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Hybrid Kalman Filtering Algorithm With Stochastic Nonlinearities and Multiple Missing Measurements

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ABSTRACT In this paper, the hybrid Kalman filter is designed for a class of special nonlinear systems where the state equation is nonlinear and the measurement equation is linear. The stochastic nonlinearities, which are described by statistical means, are considered in the system model to reflect the multiplicative stochastic disturbances. The phenomenon of multiple missing measurements is depicted by a set of the Bernoulli distributed random variables with known conditional probabilities and the missing rates of every sensor are different. We need to compute the parameters to reduce the effects of the stochastic nonlinearities and the phenomenon of multiple missing measurements. In addition then, based on the recursive projection formula and the unscented transformation approach, a new hybrid Kalman filtering algorithm is proposed such that, for the stochastic nonlinearities and multiple missing measurements, the filtering error is minimized. By solving the recursive matrix equation, the filter gain matrices and the error covariance matrices can be obtained and the proposed results can be easily verified by using the standard numerical software. We finally provide a numerical example to show the performance of the proposed approach.

INDEX TERMS Nonlinear systems, stochastic nonlinearities, multiple missing measurements, minimum mean square error, unscented transformation.

I. INTRODUCTION

Filtering is one of the most basic and important technology in the field of communication network. In the process of data transmission, the signals are often disturbed by noise or some uncertain factors. By employing the filtering technology, the useful data can be extracted from the complex signals. With the development of space technology, the Kalman filtering theory, which was applied to make the optimal state estimation for the nonstationary multiple input and multiple output random systems model [1]–[6], was proposed in [7] for the linear discrete stochastic systems by employing the principle of minimum mean square error. However, based on the traditional Kalman filtering algorithm, a large number of state estimation problems for the nonlinear systems can't be addressed. As a new approach for nonlinear filtering, unscented Kalman filter was constructed in [8] for nonlinear systems by using the unscented transformation approach. The basic idea of unscented Kalman filtering algorithm is to make use of unscented transformation approach to approximate the mean and covariance in order to satisfy the minimum mean square error principle. Thus, the unscented Kalman filtering algorithm was also attracted wide attention [9]–[13].

It is well known that the linear systems or nonlinear systems are utilized commonly to characterize the target system models. However, in many practice systems, the target system models are described by a linear equation and a nonlinear equation (linear state equation and nonlinear observation equation or nonlinear state equation and linear observation equation). In order to solving the estimation problem for this kind of systems, the hybrid filters were proposed by applying the different filtering approaches [14]–[18]. To mention just a few, in [19], a hybrid filter, which can effectively increase the filtering capability, was proposed by using the differential current control method and the hybrid filter was mainly analyzed and validated in EAST power supply system.

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An unscented Kalman-particle hybrid filter was designed in [20] for recursive Bayesian estimation of space objects, where the hybrid filtering scheme provided accurate and consistent estimates when measurements were sparse without incurring a large computational cost. Based on the quantum filtering theory and the quantum extended Kalman filter method, a filtering problem was investigated in [21] for a class of quantum systems disturbed with a classical stochastic process. In [22], by incorporating both event-based and user-based neighborhood methods into matrix factorization, a hybrid collaborative filtering model, namely, Matrix Factorization with Event-User Neighborhood (MF-EUN) model, was presented to solve the problem of social influence prediction in event-based social networks. A robust H_∞ cubature Kalman filter (CKF)/KF hybrid filter (RHCHF) was proposed in [18] to lower the computational burden and strengthen the robustness.

In recent years, along with the accelerated development of network technology and the spread of computer application, the network phenomena are taken into account in various systems [24]–[27]. Due to the long distance data transmission and unreliability of the communication network, the stochastic disturbances, which are described by the stochastic nonlinearities, may exist in the systems and the sensor measurement of the system may experience the unexpected missing measurements (packet dropout) in the transmission process [28]–[33]. Hence, it is necessary to address the problem of the stochastic nonlinearities and the missing measurements to improve the control performance for the practical systems. More concretely, in [34], a time-varying filter was constructed for a class of nonlinear stochastic systems in the presence of event-triggered transmissions and multiple missing measurements with uncertain missing probabilities, where an upper bound of the filtering error covariance was obtained and then minimized by properly designing the filter gain. In [35], the distributed H_∞-consensus filtering problem was solved for a class of discrete time-varying systems subject to stochastic nonlinearities and multiple missing measurements by applying the consensus on information approach. By using the unscented transformation approach, in [36], a modified unscented Kalman filtering scheme was proposed for a class of nonlinear systems with stochastic nonlinearities and multiple fading measurements and the sufficient conditions were obtained to ensure stochastic stability of the modified unscented Kalman filter. The problem of a secure filtering was investigated in [37] for a class of uncertain stochastic non-linear systems and the sufficient conditions were derived to provide the filtering systems φ-level security by applied the techniques of stochastic analysis. In [38], the nonlinear filtering problem was addressed for a class of nonlinear discrete time stochastic systems with missing measurements by applying the extended and unscented Kalman filtering approach, respectively and it showed that the unscented Kalman filter is more effective than the extended Kalman filter. However, to the best of authors' knowledge, these researches do not pay much attention to the problem of the hybrid Kalman filtering algorithm for hybrid discrete stochastic systems subject to stochastic nonlinearities and the phenomenon of multiple missing measurements.

Based on the above discussions, in our paper, the purpose is to solve the estimation problem for a class of special nonlinear systems subject to stochastic nonlinearities and multiple missing measurements. In the system model, the stochastic nonlinearities, which are described by statistical means, are considered to reflect the multiplicative stochastic disturbances. The measurement output may experience the missing measurements due to the unreliable communication transmissions. The phenomenon of multiple missing measurements is depicted by a set of Bernoulli distributed random variables with known conditional probabilities and the missing rates of every sensor are different. Based on the recursive projection formula and the unscented transformation approach, the new hybrid Kalman filtering algorithm is proposed which can address the effects of stochastic nonlinearities and multiple missing measurements in a unified framework. Here, we make first attempt to design the hybrid filter for systems with stochastic nonlinearities and multiple missing measurements and establish a new recursive algorithm to obtain the optimal hybrid filter. Hence, we need to compute parameters to reduce the impact of stochastic nonlinearities and multiple missing measurements. Then, we can recursively compute the filter gain matrices and the error covariance matrices by using the new algorithm and Matlab software. Finally, we finally provide a numerical example to show the performance of the proposed approach. The contribution of this paper: 1). The system model is considered a class of special nonlinear systems where the state equation is nonlinear and the measurement equation is linear. 2). We make first attempt to propose the hybrid Kalman filter for systems subject to stochastic nonlinearities and multiple missing measurements. 3). A new recursive algorithm is established to obtain the hybrid Kalman filter which is suitable for online applications.

