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An Efficient Method for Hopf Bifurcation Control in Fractional-Order Neuron Model

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ABSTRACT The nervous system contains a neural network that regulates and coordinates all physiological processes in our body, and as we all know, the damages within the system would lead to many neurological diseases, such as epilepsy, Alzheimer's disease, and Parkinson's disease or schizophrenia. The bifurcation phenomenon in the neuronal system is believed to be the cause, and thus, it is important to understand the mechanism and find effective methods to resist. Several control methods have been proved useful in the integer-order neuronal model. In this paper, we presented a novel control method based on a fractional-order washout filter with time delay for Hopf bifurcation control in a fractional-order neuron model, demonstrating and testing by a fractional-order Hodgkin–Huxley neuron model. The computer simulation shows the effectiveness of the proposed method. Furthermore, we presented the bifurcation phenomenon of fractional-order Hodgkin–Huxley neuron model with the decrease of the order and analyzed the influence of the fractional-order washout filter gain on the Hopf bifurcation of the different order Hodgkin–Huxley neuron model.

INDEX TERMS Fractional-order washout filter, Hopf bifurcation, fractional-order Hodgkin-Huxley model, time delay.

I. INTRODUCTION

Over the past decades, more and more investigators have been focusing on the bifurcation phenomenon of biological neurons because many nervous system diseases, such as Alzheimer's disease [1], epilepsy [2], Parkinson's disease [3], and attention deficit hyperactivity disorder [4], are believed result from bifurcation in cranial nerve [5]. Therefore, scientists have paid a great attention to find possible approach to control bifurcation and consequently cure diseases.

It is inspiring to see that many control methods have been developed and proved to be useful, such as TS fuzzy control [6], sliding mode control [7], adaptive fuzzy control [8], unscented Kalman filter [9], optimal control [10], washout filter [11]–[13], adaptive passive control [14], feedback control [15]–[18]. However, most previous theoretical studies on the bifurcation control of neuron have been in the form of integer-order neuron models. Recently, studies have shown that electrical properties of neuron membranes and the propagation of neural signals are well represented by differential equations of fractional order. Lundstrom *et al*. think that single neocortical pyramidal neurons adapt with a time scale that depends on the time scale of changes in stimulus statistics. This multiple time scale adaptation is consistent with fractional order differentiation [19]. Auastasio *et al*. think that the oculomotor integrator, which converts eye velocity into eye position commands, may be of fractional order [20]. In this way, it appears that fractional-order differential dynamical systems are more rational to describe the electrical properties of certain neuronal membranes. At present, there are few studies on fractional-order neuron system, mainly focusing on synchronization [21], [22], dynamic behavior analysis [23]. However, to our best knowledge, the research on Hopf bifurcation control for fractional-order neuron model is very few. For this reason, investigation of bifurcation control of the fractional-order neuron model should be performed.

In this study, first, we proposed a fractional-order washout filter with time delay for Hopf bifurcation control in fractional-order neuron model. Second, we studied bifurcation phenomenon of fractional-order Hodgkin-Huxley neuron model with the decrease of the order. Third, based on the proposed fractional-order washout filter, bifurcation control

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problem of fractional-order Hodgkin-Huxley neuron model is studied. Finally, we have analyzed the influence of the fractional-order washout filter gain on Hopf bifurcation of different order Hodgkin-Huxley neuron model. The main contributions of this paper are summarized as follows.

(1) A fractional-order washout filter with time delay for Hopf bifurcation control in fractional-order neuron model was proposed. As far as we know, this is the first time that fractional-order washout filter has been proposed.

(2) Bifurcation control problem of fractional-order Hodgkin-Huxley neuron model is studied. As far as we know, this is the first time that bifurcation control of fractional-order Hodgkin-Huxley neuron model has been studied.

(3) Bifurcation phenomenon of fractional-order Hodgkin-Huxley neuron model with the decrease of the order was researched.

(4) The effects of the fractional-order washout filter gain on Hopf bifurcation of different order Hodgkin-Huxley neuron model were analyzed.

This paper is organized as follows. We proposed a fractional-order washout filter in Section 2. In Section 3, we studied bifurcation phenomenon of fractional-order Hodgkin-Huxley neuron model with the decrease of the order. Hopf bifurcation control of the fractional-order Hodgkin-Huxley neuron model base on the fractional-order washout filter is presented in Section 4. Section 5 is the conclusions.

II. FRACTIONAL-ORDER WASHOUT FILTER

A. DEFINITION OF FRACTIONAL CALCULUS

There are several definitions of fractional calculus, such as Riemann-Liouville definition, Grünwald-Letinkov definition and Caputo definition. Because physical meaning of initial condition of the Caputo definition was clear, it was used in this paper.

The α-order differentiation of $f(t)$ in Caputo definition is defined as follows

$$ {}_{0}^{C}D_t^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau, \tag{1} $$

Here, n is the least integer that is not less than α ($n-1 < \alpha \leq n$); 0 and t are the lower and upper limits of the integral, respectively. $\Gamma(n-\alpha)$ is gamma function. $f^{(n)}(\tau)$ is *nth* derivative of the function $f(\tau)$.

