investigated the stability and the existence of Hopf bifurcation and Hopf bifurcation nature. In addition, the estimation of the length of delay to preserve stability had been analyzed. The global existence of Hopf bifurcation of system (2) was studied by $S^1$-equivariant degree ([51-52]).

Based on the analysis above, we establish a fractional-order version of model (2) as follows:

$$\begin{cases} \mathcal{D}^{\vartheta} u_1(t) = -\alpha_1 u_1(t) + \gamma_{11} h_1(v_1(t-\sigma_1)), \\ \mathcal{D}^{\vartheta} u_2(t) = -\alpha_2 u_2(t) + \gamma_{12} h_2(v_1(t-\sigma_2)), \\ \mathcal{D}^{\vartheta} v_1(t) = -\alpha_3 v_1(t) + \gamma_{13} h_3(u_1(t-\sigma_3)) \\ \qquad +\gamma_{23} h_3(u_2(t-\sigma_4)), \end{cases} \tag{3}$$

where $\vartheta \in (0, 1]$ and $\alpha_i > 0 (i = 1, 2, 3)$. All other parameters have the same implication as those in (2).

The main object of this article is to consider the impact of time delay on the stability and the existence of Hopf bifurcation of system (3).

In order to obtain the key results of system (3), the following assumptions are made:

(A1) For $i = 1, 2, 3$, $\alpha_i > 0$, $h_i \in C^1$, $h_i(0) = 0$.

(A2) $\sigma_1 + \sigma_3 = \sigma_2 + \sigma_4 = \sigma$.

The highlights of this manuscript are listed as follows:

**(a)** We extend the integer-order neural networks with multiple delays to fractional-order neural networks with multiple delays, which can describe the memory and hereditary properties of neural networks better.

**(b)** The sufficient conditions of stability and the existence of Hopf bifurcation of fractional-order neural networks with multiple delays have been established. The study shows that the delay has important influence on the stability and the existence of Hopf bifurcation of involved networks.

**(c)** Up to now, there are rare papers that investigate the Hopf bifurcation on fractional-order differential models with multiple delays. Many works focus on the Hopf bifurcation of fractional-order differential equations with single delay. Our obtained results enrich the Hopf bifurcation theory of fractional-order delayed differential equations and complete the earlier published manuscripts.

**(d)** The idea of this research can be transferred to discuss numerous other fractional-order delayed differential models.

The rest of this article is planned as follows. In segment 2, we some notations and preliminary results on fractional calculus are prepared. In segment 3, the influence of delay on stability and Hopf bifurcation of (3) has been revealed. Numerical simulations to illustrate the correctness of the theoretical findings are conducted in segment 4. At last, a brief conclusion is included.

## II. PRELIMINARY RESULTS

In this section, the related knowledge about fractional calculus will be prepared.

*Definition 1 [54]: The fractional integral of order $\theta$ for a function $h(\varsigma)$ is defined as follows*:

$$I^{\vartheta}h(\varsigma)=\frac{1}{\Gamma(\vartheta)}\int_{\varsigma_0}^{\varsigma}(\varsigma-s)^{\vartheta-1}h(s)ds,$$

*where $\varsigma\geq\varsigma_0$, $\vartheta>0$, $\Gamma(s)=\int_0^{\infty}\varsigma^{s-1}e^{-\varsigma}d\varsigma$.*

*Definition 2 [54]: Let $h(\varsigma)\in([\varsigma_0,\infty),R)$. Define the Caputo fractional-order derivative of order $\vartheta$ as follows*:

$$\mathcal{D}^{\vartheta}h(\varsigma)=\frac{1}{\Gamma(\iota-\vartheta)}\int_{\varsigma_0}^{\varsigma}\frac{h^{(\iota)}(s)}{(\varsigma-s)^{\vartheta-\iota+1}}ds,$$

*where $\varsigma\geq\varsigma_0$ and $\iota$ is a positive integer which satisfies $\iota-1\leq\vartheta<\iota$. Typically, if $0<\vartheta<1$, then*

$$\mathcal{D}^{\vartheta}h(v)=\frac{1}{\Gamma(1-\vartheta)}\int_{\varsigma_0}^{\varsigma}\frac{h'(s)}{(\varsigma-s)^{\vartheta}}ds.$$

*Lemma 1 [55]: Give the autonomous system $\mathcal{D}^{\vartheta}z=\mathcal{A}z$, $z(0)=z_0$ where $0<\vartheta<1$, $z\in R^m$, $\mathcal{A}\in R^{m\times m}$. Let $\lambda_i(i=1,2,\cdots,m)$ be the root of the characteristic equation of $\mathcal{D}^{\vartheta}z=\mathcal{A}z$. Then system $\mathcal{D}^{\vartheta}z=\mathcal{A}z$ is asymptotically stable $\Leftrightarrow|arg(\lambda_i)|>\frac{\vartheta\pi}{2}(i=1,2,\cdots,m)$. this system is stable $\Leftrightarrow|arg(\lambda_i)|>\frac{\vartheta\pi}{2}(i=1,2,\cdots,m)$ and those critical eigenvalues that satisfy $|arg(\lambda_i)|=\frac{\vartheta\pi}{2}(i=1,2,\cdots,m)$ possess geometric multiplicity one.*

## III. IMPACT OF DELAY ON HOPF BIFURCATION FOR FRACTIONAL BAM NEURAL NETWORKS

In this section, we will discuss the impact of time delay on the Hopf bifurcation of the fractional BAM neural networks. By (A1), we know that the equilibrium point of system is the zero.

Let $w_1(t)=u_1(t-\sigma_3)$, $w_2(t)=u_2(t-\sigma_4)$, $w_3(t)=v_1(t)$. Then (3) can be written as

$$\begin{cases}\mathcal{D}^{\vartheta}w_1(t)=-\alpha_1w_1(t)+\gamma_{11}h_1(w_3(t-\sigma)),\\ \mathcal{D}^{\vartheta}w_2(t)=-\alpha_2w_2(t)+\gamma_{12}h_2(w_3(t-\sigma)),\\ \mathcal{D}^{\vartheta}w_3(t)=-\alpha_3w_3(t)+\gamma_{13}h_3(w_1(t))\\ \qquad+\gamma_{23}h_3(w_2(t)).\end{cases}\tag{4}$$

The linear equation of (4) near the zero equilibrium point can be written as follows:

$$\begin{cases}\mathcal{D}^{\vartheta}w_1(t)=-\alpha_1w_1(t)+\rho_{11}w_3(t-\sigma),\\ \mathcal{D}^{\vartheta}w_2(t)=-\alpha_2w_2(t)+\rho_{12}w_3(t-\sigma),\\ \mathcal{D}^{\vartheta}w_3(t)=-\alpha_3w_3(t)+\rho_{13}w_1(t)+\rho_{23}w_2(t),\end{cases}\tag{5}$$

where $\rho_{1l}=\gamma_{1l}h_l'(0)(l=1,2,3)$, $\rho_{23}=\gamma_{23}h_3'(0)$. The corresponding characteristic equation of (5) is given by

$$\det\begin{bmatrix}s^{\vartheta}+\alpha_1 & 0 & -\rho_{11}e^{-s\sigma}\\ 0 & s^{\vartheta}+\alpha_2 & -\rho_{12}e^{-s\sigma}\\ -\rho_{13} & -\rho_{23} & s^{\vartheta}+\alpha_3\end{bmatrix}.\tag{6}$$

Then

$$s^{3\vartheta}+a_1s^{2\vartheta}+a_2s^{\vartheta}+a_3+\left(a_4s^{\vartheta}+a_5\right)e^{-s\sigma}=0,\tag{7}$$

where

$$\begin{aligned}a_1&=\alpha_1\alpha_2\alpha_3,\\ a_2&=\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3,\\ a_3&=\alpha_1\alpha_2\alpha_3,\\ a_4&=-(\rho_{11}\rho_{13}+\rho_{12}\rho_{23}),\\ a_5&=-(\rho_{11}\rho_{13}\alpha_2+\rho_{12}\rho_{23}\alpha_1).\end{aligned}$$

Denote

$$\begin{aligned}A_1(s)&=s^{3\vartheta}+a_1s^{2\vartheta}+a_2s^{\vartheta}+a_3,\\ A_2(s)&=a_4s^{\vartheta}+a_5.\end{aligned}$$

Then (7) takes the form:

$$\mathcal{A}_1(s)+\mathcal{A}_2(s)e^{-s\sigma}=0.\tag{8}$$

If $s=i\varphi=\varphi\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$ is a root of (8). By (8), one gets

$$\begin{cases}\mathcal{A}_{2R}(\varphi)\cos\varphi\sigma+\mathcal{A}_{2I}(\varphi)\sin\varphi\sigma=-\mathcal{A}_{1R}(\varphi),\\ \mathcal{A}_{2I}(\varphi)\cos\varphi\sigma-\mathcal{A}_{2R}(\varphi)\sin\varphi\sigma=-\mathcal{A}_{1I}(\varphi),\end{cases}\tag{9}$$

where $\mathcal{A}_{iR}$, $\mathcal{A}_{iI}$ are the real parts and imaginary parts of $\mathcal{A}_i(s)(i=1,2)$, respectively, which are given by

$$\begin{aligned}\mathcal{A}_{1R}(\varphi)=\varphi^{3\vartheta}\cos\frac{3\pi\vartheta}{2}+a_1\varphi^{2\vartheta}\cos\vartheta\pi\\ +a_2\varphi^{\vartheta}\cos\frac{\pi\vartheta}{2}+a_3,\end{aligned}$$

$$\mathcal{A}_{1I}(\varphi) = \varphi^{3\vartheta}\sin\frac{3\pi\vartheta}{2} + a_1\varphi^{2\vartheta}\sin\vartheta\pi + a_2\varphi^{\vartheta}\sin\frac{\pi\vartheta}{2},$$
$$\mathcal{A}_{2R}(\varphi) = a_4\varphi^{\vartheta}\cos\frac{\pi\vartheta}{2} + a_5,$$
$$\mathcal{A}_{2I}(\varphi) = a_4\varphi^{\vartheta}\sin\frac{\pi\vartheta}{2}.$$