Notation: The symbols used in the paper are standard. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. A^T represents the transpose of a matrix A. $\mathbb{E}\{x\}$ is the expectation of the random variable x. The identity matrix and the zero matrix are expressed by I and 0 with appropriate dimensions, respectively. $diag\{X_1, X_2, \cdots, X_N\}$ stands for a diagonal matrix with elements $X_1, X_2, \cdots, X_N$ in the diagonal. If the dimensions of the matrices are not definitely stated, they are considered to be well-matched for algebraic operations.

II. PROBLEM FORMULATION

We consider the following system model with stochastic nonlinearity function and multiple missing measurements:

$$x_{k+1} = f_k(x_k) + g(x_k, \eta_k) + \omega_k, \tag{1}$$

$$z_k = \Xi_k C_k x_k + D_k v_k, \tag{2}$$

where k is the sampling instant, $x_k \in \mathbb{R}^n$ is the state vector, $z_k \in \mathbb{R}^m$ is the measured output of the sensor. $f_k(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is known nonlinear function. C_k and D_k are known matrices with appropriate dimensions. $\omega_k \in \mathbb{R}^n$ and

$\nu_k \in \mathbb{R}^m$ are uncorrelated zero-mean Gaussian white noises with covariances $Q_k \geq 0$ and $R_k > 0$, respectively. The stochastic nonlinearity function $g(x_k, \eta_k) : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $g(0, \eta_k) = 0$ and is assumed to have the following properties:

$$\begin{aligned} \mathbb{E}\{g(x_k, \eta_k)|x_k\} &= 0, \\ Cov\left(g(x_k, \eta_k), g(x_j, \eta_j)\right) &= 0, \quad k \neq j, \\ Cov\left(g(x_k, \eta_k), g(x_k, \eta_k)\right) &= \sum_{i=1}^{r} \Pi_i x_k^T \Gamma_i x_k, \quad k = j, \end{aligned} \tag{3}$$

where r is a known positive integer, Π_i and Γ_i $(i = 1, 2, \cdots, r)$ are known matrices with appropriate dimensions.

Here, we choose the stochastic nonlinearity function $g(x_k, \eta_k)$ as follows:

$$\begin{aligned} g(x_k, \eta_k) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} &\Big(b_1 sign(x_{1,k}) x_{1,k} \eta_{1,k} \\ &+ b_2 sign(x_{2,k}) x_{2,k} \eta_{2,k} + \cdots \\ &+ b_n sign(x_{n,k}) x_{n,k} \eta_{n,k}\Big) \end{aligned}$$

Then, we have the following matrices Π_i and Γ_i,

$$\begin{aligned} \Pi_i &= \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}^T \\ &= \begin{bmatrix} a_1^2 & a_1 a_2 & \cdots & a_1 a_n \\ a_2 a_1 & a_2^2 & \cdots & a_2 a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n a_1 & a_n a_2 & \cdots & a_n^2 \end{bmatrix} \\ \Gamma_i &= \begin{bmatrix} b_1^2 & 0 & 0 & 0 \\ 0 & b_2^2 & 0 & 0 \\ 0 & 0 & b_3^2 & 0 \\ 0 & 0 & 0 & b_4^2 \end{bmatrix} \end{aligned}$$

The diagonal matrix $\Xi_k = diag\{\alpha_{1,k}, \alpha_{2,k}, \ldots, \alpha_{m,k}\}$ is introduced to describe the phenomena of multiple missing measurements, where $\alpha_{i,k}$ $(i = 1, 2, \ldots, m)$ obeys the Bernoulli distribution and has the following statistical properties:

$$\begin{aligned} \text{Prob}\left\{\alpha_{i,k} = 1\right\} &= \mathbb{E}\left\{\alpha_{i,k}\right\} = \alpha_i, \\ \text{Prob}\left\{\alpha_{i,k} = 0\right\} &= 1 - \mathbb{E}\left\{\alpha_{i,k}\right\} = 1 - \alpha_i, \end{aligned}$$

where $\alpha_i (i = 1, 2, \ldots, m) \in [0, 1]$ are known positive scalars, and we assume that Ξ_k and other noise signals are mutually independent.

Remark 1: In model (2), the phenomena of multiple missing measurements, which are described by the diagonal matrix Ξ_k, are taken into account. The random variables α_i $(i = 1, 2, \ldots, m)$ in the matrix Ξ_k are the missing rate of the i-th sensor. If $\alpha_i = 1$, it represents that the i-th sensor receives the data successfully at time instant k. If $\alpha_i = 0$, it stands for that the i-th sensor receives the noises of the time instant k, i.e., the sensor occurs the phenomenon of missing data.

The purpose of this paper is, based on the observation sequence $\{z_1, z_2, \cdots, z_k\}$, to construct the hybrid Kalman filter for nonlinear discrete stochastic systems (1)-(2) by employing the minimum mean square error principle and unscented transformation approach.

III. MAIN RESULTS

In this section, we aim to design a new hybrid Kalman filter to solve the estimation problem for system subject to stochastic nonlinearity function and multiple missing measurements. To begin with, based on the reference [7], the following lemmas are introduced.

Lemma 1: Based on the linear space $Z_j = \left(z_1, z_2, \cdots, z_j\right)$, which is generated by the observation sequence, the optimal estimator of the state x_k in the sense of the MMSE principle is of the following form:

$$\begin{aligned} \hat{x}_{k|j} &= \mathbb{E}\left\{x_k|Z_j\right\} \\ &= \mathbb{E}\left\{x_k\right\} + Cov\left(x_k, Z_j\right) \\ &\quad \times \left(Var\left(Z_j\right)\right)^{-1}\left(Z_j - \mathbb{E}\left\{Z_j\right\}\right). \end{aligned} \tag{4}$$

Lemma 2: Based on the linear space $Z_j = \left(z_1, z_2, \cdots, z_j\right)$, if x_k and y_k are random vectors, A_k and B_k are known matrices with appropriate dimensions, the following linear relationship holds:

$$\begin{aligned} &\mathbb{E}\left\{\left(A_k x_k + B_k y_k\right)|Z_j\right\} \\ &= A_k \mathbb{E}\left\{x_k|Z_j\right\} + B_k \mathbb{E}\left\{y_k|Z_j\right\} \\ &= A\hat{x}_{k|j} + B\hat{y}_{k|j}. \end{aligned} \tag{5}$$

Lemma 3: If the linear space is expressed as $Z_j = \left(Z_{j-1}, z_j\right)$, then the estimating $\hat{x}_{k|j}$ of the state x_k can be described as follows:

$$\begin{aligned} \hat{x}_{k|j} &= \mathbb{E}\left\{x_k|Z_j\right\} \\ &= \mathbb{E}\left\{x_k|Z_{j-1}\right\} + \mathbb{E}\left\{\tilde{x}_{k|j-1}|\tilde{z}_{j|j-1}\right\} \\ &= \hat{x}_{k|j-1} \\ &\quad + \mathbb{E}\left\{\tilde{x}_{k|j-1}\tilde{z}_{j|j-1}^T\right\}\left(\mathbb{E}\left\{\tilde{z}_{j|j-1}\tilde{z}_{j|j-1}^T\right\}\right)^{-1}\tilde{z}_{j|j-1}, \end{aligned} \tag{6}$$

where

$$\begin{aligned} \tilde{x}_{k|j-1} &= x_k - \mathbb{E}\left\{x_k|Z_{j-1}\right\}, \\ \tilde{z}_{j|j-1} &= z_j - \mathbb{E}\left\{z_j|Z_{j-1}\right\}. \end{aligned}$$

Then, by employing the above lemmas, the following definitions and calculation formulas of the parameters are given for the sake of facilitating the subsequent developments.