B. DEFINITION OF FRACTIONAL-ORDER WASHOUT FILTER

The integer-order washout filter with time delay is described as follows

$$ \begin{aligned} \frac{du(t)}{dt} &= x(t-\tau) - du(t) \\ w(t) &= k(x(t-\tau) - du(t)) \end{aligned} \tag{2} $$

Here, τ, $x(t)$, $u(t)$ and $w(t)$ represent time delay, state variable, input variable, and output variable, respectively. k is control gain. d is reciprocal of filter time constant.

In this paper, we proposed a fractional-order washout filter with time delay depicted by:

$$ \begin{aligned} \frac{d^{\alpha}u(t)}{dt^{\alpha}} &= x(t-\tau) - du(t) \\ w(t) &= k(x(t-\tau) - du(t)) \end{aligned} \tag{3} $$

Here, α is the order of fractional-order. When $\alpha = 1$, the fractional-order washout filter degenerates to an ordinary washout filter of integer-order. In this paper, we set $d = 1$.

III. FRACTIONAL-ORDER HODGKIN-HUXLEY NEURON MODEL AND ITS DYNAMICS OF BIFURCATION

Using the Caputo definition of fractional-order differentials and the definition of the fractional-order system, the fractional-order Hodgkin-Huxley neuron model reads as follows

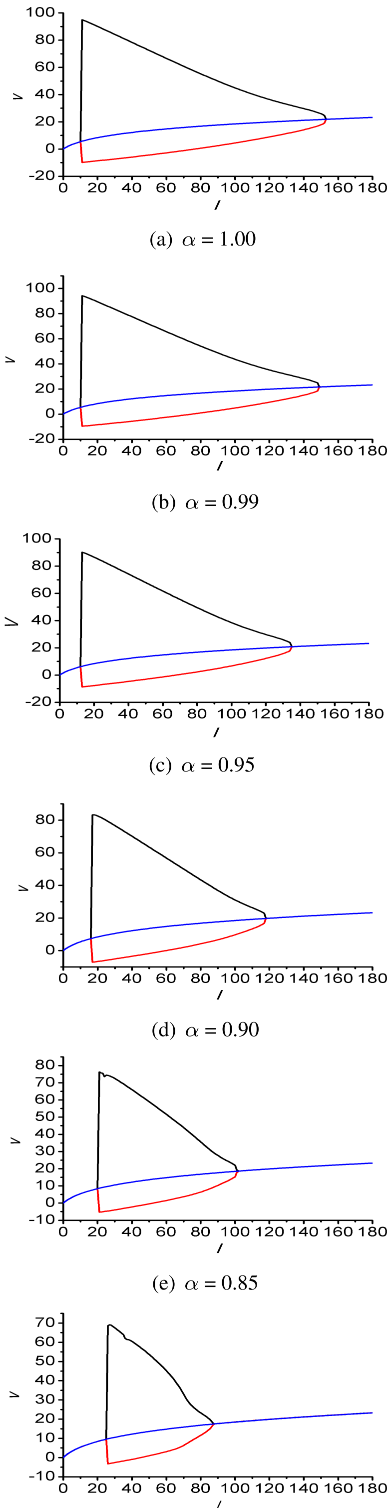
$$ \begin{aligned} \frac{d^{\alpha}v}{dt^{\alpha}} &= (I - g_{Na}m^3h(v-E_{Na}) - g_Kn^4(v-E_K) \\ &\quad - g_L(v-E_L))/C \\ \frac{d^{\alpha}m}{dt^{\alpha}} &= \alpha_m(1-m) - \beta_m m \\ \frac{d^{\alpha}h}{dt^{\alpha}} &= \alpha_h(1-h) - \beta_h h \\ \frac{d^{\alpha}n}{dt^{\alpha}} &= \alpha_n(1-n) - \beta_n n \end{aligned} \tag{4} $$

Here, v represent membrane potential, I represent external stimulus current. m, h and n represent sodium activation, sodium inactivation, and potassium activation, respectively. $g_L = 0.3$, $g_K = 36$, $g_{Na} = 120$, $E_L = 10.6$, $E_K = -12$, $E_{Na} = 115$, $C = 1$. α_m, β_m, α_h, β_h, α_n, β_n are functions of v that are defined as follows:

$$ \begin{aligned} \alpha_m(v) &= 0.1(-v+25)/(e^{-0.1v+2.5}-1) \\ \beta_m(v) &= 4e^{-v/18} \\ \alpha_h(v) &= 0.07e^{-v/20} \\ \beta_h(v) &= 1/(e^{-0.1v+3}+1) \\ \alpha_n(v) &= 0.01(-v+10)/(e^{-0.1v+1}-1) \\ \beta_n(v) &= 0.125e^{-v/80} \end{aligned} \tag{5} $$

In this section, we studied bifurcation phenomenon of fractional-order Hodgkin-Huxley neuron model with the decrease of the order. Bifurcation diagram of different order HH model is shown in **Fig. 1**. From **Fig. 1**, we can see that as the external stimulus current I increases, the neuron undergoes a Hopf bifurcation from quiescence to periodic spiking at I_{left}. When the external stimulus current I further increases, the amplitude of the stable periodic oscillation decreases, and the periodic oscillation terminates at another Hopf bifurcation I_{right}. In addition, as the order α decreases, the left Hopf bifurcation I_{left} increased, the right Hopf bifurcation I_{right} decreased, and the amplitude of the stable periodic oscillation decreases.

