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Fractal Generation via CR Iteration Scheme With S-Convexity

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ABSTRACT The visual beauty, self-similarity, and complexity of Mandelbrot sets and Julia sets have made an attractive field of research. One can find many generalizations of these sets in the literature. One such generalization is the use of results from fixed-point theory. The aim of this paper is to provide escape criterion and generate fractals (Julia sets and Mandelbrot sets) via CR iteration scheme with s-convexity. Many graphics of Mandelbrot sets and Julia sets of the proposed three-step iterative process with s-convexity are presented. We think that the results of this paper can inspire those who are interested in generating automatically aesthetic patterns.

INDEX TERMS Iteration schemes, Julia set, Mandelbrot set, escape criterion, s-convexity.

I. INTRODUCTION

Complex graphics of nonlinear dynamical systems is an inspiring field of interest with various applications in sciences, art, textile industries, engineering and many other areas of human activity. In 1918, Julia [1] tried to get the iteration procedure of complex function $f(z) = z^2 + c$ where c be a complex number and obtained a Julia set. Julia sets are astonishing examples of computational research that were far ahead of its time. These mathematical objects were seen when computer graphics became available [2]. On the other hand, Benoit Mandelbrot introduced the Mandelbrot set in 1979 by taking c as a complex parameter in complex quadratic function [3]. Mandelbrot sets and Julia sets are some of the best known illustrations of a highly complicated chaotic systems generated by a very simple mathematical process. Mandelbrot introduced the name "Fractal" for such self similar structures. Fractals are not just complex shapes and pretty pictures generated by computers. Anything that appears random and irregular can be a fractal. Fractals are the exclusive, random patterns left behind by the unpredictable movements of the chaotic world at work. The most important use of fractals in computer science is the fractal image compression. This type of compression uses the fact that the real world is well described by fractal geometry. In this way, images are compressed much more than by usual ways (e.g., JPEG or GIF file formats). Another advantage of fractal compression is that when the image is enlarged, there is no pixelization. The picture seems very often better when its size is increased.

Julia sets and Mandelbrot sets have been generalized in several different manners. One of these generalizations is the use of various iteration processes from the fixed point theory. In the fixed point theory there exist many approximate methods of finding fixed points of a given mapping that are based on the use of different feedback iteration processes. These methods can be used in the generalization of Julia sets and Mandelbrot sets. In 2004, superior Julia sets and Mandelbrot sets introduced by Rani and Kumar [4], [5] by using Mann iteration scheme which is one-step fixed point iterative process. Rana *et al.* [6] and Chauhan *et al.* [7] presented relative superior Julia sets and Mandelbrot sets via Ishikawa iteration scheme which is two-step fixed point iterative procedure. Kang *et al*. [8], [9] introduced relative superior Mandelbrot sets and tricorn & multicorns via *S*-

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iteration scheme. Also, discussed the method of generating fractal images for S-iteration procedure and proved that S-iteration scheme converges faster than Ishikawa iteration scheme in complex plane. Julia sets and Mandelbrot sets via Noor iteration process, which is a three-step iterative procedure, are presented in [10].

Convexity and its generalization performs an important role in many fields of mathematics, essentially in optimization theory. A common generalization of s-convexity, approximate convexity and consequences of Bernstein and Doetsch [11] are dealt with presented paper. Breckner and Orbán [12] introduced the notion of s-convexity and rational s-convexity. Some new results about Hadamard's inequality for s-convex functions are discussed in [**?**], [13], [14]. Hudzik and Maligranda [15] discussed a few results connecting with s-convex functions in second sense. Takahashi [16] first introduced concept of convex metric space, which is more general space, and each linear normed space is a special example of the space.

The junction of a s-convex combination [17] and various iteration schemes was studied in numerous papers. Mishra *et al*. [18], [19] established fixed point results for relative superior Julia sets and tricorn & multicorns by using Ishikawa iteration with s-convexity. Fixed point results with s-convexity have been studied extensively by Kang *et al*. [20], Nazeer *et al*. [21], Goyal and Prasad [22] and Cho *et al*. [23] for several fixed point iterative schemes. Recently, Kwun *et al*. [24], [25] generated fractals via Jungck-CR and Modified Jungck-S iterations with s-convexity. In these recently published papers the authors used Jungck-type iterative procedures to establish escape criterion and generated few patterns of Julia sets and Mandelbrot sets. Chugh *et al*. [26] introduced and proved that CR iteration converges to a fixed point faster than Mann, Ishikawa, Noor and S-iteration procedures. A pair of maps is used in Jungck-type iterative procedures whereas a single map is used in CR iteration that is main difference between these iterative processes. In this paper we established escape criterion which perform important role to generate Mandelbrot sets and Julia sets in CR orbit with s-convexity and obtained large variety of these sets that are quite different from those existed in the literature.

The paper is organized as follows: In Sec. II we introduce some basic definitions. In Sec. III we established the escape criterion for quadratic, cubic and $(k+1)$th degree polynomials. Moreover, we present generalized Mandelbrot sets and Julia sets in Sec. IV. Finally, in Sec. V we give some concluding remarks.

II. PRELIMINARIES

Definition 1: (see [27], p255,256) Let $f : C \longrightarrow C$ symbolize a polynomial of degree ≥ 2. Let F_f be the set of points in C whose orbits do not converge to the point at infinity. That is, $F_f = \{x \in C : \{|f^n(x)|\,, n\text{varies from}0\text{to}\infty\}\text{is bounded}\}$. F_f is called as filled Julia set of the polynomial f. The boundary points of F_f are called as the points of Julia set of the polynomial f or simply the Julia set.