Definition 1: Define the one-step prediction error $\tilde{x}_{k+1|k} = x_{k+1} - \hat{x}_{k+1|k}$, where $\hat{x}_{k+1|k}$ is the one-step prediction and the error covariance matrix of one-step prediction $P_{k+1|k} = \mathbb{E}\left\{\tilde{x}_{k+1|k}\tilde{x}_{k+1|k}^T\right\}$. Similarly, define the state estimation error $\tilde{x}_{k+1|k+1} = x_{k+1} - \hat{x}_{k+1|k+1}$, where $\hat{x}_{k+1|k+1}$ is the state estimation and the error covariance matrix of state estimation $P_{k+1|k+1} = \mathbb{E}\left\{\tilde{x}_{k+1|k+1}\tilde{x}^T_{k+1|k+1}\right\}$.

Lemma 4: Let the state covariance as $X_{k+1} = \mathbb{E}\{x_{k+1}x^T_{k+1}\}$. Then, by using the derivation method in the reference [39], we have the following equation:

$$X_{k+1} = P_{x_{k|k}x_{k|k}} + f_k\left(\hat{x}_{k|k}\right) f_k\left(\hat{x}_{k|k}\right)^T + \sum_{i=1}^{r} \Pi_i tr\left(X_k\Gamma_i\right) + Q_k, \tag{7}$$

where

$$\begin{aligned}\mathbb{E}\left\{f_k(x_k)|Z_k\right\} &\triangleq f_k\left(\hat{x}_{k|k}\right)\\ P_{x_{k|k}x_{k|k}} &= \mathbb{E}\Big\{\left(f_k\left(x_k\right) - f_k\left(\hat{x}_{k|k}\right)\right)\\ &\quad\times\left(f_k\left(x_k\right) - f_k\left(\hat{x}_{k|k}\right)\right)^T\Big\}.\end{aligned} \tag{8}$$

Proof: By using the equation (1) and (3), X_{k+1} can be computed as follows:

$$\begin{aligned}
&X_{k+1}\\
&= \mathbb{E}\left\{x_{k+1}x^T_{k+1}\right\}\\
&= \mathbb{E}\Big\{\left(f_k(x_k) + g(x_k, \eta_k) + \omega_k\right)\\
&\quad\times\left(f_k(x_k) + g(x_k, \eta_k) + \omega_k\right)^T\Big\}\\
&= \mathbb{E}\left\{f_k(x_k)f_k(x_k)^T\right\} + \mathbb{E}\left\{f_k(x_k)g(x_k, \eta_k)^T\right\}\\
&\quad+\mathbb{E}\left\{f_k(x_k)\omega^T_k\right\} + \mathbb{E}\left\{g(x_k, \eta_k)f_k(x_k)^T\right\}\\
&\quad+\mathbb{E}\left\{g(x_k, \eta_k)g(x_k, \eta_k)^T\right\} + \mathbb{E}\left\{g(x_k, \eta_k)\omega^T_k\right\}\\
&\quad+\mathbb{E}\left\{\omega_k f_k(x_k)^T\right\} + \mathbb{E}\left\{\omega_k g(x_k, \eta_k)^T\right\} + \mathbb{E}\left\{\omega_k\omega^T_k\right\}\\
&= \mathbb{E}\left\{\left(f_k\left(x_k\right) - f_k\left(\hat{x}_{k|k}\right)\right)\left(f_k\left(x_k\right) - f_k\left(\hat{x}_{k|k}\right)\right)^T\right\}\\
&\quad+f_k\left(\hat{x}_{k|k}\right)f_k\left(\hat{x}_{k|k}\right)^T + \sum_{i=1}^{r}\Pi_i tr\left(X_k\Gamma_i\right) + Q_k\\
&= P_{x_{k|k}x_{k|k}} + f_k\left(\hat{x}_{k|k}\right)f_k\left(\hat{x}_{k|k}\right)^T\\
&\quad+\sum_{i=1}^{r}\Pi_i tr\left(X_k\Gamma_i\right) + Q_k,
\end{aligned} \tag{9}$$

where

$$\begin{aligned}\mathbb{E}\left\{f_k(x_k)|Z_k\right\} &\triangleq f_k\left(\hat{x}_{k|k}\right)\\ P_{x_{k|k}x_{k|k}} &= \mathbb{E}\Big\{\left(f_k\left(x_k\right) - f_k\left(\hat{x}_{k|k}\right)\right)\\ &\quad\times\left(f_k\left(x_k\right) - f_k\left(\hat{x}_{k|k}\right)\right)^T\Big\}.\end{aligned} \tag{10}$$

It follows from (9) and (10) that (7) and (8) hold. Then, the proof of this lemma is complete.

Lemma 5: The one-step prediction $\hat{x}_{k+1|k}$ and the error covariance matrix of one-step prediction $P_{k+1|k}$ obey the following equations:

$$\hat{x}_{k+1|k} = f_k\left(\hat{x}_{k|k}\right), \tag{11}$$

$$P_{k+1|k} = P_{x_{k|k}x_{k|k}} + \sum_{i=1}^{r}\Pi_i tr\left(X_k\Gamma_i\right) + Q_k, \tag{12}$$

where the parameter $P_{x_{k|k}x_{k|k}}$ is calculated by the equation (8).