By (9), one has

$$\begin{cases}\cos\varphi\sigma = -\dfrac{\mathcal{A}_{1R}(\varphi)\mathcal{A}_{2R}(\varphi) + \mathcal{A}_{1I}(\varphi)\mathcal{A}_{2I}(\varphi)}{(\mathcal{A}_{2R}(\varphi))^2 + (\mathcal{A}_{2I}(\varphi))^2},\\ \sin\varphi\sigma = \dfrac{\mathcal{A}_{1I}(\varphi)\mathcal{A}_{2R}(\varphi) - \mathcal{A}_{1R}(\varphi)\mathcal{A}_{2I}(\varphi)}{(\mathcal{A}_{2R}(\varphi))^2 + (\mathcal{A}_{2I}(\varphi))^2}.\end{cases} \tag{10}$$

By (10), we get

$$[\mathcal{A}_{1R}(\varphi)\mathcal{A}_{2R}(\varphi) + \mathcal{A}_{1I}(\varphi)\mathcal{A}_{2I}(\varphi)]^2 + [\mathcal{A}_{1I}(\varphi)\mathcal{A}_{2R}(\varphi) - \mathcal{A}_{1R}(\varphi)\mathcal{A}_{2I}(\varphi)]^2 = [(\mathcal{A}_{2R}(\varphi))^2 + (\mathcal{A}_{2I}(\varphi))^2]^2. \tag{11}$$

Notice that

$$[\mathcal{A}_{1R}(\varphi)\mathcal{A}_{2R}(\varphi) + \mathcal{A}_{1I}(\varphi)\mathcal{A}_{2I}(\varphi)]^2 = b_1\varphi^{8\vartheta} + b_2\varphi^{7\vartheta} + b_3\varphi^{6\vartheta} + b_4\varphi^{5\vartheta} + b_5\varphi^{4\vartheta} + b_6\varphi^{3\vartheta} + b_7\varphi^{2\vartheta} + b_8\varphi^{\vartheta} + b_9, \tag{12}$$

$$[\mathcal{A}_{1I}(\varphi)\mathcal{A}_{2R}(\varphi) - \mathcal{A}_{1R}(\varphi)\mathcal{A}_{2I}(\varphi)]^2 = c_1\varphi^{8\vartheta} + c_2\varphi^{7\vartheta} + c_3\varphi^{6\vartheta} + c_4\varphi^{5\vartheta} + c_5\varphi^{4\vartheta} + c_6\varphi^{3\vartheta} + c_7\varphi^{2\vartheta}, \tag{13}$$

$$[(\mathcal{A}_{2R}(\varphi))^2 + (\mathcal{A}_{2I}(\varphi))^2]^2 = d_1\varphi^{2\vartheta} + d_2\varphi^{\vartheta} + d_3, \tag{14}$$

where

$$b_1 = a_4^2\left(\cos\frac{3\pi\vartheta}{2}\cos\frac{\pi\vartheta}{2} + \sin\frac{3\pi\vartheta}{2}\sin\frac{\pi\vartheta}{2}\right)^2,$$
$$b_2 = 2a_4\left(\cos\frac{3\pi\vartheta}{2}\cos\frac{\pi\vartheta}{2} + \sin\frac{3\pi\vartheta}{2}\sin\frac{\pi\vartheta}{2}\right) \times\left[a_1a_4\left(\cos\pi\vartheta\cos\frac{\pi\vartheta}{2} + \sin\pi\vartheta\sin\frac{\pi\vartheta}{2}\right) + a_5\cos\frac{3\pi\vartheta}{2}\right],$$
$$b_3 = 2a_4\left(\cos\pi\vartheta\cos\frac{\pi\vartheta}{2} + \sin\pi\vartheta\sin\frac{\pi\vartheta}{2}\right) \times(a_2a_4 + a_1a_5\cos\pi\vartheta) \times\left[a_1a_4\left(\cos\pi\vartheta\cos\frac{\pi\vartheta}{2} + \sin\pi\vartheta\sin\frac{\pi\vartheta}{2}\right) + a_5\cos\frac{3\pi\vartheta}{2}\right]^2,$$
$$b_4 = 2\left[a_1a_4\left(\cos\pi\vartheta\cos\frac{\pi\vartheta}{2} + \sin\pi\vartheta\sin\frac{\pi\vartheta}{2}\right) + a_5\cos\frac{3\pi\vartheta}{2}\right](a_2a_4 + a_1a_5\cos\pi\vartheta),$$
$$b_5 = (a_2a_4 + a_1a_5\cos\pi\vartheta)^2 + 2a_3a_4a_5 \times\left(\cos\frac{\pi\vartheta}{2}\cos\frac{3\pi\vartheta}{2} + \sin\frac{\pi\vartheta}{2}\sin\frac{3\pi\vartheta}{2}\right) \times2\left[a_1a_4\left(\cos\pi\vartheta\cos\frac{\pi\vartheta}{2} + \sin\pi\vartheta\sin\frac{\pi\vartheta}{2}\right) + a_5\cos\frac{3\pi\vartheta}{2}\right]\left(a_3a_4 + a_2a_5\cos\frac{\pi\vartheta}{2}\right),$$
$$b_6 = 2a_3a_5(a_1a_4\cos\pi\vartheta\cos\frac{\pi\vartheta}{2} + a_5\cos\frac{3\pi\vartheta}{2} + a_1a_4\sin\pi\vartheta\sin\frac{\pi\vartheta}{2} + 2(a_2a_4 + a_1a_5\cos\pi\vartheta)(a_3a_4 + a_2a_5)\cos\frac{\pi\vartheta}{2},$$
$$b_7 = 2a_3a_5(a_2a_4 + a_1a_5\cos\pi\vartheta),$$
$$b_8 = 2a_3a_5(a_3a_5 + a_2a_4)\cos\frac{\pi\vartheta}{2},$$
$$b_9 = (a_3a_5)^2,$$
$$c_1 = \left[a_4\sin\frac{3\pi\vartheta}{2}\left(\cos\frac{\pi\vartheta}{2} - \sin\frac{\pi\vartheta}{2}\right)\right]^2,$$
$$c_2 = 2a_4\sin\frac{3\pi\vartheta}{2}\left(\cos\frac{\pi\vartheta}{2} - \sin\frac{\pi\vartheta}{2}\right) \times\left[a_1a_4\cos\pi\vartheta\left(\cos\frac{\pi\vartheta}{2} - \sin\frac{\pi\vartheta}{2}\right) + a_5\sin\frac{3\pi\vartheta}{2}\right],$$
$$c_3 = \left[a_1a_4\sin\pi\vartheta\left(\cos\frac{\pi\vartheta}{2} - \sin\frac{\pi\vartheta}{2}\right) + a_5\sin\frac{3\pi\vartheta}{2}\right]^2 + 2a_4\sin\frac{3\pi\vartheta}{2}\left(\cos\frac{\pi\vartheta}{2} - \sin\frac{\pi\vartheta}{2}\right) \times\left[a_2a_4\left(\cos\frac{\pi\vartheta}{2}\sin\frac{\pi\vartheta}{2} - \sin^2\frac{\pi\vartheta}{2}\right) + a_1a_5\sin\pi\vartheta\right],$$
$$c_4 = 2a_2a_4a_5\sin\frac{\pi\vartheta}{2}\sin\frac{3\pi\vartheta}{2}\left(\cos\frac{\pi\vartheta}{2} - \sin\frac{\pi\vartheta}{2}\right) + 2\left[a_1a_4\sin\pi\vartheta\left(\cos\frac{\pi\vartheta}{2} - \sin\frac{\pi\vartheta}{2}\right) + a_5\sin\frac{3\pi\vartheta}{2}\right] \times\left[a_2a_4\left(\cos\frac{\pi\vartheta}{2}\sin\frac{\pi\vartheta}{2} - \sin^2\frac{\pi\vartheta}{2}\right) + a_1a_5\sin\pi\vartheta\right],$$
$$c_5 = \left[a_2a_4\left(\cos\frac{\pi\vartheta}{2}\sin\frac{\pi\vartheta}{2} - \sin^2\frac{\pi\vartheta}{2}\right) + a_1a_5\sin\pi\vartheta\right]^2 + 2\left[a_1a_4\sin\pi\vartheta\left(\cos\frac{\pi\vartheta}{2} - \sin\frac{\pi\vartheta}{2}\right) + a_5\sin\frac{3\pi\vartheta}{2}\right]a_2a_5\sin\frac{\pi\vartheta}{2},$$
$$c_6 = 2\left[a_2a_4\left(\cos\frac{\pi\vartheta}{2}\sin\frac{\pi\vartheta}{2} - \sin^2\frac{\pi\vartheta}{2}\right)\right.$$

$$+a_1a_5\sin\pi\vartheta]\,a_2a_5\sin\frac{\pi\vartheta}{2},$$
$$c_7=\left(a_2a_5\sin\frac{\pi\vartheta}{2}\right)^2,$$
$$d_1=a_4^2,$$
$$d_2=2a_4a_5\cos\frac{\pi\vartheta}{2},$$
$$d_3=a_5^2.$$

It follows from (11) that

$$K_1\varphi^{8\vartheta}+K_2\varphi^{7\vartheta}+K_3\varphi^{6\vartheta}+K_4\varphi^{5\vartheta}+K_5\varphi^{4\vartheta}+K_6\varphi^{3\vartheta}+K_7\varphi^{2\vartheta}+K_8\varphi^{\vartheta}+K_9=0, \quad (15)$$

where $K_i=b_i+c_i(i=1,2,3,4,5,6), K_7=b_7+c_7-d_1, K_8=b_8-d_2, K_9=b_9-d_3$.