In this paper, we just focus our discussion on the left Hopf bifurcation I_{left} because external stimulus current can't be too



(a) $\alpha = 1.00$

(b) $\alpha = 0.99$

(c) $\alpha = 0.95$

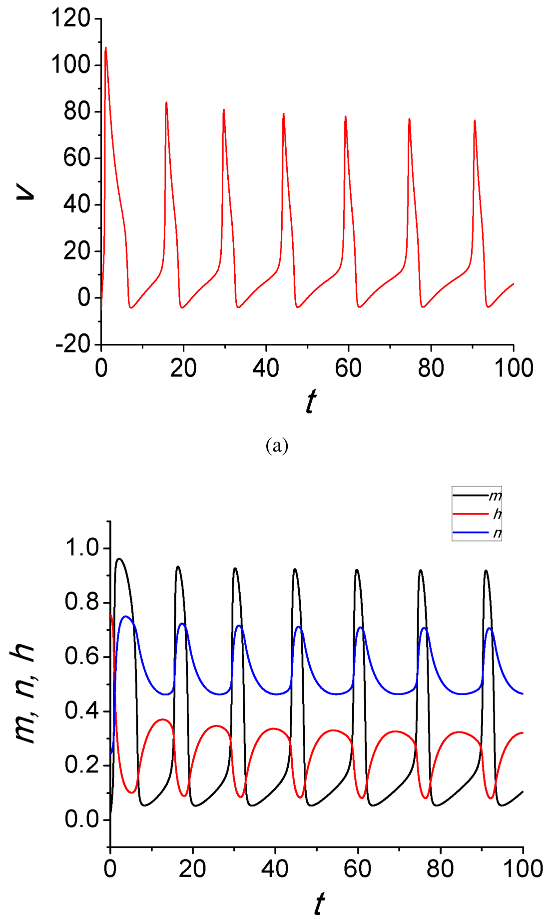
(d) $\alpha = 0.90$

(e) $\alpha = 0.85$

(f) $\alpha = 0.80$

FIGURE 1. Bifurcation diagram of different order Hodgkin-Huxley model. The blue line, black line and red line represent equilibrium points, maxima limit cycles and minima limit cycles, respectively. (a) $\alpha = 1.00$. (b) $\alpha = 0.99$. (c) $\alpha = 0.95$. (d) $\alpha = 0.90$. (e) $\alpha = 0.85$. (f) $\alpha = 0.80$.

large, otherwise it would destroy physiological nerve structure. At $\alpha = 0.80$, the left Hopf bifurcation $I_{left} = 25.666$, dynamical responses to the fractional-order HH system with



(a)

(b)

FIGURE 2. Dynamical responses to 0.8-order Hodgkin-Huxley model. (a) State of *v*. (b) State of *m, n, h*.

initial conditions

$$(v_0, m_0, h_0, n_0) = (-5, \frac{\alpha_m(v_0)}{\alpha_m(v_0) + \beta_m(v_0)}, \frac{\alpha_h(v_0)}{\alpha_h(v_0) + \beta_h(v_0)}, \frac{\alpha_n(v_0)}{\alpha_n(v_0) + \beta_n(v_0)}) \tag{6}$$

are shown in **Fig. 2**.

IV. HOPF BIFURCATION CONTROL OF THE FRACTIONAL-ORDER HH MODEL

A general fractional-order nonlinear autonomous system can be described by the following equation

$$\frac{d^\alpha x}{dt^\alpha} = f(x) \tag{7}$$

Here, $x \in R^n$ represent the state vector. Assume that $\bar{x} \in R^n$ as the equilibrium point of system (7), and $f(x)$ have first-order partial derivative at equilibrium point $\bar{x}$. Let J be the Jacobia matrix of system (7) at equilibrium point $\bar{x}$. Then, the linear topological equivalence of system (7) in a sufficient small neighborhood of $\bar{x}$ has the following form:

$$\frac{d^\alpha x}{dt^\alpha} = Jx \tag{8}$$

Lemma 1: Let λ be the eigenvalue of the matrix J, the conclusions as following:

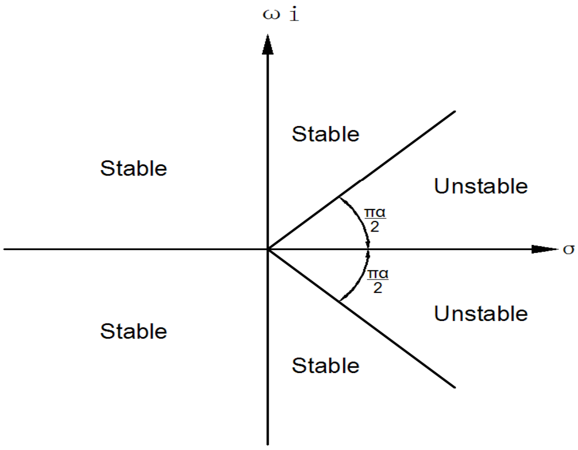


FIGURE 3. **Stable and unstable regions of a linear fractional-order system of order α.**

(1) If $|\arg(\lambda)| > \frac{\alpha\pi}{2}$, then the equilibrium point $\bar{x}$ is locally asymptotically stable.