*Proof:*Based on the method in the reference [7], $\hat{x}_{k+1|k}$ can be calculated as follows:

$$\begin{aligned}\hat{x}_{k+1|k} &= \mathbb{E}\left\{x_{k+1}|Z_k\right\}\\ &= \mathbb{E}\left\{\left(f_k(x_k) + g(x_k, \eta_k) + \omega_k\right)|Z_k\right\}\\ &= \mathbb{E}\left\{f_k(x_k)|Z_k\right\}\\ &= f_k\left(\hat{x}_{k|k}\right).\end{aligned} \tag{13}$$

Then, we have

$$\begin{aligned}\tilde{x}_{k+1|k} &= x_{k+1} - \hat{x}_{k+1}\\ &= f_k(x_k) - f_k\left(\hat{x}_{k|k}\right) + g(x_k, \eta_k) + \omega_k.\end{aligned} \tag{14}$$

From **Definition 1**, one has

$$\begin{aligned}
&P_{k+1|k}\\
&= \mathbb{E}\left\{\tilde{x}_{k+1|k}\tilde{x}^T_{k+1|k}\right\}\\
&= \mathbb{E}\Big\{\left(f_k(x_k) - f_k\left(\hat{x}_{k|k}\right) + g(x_k, \eta_k) + \omega_k\right)\\
&\quad\times\left(f_k(x_k) - f_k\left(\hat{x}_{k|k}\right) + g(x_k, \eta_k) + \omega_k\right)^T\Big\}\\
&= \mathbb{E}\left\{\left(f_k(x_k) - f_k\left(\hat{x}_{k|k}\right)\right)\left(f_k(x_k) - f_k\left(\hat{x}_{k|k}\right)\right)^T\right\}\\
&\quad+\mathbb{E}\left\{\left(f_k(x_k) - f_k\left(\hat{x}_{k|k}\right)\right)g(x_k, \eta_k)^T\right\}\\
&\quad+\mathbb{E}\left\{\left(f_k(x_k) - f_k\left(\hat{x}_{k|k}\right)\right)\omega^T_k\right\}\\
&\quad+\mathbb{E}\left\{g(x_k, \eta_k)\left(f_k(x_k) - f_k\left(\hat{x}_{k|k}\right)\right)^T\right\}\\
&\quad+\mathbb{E}\left\{g(x_k, \eta_k)g(x_k, \eta_k)^T\right\}\\
&\quad+\mathbb{E}\left\{g(x_k, \eta_k)\omega^T_k\right\}\\
&\quad+\mathbb{E}\left\{\omega_k\left(f_k(x_k) - f_k\left(\hat{x}_{k|k}\right)\right)^T\right\}\\
&\quad+\mathbb{E}\left\{\omega_k g(x_k, \eta_k)^T\right\} + \mathbb{E}\left\{\omega_k\omega^T_k\right\}\\
&= \mathbb{E}\left\{\left(f_k(x_k) - f_k\left(\hat{x}_{k|k}\right)\right)\left(f_k(x_k) - f_k\left(\hat{x}_{k|k}\right)\right)^T\right\}\\
&\quad+\sum_{i=1}^{r}\Pi_i tr\left(X_k\Gamma_i\right) + Q_k\\
&= P_{x_{k|k}x_{k|k}} + \sum_{i=1}^{r}\Pi_i tr\left(X_k\Gamma_i\right) + Q_k,
\end{aligned} \tag{15}$$

where the parameter $P_{x_{k|k}x_{k|k}}$ is calculated by the equation (8). Then, the equation (12) is true.

Lemma 6: [40] Let $T = \left[t_{ij}\right]_{p\times p}$ be a real matrix and $S = diag\left\{s_1, s_2, \cdots, s_p\right\}$ be a diagonal random matrix. Then

$$\mathbb{E}\left\{STS^T\right\} = \begin{bmatrix} \mathbb{E}\left\{s_1^2\right\} & \mathbb{E}\left\{s_1 s_2\right\} & \cdots & \mathbb{E}\left\{s_1 s_p\right\} \\ \mathbb{E}\left\{s_2 s_1\right\} & \mathbb{E}\left\{s_2^2\right\} & \cdots & \mathbb{E}\left\{s_2 s_p\right\} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{E}\left\{s_p s_1\right\} & \mathbb{E}\left\{s_p s_2\right\} & \cdots & \mathbb{E}\left\{s_p^2\right\} \end{bmatrix} \circ T$$

where $\circ$ is the Hadamard product.

Now, based on the observation sequence $\{z_1, z_2, \cdots, z_k\}$, we are ready to design the optimal hybrid Kalman filter for nonlinear discrete stochastic systems (1)-(2) by employing the projection theory and the unscented transformation approach.

Theorem 1: The recursive hybrid Kalman filter for system (1)-(2) is given as follows:

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1}\tilde{z}_{k+1|k}, \tag{16}$$

$$\tilde{z}_{k+1|k} = z_{k+1} - \Xi C_{k+1}, \tag{17}$$

$$K_{k+1} = P_{k+1|k}C_{k+1}^T \Xi Q_{\tilde{z}_{k+1|k}}^{-1}, \tag{18}$$

$$Q_{\tilde{z}_{k+1|k}} = \Xi C_{k+1}P_{k+1|k}C_{k+1}^T \Xi + \mathscr{H} + R_{k+1}, \tag{19}$$

$$P_{k+1|k+1} = P_{k+1|k} - K_{k+1}\Xi C_{k+1}P_{k+1|k}^T, \tag{20}$$

where

$$\mathscr{H} = \Xi(I - \Xi) \circ C_{k+1}X_{k+1}C_{k+1}^T, \tag{21}$$

$\tilde{z}_{k+1|k}$ is the innovation sequence and K_{k+1} is the filter gain matrix to be determined. The parameters $\hat{x}_{k+1|k}$ and $P_{k+1|k}$ are computed by **Lemma 5**. $\mathscr{H}$ can be calculated by **Lemma 6**.

Proof: Based on the linear space $Z_{k+1} = \{z_1, z_2, \cdots, z_{k+1}\}$, which is generated by the observation sequence, the optimal estimator $\hat{x}_{k+1|k+1}$ in the sense of the minimum mean square error principle is the following form by using the method in the reference [7]:

$$\begin{aligned}\hat{x}_{k+1|k+1} &= \mathbb{E}\left\{x_{k+1}|Z_{k+1}\right\} \\ &= \mathbb{E}\left\{x_{k+1}|Z_k\right\} + \mathbb{E}\left\{\tilde{x}_{k+1|k}|\tilde{z}_{k+1|k}\right\} \\ &= \hat{x}_{k+1|k} + \mathbb{E}\left\{\tilde{x}_{k+1|k}\tilde{z}_{k+1|k}^T\right\} \\ &\quad\times\left(\mathbb{E}\left\{\tilde{z}_{k+1|k}\tilde{z}_{k+1|k}^T\right\}\right)^{-1}\tilde{z}_{k+1|k},\end{aligned} \tag{22}$$

where $\tilde{x}_{k+1|k}$ is calculated by **Definition 1** and $\tilde{z}_{k+1|k}$ is derived as follows:

$$\begin{aligned}\tilde{z}_{k+1|k} &= z_{k+1} - \hat{z}_{k+1|k} \\ &= z_{k+1} - \Xi C_{k+1}\hat{x}_{k+1|k} \\ &= \Xi_k C_{k+1}x_{k+1} - \Xi C_{k+1}\hat{x}_{k+1|k} + \nu_{k+1} \\ &= (\Xi_k - \Xi)\, C_{k+1}x_{k+1} \\ &\quad + \Xi C_{k+1}\left(x_{k+1} - \hat{x}_{k+1|k}\right) + \nu_{k+1} \\ &= (\Xi_k - \Xi)\, C_{k+1}x_{k+1} + \Xi C_{k+1}\tilde{x}_{k+1|k} + \nu_{k+1}.\end{aligned} \tag{23}$$