Definition 2: (see [28], Mandelbrot set) The Mandelbrot set M consists of all parameters c for which the filled Julia set of Q_c is connected, that is

$$M = \{c \in C : \ K(Q_c) \text{ is connected }\}. \tag{1}$$

In fact, M contains an enormous amount of information about the structure of Julia sets. The Mandelbrot set M for the quadratic $Q_c(z) = z^2 + c$ is defined as the collection of all $c \in C$ for which the orbit of the point 0 is bounded, that is

$$M = \{c \in C : \{Q_c^n(0)\}; n = 0, 1, 2, ...\text{is bounded}\}. \tag{2}$$

We choose the initial point 0, as 0 is the only critical point of Q_c.

Definition 3: Let C be a nonempty set and $T : C \to C$ be a mapping. For any point $z_0 \in C$, the Picard's orbit is defined as the set of iterates of a point z_0, that is;

$$O(T, z_0) = \{z_n; z_n = T(z_{n-1}).n = 1, 2, 3, ...\}, \tag{3}$$

where the orbit $O(T, z_0)$ of T at the initial point z_0 is the sequence $\{T^n z_0\}$.

Definition 4: (see [26], CR orbit) Let C be a nonempty set and $T : C \to C$ be a mapping. Consider a sequence $\{z_n\}$ of iterates for initial point $z_0 \in C$ such that

$$\begin{cases} z_{n+1} = (1-\lambda_n^{(1)})t_n + \lambda_n^{(1)} T t_n; \\ t_n = (1-\lambda_n^{(2)}) T t_n + \lambda_n^{(2)} T w_n; \\ w_n = (1-\lambda_n^{(3)}) z_n + \lambda_n^{(3)} T z_n; \quad n \geq 0, \end{cases} \tag{4}$$

where $\lambda_n^{(1)}, \lambda_n^{(2)} \in (0, 1]$, $\lambda_n^{(3)} \in [0, 1]$ and $\{\lambda_n^{(1)}\}, \{\lambda_n^{(2)}\}, \{\lambda_n^{(3)}\}$ are sequences of positive numbers. The above sequence of iterates is called as CR orbit, which is a function of five tuples $(T, z_0, \lambda_n^{(1)}, \lambda_n^{(2)}, \lambda_n^{(3)})$.

Definition 5: (s-convex combination [17]) Let $z_1, z_2, ..., z_n \in C$ and $s \in (0, 1]$. The s-convex combination is defined in the following way:

$$\lambda_1^s z_1 + \lambda_2^s z_2 + ... + \lambda_n^s z_n, \tag{5}$$

where $\lambda_k \geq 0$ for $k \in \{1, 2, ..., n\}$ and $\sum_{k=1}^{n} \lambda_k = 1$.

It is noticed that for $s = 1$ the s-convex combination arrange to the normal convex combination. We shall write the s-convex combination in the CR iteration. We take $z_0 = z \in C$, $\lambda_k^{(1)} = \lambda_1$, $\lambda_k^{(2)} = \lambda_2$ and $\lambda_k^{(3)} = \lambda_3$ then we can write CR iteration scheme with s-convexity in the following way where $Q_c(z_n)$ be a quadratic, cubic or $(k+1)$th degree polynomial,

$$\begin{cases} z_{n+1} = (1-\lambda_1)^s t_n + \lambda_1^s Q_c(t_n) \\ t_n = (1-\lambda_2)^s Q_c(z_n) + \lambda_2^s Q_c(w_n) \\ w_n = (1-\lambda_3)^s z_n + \lambda_3^s Q_c(z_n), \quad n \geq 0, \end{cases} \tag{6}$$

where $\lambda_1, \lambda_2, s \in (0, 1]$ and $\lambda_3 \in [0, 1]$.

III. MAIN RESULT

Escape criterion perform an important role in the analysis and generation of Mandelbrot sets and Julia sets. Now, we define escape criterion for Mandelbrot sets and Julia sets in CR orbit with s-convexity.

A. ESCAPE CRITERION FOR QUADRATIC POLYNOMIAL

Theorem 1: Suppose that $|z| \geq |c| > \frac{2}{s\lambda_1}$, $|z| \geq |c| > \frac{2}{s\lambda_2}$ *and* $|z| \geq |c| > \frac{2}{s\lambda_3}$ *here c be a complex number. Let* $t_\circ = t$, $w_\circ = w$ *and* $z_\circ = z$ *then sequence* $\{z_n\}$ *define as*

$$\begin{cases} z_{n+1} = (1-\lambda_1)^s t_n + \lambda_1^s Q_c(t_n) \\ t_n = (1-\lambda_2)^s Q_c(z_n) + \lambda_2^s Q_c(w_n) \\ w_n = (1-\lambda_3)^s z_n + \lambda_3^s Q_c(z_n), n \geq 0, \end{cases} \tag{7}$$

where $\lambda_1, \lambda_2, s \in (0,1]$, $\lambda_3 \in [0,1]$ *and* $Q_c(z) = z^2 + c$. *Then* $|z_n| \to \infty$ *as* $n \to \infty$.