Denote

$$g(\varphi)=K_1\varphi^{8\vartheta}+K_2\varphi^{7\vartheta}+K_3\varphi^{6\vartheta}+K_4\varphi^{5\vartheta}+K_5\varphi^{4\vartheta}+K_6\varphi^{3\vartheta}+K_7\varphi^{2\vartheta}+K_8\varphi^{\vartheta}+K_9. \quad (16)$$

and

$$l(\mu)=K_1\mu^8+K_2\mu^7+K_3\mu^6+K_4\mu^5+K_5\mu^4+K_6\mu^3+K_7\mu^2+K_8\mu+K_9. \quad (17)$$

The following hypothesis is given:

(A3) $\alpha_1\alpha_2\alpha_3-(\rho_{11}\rho_{13}\alpha_2+\rho_{12}\rho_{23}\alpha_1)\neq 0$.

*Lemma 2: For (7), the following results are true:*

*(a) If $K_i>0(i=1,2,\cdots,9)$ holds, then (7) has no root with zero real parts.*

*(b) If $K_9<0$, then (7) has at least pair of purely imaginary roots.*

*(c) If $K_9>0$ and $\exists\,\theta_0>0$ which satisfies $l(\theta_0)<0$, then (3.4) has at least two pairs of purely imaginary roots.*

*(d) If $K_i>0(i=1,2,\cdots,7), K_8<0, K_9>0$ and $\exists\,\theta_0>0$ which satisfies $l(\theta_0)<0$, then (7) has at least two pairs of purely imaginary roots.*

*(e) If $K_i>0(i=1,2,\cdots,6), K_7<0, K_9>0$ and $\exists\,\theta_0>0$ such that $l(\theta_0)<0$, then (7) has at least two pairs of purely imaginary roots.*

*Proof:* We will prove the five cases, respectively.

**(a)** It follows from (16) that

$$\frac{dg(\varphi)}{d\varphi}=8\vartheta K_1\varphi^{8\vartheta-1}+7\vartheta K_2\varphi^{7\vartheta-1}+6\vartheta K_3\varphi^{6\vartheta}+5\vartheta K_4\varphi^{5\vartheta-1}+4\vartheta K_5\varphi^{4\vartheta-1}+3\vartheta K_6\varphi^{3\vartheta-1}+2\vartheta K_7\varphi^{2\vartheta-1}+\vartheta K_8\varphi^{\vartheta-1}.$$

In view of $K_i>0(i=1,2,\cdots,9)$, we know that $\frac{dg(\varphi)}{d\varphi}>0$ $\forall\,\varphi>0$. Notice that $g(0)=K_9>0$, then Eq. (15) has no positive real root. By (A3), one knows that $a_3+a_5\neq 0$, then $s=0$ is not the root of (7). The proof of (a) is complete.

**(b)** Obviously, $g(0)=K_9<0$ and $\lim_{\varphi\to+\infty}g(\varphi)=+\infty$. Then (15) has at least one positive real root. Thus (7) has at least one pair of purely imaginary roots. The proof of (b) is complete.

**(c)** Since $l(0)=K_9>0, l(\theta_0)<0(\theta_0>0)$ and $\lim_{\mu\to+\infty}l(\mu)=+\infty$, then $\exists\,\theta_{01}\in(0,\theta_0)$ and $\theta_{02}\in(\theta_0,+\infty)$ which satisfy $l(\theta_{01})=l(\theta_{02})=0$, Then Eq. (15) has at least two positive real roots. Thus (7) has at least two pairs of purely imaginary roots. The proof of (c) ends.

**(d)** Clearly, in view of $K_i>0(i=1,2,\cdots,7)$, one has $\frac{d^2l(\mu)}{d\mu}>0$, $\forall\,\mu>0$. In addition, $\frac{dl(\mu)}{d\mu}|_{\mu=0}=K_8<0$ and $\lim_{\mu\to+\infty}\frac{dl(\mu)}{d\mu}=+\infty$, then $\frac{dl(\mu)}{d\mu}=0$ has only one positive real root. So $l(\mu)$ has a unique stationary point for $\mu>0$. Since $l(0)=K_9>0, l(\theta_0)<0(\theta_0>0)$ and $\lim_{\mu\to+\infty}l(\mu)=+\infty$, one can conclude that $l(\mu)=0$ has two positive real roots. Hence $g(\varphi)$ has two positive real roots. Thus (7) has two pairs of purely imaginary roots. The proof of (d) is complete.

**(e)** According to $l(0)=K_9>0$ and $l(\theta_0)<0$, $\forall\,\theta_0>0$, $\exists\,\theta_1\in(0,\theta_0)$ which satisfies $\frac{dl(\mu)}{d\mu}|_{\mu=\theta_1}<0$. According to $\lim_{\mu\to+\infty}\frac{dl(\mu)}{d\mu}=+\infty$, one has that $\frac{dl(\mu)}{d\mu}=0$ has at least one positive real root. Notice that $\frac{d^3l(\mu)}{d\mu}>0(\mu>0)$, $\frac{d^2l(\mu)}{d\mu}|_{\mu=0}=2K_7<0$ and $\lim_{\mu\to+\infty}\frac{d^2l(\mu)}{d\mu}=+\infty$. Thus $\frac{d^2l(\mu)}{d\mu}=0$ has only one positive real root. Then $\frac{dl(\mu)}{d\mu}$ has only the unique stationary point for $\mu>0$, which implies that $\frac{dl(\mu)}{d\mu}=0$ has at most two positive real roots and $l(\mu)=0$ has at most three positive real roots. In addition, in view of $l(0)=K_9>0, l(\theta_0)<0(\theta_0>0)$ and $\lim_{\mu\to+\infty}\frac{dl(\mu)}{d\mu}=+\infty$, one has that $l(\mu)=0$ may have $2j(j=1,2,3,4)$ positive real roots. Therefore $l(\mu)=0$ has two positive real roots and $g(\varphi)=0$ has two positive real roots. So (7) has two pairs of purely imaginary roots. The proof of (e) ends.

Without loss of generality, if (15) has nine positive real roots $\varphi_j(j=1,2,\cdots,9)$. By (10), one gets

$$\sigma_j^k=\frac{1}{\varphi_j}\left[\arccos\left(-\frac{\mathcal{A}_{1R}(\varphi)\mathcal{A}_{2R}(\varphi)+\mathcal{A}_{1I}(\varphi)\mathcal{A}_{2I}(\varphi)}{(\mathcal{A}_{2R}(\varphi))^2+(\mathcal{A}_{2I}(\varphi))^2}\right)+2l\pi\right], \quad (18)$$

where $l=0,1,2,\cdots,j=1,2,\cdots,9$. Denote

$$\sigma_0=\min_{j=1,2,\cdots,9}\{\sigma_j^0\},\ \varphi_0=\varphi|_{\sigma=\sigma_0}. \quad (19)$$

Now we give the following hypotheses:

(A4) $\mathcal{L}_1\mathcal{S}_1+\mathcal{L}_2\mathcal{S}_2>0$, where

$$\mathcal{L}_1=3\vartheta\varphi_0^{3\vartheta-1}\cos\frac{(3\vartheta-1)\pi}{2}+2a_1\vartheta\varphi_0^{2\vartheta-1}\times\cos\frac{(2\vartheta-1)\pi}{2}+a_2\vartheta\varphi_0^{(\vartheta-1)}\cos\frac{(\vartheta-1)\pi}{2}+a_4\vartheta\varphi_0^{\vartheta-1}\left[\cos\frac{(\vartheta-1)\pi}{2}\cos\varphi_0\sigma_0+\cos\frac{(\vartheta-1)\pi}{2}\cos\varphi_0\sigma_0\right],$$

$$\mathcal{L}_2=3\vartheta\varphi_0^{3\vartheta-1}\sin\frac{(3\vartheta-1)\pi}{2}+2a_1\vartheta\varphi_0^{2\vartheta-1}\times\sin\frac{(2\vartheta-1)\pi}{2}+a_2\vartheta\varphi_0^{(\vartheta-1)}\sin\frac{(\vartheta-1)\pi}{2}$$

$$+a_4\vartheta\varphi_0^{\vartheta-1}\left[\cos\frac{(\vartheta-1)\pi}{2}\sin\varphi_0\sigma_0 - \sin\frac{(\vartheta-1)\pi}{2}\cos\varphi_0\sigma_0\right],$$

$$\mathcal{S}_1 = -a_4\varphi_0^{\vartheta+1}\sin\frac{\vartheta\pi}{2},$$

$$\mathcal{S}_2 = \varphi_0\left(a_4\varphi_0^{\vartheta}\cos\frac{\vartheta\pi}{2}+a_5\right).$$