(2) If $|\arg(\lambda)| \geq \frac{\alpha\pi}{2}$, then the equilibrium point $\bar{x}$ is locally stable.

(3) If $|\arg(\lambda)| < \frac{\alpha\pi}{2}$, then the equilibrium point $\bar{x}$ is unstable.

Lemma 2 (Routh-Hurwitz stability criterion):

Define characteristic polynomial of J is

$$\begin{aligned} g(\lambda) &= det(\lambda I_n - J) \\ &= a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n \end{aligned} \tag{9}$$

Here, $a_0 = 1, a_k \in R, k = 1, \cdots, n$.

And define a matrix:

$$\begin{aligned} H_a &= H_a(a_{ij}) \\ &= \begin{pmatrix} a_1 & a_0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & \cdots & 0 \\ a_5 & a_4 & a_3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{2n-3} & a_{2n-4} & \cdots & a_{n-1} & a_{n-2} \\ a_{2n-1} & a_{2n-2} & \cdots & a_{n+1} & a_n \end{pmatrix} \end{aligned} \tag{10}$$

Here, $a_{ij} = a_{2i-j}$, (i, j=1, 2, $\cdots$,n; where $a_k = 0$, if $k < 0$ or $k > n$).

Then all roots of g(λ) have negative real parts if and only if order principal minor determinant of H_a are all positive. That is, all eigenvalues λ of the Jacobian matrix J have negative real parts if and only if order principal minor determinant of H_a are all positive.

Theorem 1: The equilibrium point $\bar{x}$ is locally asymptotically stable if order principal minor determinant of H_a are all positive.

Proof: According to **Lemma 1**, the region of stability in the equilibrium point $\bar{x}$ is shown in **Fig. 3**.

Obviously, the equilibrium point $\bar{x}$ is locally asymptotically stable if all eigenvalues λ of Jacobian matrix J have negative real parts.

Then, according to **Lemma 2**, all eigenvalues λ of the Jacobian matrix J have negative real parts if and only if order principal minor determinant of H_a are all positive.

Since, the equilibrium point $\bar{x}$ is locally asymptotically stable if order principal minor determinant of H_a are all positive. □

We study Hopf bifurcation control in 0.8-order Hodgkin-Huxley model, and $I = 25.666$ is the left Hopf bifurcation. The equilibrium points of system (4) at $I = 25.666$ are solutions (v, m, h, n) to

$$\begin{aligned} &(I - g_{Na}m^3h(v - E_{Na}) - g_Kn^4(v - E_K) \\ &\quad - g_L(v - E_L))/C = 0 \\ &\quad \alpha_m(1 - m) - \beta_m m = 0 \\ &\quad \alpha_h(1 - h) - \beta_h h = 0 \\ &\quad \alpha_n(1 - n) - \beta_n n = 0 \end{aligned} \tag{11}$$

We only consider the equilibrium point

$$(v_e, m_e, h_e, n_e) = (9.6807, 0.15311, 0.27126, 0.47050) \tag{12}$$

We add the fractional-order washout filter with time delay to system (4), and get controlled fractional-order Hodgkin-Huxley model as follows:

$$\begin{aligned} \frac{d^\alpha v(t)}{dt^\alpha} &= (I - g_{Na}m^3(t)h(t)(v(t) - E_{Na}) \\ &\quad - g_Kn^4(t)(v(t) - E_K) - g_L(v(t) - E_L))/C \\ &\quad + k(v(t-\tau) - du(t)) \\ \frac{d^\alpha m(t)}{dt^\alpha} &= \alpha_{m(t)}(1 - m(t)) - \beta_m m(t) \\ \frac{d^\alpha h(t)}{dt^\alpha} &= \alpha_{h(t)}(1 - h(t)) - \beta_h h(t) \\ \frac{d^\alpha n(t)}{dt^\alpha} &= \alpha_{n(t)}(1 - n(t)) - \beta_n n(t) \\ \frac{d^\alpha u(t)}{dt^\alpha} &= v(t-\tau) - du(t) \end{aligned} \tag{13}$$

Here, $\tau \geq 0$ is time delay.

The Jacobian matrix for the system (13) at equilibrium point (v_e, m_e, h_e, n_e) is

$$J = \begin{pmatrix} J_{11} & J_{12} & J_{13} & J_{14} & -kd \\ J_{21} & J_{22} & 0 & 0 & 0 \\ J_{31} & 0 & J_{33} & 0 & 0 \\ J_{41} & 0 & 0 & J_{44} & 0 \\ e^{(-\lambda\tau)} & 0 & 0 & 0 & -d \end{pmatrix} \tag{14}$$

where

$$\begin{aligned} J_{11} &= -\frac{(g_{Na}m_e^3h_e - g_Kn_e^4 - gL)}{C} + ke^{(-\lambda\tau)} \\ J_{12} &= -\frac{3g_{Na}m_e^2h_e(v_e - E_{Na})}{C} \\ J_{13} &= -\frac{g_{Na}m^3(v_e - E_{Na})}{C} \\ J_{14} &= -\frac{4g_Kn_e^3(v_e - E_{Na})}{C} \\ J_{21} &= \frac{(0.15 - 0.01v_e)e^{-0.1v_e+2.5} + 0.1}{(e^{-0.1v_e+2.5} - 1)^2}(1 - m_e) \end{aligned}$$

$$
\begin{aligned}
&+\frac{2}{9}m_e e^{-\frac{v_e}{18}}\\
J_{22} &= -(\alpha_m+\beta_m)\\
J_{31} &= -0.0035e^{-\frac{v_e}{20}(1-h_e)} - \frac{0.1e^{-0.1v_e+3}}{(e^{-0.1v_e+3}+1)^2}h_e\\
J_{33} &= -(\alpha_h+\beta_h)\\
J_{41} &= \frac{0.01-0.001v_e e^{-0.1v_e+1}}{(e^{-0.1v_e+1}-1)^2}(1-n_e)\\
&\quad +0.0015625ne^{-\frac{v_e}{80}}\\
J_{44} &= -(\alpha_n+\beta_n)
\end{aligned} \tag{15}
$$