Proof: Consider

$$|w| = \left|(1-\lambda_3)^s z + \lambda_3^s Q_c(z)\right|.$$

For $Q_c(z) = z^2 + c$,

$$\begin{aligned} |w| &= \left|(1-\lambda_3)^s z + \lambda_3^s (z^2 + c)\right| \\ &= \left|(1-\lambda_3)^s z + (1-(1-\lambda_3))^s (z^2+c)\right|. \end{aligned}$$

As $1 - s + s\lambda_3 \geq s\lambda_3$ and $|z| \geq |c|$, so by binomial expansion upto linear terms of λ_3 and $(1-\lambda_3)$, we obtain

$$\begin{aligned} |w| &\geq \left|(1-s\lambda_3)z + (1-s(1-\lambda_3))(z^2+c)\right|, \\ &\geq \left|(1-s\lambda_3)z + (1-s+s\lambda_3)(z^2+c)\right|, \\ &\geq \left|(1-s\lambda_3)z + s\lambda_3(z^2+c)\right|, \\ &\geq \left|s\lambda_3 z^2 + (1-s\lambda_3)z\right| - |s\lambda_3 c|, \\ &\geq \left|s\lambda_3 z^2 + (1-s\lambda_3)z\right| - |s\lambda_3 z|, \\ &\geq \left|s\lambda_3 z^2\right| - |(1-s\lambda_3)z| - |s\lambda_3 z|, \\ &\geq \left|s\lambda_3 z^2\right| - |z| + |s\lambda_3 z| - |s\lambda_3 z|, \\ &\geq |z|\,(s\lambda_3 |z| - 1). \end{aligned} \tag{8}$$

And

$$\begin{aligned} |t| &= \left|(1-\lambda_2)^s Q_c(z) + \lambda_2^s Q_c(w)\right| \\ &= \left|(1-\lambda_2)^s (z^2+c) + (1-(1-\lambda_2))^s (w^2+c)\right|. \end{aligned} \tag{9}$$

As $1 - s + s\lambda_2 \geq s\lambda_2$, so by binomial expansion upto linear terms of λ_2 and $(1-\lambda_2)$, we obtain

$$\begin{aligned} |t| &\geq \left|(1-s\lambda_2)(z^2+c) + (1-s(1-\lambda_2))(w^2+c)\right|, \\ &\geq \left|(1-s\lambda_2)(z^2+c) + (1-s+s\lambda_2)(w^2+c)\right|, \\ &\geq \left|(1-s\lambda_2)(z^2+c) + s\lambda_2((|z|\,(s\lambda_3|z|-1))^2+c)\right|. \end{aligned} \tag{10}$$

Since $|z| > \frac{2}{s\lambda_3}$ implies $s\lambda_3 |z| - 1 > 1$ and $|z|^2 (s\lambda_3 |z| - 1)^2 > |z|^2$ using this in (10) and $|z| \geq |c|$ we have

$$\begin{aligned} |t| &\geq \left|(1-s\lambda_2)(z^2+c) + s\lambda_2(|z|^2+c)\right|, \\ &\geq \left|(1-s\lambda_2)z^2 + (1-s\lambda_2)c + s\lambda_2 |z|^2 + s\lambda_2 c\right|, \\ &\geq \left|z^2 + c\right|, \\ &\geq \left|z^2\right| - |c|, \\ &\geq \left|z^2\right| - |z|, \\ &\geq |z|\,(|z| - 1). \end{aligned} \tag{11}$$

Also for

$$\begin{aligned} z_1 &= (1-\lambda_1)^s t + \lambda_1^s Q_c(t) \\ |z_1| &= \left|(1-\lambda_1)^s t + (1-(1-\lambda_1))^s (t^2+c)\right|. \end{aligned}$$

As $1 - s + s\lambda_1 \geq s\lambda_1$ and $|z| \geq |c|$, so by binomial expansion upto linear terms of λ_1 and $(1-\lambda_1)$, we obtain

$$\begin{aligned} |z_1| &= \left|(1-s\lambda_1)t + (1-s(1-\lambda_1))(t^2+c)\right| \\ &\geq \Big|(1-s\lambda_1)\,|z|\,(|z|-1) \\ &\quad + (1-s+s\lambda_1)((|z|\,(|z|-1))^2+c)\Big| \\ &\geq \left|(1-s\lambda_1)\,|z| + s\lambda_1(|z|^2+c)\right|, \\ &\geq \left|(1-s\lambda_1)\,|z| + s\lambda_1\,|z|^2\right| - |s\lambda_1 c| \\ &\geq \left|s\lambda_1\,|z|^2 + (1-s\lambda_1)\,|z|\right| - |s\lambda_1 z|, \\ &\geq \left|s\lambda_1\,|z|^2\right| - |(1-s\lambda_1)\,|z|| - |s\lambda_1 z| \\ &\geq s\lambda_1 \left|z^2\right| - |z| + s\lambda_1 |z| - s\lambda_1 |z| \\ &\geq |z|\,(s\lambda_1 |z| - 1). \end{aligned}$$

Since $|z| > \frac{2}{s\lambda_1}$ implies $s\lambda_1 |z| - 1 > 1$, there exist a number $\delta > 0$, such that $s\lambda_1 |z| - 1 > 1 + \delta > 1$. Therefore

$$\begin{aligned} |z_1| &> (1+\delta)\,|z|, \\ |z_2| &> (1+\delta)^2\,|z|, \\ &\vdots \\ |z_n| &> (1+\delta)^n\,|z|. \end{aligned}$$

Hence $|z_n| \longrightarrow \infty$ as $n \to \infty$ and proved. □

Corollary 1: Suppose that $|c| > \frac{2}{s\lambda_1}$, $|c| > \frac{2}{s\lambda_2}$ *and* $|c| > \frac{2}{s\lambda_3}$, *then, the orbit CRSO*$(Q_c, 0, s\lambda_1, s\lambda_2, s\lambda_3)$ *escapes to infinity.*

Corollary 2: (Escape Criterion): Suppose that $|z| > \max\{|c|, \frac{2}{s\lambda_1}, \frac{2}{s\lambda_2}, \frac{2}{s\lambda_3}\}$, *then* $|z_n| > (1+\lambda)^n |z|$ *and* $|z_n| \longrightarrow \infty$ *as* $n \to \infty$.