*Lemma 3: If $s(\sigma) = \mu(\sigma) + i\varphi(\sigma)$ is the root of (7) around $\sigma = \sigma_0$ satisfying $\mu(\sigma_0) = 0, \varphi(\sigma_0) = \varphi_0$, then* $Re\left[\frac{ds}{d\sigma}\right]\Big|_{\sigma=\sigma_0,\varphi=\varphi_0} \neq 0.$

*Proof:* By (7), one gets

$$\left[\frac{ds}{d\sigma}\right]^{-1} = \frac{p(s)}{q(s)} - \frac{\sigma}{s}, \tag{20}$$

where

$$p(s) = 3\vartheta s^{3\vartheta-1} + 2a_1\vartheta s^{2\vartheta-1} + a_2\vartheta s^{\vartheta-1} + a_4\vartheta s^{\vartheta-1}e^{-s\sigma},$$
$$q(s) = s(a_4 s^{\vartheta} + a_5).$$

Then

$$\text{Re}\left\{\left[\frac{ds}{d\sigma}\right]^{-1}\right\} = \text{Re}\left\{\frac{p(s)}{q(s)}\right\}. \tag{21}$$

Therefore

$$\text{Re}\left\{\left[\frac{ds}{d\sigma}\right]^{-1}\right\}\Big|_{\sigma=\sigma_0,\varphi=\varphi_0} = \text{Re}\left\{\frac{p(s)}{q(s)}\right\}\Big|_{\sigma=\sigma_0,\varphi=\varphi_0} = \frac{\mathcal{L}_1\mathcal{S}_1+\mathcal{L}_2\mathcal{S}_2}{\mathcal{S}_1^2+\mathcal{S}_2^2}.$$

By (A4), one has

$$\text{Re}\left\{\left[\frac{ds}{d\sigma}\right]^{-1}\right\}\Big|_{\sigma=\sigma_0,\varphi=\varphi_0} > 0.$$

Then the transversality condition holds. This ends the proof of Lemma 2.

Now we consider the stability of (1.3) with $\sigma = 0$. The following assumption is given:

(A5) $a_1 > 0, a_1(a_2+a_4) > a_3+a_5, a_1(a_2+a_4)(a_3+a_5) > (a_3+a_5)^2$.

It is not difficult to obtain the following lemma.

*Lemma 3: If $\sigma = 0$ and (A5) are true, then model (3) is asymptotically stable.*

*Proof:* Let $\sigma = 0$, then (7) can be written as follows:

$$\lambda^3 + a_1\lambda^2 + (a_2+a_4)\lambda + (a_3+a_5) = 0. \tag{22}$$

By (A5), one has that all the roots $\lambda_i$ of (21) satisfies $|\arg(\lambda_i)| > \frac{\vartheta\pi}{2}(i = 1, 2, 3, 4)$ According to Lemma 2, one can conclude that model (3) with $\sigma = 0$ is asymptotically stable. This ends the proof of Lemma 3.

*Theorem 1: For model (3), if (A1)-(A5) hold true, then the zero equilibrium point is global asymptotically stable when $\sigma \in [0, \sigma_0)$ and a Hopf bifurcation appears around the zero equilibrium point for $\sigma = \sigma_0$.*

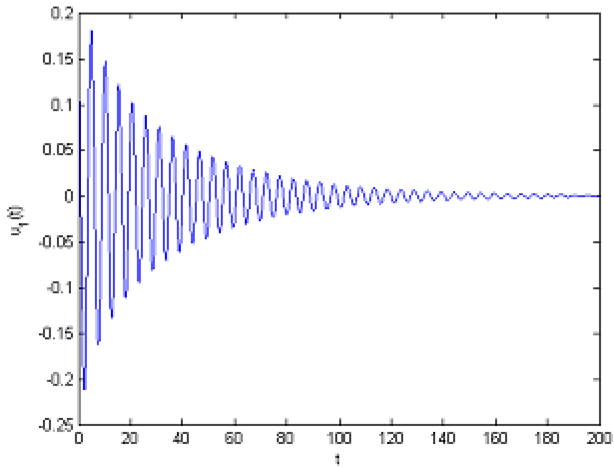



**FIGURE 1.** The relation of $t$ and $u_1(t)$ when $\sigma_1 = 1, \sigma_2 = 0.7, \sigma_3 = 0.5, \sigma_4 = 0.8, \sigma = \sigma_1 + \sigma_3 = \sigma_1 + \sigma_3 = 1.5 < \sigma_0 = 1.7664$.

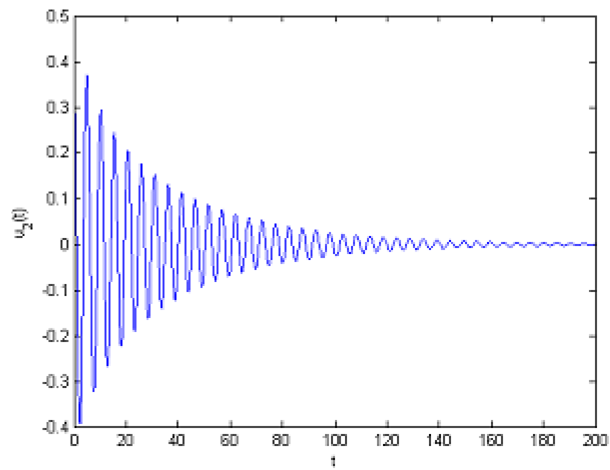



**FIGURE 2.** The relation of $t$ and $u_2(t)$ when $\sigma_1 = 1, \sigma_2 = 0.7, \sigma_3 = 0.5, \sigma_4 = 0.8, \sigma = \sigma_1 + \sigma_3 = \sigma_1 + \sigma_3 = 1.5 < \sigma_0 = 1.7664$.

*Remark 1: In [38]–[44], [56], the authors studied the stability and Hopf bifurcation of integer-order neural networks delays. In this article, we study the stability and Hopf bifurcation of fractional-order neural networks with four different delays. All the derived results in [38]–[44], [56] can not be applied to (3) to obtain the stability and the existence of Hopf bifurcation for (3). Based on the reason, the main results of this article on the stability and the existence of Hopf bifurcation for (3) are completely new and complete previous publications.*

*Remark 2: Huang and Cao [45] discussed the effect of leakage delay on bifurcation in high-order fractional BAM neural networks. In [47], the authors investigated the control problem of bifurcation for a delayed fractional gene regulatory network. In [48], the authors studied the bifurcation behavior on fractional complex-valued neural network. In [49], Huang et al. discussed the effect of leakage delay*

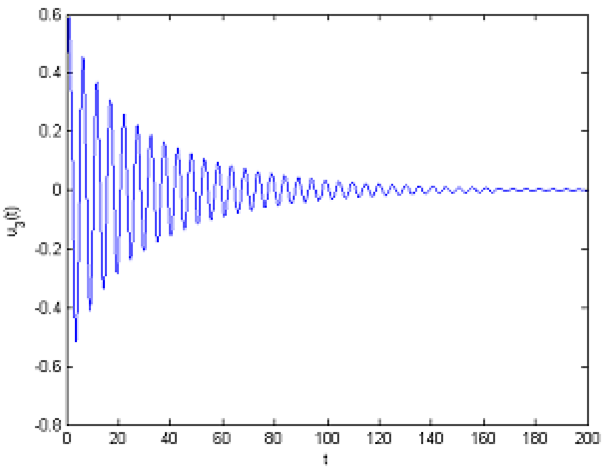



**FIGURE 3.** The relation of $t$ and $u_3(t)$ when $\sigma_1 = 1, \sigma_2 = 0.7, \sigma_3 = 0.5, \sigma_4 = 0.8, \sigma = \sigma_1 + \sigma_3 = \sigma_1 + \sigma_3 = 1.5 < \sigma_0 = 1.7664$.

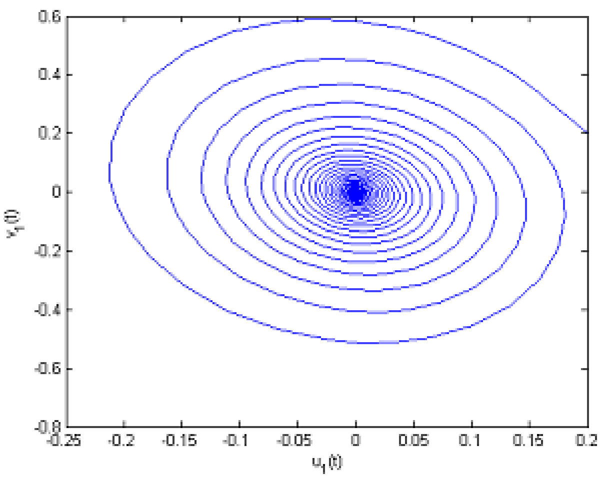



**FIGURE 4.** The relation of $u_1(t)$ and $v_1(t)$ when $\sigma_1 = 1, \sigma_2 = 0.7, \sigma_3 = 0.5, \sigma_4 = 0.8, \sigma = \sigma_1 + \sigma_3 = \sigma_1 + \sigma_3 = 1.5 < \sigma_0 = 1.7664$.