Then, the linear topological equivalence of system (13) in a sufficient small neighborhood of equilibrium point (v_e, m_e, h_e, n_e) has the following form:

$$
\begin{aligned}
\frac{d^\alpha v}{dt_\alpha} &= J_{11}v(t)+J_{12}n(t)+J_{13}h(t)+J_{14}m(t)\\
&\quad -kdu(t)\\
\frac{d^\alpha m}{dt_\alpha} &= J_{21}v(t)+J_{22}m(t)\\
\frac{d^\alpha h}{dt_\alpha} &= J_{31}v(t)+J_{33}h(t)\\
\frac{d^\alpha n}{dt_\alpha} &= J_{41}v(t)+J_{44}n(t)\\
\frac{d^\alpha u}{dt_\alpha} &= e^{(-\lambda\tau)}v(t)-du(t)
\end{aligned} \tag{16}
$$

Then the characteristic equation for the linearized system of (16) is

$$
\begin{aligned}
g(\lambda) &= det(\lambda I_5 - J)\\
&= a_0\lambda^5+a_1\lambda^4+a_2\lambda^3+a_3\lambda^2+a_4\lambda+a_5=0
\end{aligned} \tag{17}
$$

Here, I_5 is 5-order identity matrix.

We can get the coefficients of as follows:

$$
\begin{aligned}
a_0 &= 1\\
a_1 &= -ke^{(-\lambda\tau)}+6.3076\\
a_2 &= -3.1266ke^{(-\lambda\tau)}+4.2726\\
a_3 &= -1.0489ke^{(-\lambda\tau)}+1.2906\\
a_4 &= -0.091757ke^{(-\lambda\tau)}+2.7681\\
a_5 &= 0.44247
\end{aligned} \tag{18}
$$

Based on **Theorem 1**, we deduce control gain k of the fractional-order washout filter for $\tau = 0$ and $\tau \neq 0$, respectively.

A. $\tau = 0$

When $\tau = 0$, the coefficients $a_0, a_1, a_2, a_3, a_4, a_5$ as follows:

$$
\begin{aligned}
a_0 &= 1\\
a_1 &= -k+6.3076\\
a_2 &= -3.1266k+4.2726\\
a_3 &= -1.0489k+1.2906\\
a_4 &= -0.091757k+2.7681\\
a_5 &= 0.44247
\end{aligned} \tag{19}
$$

Define a matrix:

$$
H_a = \begin{pmatrix} a_1 & a_0 & 0 & 0 & 0\\ a_3 & a_2 & a_1 & a_0 & 0\\ a_5 & a_4 & a_3 & a_2 & a_1\\ 0 & 0 & a_5 & a_4 & a_3\\ 0 & 0 & 0 & 0 & a_5 \end{pmatrix} \tag{20}
$$

According to **Theorem 1**, we can get the stability condition of the equilibrium point (v_e, m_e, h_e, n_e)

$$
\begin{aligned}
D_1 &= -k+6.3076>0\\
D_2 &= 3.1266k^2-22.945k+25.659>0\\
D_3 &= -3.1878k^3+24.177k^2-18.399k\\
&\quad -74.228>0\\
D_4 &= 0.29250k^4-6.7173k^3+31.002k^2\\
&\quad +33.274k-246.46>0\\
D_5 &= 0.12942k^4-2.9723k^3+13.718k^2\\
&\quad +14.723k-109.05>0
\end{aligned} \tag{21}
$$

Solving the above inequalities, we can get

$$
k < -2.5715 \tag{22}
$$

In this we set $k = -3$. The dynamical response to the controlled 0.8-order HH model at the 0.8-order washout filter without time delay and comparisons with integer-order washout filter without time delay are shown in **Fig.4**. From the **Fig.4**, we can see that the 0.8-order washout filter takes a shorter than the integer-order washout filter for the controlled 0.8-order HH model to be stabilized.