B. ESCAPE CRITERION FOR CUBIC POLYNOMIALS

We shall prove the following result for the cubic polynomial $Q_c(z) = z^3 + c$, where c be a complex number, in CR orbit with s-convexity.

Theorem 2: Suppose that $|z| \geq |c| > (\frac{2}{s\lambda_1})^{\frac{1}{2}}$, $|z| \geq |c| > (\frac{2}{s\lambda_2})^{\frac{1}{2}}$ and $|z| \geq |c| > (\frac{2}{s\lambda_3})^{\frac{1}{2}}$ here c be a complex number. Let $t_\circ = t$, $w_\circ = w$ and $z_\circ = z$ then sequence $\{z_n\}$ define as

$$\begin{cases} z_{n+1} = (1-\lambda_1)^s t_n + \lambda_1^s Q_c(t_n) \\ t_n = (1-\lambda_2)^s Q_c(z_n) + \lambda_2^s Q_c(w_n) \\ w_n = (1-\lambda_3)^s z_n + \lambda_3^s Q_c(z_n), \quad n \geq 0, \end{cases} \tag{12}$$

where $\lambda_1, \lambda_2, s \in (0,1]$, $\lambda_3 \in [0,1]$ and $Q_c(z) = z^3 + c$. Then $|z_n| \to \infty$ as $n \to \infty$.

Proof: Consider

$$|w| = \left|(1-\lambda_3)^s z + \lambda_3^s Q_c(z)\right|.$$

For $Q_c(z) = z^3 + c$,

$$\begin{aligned} |w| &= \left|(1-\lambda_3)^s z + \lambda_3^s (z^3 + c)\right| \\ &= \left|(1-\lambda_3)^s z + (1-(1-\lambda_3))^s (z^3 + c)\right|. \end{aligned}$$

As $1 - s + s\lambda_3 \geq s\lambda_3$ and $|z| \geq |c|$, so by binomial expansion upto linear terms of λ_3 and $(1-\lambda_3)$, we obtain

$$\begin{aligned} |w| &\geq \left|(1-s\lambda_3)z + (1-s(1-\lambda_3))(z^3+c)\right| \\ &\geq \left|(1-s\lambda_3)z + (1-s+s\lambda_3)(z^3+c)\right| \\ &\geq \left|(1-s\lambda_3)z + s\lambda_3(z^3+c)\right|, \\ &\geq \left|s\lambda_3 z^3 + (1-s\lambda_3)z\right| - |s\lambda_3 c| \\ &\geq \left|s\lambda_3 z^3 + (1-s\lambda_3)z\right| - |s\lambda_3 z|, \\ &\geq \left|s\lambda_3 z^3\right| - |(1-s\lambda_3)z| - |s\lambda_3 z| \\ &\geq \left|s\lambda_3 z^3\right| - |z| + |s\lambda_3 z| - |s\lambda_3 z| \\ &\geq |z|\,(s\lambda_3\,|z|^2 - 1). \end{aligned} \tag{13}$$

And

$$\begin{aligned} |t| &= \left|(1-\lambda_2)^s Q_c(z) + \lambda_2^s Q_c(w)\right| \\ &= \left|(1-\lambda_2)^s (z^3+c) + (1-(1-\lambda_2))^s (w^3+c)\right|. \end{aligned} \tag{14}$$

As $1 - s + s\lambda_2 \geq s\lambda_2$, so by binomial expansion upto linear terms of λ_2 and $(1-\lambda_2)$, we obtain

$$\begin{aligned} |t| &\geq \left|(1-s\lambda_2)(z^3+c) + (1-s(1-\lambda_2))(w^3+c)\right| \\ &\geq \left|(1-s\lambda_2)(z^3+c) + (1-s+s\lambda_2)(w^3+c)\right| \\ &\geq \left|(1-s\lambda_2)(z^3+c) + s\lambda_2((|z|\,(s\lambda_3\,|z|^2-1))^3+c)\right|. \end{aligned} \tag{15}$$

Since $|z| > (\frac{2}{s\lambda_3})^{\frac{1}{2}}$ implies $s\lambda_3\,|z|^2 - 1 > 1$ and $|z|^3\,(s\lambda_3\,|z|^2 - 1)^3 > |z|^3$ using this in (15) and $|z| \geq |c|$ we have

$$\begin{aligned} |t| &\geq \left|(1-s\lambda_2)(z^3+c) + s\lambda_2(|z|^3+c)\right| \\ &\geq \left|(1-s\lambda_2)z^3 + (1-s\lambda_2)c + s\lambda_2\,|z|^3 + s\lambda_2 c\right| \\ &\geq \left|z^3 + c\right| \\ &\geq \left|z^3\right| - |c| \\ &\geq \left|z^3\right| - |z|, \\ &\geq |z|\,(|z|^2 - 1). \end{aligned} \tag{16}$$

Also for

$$\begin{aligned} z_1 &= (1-\lambda_1)^s t + \lambda_1^s Q_c(t) \\ |z_1| &= \left|(1-\lambda_1)^s t + (1-(1-\lambda_1))^s (t^3+c)\right|. \end{aligned}$$