Note that $\mathbb{E}\{\Xi_k - \Xi\} = 0$ and ν_{k+1} is uncorrelated with other terms. Then, we have

$$\begin{aligned}&Q_{\tilde{z}_{k+1|k}} \\ &= \mathbb{E}\left\{\tilde{z}_{k+1|k}\tilde{z}_{k+1|k}^T\right\} \\ &= \mathbb{E}\Big\{\left((\Xi_k - \Xi)\, C_{k+1}x_{k+1} + \Xi C_{k+1}\tilde{x}_{k+1|k} + \nu_{k+1}\right) \\ &\quad\times\left((\Xi_k - \Xi)\, C_{k+1}x_{k+1} + \Xi C_{k+1}\tilde{x}_{k+1|k} + \nu_{k+1}\right)^T\Big\} \\ &= \mathbb{E}\left\{(\Xi_k - \Xi)\, C_{k+1}x_{k+1}\left((\Xi_k - \Xi)\, C_{k+1}x_{k+1}\right)^T\right\} \\ &\quad+\mathbb{E}\left\{(\Xi_k - \Xi)\, C_{k+1}x_{k+1}\left(\Xi C_{k+1}\tilde{x}_{k+1|k}\right)^T\right\} \\ &\quad+\mathbb{E}\left\{(\Xi_k - \Xi)\, C_{k+1}x_{k+1}\nu_{k+1}^T\right\} \\ &\quad+\mathbb{E}\left\{\Xi C_{k+1}\tilde{x}_{k+1|k}\left((\Xi_k - \Xi)\, C_{k+1}x_{k+1}\right)^T\right\} \\ &\quad+\mathbb{E}\left\{\Xi C_{k+1}\tilde{x}_{k+1|k}\left(\Xi C_{k+1}\tilde{x}_{k+1|k}\right)^T\right\} \\ &\quad+\mathbb{E}\left\{\Xi C_{k+1}\tilde{x}_{k+1|k}\nu_{k+1}^T\right\} \\ &\quad+\mathbb{E}\left\{\nu_{k+1}\left((\Xi_k - \Xi)\, C_{k+1}x_{k+1}\right)^T\right\} \\ &\quad+\mathbb{E}\left\{\nu_{k+1}\left(\Xi C_{k+1}\tilde{x}_{k+1|k}\right)^T\right\} + \mathbb{E}\left\{\nu_{k+1}\nu_{k+1}^T\right\} \\ &= \mathbb{E}\left\{(\Xi_k - \Xi)\, C_{k+1}x_{k+1}x_{k+1}^T C_{k+1}^T (\Xi_k - \Xi)^T\right\} \\ &\quad+\Xi C_{k+1}\mathbb{E}\left\{\tilde{x}_{k+1|k}\tilde{x}_{k+1|k}^T\right\}C_{k+1}^T\Xi^T \\ &\quad+\mathbb{E}\left\{\nu_{k+1}\nu_{k+1}^T\right\} \\ &= \Xi C_{k+1}P_{k+1|k}C_{k+1}^T\Xi + \mathscr{H} + R_{k+1},\end{aligned} \tag{24}$$

where

$$\mathscr{H} = \mathbb{E}\left\{(\Xi_k - \Xi)\, C_{k+1}x_{k+1} \times x_{k+1}^T C_{k+1}^T (\Xi_k - \Xi)^T\right\}. \tag{25}$$

By applying **Lemma 6**, the parameter $\mathscr{H}$ can be calculated by formula (26), as shown at the top of the next page. Therefore, the equations (17), (19) and (21) can be obtained by (23)-(26).

By using the equations (14), (23) and $\mathbb{E}\{\Xi_k - \Xi\} = 0$, $\mathbb{E}\{\nu_{k+1}\} = 0$, it can be easily deduced that

$$\begin{aligned}&\mathbb{E}\left\{\tilde{x}_{k+1|k}\tilde{z}_{k+1|k}^T\right\} \\ &= \mathbb{E}\Big\{\tilde{x}_{k+1|k}\Big((\Xi_k - \Xi)\, C_{k+1}x_{k+1} \\ &\quad+\Xi C_{k+1}\tilde{x}_{k+1|k} + \nu_{k+1}\Big)^T\Big\} \\ &= \mathbb{E}\left\{\tilde{x}_{k+1|k}x_{k+1}^T C_{k+1}^T (\Xi_k - \Xi)^T\right\} \\ &\quad+\mathbb{E}\left\{\tilde{x}_{k+1|k}\tilde{x}_{k+1|k}^T\right\}C_{k+1}^T\Xi^T \\ &\quad+\mathbb{E}\left\{\tilde{x}_{k+1|k}\nu_{k+1}^T\right\} \\ &= P_{k+1|k}C_{k+1}^T\Xi^T.\end{aligned} \tag{27}$$

$$\begin{aligned}
\mathscr{H} &= \mathbb{E}\left\{(\Xi_{k+1}-\Xi)C_{k+1}x_{k+1}x_{k+1}^T C_{k+1}^T(\Xi_{k+1}-\Xi)^T\right\} \\
&= \begin{bmatrix} \mathbb{E}\left\{(\alpha_{1,k+1}-\alpha_1)^2\right\} & \cdots & \mathbb{E}\left\{(\alpha_{1,k+1}-\alpha_1)(\lambda_{m,k+1}-\alpha_m)\right\} \\ \vdots & \ddots & \vdots \\ \mathbb{E}\left\{(\alpha_{m,k+1}-\alpha_m)(\alpha_{1,k+1}-\alpha_1)\right\} & \cdots & \mathbb{E}\left\{(\alpha_{m,k+1}-\alpha_m)^2\right\} \end{bmatrix} \circ C_{k+1}\mathbb{E}\left\{x_{k+1}x_{k+1}^T\right\}C_{k+1}^T \\
&= diag\left\{\alpha_1(1-\alpha_1), \alpha_2(1-\alpha_2), \cdots, \alpha_m(1-\alpha_m)\right\} \circ C_{k+1}X_{(k+1,k+1)}C_{k+1}^T \\
&= \left(diag\left\{\alpha_1, \alpha_2, \cdots, \alpha_m\right\} diag\left\{1-\alpha_1, 1-\alpha_2, \cdots, 1-\alpha_m\right\}\right) \circ C_{k+1}X_{(k+1,k+1)}C_{k+1}^T \\
&= \Xi(I-\Xi) \circ C_{k+1}X_{(k+1,k+1)}C_{k+1}^T. \qquad (26)
\end{aligned}$$

We define the gain matrix K_{k+1} as follows:

$$K_{k+1} = \mathbb{E}\left\{\tilde{x}_{k+1|k}\tilde{z}_{k+1|k}^T\right\}\left(\mathbb{E}\left\{\tilde{z}_{k+1|k}\tilde{z}_{k+1|k}^T\right\}\right)^{-1}. \qquad (28)$$

From the equations (24) and (27), we know

$$K_{k+1} = P_{k+1|k}C_{k+1}^T \Xi Q_{\tilde{z}_{k+1|k}}^{-1}, \qquad (29)$$

where $Q_{\tilde{z}_{k+1|k}} = \mathbb{E}\left\{\tilde{z}_{k+1|k}\tilde{z}_{k+1|k}^T\right\}$. Substituting the equation (28) into (22) yields (16).