B. $\tau \neq 0$

In this subsectoion, we deduce control gain k of the fractional-order washout filter for $\tau = 1$. When $\tau = 1$, the coefficients $a_0, a_1, a_2, a_3, a_4, a_5$ as follows:

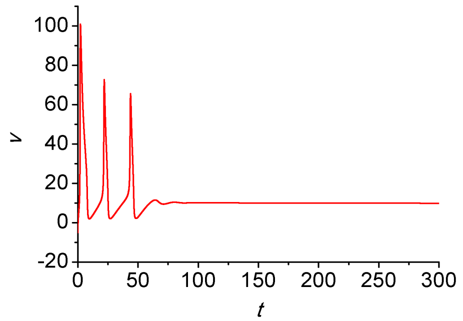
$$
\begin{aligned}
a_0 &= 1\\
a_1 &= -ke^{-\lambda}+6.3076\\
a_2 &= -3.1266ke^{-\lambda}+4.2726\\
a_3 &= -1.0489ke^{-\lambda}+1.2906\\
a_4 &= -0.091757ke^{-\lambda}+2.7681\\
a_5 &= 0.44247
\end{aligned} \tag{23}
$$

Padé approximation of order (m, n) to $e^{(-\lambda\tau)}$ is define as a rational function $P_{m,n}(\tau,\lambda)$ expressed in the form of [24], [25]

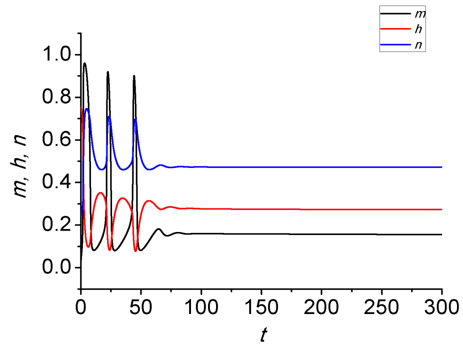
$$
P_{m,n}(\tau,\lambda) = \frac{q_0+q_1(\tau\lambda)+\cdots+q_m(\tau\lambda)^m}{r_0+r_1(\tau\lambda)+\cdots+r_n(\tau\lambda)^n} \tag{24}
$$

where

$$
\begin{aligned}
q_i &= (-1)^i\frac{(m+n-i)!m!}{(m+n)!(m-i)!i!}, i\in\{0,1,2,\cdots,m\}\\
r_j &= \frac{(m+n-j)!n!}{(m+n)!(n-j)!j!}j\in\{0,1,2,\cdots,n\}
\end{aligned} \tag{25}
$$



(a)



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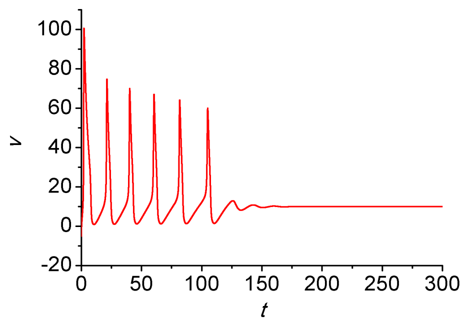

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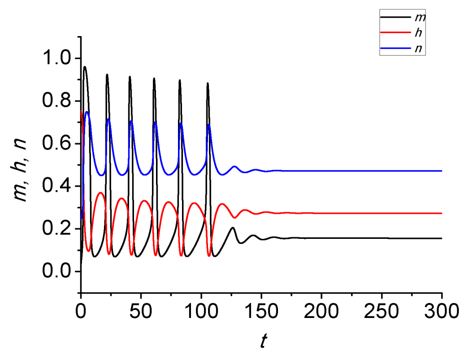

(d)

FIGURE 4. **Dynamical responses to the 0.8-order Hodgkin-Huxley model. (a) State response of *v* under 0.8-order washout filter without time delay. (b) State response of *m, n, h* under 0.8-order washout filter without time delay. (c) State response of *v* under integer-order washout filter without time delay. (d) State response of *m, n, h* under integer-order washout filter without time delay.**

In this $m = 3$, $n = 3$.

$$e^{(-\lambda)} = \frac{a_{pade}}{b_{pade}} \tag{26}$$

where

$$\begin{aligned} a_{pade} &= 120 - 60\lambda + 12\lambda^2 - \lambda^3 \\ b_{pade} &= 120 + 60\lambda + 12\lambda^2 + \lambda^3 \end{aligned} \tag{27}$$

Then characteristic equation (17), multiply both sides by b_{pade}:

$$\begin{aligned} p_0\lambda^8 + p_1\lambda^7 + p_2\lambda^6 + p_3\lambda^5 + p_4\lambda^4 \\ + p_5\lambda^3 + p_6\lambda^2 + p_7\lambda + p_8\lambda = 0 \end{aligned} \tag{28}$$

where

$$\begin{aligned} p_0 &= 1 \\ p_1 &= k + 18.307 \\ p_2 &= -8.8733k + 139.96 \\ p_3 &= 23.529k + 551.02 \\ p_4 &= 55.102k + 1031.5 \\ p_5 &= -313.36k + 623.81 \\ p_6 &= -120.36k + 326.28 \\ p_7 &= -11.011k + 358.73 \\ p_8 &= 53.097 \end{aligned} \tag{29}$$

Define a matrix:

$$H_{pade} = \begin{pmatrix} p_1 & p_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ p_3 & p_2 & p_1 & p_0 & 0 & 0 & 0 & 0 \\ p_5 & p_4 & p_3 & p_2 & p_1 & p_0 & 0 & 0 \\ p_7 & p_6 & p_5 & p_4 & p_3 & p_2 & p_1 & p_0 \\ 0 & p_8 & p_7 & p_6 & p_5 & p_4 & p_3 & p_2 \\ 0 & 0 & 0 & p_8 & p_7 & p_6 & p_5 & p_4 \\ 0 & 0 & 0 & 0 & 0 & p_8 & p_7 & p_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_8 \end{pmatrix} \tag{30}$$