Since $1 - s + s\lambda \geq s\lambda$ and $|z| \geq |c|$, so by binomial expansion upto linear terms of λ_1 and $(1-\lambda_1)$, we obtain

$$\begin{aligned} |z_1| &= \left|(1-s\lambda_1)t + (1-s(1-\lambda_1))(t^3+c)\right| \\ &\geq \left|(1-s\lambda_1)\,|z|\,(|z|-1) + (1-s+s\lambda_1)((|z|\,(|z|-1))^3+c)\right| \\ &\geq \left|(1-s\lambda_1)\,|z| + s\lambda_1(|z|^3+c)\right|, \\ &\geq \left|(1-s\lambda_1)\,|z| + s\lambda_1\,|z|^3\right| - |s\lambda_1 c| \\ &\geq \left|s\lambda_1\,|z|^3 + (1-s\lambda_1)\,|z|\right| - |s\lambda_1 z|, \\ &\geq \left|s\lambda_1\,|z|^3\right| - |(1-s\lambda_1)\,|z|| - |s\lambda_1 z| \\ &\geq s\lambda_1 \left|z^3\right| - |z| + s\lambda_1\,|z| - s\lambda_1\,|z| \\ &\geq |z|\,(s\lambda_1\,|z|^2 - 1). \end{aligned}$$

Since $|z| > (\frac{2}{s\lambda_1})^{\frac{1}{2}}$ implies $s\lambda_1\,|z|^2 - 1 > 1$, there exist a number $\delta > 0$, such that $s\lambda_1|z|^2 - 1 > 1 + \delta > 1$. Therefore

$$\begin{aligned} |z_1| &> (1+\delta)\,|z|, \\ |z_2| &> (1+\delta)^2\,|z|, \\ &\vdots \\ |z_n| &> (1+\delta)^n\,|z|. \end{aligned}$$

Hence $|z_n| \longrightarrow \infty$ as $n \to \infty$ and proved. □

Corollary 3: (Escape criterion): Let $Q_c(z) = z^3 + c$, where c be a complex number. Suppose $|z| > \max\{|c|, (\frac{2}{s\lambda_1})^{\frac{1}{2}}, (\frac{2}{s\lambda_2})^{\frac{1}{2}}, (\frac{2}{s\lambda_3})^{\frac{1}{2}}\}$ then $|z_n| \to \infty$ as $n \to \infty$.

C. ESCAPE CRITERION FOR HIGHER DEGREE POLYNOMIALS

We shall obtain escape criterion for higher degree polynomials of the form $Q_c(z) = z^{k+1} + c$, $k = 1, 2, 3, \ldots$ in CR orbit with s-convexity.

Theorem 3: Suppose that $|z| \geq |c| > (\frac{2}{s\lambda_1})^{\frac{1}{k}}$, $|z| \geq |c| > (\frac{2}{s\lambda_2})^{\frac{1}{k}}$ and $|z| \geq |c| > (\frac{2}{s\lambda_3})^{\frac{1}{k}}$ here c be a complex number. Let $t_\circ = t$, $w_\circ = w$ and $z_\circ = z$ then sequence $\{z_n\}$ define as

$$\begin{cases} z_{n+1} = (1-\lambda_1)^s t_n + \lambda_1^s Q_c(t_n) \\ t_n = (1-\lambda_2)^s Q_c(z_n) + \lambda_2^s Q_c(w_n) \\ w_n = (1-\lambda_3)^s z_n + \lambda_3^s Q_c(z_n), \ n \geq 0, \end{cases} \tag{17}$$

where $\lambda_1, \lambda_2, s \in (0,1]$, $\lambda_3 \in [0,1]$ and $Q_c(z) = z^{k+1} + c$, $k = 1, 2, 3, \ldots$. Then $|z_n| \to \infty$ as $n \to \infty$.

Proof: Consider

$$|w| = \left|(1-\lambda_3)^s z + \lambda_3^s Q_c(z)\right|.$$

For $Q_c(z) = z^{k+1} + c, k = 1, 2, 3, \ldots$

$$\begin{aligned}|w| &= \left|(1-\lambda_3)^s z + \lambda_3^s(z^{k+1}+c)\right| \\ &= \left|(1-\lambda_3)^s z + (1-(1-\lambda_3))^s(z^{k+1}+c)\right|.\end{aligned}$$

Since $1 - s + s\lambda_3 \geq s\lambda_3$ and $|z| \geq |c|$, so by binomial expansion upto linear terms of λ_3 and $(1-\lambda_3)$, we obtain

$$\begin{aligned}|w| &\geq \left|(1-s\lambda_3)z + (1-s(1-\lambda_3))(z^{k+1}+c)\right| \\ &\geq \left|(1-s\lambda_3)z + (1-s+s\lambda_3)(z^{k+1}+c)\right| \\ &\geq \left|(1-s\lambda_3)z + s\lambda_3(z^{k+1}+c)\right|, \\ &\geq \left|s\lambda_3 z^{k+1} + (1-s\lambda_3)z\right| - |s\lambda_3 c| \\ &\geq \left|s\lambda_3 z^{k+1} + (1-s\lambda_3)z\right| - |s\lambda_3 z|, \\ &\geq \left|s\lambda_3 z^{k+1}\right| - |(1-s\lambda_3)z| - |s\lambda_3 z| \\ &\geq \left|s\lambda_3 z^{k+1}\right| - |z| + |s\lambda_3 z| - |s\lambda_3 z| \\ &\geq |z|\,(s\lambda_3\,|z|^k - 1). \end{aligned} \tag{18}$$

And

$$\begin{aligned}|t| &= \left|(1-\lambda_2)^s Q_c(z) + \lambda_2^s Q_c(w)\right| \\ &= \left|(1-\lambda_2)^s(z^{k+1}+c) + (1-(1-\lambda_2))^s(w^{k+1}+c)\right|. \end{aligned} \tag{19}$$