Subsequently, the following derivations are given to obtain $P_{k+1|k+1}$. By **Definition 1** and the equation (16), it has

$$\begin{aligned}
\tilde{x}_{k+1|k+1} &= x_{k+1} - \hat{x}_{k+1|k+1} \\
&= x_{k+1} - \hat{x}_{k+1|k} - K_{k+1}\tilde{z}_{k+1|k} \\
&= \tilde{x}_{k+1|k} - K_{k+1}\tilde{z}_{k+1|k}. \qquad (30)
\end{aligned}$$

Then, we have

$$\begin{aligned}
P_{k+1|k+1} &= \mathbb{E}\left\{\tilde{x}_{k+1|k+1}\tilde{x}_{k+1|k+1}^T\right\} \\
&= \mathbb{E}\left\{(\tilde{x}_{k+1|k} - K_{k+1}\tilde{z}_{k+1|k}) \times (\tilde{x}_{k+1|k} - K_{k+1}\tilde{z}_{k+1|k})^T\right\} \\
&= \mathbb{E}\left\{\tilde{x}_{k+1|k}\tilde{x}_{k+1|k}^T\right\} - \mathbb{E}\left\{\tilde{x}_{k+1|k}\tilde{z}_{k+1|k}^T\right\}K_{k+1}^T \\
&\quad - K_{k+1}\mathbb{E}\left\{\tilde{z}_{k+1|k}\tilde{x}_{k+1|k}^T\right\} + K_{k+1}\mathbb{E}\left\{\tilde{z}_{k+1|k}\tilde{z}_{k+1|k}^T\right\}K_{k+1}^T \\
&= P_{k+1|k} - K_{k+1}\left(\mathbb{E}\left\{\tilde{x}_{k+1|k}\tilde{z}_{k+1|k}^T\right\}\right)^T \\
&= P_{k+1|k} - \alpha K_{k+1}\Xi C_{k+1}P_{k+1|k}^T. \qquad (31)
\end{aligned}$$

Hence, the equation (20) is true. The proof of this theorem is now complete.

According to **Theorem 1**, a new recursive algorithm can be established to obtain the optimal hybrid Kalman filter for the addressed discrete stochastic systems subject to stochastic nonlinearity function and multiple missing measurements. The following algorithm shows how to design the hybrid Kalman filter in **Theorem 1**.

Algorithm The steps of the design of the hybrid Kalman filter are shown as follows:

Step 1: Choose the sigma points.

We choose $2n+1$ points as a sigma points set, i.e.

$$\begin{aligned}
\chi_{k|k}^0 &= \hat{x}_{k|k}, \quad s = 0, \\
\chi_{k|k}^s &= \hat{x}_{k|k} + \left(\sqrt{(n+\kappa)P_{k|k}}\right)_s, \quad s = 1, \cdots, n, \\
\chi_{k|k}^s &= \hat{x}_{k|k} - \left(\sqrt{(n+\kappa)P_{k|k}}\right)_{s-n}, \quad s = n+1, \cdots, 2n,
\end{aligned}$$

where κ is the scaling factor and $\left(\sqrt{(n+\kappa)P_{k|k}}\right)_s$ is either the s-th row or the s-th column of the matrix square root of $(n+\kappa)P_{k|k}$.

Compute the transformed values of the sigma points by using the nonlinear functions $f_k(x_k)$.

$$\chi_{k+1|k}^s = f_k\left(\chi_{k|k}^s\right), \quad s = 0, 1, \cdots, 2n, \qquad (32)$$

Step 2: Compute the value of parameters.

The one-step prediction $\hat{x}_{k+1|k}$ can be calculated by recombining the weighted sigma points as follows:

$$\hat{x}_{k+1|k} = \sum_{s=0}^{2n} W^s \chi_{k+1|k}^s, \qquad (33)$$

according to weights

$$W^s = \begin{cases} \dfrac{\kappa}{n+\kappa}, & s = 0, \\ \dfrac{1}{2(n+\kappa)}, & s = 1, 2, \cdots, 2n. \end{cases} \qquad (34)$$

The parameters $P_{x_{k|k}x_{k|k}}$ is obtained as follows:

$$\begin{aligned}
&P_{x_{k|k}x_{k|k}} \\
&= \mathbb{E}\left\{\left(f_k(x_k) - f_k\left(\hat{x}_{k|k}\right)\right)\left(f_k(x_k) - f_k\left(\hat{x}_{k|k}\right)\right)^T\right\} \\
&= \sum_{s=0}^{2n} W^s\left(\chi_{k+1|k}^s - \hat{x}_{k+1|k}\right)\left(\chi_{k+1|k}^s - \hat{x}_{k+1|k}\right)^T. \qquad (35)
\end{aligned}$$

Then, the state covariance X_{k+1} and the error covariance matrix $P_{k+1|k}$ satisfy the following equations:

$$\begin{aligned} X_{k+1} &= \sum_{s=0}^{2n} W^s \left(\chi^s_{k+1|k} - \hat{x}_{k+1|k}\right)\left(\chi^s_{k+1|k} - \hat{x}_{k+1|k}\right)^T \\ &\quad + \sum_{s=0}^{2n} W^s \chi^s_{k+1|k}\left(\chi^s_{k+1|k}\right)^T \\ &\quad + \sum_{i=1}^{r} \Pi_i tr\left(X_k \Gamma_i\right) + Q_k, \end{aligned} \tag{36}$$

$$\begin{aligned} P_{k+1|k} &= \sum_{s=0}^{2n} W^s \left(\chi^s_{k+1|k} - \hat{x}_{k+1|k}\right)\left(\chi^s_{k+1|k} - \hat{x}_{k+1|k}\right)^T \\ &\quad + \sum_{i=1}^{r} \Pi_i tr\left(X_k \Gamma_i\right) + Q_k. \end{aligned} \tag{37}$$

Step 3: Compute the value of estimation.

1). Substituting the equation (36) into (21), we can obtain $\mathscr{H}$.

2). Computing $Q_{\tilde{z}_{k+1|k}}$ by substituting the equations (21) and (37) into (19).

3). Substituting (10), (11) and (14) into (9), we have K_{k+1}.

4). The optimal estimation $\hat{x}_{k+1|k+1}$ can be computed by substituting the equations (17), (18) and (33) into (16).

5). Substituting the equations (18) and (37) into (20), the error covariance matrix of state estimation $P_{k+1|k+1}$ can be obtained.