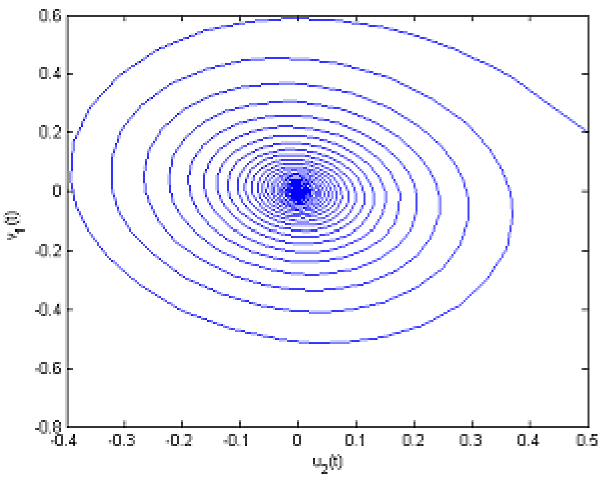



**FIGURE 5.** The relation of $u_2(t)$ and $v_1(t)$ when $\sigma_1 = 1, \sigma_2 = 0.7, \sigma_3 = 0.5, \sigma_4 = 0.8, \sigma = \sigma_1 + \sigma_3 = \sigma_1 + \sigma_3 = 1.5 < \sigma_0 = 1.7664$.

*on bifurcation for fractional BAM neural networks. All the publications [45], [47]–[49] only involve a single delay. In this article, we investigate the Hopf bifurcation of involved*

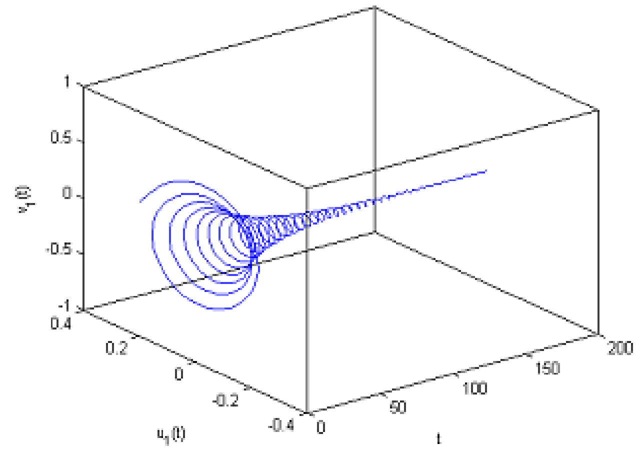



**FIGURE 6.** The relation of $t$, $u_1(t)$ and $v_1(t)$ when $\sigma_1 = 1, \sigma_2 = 0.7, \sigma_3 = 0.5, \sigma_4 = 0.8, \sigma = \sigma_1 + \sigma_3 = \sigma_1 + \sigma_3 = 1.5 < \sigma_0 = 1.7664$.

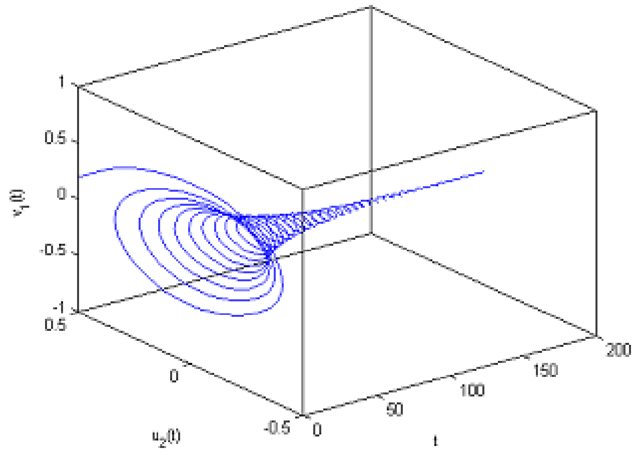



**FIGURE 7.** The relation of $t$, $u_2(t)$ and $v_1(t)$ when $\sigma_1 = 1, \sigma_2 = 0.7, \sigma_3 = 0.5, \sigma_4 = 0.8, \sigma = \sigma_1 + \sigma_3 = \sigma_1 + \sigma_3 = 1.5 < \sigma_0 = 1.7664$.

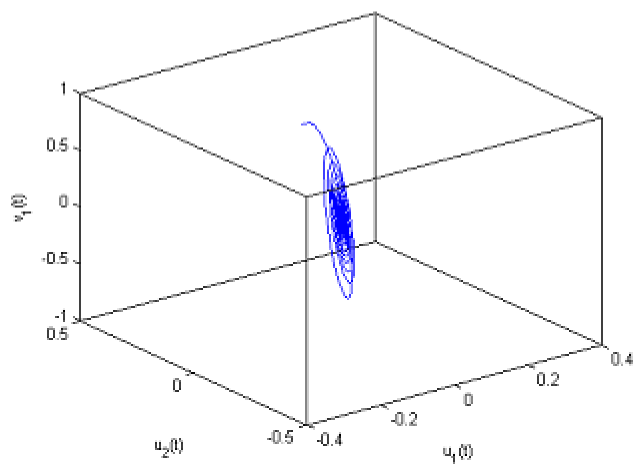



**FIGURE 8.** The relation of $u_1(t)$, $u_2(t)$ and $v_1(t)$ when $\sigma_1 = 1, \sigma_2 = 0.7, \sigma_3 = 0.5, \sigma_4 = 0.8, \sigma = \sigma_1 + \sigma_3 = \sigma_1 + \sigma_3 = 1.5 < \sigma_0 = 1.7664$.

*models with four different delays by choosing the sum of two different delays as bifurcation parameter. Up to now, there are few results on Hopf bifurcation of fractional-order systems with multiple delays. From the viewpoint, the main results of*

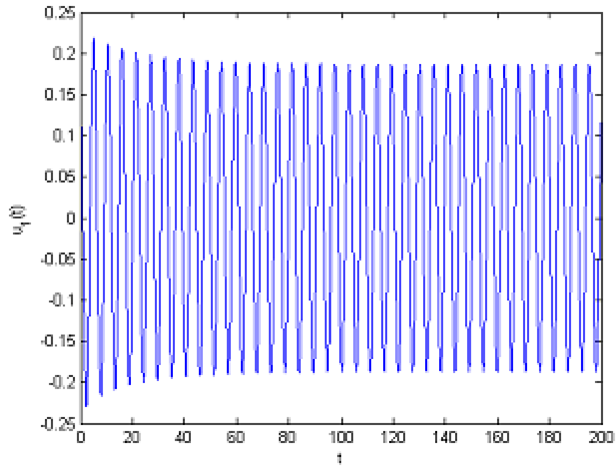



**FIGURE 9.** The relation of $t$ and $u_1(t)$ when $\sigma_1 = 1, \sigma_2 = 1.2, \sigma_3 = 0.8, \sigma_4 = 0.6, \sigma = \sigma_1 + \sigma_3 = \sigma_1 + \sigma_3 = 1.8 > \sigma_0 = 1.7664$.

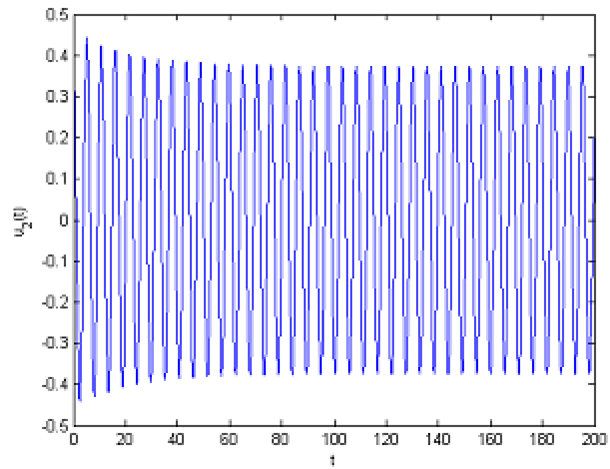



**FIGURE 10.** The relation of $t$ and $u_2(t)$ when $\sigma_1 = 1, \sigma_2 = 1.2, \sigma_3 = 0.8, \sigma_4 = 0.6, \sigma = \sigma_1 + \sigma_3 = \sigma_1 + \sigma_3 = 1.8 > \sigma_0 = 1.7664$.