Since $1-s+s\lambda_2 \geq s\lambda_2$ so, by binomial expansion upto linear terms of λ_2 and $(1-\lambda_2)$, we obtain

$$\begin{aligned}|t| &\geq \left|(1-s\lambda_2)(z^{k+1}+c) + (1-s(1-\lambda_2))(w^{k+1}+c)\right| \\ &\geq \left|(1-s\lambda_2)(z^{k+1}+c) + (1-s+s\lambda_2)(w^{k+1}+c)\right| \\ &\geq \left|(1-s\lambda_2)(z^{k+1}+c) + s\lambda_2((|z|\,(s\lambda_3\,|z|-1))^{k+1}+c)\right|. \end{aligned} \tag{20}$$

Since $|z| > (\frac{2}{s\lambda_3})^{\frac{1}{k}}$ implies $s\lambda_3\,|z|^k - 1 > 1$ and $|z|^{k+1}\,(s\lambda_3\,|z|-1)^{k+1} > |z|^{k+1}$ using this in (20) and $|z| \geq |c|$, we have

$$\begin{aligned}|t| &\geq \left|(1-s\lambda_2)(z^{k+1}+c) + s\lambda_2(|z|^{k+1}+c)\right| \\ &\geq \left|(1-s\lambda_2)z^{k+1} + (1-s\lambda_2)c + s\lambda_2\,|z|^{k+1} + s\lambda_2 c\right| \\ &\geq \left|z^{k+1}+c\right| \\ &\geq \left|z^{k+1}\right| - |c| \\ &\geq \left|z^{k+1}\right| - |z|, \\ &\geq |z|\,(|z|^k - 1). \end{aligned} \tag{21}$$

Also for

$$\begin{aligned}z_1 &= (1-\lambda_1)^s t + \lambda_1^s Q_c(t) \\ |z_1| &= \left|(1-\lambda_1)^s t + (1-(1-\lambda_1))^s(t^{k+1}+c)\right|.\end{aligned}$$

Since $1 - s + s\lambda_1 \geq s\lambda_1$ and $|z| \geq |c|$, so by binomial expansion upto linear terms of λ_1 and $(1-\lambda_1)$, we obtain

$$\begin{aligned}|z_1| &= \left|(1-s\lambda_1)t + (1-s(1-\lambda_1))(t^{k+1}+c)\right| \\ &\geq \Big|(1-s\lambda_1)\,|z|\,(|z|-1) \\ &\quad + (1-s+s\lambda_1)((|z|\,(|z|-1))^{k+1}+c)\Big| \\ &\geq \left|(1-s\lambda_1)\,|z| + s\lambda_1(|z|^{k+1}+c)\right|, \\ &\geq \left|(1-s\lambda_1)\,|z| + s\lambda_1\,|z|^{k+1}\right| - |s\lambda_1 c| \\ &\geq \left|s\lambda_1\,|z|^{k+1} + (1-s\lambda_1)\,|z|\right| - |s\lambda_1 z|, \\ &\geq \left|s\lambda_1\,|z|^{k+1}\right| - |(1-s\lambda_1)\,|z|| - |s\lambda_1 z| \\ &\geq s\lambda_1\,|z|^{k+1} - |z| + s\lambda_1\,|z| - s\lambda_1\,|z| \\ &\geq |z|\,(s\lambda_1\,|z|^k - 1).\end{aligned}$$

Since $|z| > (\frac{2}{s\lambda_1})^{\frac{1}{k}}$ implies $s\lambda_1\,|z|^k - 1 > 1$, there exist a number $\delta > 0$, such that $s\lambda_1|z|^k - 1 > 1 + \delta > 1$. Therefore

$$\begin{aligned}|z_1| &> (1+\delta)\,|z|, \\ |z_2| &> (1+\delta)^2\,|z|, \\ &\vdots \\ |z_n| &> (1+\delta)^n\,|z|.\end{aligned}$$

Hence $|z_n| \longrightarrow \infty$ as $n \to \infty$ and proved. □

Corollary 4: Suppose that $|c| > (\frac{2}{s\lambda_1})^{\frac{1}{k}}, |c| > (\frac{2}{s\lambda_2})^{\frac{1}{k}}$ *and* $|c| > (\frac{2}{s\lambda_3})^{\frac{1}{k}}$ *exists, then the orbit* $CRSO(Q_c, 0, s\lambda_1, s\lambda_2, s\lambda_3)$ *escape to infinity.*

This corollary gives an algorithm for computing the Julia sets and Mandelbrot sets for the functions of the form $G_c(z) = z^{k+1} + c, k = 1, 2, 3, \ldots$

Corollary 5: (Escape criterion): Let $Q_c(z) = z^{k+1} + c, k = 1, 2, 3, \ldots,$ *where c be a complex number. Suppose* $|z| > \max\{|c|, (\frac{2}{s\lambda_1})^{\frac{1}{k}}, (\frac{2}{s\lambda_2})^{\frac{1}{k}}, (\frac{2}{s\lambda_3})^{\frac{1}{k}}\}$ *then* $|z_n| \to \infty$ *as* $n \to \infty$.

IV. GENERATION OF FRACTALS

In this section Mandelbrot sets are presented for quadratic, cubic and bi-quadratic functions. Also, Julia sets are presented for quadratic and cubic functions. Pseudo code of the Mandelbrot set generation algorithm is presented in Algorithm 1, whereas Algorithm 2 presents the pseudo code for the Julia set generation algorithm. To generate the images we used the escape time algorithm with the escape criterion attained in above section and implemented in Mathematica 9.