According to **Theorem 1**, we can get the stability condition of the equilibrium point (v_e, m_e, h_e, n_e)

$$\begin{aligned} D_1 &= k + 18.308 > 0 \\ D_2 &= -8.8734k^2 - 46.015k + 2011.4 > 0 \\ D_3 &= -2.6389k^3 - 93.346k^2 - 393.80k + 7740.1 > 0 \\ D_4 &= 0.11258k^4 - 10.921k^3 - 197.99k^2 + 971.32k \\ &\quad + 6454.8 > 0 \\ D_5 &= -0.046368k^5 + 3.0067k^4 + 50.073k^3 \\ &\quad - 403.13k^2 - 1005.3k + 3080.8 > 0 \\ D_6 &= 0.0056751k^6 - 0.38506k^5 - 5.1548k^4 \\ &\quad + 70.919k^3 + 59.025k^2 - 1128.5k - 1114.4 > 0 \\ D_7 &= -0.00062467k^7 + 0.055037k^6 - 0.29638k^5 \\ &\quad - 18.936k^4 + 162.80k^3 + 282.64k^2 \\ &\quad - 2972.0k - 4406.9 > 0 \\ D_8 &= -0.00033179k^7 + 0.029223k^6 - 0.15737k^5 \\ &\quad - 10.055k^4 + 86.442k^3 + 150.07k^2 \\ &\quad - 1578.0k - 2340.0 > 0 \end{aligned} \tag{31}$$

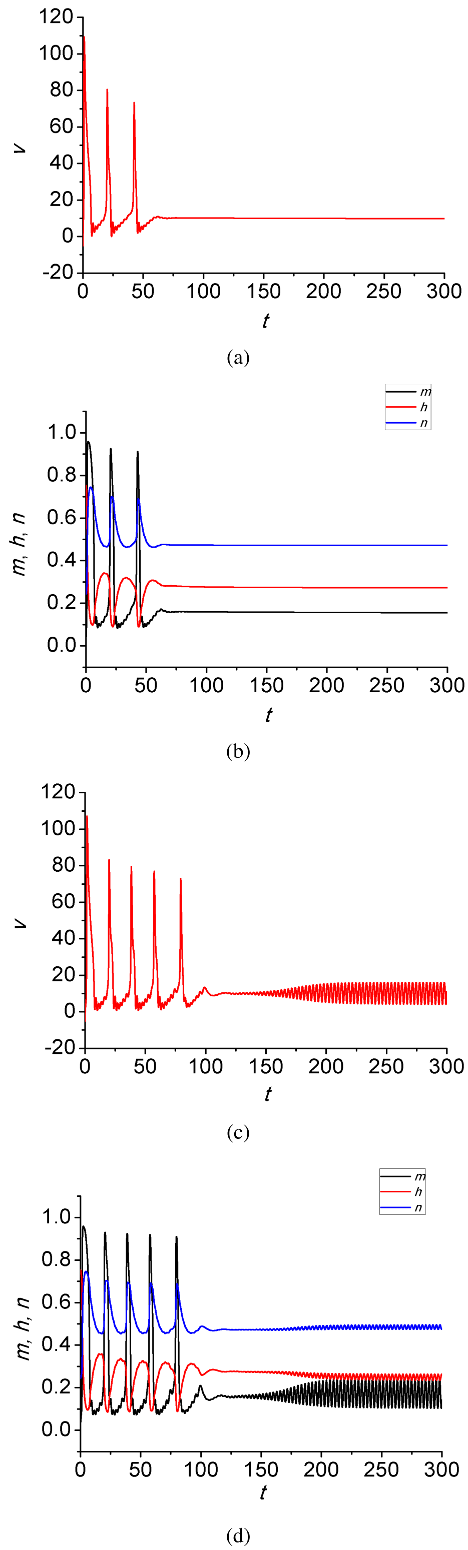

FIGURE 5. **Dynamical responses to the 0.8-order Hodgkin-Huxley model. (a) State response of *v* under 0.8-order washout filter with time delay. (b) State response of *m, n, h* under 0.8-order washout filter with time delay. (c) State response of *v* under integer-order washout filter with time delay. (d) State response of *m, n, h* under integer-order washout filter with time delay.**

Solving the above inequalities, we can get

$$-3.4728 < k < -1.4821 \tag{32}$$

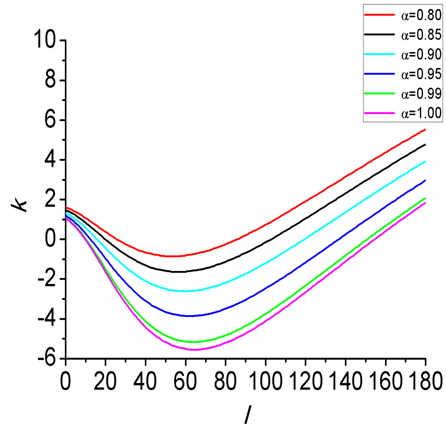

FIGURE 6. **The curve denotes the Hopf bifurcation boundary. The region of below and above the curve represent stable region and unstable region.**