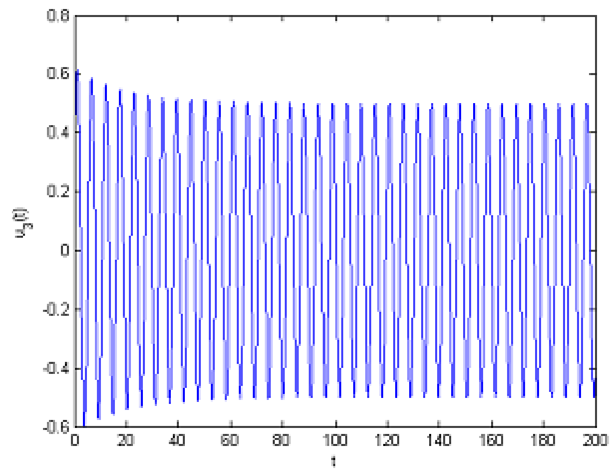



**FIGURE 11.** The relation of $t$ and $u_3(t)$ when $\sigma_1 = 1, \sigma_2 = 1.2, \sigma_3 = 0.8, \sigma_4 = 0.6, \sigma = \sigma_1 + \sigma_3 = \sigma_1 + \sigma_3 = 1.8 > \sigma_0 = 1.7664$.

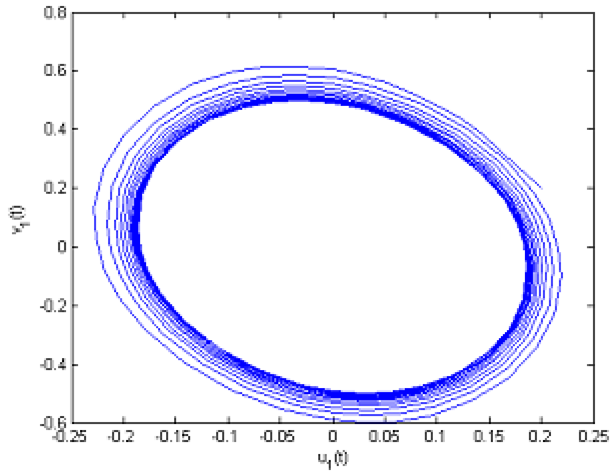



**FIGURE 12.** The relation of $u_1(t)$ and $v_1(t)$ when $\sigma_1 = 1, \sigma_2 = 1.2, \sigma_3 = 0.8, \sigma_4 = 0.6, \sigma = \sigma_1 + \sigma_3 = \sigma_1 + \sigma_3 = 1.8 > \sigma_0 = 1.7664$.

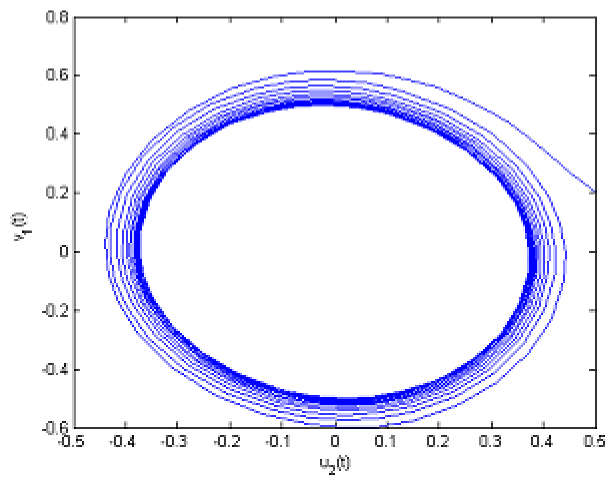



**FIGURE 13.** The relation of $u_2(t)$ and $v_1(t)$ when $\sigma_1 = 1, \sigma_2 = 1.2, \sigma_3 = 0.8, \sigma_4 = 0.6, \sigma = \sigma_1 + \sigma_3 = \sigma_1 + \sigma_3 = 1.8 > \sigma_0 = 1.7664$.

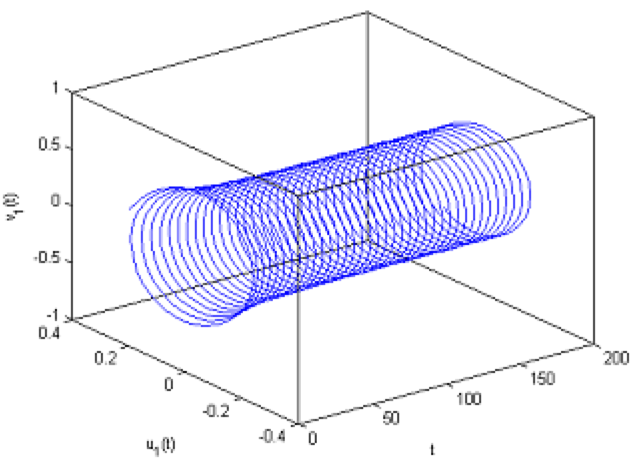



**FIGURE 14.** The relation of $t$, $u_1(t)$ and $v_1(t)$ when $\sigma_1 = 1, \sigma_2 = 1.2, \sigma_3 = 0.8, \sigma_4 = 0.6, \sigma = \sigma_1 + \sigma_3 = \sigma_1 + \sigma_3 = 1.8 > \sigma_0 = 1.7664$.

*this article on stability and existence of Hopf bifurcation of neural networks are essentially innovative.*

*Remark 3: The results on the considered fractional-order neural networks with four delays can not extend the result with n-delays since the characteristic equation of fractional-order neural networks with n-delays is more complex.*

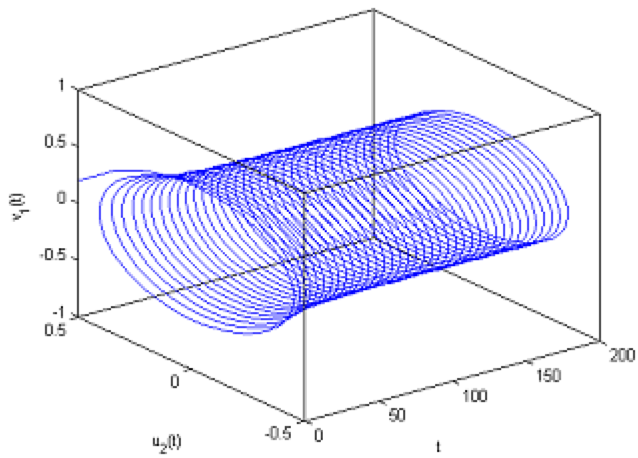



**FIGURE 15.** The relation of $t$, $u_2(t)$ and $v_1(t)$ when $\sigma_1 = 1, \sigma_2 = 1.2, \sigma_3 = 0.8, \sigma_4 = 0.6, \sigma = \sigma_1 + \sigma_3 = \sigma_1 + \sigma_3 = 1.8 > \sigma_0 = 1.7664$.

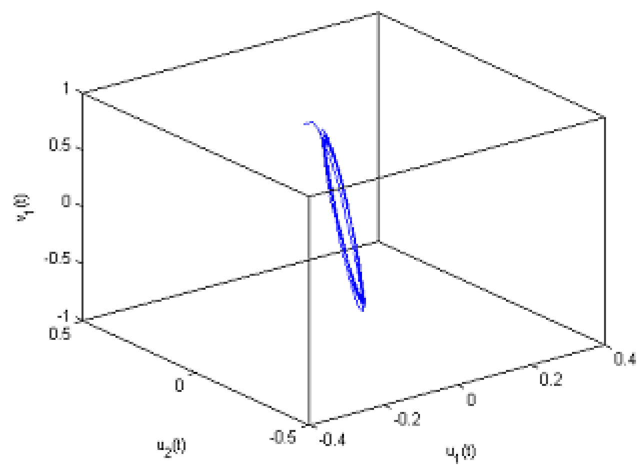



**FIGURE 16.** The relation of $u_1(t)$, $u_2(t)$ and $v_1(t)$ when $\sigma_1 = 1, \sigma_2 = 1.2, \sigma_3 = 0.8, \sigma_4 = 0.6, \sigma = \sigma_1 + \sigma_3 = \sigma_1 + \sigma_3 = 1.8 > \sigma_0 = 1.7664$.

*Remark 4: In order to obtain the stability and the existence of Hopf bifurcation of model (3), we carried out some complex computation. If the parameters are given, we can compute corresponding value by computer.*

## IV. COMPUTER SIMULATIONS

Consider the following fractional-order model:

$$\begin{cases} \mathcal{D}^{\vartheta} u_1(t) = -0.5u_1(t) - 0.5\tanh(v_1(t-\sigma_1)), \\ \mathcal{D}^{\vartheta} u_2(t) = -0.5u_2(t) - \tanh(v_1(t-\sigma_2)), \\ \mathcal{D}^{\vartheta} v_1(t) = -0.5v_1(t) + 2\tanh(u_1(t-\sigma_3)) \\ \qquad +0.6\tanh(u_2(t-\sigma_4)). \end{cases} \tag{23}$$

All the coefficients and functions are same as those in Xu [38]). Set $\vartheta = 0.82$. Then $\varphi_0 = 0.6759$ and $\sigma_0 = 1.7664$. We can verify that (A1)-(A5) of Theorem 1 are fulfilled. Let $\sigma_1 = 1, \sigma_2 = 0.7, \sigma_3 = 0.5, \sigma_4 = 0.8$, then $\sigma = \sigma_1 + \sigma_3 = \sigma_1 + \sigma_3 = 1.5 < \sigma_0 = 1.7664$. In this case, we can conclude that the zero equilibrium point of model (23) is locally asymptotically stable. Figures 1-8 indicate that the zero equilibrium point of model (23) is locally asymptotically stable for $\sigma \in [0, \sigma_0)$. Let $\sigma_1 = 1, \sigma_2 = 1.2, \sigma_3 = 0.8, \sigma_4 = 0.6$, then $\sigma = \sigma_1 + \sigma_3 = \sigma_1 + \sigma_3 = 1.8 > \sigma_0 = 1.7664$. In this case, model (23) becomes unstable and a Hopf bifurcation exists. Figures 9-16 manifest that model (23) becomes unstable and a Hopf bifurcation takes place for $\sigma \in [\sigma_0, +\infty)$. In addition, we can see that the order can postpone the onset of Hopf bifurcation (compared with Xu [38]).