A. MANDELBROT SETS FOR THE QUADRATIC FUNCTION $Q_C(z) = z^2 + c$

In Figs. 1–6, quadratic Mandelbrot sets are presented in CR orbit with s-convexity by using maximum number of iterations 50, $s = 0.5$ and varying parameters are following:

- Fig. 1: $\lambda_3 = \lambda_2 = \lambda_1 = 0.3$ and $A = [-3.1, 0.9] \times [-2.4, 2.4]$,
- Fig. 2: $\lambda_3 = 0.6, \lambda_2 = 0.5, \lambda_1 = 0.4$ and $A = [-2.8, 0.6] \times [-1.8, 1.8]$,
- Fig. 3: $\lambda_3 = 0.3, \lambda_2 = 0.2, \lambda_1 = 0.6$ and $A = [-2.8, 0.6] \times [-1.8, 1.8]$,
- Fig. 4: $\lambda_3 = \lambda_2 = \lambda_1 = 0.6$ and $A = [-3.1, 0.9] \times [-2.3, 2.3]$,

Algorithm 1 Mandelbrot Set Generation

Input: $Q_c(z) = z^{k+1} + c$, where $c \in \mathbb{C}$ and $k = 1, 2, \ldots$, $A \subset \mathbb{C}$ – area, K – the maximum number of iterations, $\lambda_1, \lambda_2, s \in (0, 1]$, $\lambda_3 \in [0, 1]$ – parameters for the CR iteration with s-convexity, $colourmap[0..C-1]$ – with C colours.
Output: Mandelbrot set for the area A.

for $c \in A$ **do**
  $R = \max\{|c|, (2/(s\lambda_1))^{\frac{1}{k}}, (2/(s\lambda_2))^{\frac{1}{k}}, (2/(s\lambda_3))^{\frac{1}{k}}\}$
  $n = 0$
  $z_0 = 0$
  while $n \leq K$ **do**
    $w_n = (1-\lambda_3)^s z_n + \lambda_3^s Q_c(z_n)$
    $t_n = (1-\lambda_2)^s Q_c(z_n) + \lambda_2^s Q_c(w_n)$
    $z_{n+1} = (1-\lambda_1)^s t_n + \lambda_1^s Q_c(t_n)$
    if $|z_{n+1}| > R$ **then**
      break
    $n = n + 1$
  $i = \lfloor (C-1)\frac{n}{K} \rfloor$
  color c with $colourmap[i]$

Algorithm 2 Julia Set Generation

Input: $Q_c(z) = z^{k+1} + c$, where $k = 1, 2, \ldots$, $c \in \mathbb{C}$ – parameter, $A \subset \mathbb{C}$ – area, K – the maximum number of iterations, $\lambda_1, \lambda_2, s \in (0, 1]$, $\lambda_3 \in [0, 1]$ – parameters for the CR iteration with s-convexity, $colourmap[0..C-1]$ – with C colours.
Output: Julia set for the area A.

$R = \max\{|c|, (2/(s\lambda_1))^{\frac{1}{k}}, (2/(s\lambda_2))^{\frac{1}{k}}, (2/(s\lambda_3))^{\frac{1}{k}}\}$
for $z_0 \in A$ **do**
  $n = 0$
  while $n \leq K$ **do**
    $w_n = (1-\lambda_3)^s z_n + \lambda_3^s Q_c(z_n)$
    $t_n = (1-\lambda_2)^s Q_c(z_n) + \lambda_2^s Q_c(w_n)$
    $z_{n+1} = (1-\lambda_1)^s t_n + \lambda_1^s Q_c(t_n)$
    if $|z_{n+1}| > R$ **then**
      break
    $n = n + 1$
  $i = \lfloor (C-1)\frac{n}{K} \rfloor$
  color z_0 with $colourmap[i]$

- Fig. 5: $\lambda_3 = \lambda_2 = \lambda_1 = 0.8$ and $A = [-2.5, 0.6] \times [-1.8, 1.8]$,
- Fig. 6: $\lambda_3 = \lambda_2 = \lambda_2 = 0.9$ and $A = [-2.5, 0.6] \times [-1.8, 1.8]$.

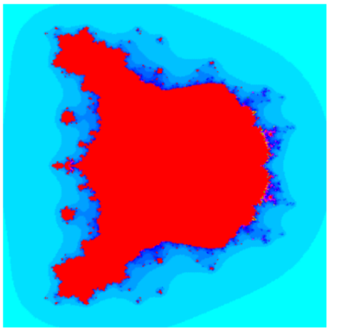

FIGURE 1. Quadratic Mandelbrot set generated via CR iteration with s-convexity.

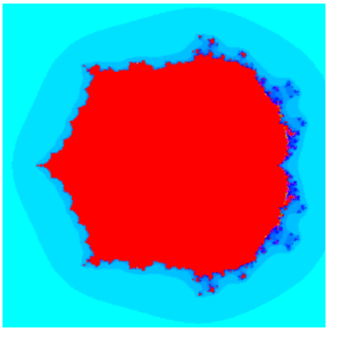

FIGURE 2. Quadratic Mandelbrot set generated via CR iteration with s-convexity.

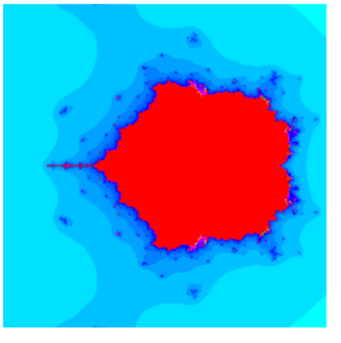

FIGURE 3. Quadratic Mandelbrot set generated via CR iteration with s-convexity.