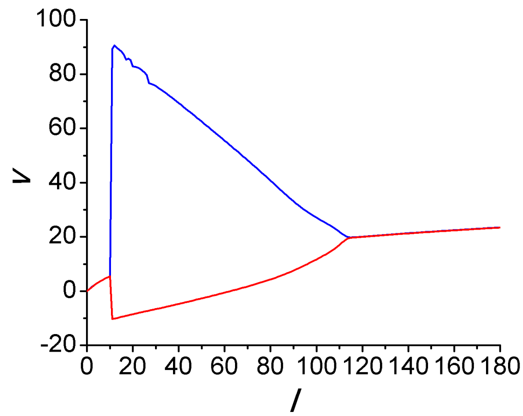

FIGURE 7. **Bifurcation diagram of the controlled HH model with $k = 1.06892$.**

For comparison with $\tau = 0$, it is also set $k = -3$. The dynamical response to the controlled 0.8-order HH model at the 0.8-order washout filter with time delay and comparisons with integer-order washout filter with time delay are shown in **Fig.5**. From the **Fig.5**, the controlled 0.8-order HH model become stable under the action of the 0.8-order washout filter, or form a limit cycle under the action of the integer-order washout filter. In addition, comparing **Fig.4** and **Fig.5**, we can see that time delay increases the fluctuation of the controlled 0.8-order HH model.

C. EFFECTS OF FRACTIONAL-ORDER WASHOUT FILTER ON HOPF BIFURCATION

In this subsection, we consider the influence of the fractional-order washout filter gain k on Hopf bifurcation of different order Hodgkin-Huxley neuron model. External stimulus current I increases from 0 to 180, based on **Lemma 1**, we deduce controller gain k to make different order Hodgkin-Huxley neuron model stable at its corresponding equilibrium point, and then get the stable region on the two-dimension parameter plane (I, k), as shown in **Fig.6**.

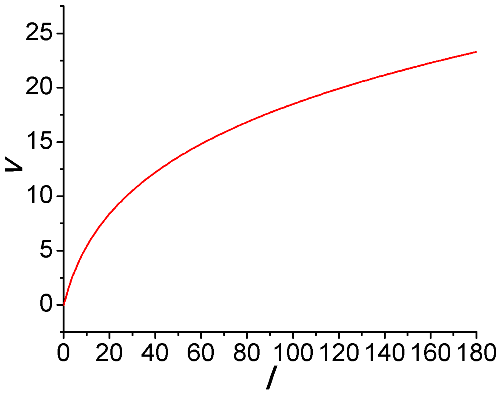

FIGURE 8. **Stability of equilibrium point in controlled fractional-order HH model with $k = -1$.**

From the **Fig.6**, for any order, there always exits k to make the controlled system stable on the domain $I \in [0, 180]$, thus, the Hopf bifurcation point can be move to any target location by choosing gain k properly. In addition, with order fallen from 1 to 0.8, the I-k curve move in positive direction of k axis, thus, the decrease of order may help improves system stability.

Further analysis the effects of fractional-order washout filter on Hopf bifurcation when $\alpha = 0.80$. From the **Fig.6**, the equilibrium point of the controlled system is stable on the domain $I \in [0, 180]$ when $k < -0.85435$. To move the bifurcation point to $I = 10$, based on **Lemma 1**, we can get the controller gain $k = 1.0689$. **Fig.7** shows the Hopf bifurcation diagram of the controlled fractional-order HH model when $k = 1.0689$. From the **Fig.7**, we can see that the Hopf bifurcation point has been successfully moved to $I = 10$. **Fig.8** shows the stability of equilibrium point in the controlled fractional-order HH model when $k = -1$ ($k < -0.85435$). From the **Fig.8**, we can see that the Hopf bifurcation has been successfully removed. The reason is that the system (4) is stable for any external stimulus current I when $k < -0.85435$. From the physical sense, washout filter likely decreases neuronal discharge when $k < 0$, or increases neuronal discharge when $k > 0$. Thus, choosing appropriate k can completely inhibit neuronal discharge, in other words, Hopf bifurcation disappear [26].

V. CONCLUSION

In this paper, fractional-order washout filter for fractional-order neuron model bifurcation control was proposed for the first time. On the other hand, based on the proposed fractional-order washout filter, bifurcation control problem of fractional-order Hodgkin-Huxley neuron model is studied. Conclusions are as follows. (1) The proposed fractional-order washout filter is effective and reasonable. (2) The fractional-order washout filter with time delay increases fluctuation of the controlled fractional-order Hodgkin-Huxley neuron model comparison with the fractional-order washout filter without time delay.

In addition, we presented bifurcation phenomenon of fractional-order Hodgkin-Huxley neuron model with the decrease of the order and analyzed the influence of the fractional-order washout filter gain on Hopf bifurcation of different order Hodgkin-Huxley neuron model. Conclusions are as follows. (1) The order of Hodgkin-Huxley neuron model will affect its bifurcation behavior. (2) The decrease of order of Hodgkin-Huxley neuron model may help improves system stability.

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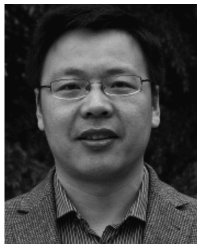

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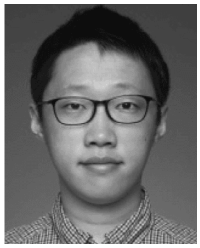

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