Remark 2: It is worth noting that, in **Theorem 1**, the recursive optimal hybrid Kalman-type filter is designed for the addressed nonlinear discrete stochastic systems with stochastic nonlinearity function and multiple missing measurements. In our paper, we consider a class of special nonlinear systems where the state equation is nonlinear and the measurement equation is linear. In order to improve the accuracy of filtering algorithm, we not only employ the unscented transformation approach, but also use the linear filtering method to computed the parameters. For example, the innovation sequence $\tilde{z}_{k+1|k}$ is calculated by using the projective theorem (the linear filtering method); the one-step prediction $\hat{x}_{k+1|k}$ is computed by utilizing the unscented transformation approach. In addition, some parameters are computed by combining the linear filtering method with the unscented transformation approach, such as X_{k+1}, $P_{x_{k|k}x_{k|k}}$ and P_{k+1}. Owing to the long distance transmission and unreliability of the communication network, the stochastic nonlinearity function and the phenomenon of multiple missing measurements exist commonly. Therefore, in this paper, we consider the nonlinear discrete stochastic systems subject to stochastic nonlinearity function and multiple missing measurements. Here, we need to compute the parameters $\sum_{i=1}^{r} \Pi_i tr\left(X_k \Gamma_i\right)$ and $\mathscr{H}$ to reduce the impact of stochastic nonlinearity function and multiple missing measurements. In the following, an illustrative example will be provided to show the feasibility of the proposed filtering scheme.

IV. AN ILLUSTRATIVE EXAMPLE

In this section, we provide a numerical example to show the performance of the proposed approach.

Consider the following nonlinear discrete stochastic systems:

$$\begin{aligned} x_{k+1} &= \begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \\ x_{3,k+1} \end{bmatrix} \\ &= \begin{bmatrix} -2\sin(x_{2,k}) + \cos(x_{3,k}) \\ 1.5\cos(x_{2,k}) + \sin(x_{1,k}) \\ \dfrac{x_{2,k}}{1 + x_{3,k}^2} \end{bmatrix} + g(x_k, \eta_k) + \omega_k, \\ z_k &= \Xi_k C_k x_k + D_k \nu_k, \end{aligned}$$

where ω_k and ν_k are uncorrelated Gaussian white noises with covariances $Q_k = 0.4I_3$, $R_k = 0.1I_1$.

We choose the stochastic nonlinear function $g(x_k, \eta_k)$ as follows:

$$\begin{aligned} g(x_k, \eta_k) = \begin{bmatrix} 0.3 \\ 0.2 \\ 0.1 \end{bmatrix} \big(&0.3 sign(x_{1,k}) x_{1,k} \eta_{1,k} \\ &+ 0.2 sign(x_{2,k}) x_{2,k} \eta_{2,k} \\ &+ 0.1 sign(x_{3,k}) x_{3,k} \eta_{3,k}\big) \end{aligned}$$

where $x_{i,k}$ $(i = 1, 2, 3)$ denotes the i-th element of the system state, and $\eta_{i,k}$ are zero mean, uncorrelated Gaussian white noises with unity covariances. It is easy to known that the stochastic nonlinear function satisfies the following equations:

$$\mathbb{E}\{g(x_k, \eta_k)|x_k\} = 0,$$

and

$$\begin{aligned} \Pi_i &= \begin{bmatrix} 0.3 \\ 0.2 \\ 0.1 \end{bmatrix} \begin{bmatrix} 0.3 \\ 0.2 \\ 0.1 \end{bmatrix}^T \\ &= \begin{bmatrix} 0.09 & 0.06 & 0.03 \\ 0.06 & 0.04 & 0.02 \\ 0.03 & 0.02 & 0.01 \end{bmatrix}, \\ \Gamma_i &= \begin{bmatrix} 0.09 & 0 & 0 \\ 0 & 0.04 & 0 \\ 0 & 0 & 0.01 \end{bmatrix}. \end{aligned}$$

Let

$$\begin{aligned} C_k &= \begin{bmatrix} 1.5 & -1 & 0.5 \end{bmatrix}, \\ D_k &= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \\ x_0 &= \begin{bmatrix} -0.5 & 1 & 1 \end{bmatrix}^T, \\ \hat{x}_{0|0} &= \begin{bmatrix} 0.5 & 2 & 4 \end{bmatrix}^T, \\ P_{0|0} &= I_3, \\ \Xi &= \alpha I_3 = 0.9 I_3, \end{aligned}$$

and $e_{i,k}$ denote the error for the estimation of $x_{i,k}$, i.e., $e_{i,k} = x_{i,k} - \hat{x}_{i,k|k}$, where $i = 1, 2, 3$.

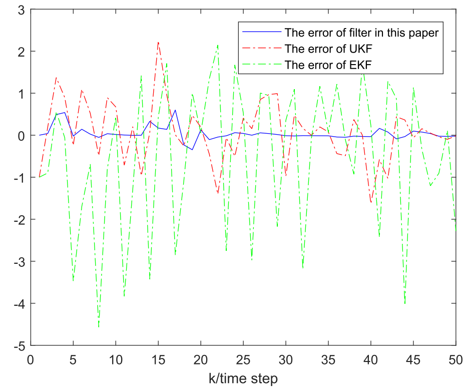

FIGURE 1. The error $e_{1,k}$.

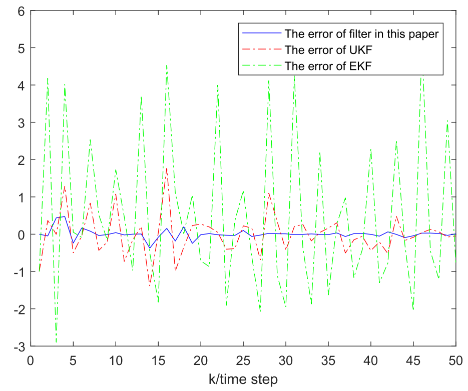

FIGURE 2. The error $e_{2,k}$.

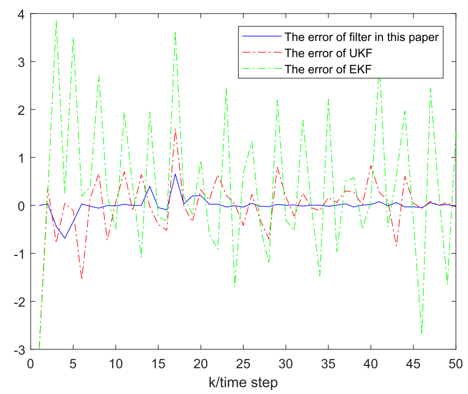

FIGURE 3. The error $e_{3,k}$.