## V. CONCLUSIONS

The bifurcation issue of fractional order delayed BAM neural networks has been analyzed in details. By discussing the distribution of characteristic roots of corresponding characteristic equation of involved system and choosing the sum of different delays as bifurcation parameter, we establish a set of sufficient conditions to ensure the stability and the existence of Hopf bifurcation of considered neural networks. The study reveals that the delay has important effect on the stability and Hopf bifurcation of involved neural networks. We find that when the sum of delays remain the suitable interval, then the equilibrium point of involved system is locally asymptotically stable, while the Hopf bifurcation appears when the sum of delays exceed some critical values. The obtained results on the stability and the existence of Hopf bifurcation of fractional order delayed BAM neural networks can be applied to design and optimize neural networks. It will be widely applied in artificial intelligence, automatic control, disease diagnosis, image processing, etc. So we can say that the study of this manuscript plays an important role in serving human beings.

## ACKNOWLEDGMENT

The authors would like to thank the referees and the editor for helpful suggestions incorporated into this paper.

## REFERENCES

[1] J. Wang, L. Huang, and Z. Guo, "Dynamical behavior of delayed Hopfield neural networks with discontinuous activations," *Appl. Math. Model.*, vol. 33, no. 4, pp. 1793–1802, Apr. 2009.

[2] C. M. Marcus and R. M. Westervelt, "Stability of analog neural networks with delay," *Phys. Rev. A, Gen. Phys.*, vol. 39, no. 1, pp. 347–359, Jan. 1989.

[3] H. Li, C. Li, and T. Huang, "Periodicity and stability for variable-time impulsive neural networks," *Neural Netw.*, vol. 94, pp. 24–33, Oct. 2017.

[4] Y. Li and L. Yang, "Almost automorphic solution for neutral type high-order Hopfield neural networks with delays in leakage terms on time scales," *Appl. Math. Comput.*, vol. 242, pp. 679–693, Sep. 2014.

[5] Y. Li and Y. Li, "Existence and exponential stability of almost periodic solution for neutral delay BAM neural networks with time-varying delays in leakage terms," *J. Franklin Inst.*, vol. 350, no. 9, pp. 2808–2825, Feb. 2013.

[6] C. Aouiti, I. Ben Gharbia, J. D. Cao, M. S. M'Hamdi, and A. Alsaedi, "Existence and global exponential stability of pseudo almost periodic solution for neutral delay BAM neural networks with time-varying delay in leakage terms," *Chaos, Solitons Fractals*, vol. 107, pp. 111–127, Feb. 2018.

[7] C. Xu and P. Li, "Global exponential convergence of neutral-type Hopfield neural networks with multi-proportional delays and leakage delays," *Chaos, Solitons Fractals*, vol. 96, pp. 139–144, Mar. 2017.

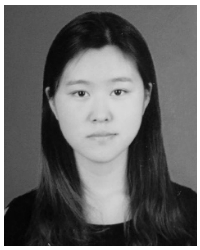
[8] C. Xu and P. Li, "Global exponential convergence of fuzzy cellular neural networks with leakage delays, distributed delays and proportional delays," *Circuits, Syst., Signal Process.*, vol. 37, pp. 163–177, Jan. 2018.
[9] Z. Long, "New results on anti-periodic solutions for SICNNs with oscillating coefficients in leakage terms," *Neurocomputing*, vol. 171, pp. 503–509, Jan. 2016.
[10] G. Zhang, Z. Zeng, and J. Hu, "New results on global exponential dissipativity analysis of memristive inertial neural networks with distributed time-varying delays," *Neural Netw.*, vol. 97, pp. 183–191, Jan. 2017.
[11] C. Xu, "Global exponential periodicity for fuzzy cellular neural networks with proportional delays," *J. Intell. Fuzzy Syst.*, vol. 33, no. 2, pp. 829–839, Mar. 2017.
[12] P. Jiang, Z. Zeng, and J. Chen, "On the periodic dynamics of memristor-based neural networks with leakage and time-varying delays," *Neurocomputing*, vol. 219, pp. 163–173, Jan. 2017.
[13] Y. Xu, "Exponential stability of weighted pseudo almost periodic solutions for HCNNs with mixed delays," *Neural Process. Lett.*, vol. 46, no. 2, pp. 507–519, Feb. 2017.
[14] F. Zhang and Z. Zeng, "Multistability and instability analysis of recurrent neural networks with time-varying delays," *Neural Netw.*, vol. 97, pp. 116–126, Jan. 2018.
[15] C. Xu and Q. Zhang, "On antiperiodic solutions for Cohen–Grossberg shunting inhibitory neural networks with time-varying delays and impulses," *Neural Comput.*, vol. 26, no. 10, pp. 2328–2349, Oct. 2014.
[16] C. Xu, P. Li, and Y. Pang, "Global exponential stability for interval general bidirectional associative memory (BAM) neural networks with proportional delays," *Math. Methods Appl. Sci.*, vol. 39, no. 8, pp. 5720–5731, Dec. 2016.
[17] C. Xu, Q. Zhang, and Y. Wu, "Bifurcation analysis of two-neuron networks with discrete and distributed delays," *Cognit. Comput.*, vol. 8, no. 6, pp. 1103–1118, Dec. 2016.
[18] C. Xu and Q. Zhang, "Existence and global exponential stability of anti-periodic solutions for BAM neural networks with inertial term and delay," *Neurocomputing*, vol. 153, pp. 108–116, Apr. 2015.
[19] C. Xu, Y. Wu, and Q. Zhang, "Existence and stability of pseudo almost periodic solutions for shunting inhibitory cellular neural networks with neutral type delays and time-varying leakage delays," *Netw., Comput. Neural Syst.*, vol. 25, no. 4, pp. 168–192, Oct. 2014.
[20] C. Xu, P. Li, and Y. Pang, "Exponential stability of almost periodic solutions for memristor-based neural networks with distributed leakage delays," *Neural Comput.*, vol. 28, no. 11, pp. 2726–2756, Dec. 2016.
[21] C. Xu and P. Li, "Pseudo almost periodic solutions for high-order hopfield neural networks with time-varying leakage delays," *Neural Process. Lett.*, vol. 46, no. 1, pp. 41–58, Aug. 2017.
[22] Z. Zhang and T. Zheng, "Global asymptotic stability of periodic solutions for delayed complex-valued CohenGrossberg neural networks by combining coincidence degree theory with LMI method," *Neurocomputing*, vol. 289, pp. 220–230, May 2018.
[23] S. Lakshmanan, C. P. Lim, M. Prakash, S. Nahavandi, and P. Balasubramaniam, "Neutral-type of delayed inertial neural networks and their stability analysis using the LMI Approach," *Neurocomputing*, vol. 230, pp. 243–250, Mar. 2017.
[24] P. Balasubramaniam and S. Lakshmanan, "Delay-range dependent stability criteria for neural networks with Markovian jumping parameters," *Nonlinear Anal., Hybrid Syst.*, vol. 3, no. 4, pp. 749–756, Nov. 2009.
[25] J. J. Oliveira, "Convergence of asymptotic systems of non-autonomous neural network models with infinite distributed delays," *J. Nonlinear Sci.*, vol. 27, pp. 1463–1486, Oct. 2017.
[26] X. Li and S. Song, "Impulsive control for existence, uniqueness, and global stability of periodic solutions of recurrent neural networks with discrete and continuously distributed delays," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 24, no. 6, pp. 868–877, Jun. 2013.
[27] M. Li and J. Wang, "Exploring delayed Mittag–Leffler type matrix functions to study finite time stability of fractional delay differential equations," *Appl. Math. Comput.*, vol. 324, pp. 254–265, May 2018.
[28] Y. Guo, "Globally robust stability analysis for stochastic Cohen–Grossberg neural networks with impulse control and time-varying delays," *Ukrainian Math. J.*, vol. 69, no. 8, pp. 1220–1233, Jan. 2017.
[29] Y. Li, Y. Sun, and F. Meng, "New criteria for exponential stability of switched time-varying systems with delays and nonlinear disturbances," *Nonlinear Anal., Hybrid Syst.*, vol. 26, pp. 284–291, Nov. 2017.
[30] Z. Ding, Y. Shen, and L. Wang, "Global Mittag–Leffler synchronization of fractional-order neural networks with discontinuous activations," *Neural Netw.*, vol. 73, pp. 77–85, Jan. 2016.
[31] X. J. Yang, Q. K. Song, Y. R. Liu, and Z. J. Zhao, "Finite-time stability analysis of fractional-order neural networks with delay," *Neurocomputing*, vol. 152, pp. 19–26, Mar. 2015.
[32] J. Chen, Z. Zeng, and P. Jiang, "Global Mittag–Leffler stability and synchronization of memristor-based fractional-order neural networks," *Neural Netw.*, vol. 51, pp. 1–8, Mar. 2014.
[33] E. Kaslik and I. R. Rădulescu, "Dynamics of complex-valued fractional-order neural networks," *Neural Netw.*, vol. 89, pp. 39–49, May 2017.
[34] A. Pratap, R. Raja, C. Sowmiya, O. Bagdasar, J. Cao, and G. Rajchakit, "Robust generalized Mittag–Leffler synchronization of fractional order neural networks with discontinuous activation and impulses," *Neural Netw.*, vol. 103, pp. 128–141, Jul. 2018.
[35] S. Zhang, Y. Yu, and Y. Gu, "Global attractivity of memristor-based fractional-order neural networks," *Neurocomputing*, vol. 227, pp. 64–73, Mar. 2017.
[36] X. Zhang, P. Niu, Y. Ma, Y. Wei, and G. Li, "Global Mittag–Leffler stability analysis of fractional-order impulsive neural networks with one-side Lipschitz condition," *Neural Netw.*, vol. 94, pp. 67–75, Oct. 2017.
[37] L. Chen, Y. Chai, R. Wu, T. Ma, and H. Zhai, "Dynamic analysis of a class of fractional-order neural networks with delay," *Neurocomputing*, vol. 111, pp. 190–194, Jul. 2013.
[38] C. Xu, "Local and global Hopf bifurcation analysis on simplified bidirectional associative memory neural networks with multiple delays," *Mathe. Comput. Simul.*, vol. 149, pp. 69–90, Jul. 2018.
[39] C. Sun, M. Han, and X. Pang, "Global Hopf bifurcation analysis on a BAM neural network with delays," *Phys. Lett. A*, vol. 360, pp. 689–695, Jan. 2007.
[40] J. Ge and J. Xu, "Stability and Hopf bifurcation on four-neuron neural networks with inertia and multiple delays," *Neurocomputing*, vol. 287, pp. 34–44, Feb. 2018.
[41] L. Li, Z. Wang, Y. Li, H. Shen, and J. Lu, "Hopf bifurcation analysis of a complex-valued neural network model with discrete and distributed delays," *Appl. Math. Comput.*, vol. 330, pp. 152–169, Aug. 2018.
[42] C. Xu, Y. Shao, and P. Li, "Bifurcation behavior for an electronic neural network model with two different delays," *Neural Process. Lett.*, vol. 42, no. 3, pp. 541–561, Dec. 2015.
[43] C. Xu and Q. Zhang, "Bifurcation analysis of a tri-neuron neural network model in the frequency domain," *Nonlinear Dyn.*, vol. 76, no. 1, pp. 33–46, Apr. 2014.
[44] C. Xu, X. Tang, and M. Liao, "Stability and bifurcation analysis of a six-neuron BAM neural network model with discrete delays," *Neurocomputing*, vol. 74, no. 5, pp. 689–707, 2011.
[45] C. Huang and J. Cao, "Impact of leakage delay on bifurcation in high-order fractional BAM neural networks," *Neural Netw.*, vol. 98, pp. 223–235, Feb. 2018.
[46] C. Huang, J. Cao, M. Xiao, A. Alsaedi, and T. Hayat, "Effects of time delays on stability and Hopf bifurcation in a fractional ring-structured network with arbitrary neurons," *Commun. Nonlinear Sci. Numer. Simul.*, vol. 57, pp. 1–13, Apr. 2018.
[47] C. Huang, J. Cao, and M. Xiao, "Hybrid control on bifurcation for a delayed fractional gene regulatory network," *Chaos, Solitons Fractals*, vol. 87, pp. 19–29, Jun. 2016.
[48] C. Huang, J. Cao, M. Xiao, A. Alsaedi, and T. Hayat, "Bifurcations in a delayed fractional complex-valued neural network," *Appl. Math. Comput.*, vol. 229, pp. 210–227, Jan. 2017.
[49] C. Huang, Y. Meng, J. Cao, A. Alsaedi, and F. E. Alsaadi, "New bifurcation results for fractional BAM neural network with leakage delay," *Chaos, Solitons Fractals*, vol. 100, pp. 31–44, Jul. 2017.
[50] L. Zhao, J. Cao, C. Huang, A. Alsaedi, A. Al-Barakati, and H. M. Fardoun, "Bifurcation control in a delayed two-neuron fractional network," *Int. J. Control, Automat. Syst.*, vol. 15, no. 3, pp. 1134–1144, Jun. 2017.
[51] M. Xiao, W. X. Zheng, G. P. Jiang, and J. D. Cao, "Undamped oscillations generated by Hopf bifurcations in fractional-order recurrent neural networks with Caputo derivative," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 26, no. 12, pp. 3201–3214, Dec. 2015.
[52] M. Xiao, W. X. Zheng, J. Lin, G. Jiang, J. Cao, and L. Zhao, "Fractional-order PD control at Hopf bifurcations in delayed fractional-order small-world networks," *J. Franklin Inst.*, vol. 354, no. 17, pp. 7643–7667, Nov. 2017.
[53] X. Tian and R. Xu, "Stability and Hopf bifurcation of time fractional Cohen–Grossberg neural networks with diffusion and time delays in leakage terms," *Neural Process. Lett.*, vol. 45, pp. 593–614, Apr. 2017.
[54] I. Podlubny, *Fractional Differential Equations*. New York, NY, USA: Academic, 1999.

[55] D. Matignon, "Stability results for fractional differential equations with applications to control processing," in *Proc. IEEE-SMC Comput. Eng. Syst. Appl. Multi-Conf. (IMACS)*, Lille, France, vol. 2, Jul. 1996, pp. 963–968.
[56] T. Dong, X. Liao, and A. Wang, "Stability and Hopf bifurcation of a complex-valued neural network with two time delays," *Nonlinear Dyn.*, vol. 82, nos. 1–2, pp. 173–184, Oct. 2015.
[57] K. Zhang, "On a sign-changing solution for some fractional differential equations," *Boundary Value Problems*, vol. 2017, p. 59, Dec. 2017.
[58] Y. Wang and L. S. Liu, "Uniqueness and existence of positive solutions for the fractional integro-differential equation," *Bound. Value Prob.*, vol. 2017, p. 12, Dec. 2017.
[59] Y. Wang and J. Jiang, "Existence and nonexistence of positive solutions for the fractional coupled system involving generalized $p$-Laplacian," *Adv. Difference Equ.*, vol. 337, p. 337, Dec. 2017.
[60] X. Hao, H. Wang, L. Liu, and Y. Cui, "Positive solutions for a system of nonlinear fractional nonlocal boundary value problems with parameters and $p$-Laplacian operator," *Boundary Value Problems*, vol. 182, pp. 1–18, Dec. 2017.
[61] P. Selvaraj, O. M. Kwon, and R. Sakthivel, "Disturbance and uncertainty rejection performance for fractional-order complex dynamical networks," *Neural Netw.*, vol. 112, pp. 73–84, Apr. 2019.
[62] P. Selvaraj, R. Sakthivel, and O. M. Kwon, "Synchronization of fractional-order complex dynamical network with random coupling delay, actuator faults and saturation," *Nonlinear Dyn.*, vol. 94, no. 4, pp. 3101–3116, Aug. 2018.
[63] R. Sakthivel, P. Selvaraj, Y. Lim, and H. R. Karimi, "Adaptive reliable output tracking of networked control systems against actuator faults," *J. Franklin Inst.*, vol. 354, no. 9, pp. 3813–3837, Jun. 2017.

**CHANGJIN XU** was born in 1970. He graduated from Huaihua University, China, in 1994. He received the M.S. degree from the Kunming University of Science and Technology, in 2004, and the Ph.D. degree from Central South University, China, in 2010. He is currently a Professor with the Guizhou Key Laboratory of Economics System Simulation, Guizhou University of Finance and Economics. He has published about 100 refereed journal papers. He is a Reviewer of *Mathematical Reviews* and *Zentralbatt Math*. His research interests include nonlinear systems, neural networks, anti-periodic solution, stability, and bifurcation theory.

**MAOXIN LIAO** was born in 1969. He graduated from the Changsha University of Science and Technology, China, in 1992. He received the M.S. degree from the Changsha University of Science and Technology, in 2005, and the Ph.D. degree from Central South University, China, in 2010. He is currently a Professor with the School of Mathematics and Physics, University of South China, Hengyang, China. His research interests include the stability and bifurcation theory of delayed differential and equation.

**PEILUAN LI** was born in 1979. He graduated from Wuhan University, China, in 2001. He received the M.S. degree from Wuhan University, in 2004, and the Ph.D. degree from Central South University, China, in 2010. He held a postdoctoral position with Hunan Normal University, China, from 2011 to 2013. He is currently an Associate Professor with the School of Mathematics and Statistics, Henan University of Science and Technology. He has published about 70 refereed journal papers. His research interests include nonlinear systems, functional differential equations, and boundary value problems.



**JINLING YAN** was born in 1998. She graduated from the Henan University of Science and Technology, China, in 2019. She received the B.S. degree from the Henan University of Science and Technology, where she is currently pursuing the master's degree. Her research interests include statistical analysis and neural networks.