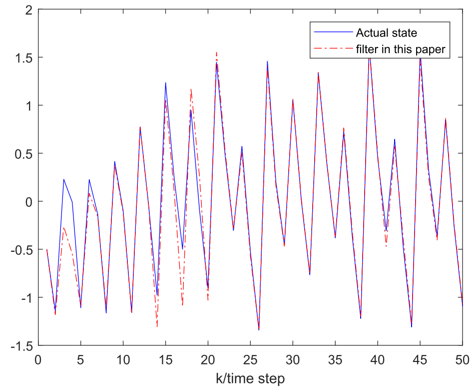

FIGURE 4. The trajectories of $x_{1,k}$ and $\hat{x}_{1,k|k}$.

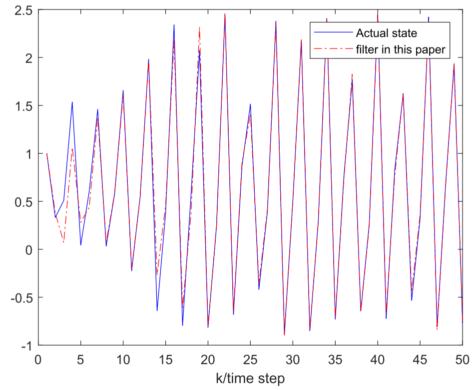

FIGURE 5. The trajectories of $x_{2,k}$ and $\hat{x}_{2,k|k}$.

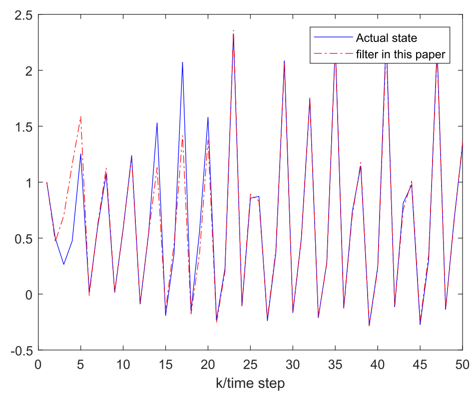

FIGURE 6. The trajectories of $x_{3,k}$ and $\hat{x}_{3,k|k}$.

According to Theorem 1, the optimal hybrid Kalman filter can be constructed for a class of hybrid systems with stochastic nonlinearities and multiple missing measurements by applying the recursive projection formula and the unscented transformation approach. Based on the given hybrid Kalman filtering algorithm and Matlab software, the filter gain matrices K_{k+1} and the error covariance matrices $P_{k+1|k+1}$ at every time step can be recursively computed. The results are shown in Figs. 1-6. Fig. 1 and Fig. 3 plot the filtering errors $e_{i,k}$ $(i = 1, 2, 3)$. From the simulations, we can see that the range of error fluctuation in our paper is relatively small compared with the error of UKF and EKF. The actual system states $x_{i,k}$ and their estimates $\hat{x}_{i,k|k}$ $(i = 1, 2, 3)$ are plotted in Fig. 4 and Fig. 6. It is easily seen that, due to making a lot of efforts to reduce the effects from stochastic nonlinearities and multiple missing measurements, the proposed filter can estimate the system state effectively and the recursive algorithm is feasible.

In our paper, we compare the filter estimation performance with the different rates of missing measurements (i.e., $\alpha = 0.9, 0.6, 0.3$). The corresponding simulation results are given in Figs. 7-9. Based on the simulation results, we can conclude that the estimation performance of the filter becomes worse along with the missing rate decreases. Hence, the

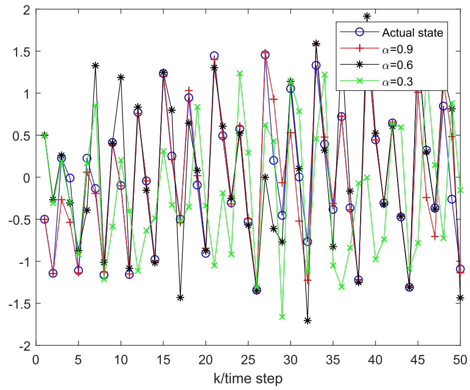

FIGURE 7. The trajectories of $x_{1,k}$ and $\hat{x}_{1,k|k}$.

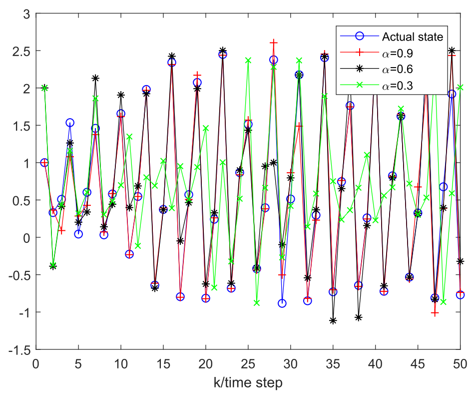

FIGURE 8. The trajectories of $x_{2,k}$ and $\hat{x}_{2,k|k}$.

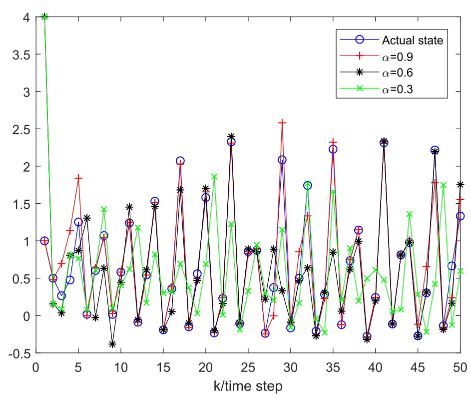

FIGURE 9. The trajectories of $x_{3,k}$ and $\hat{x}_{3,k|k}$.

different rates of missing measurements have a great impact on the accuracy of the filtering algorithm. Compared with unscented Kalman filtering, the filtering algorithm which proposed by us can estimate the system states more accurately and effectively. The reason is that we combine the linear recursive projection formula with the unscented transformation approach to compute the parameters of filter and the variables $\sum_{i=1}^{r} \Pi_i tr\,(X_k \Gamma_i)$ and $\mathscr{H}$ are calculated to reduce the impact of stochastic nonlinearity function and multiple missing measurements on the performance of the filter which can improve the accuracy of the hybrid unscented Kalman filtering algorithm.

V. CONCLUSION

The state estimation problem been investigated for a class of special nonlinear systems where the state equation is nonlinear and the measurement equation is linear. The stochastic nonlinearities are considered in the system model to reflect the multiplicative stochastic disturbances. Due to taking the phenomenon of multiple missing measurements into account, we need to compute parameters to reduce the effects of stochastic nonlinearities and multiple missing measurements. Then, based on the recursive projection formula and the unscented transformation approach, we make first attempt to propose the hybrid Kalman filter for systems subject to stochastic nonlinearities and multiple missing measurements which can estimate the system state effectively. Also, a recursive algorithm has been given to design the hybrid filter and a simulation example has been given to show the feasibility and usefulness of the proposed approach.

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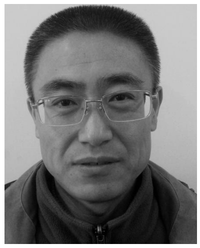

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