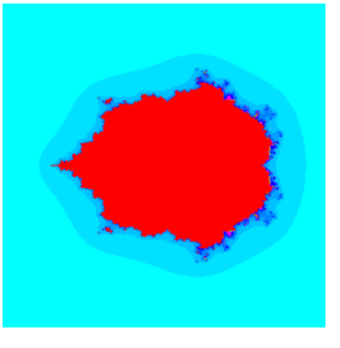

FIGURE 4. Quadratic Mandelbrot set generated via CR iteration with s-convexity.

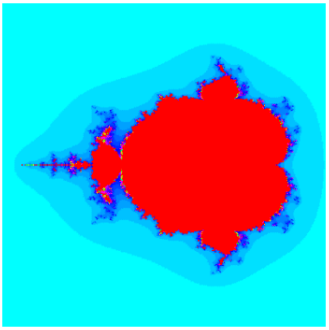

FIGURE 5. Quadratic Mandelbrot set generated via CR iteration with s-convexity.

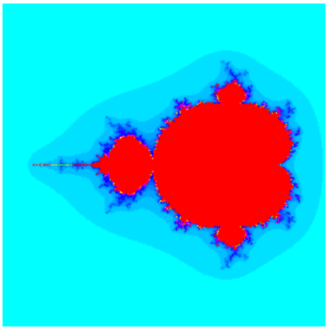

FIGURE 6. Quadratic Mandelbrot set generated via CR iteration with s-convexity.

B. MANDELBROT SETS FOR THE CUBIC FUNCTION $Q_C(z) = z^3 + c$

In Figs. 7–12, cubic Mandelbrot sets are presented in CR orbit with s-convexity by using maximum number of iterations 50, $s = 0.7$ and varying parameters are following:

- Fig. 7: $\lambda_3 = \lambda_2 = \lambda_1 = 0.2$ and $A = [-1.4, 1.4] \times [-2, 2]$,
- Fig. 8: $\lambda_3 = \lambda_2 = \lambda_1 = 0.4$ and $A = [-1.4, 1.4] \times [-2.3, 2.3]$,
- Fig. 9: $\lambda_3 = \lambda_2 = \lambda_1 = 0.5$ and $A = [-1.4, 1.4] \times [-2.3, 2.3]$,

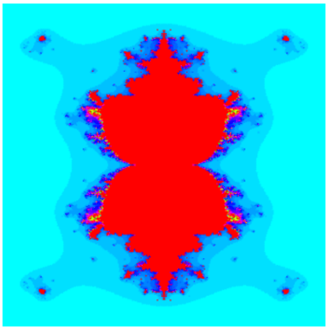

FIGURE 7. Cubic Mandelbrot set generated via CR iteration with s-convexity.

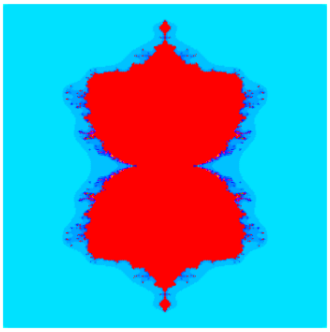

FIGURE 8. Cubic Mandelbrot set generated via CR iteration with s-convexity.

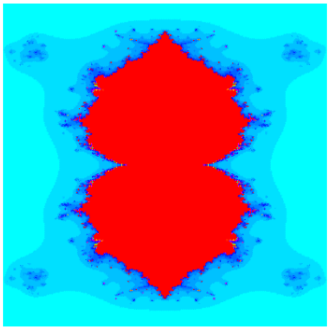

FIGURE 9. Cubic Mandelbrot set generated via CR iteration with s-convexity.

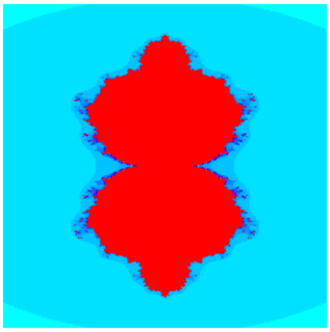

FIGURE 10. Cubic Mandelbrot set generated via CR iteration with s-convexity.

- Fig. 10: $\lambda_3 = 0.3, \lambda_2 = 0.2, \lambda_1 = 0.6$ and $A = [-1.1, 1.1] \times [-1.8, 1.8]$,
- Fig. 11: $\lambda_3 = \lambda_2 = \lambda_1 = 0.7$ and $A = [-1.2, 1.2] \times [-2, 2]$,
- Fig. 12: $\lambda_3 = \lambda_2 = \lambda_1 = 0.8$ and $A = [-1.2, 1.2] \times [-2, 2]$.

C. MANDELBROT SETS FOR THE FUNCTION $Q_C(z) = z^{k+1} + c$ WHERE $k = 3$

In Figs. 13–18, Mandelbrot sets for the function $Q_c(z) = z^{k+1} + c$ where $k = 3$ are presented in CR orbit with

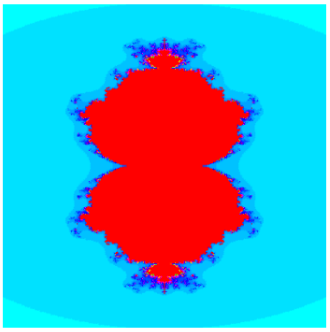

FIGURE 11. Cubic Mandelbrot set generated via CR iteration with s-convexity.

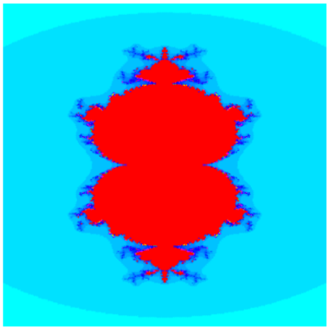

FIGURE 12. Cubic Mandelbrot set generated via CR iteration with s-convexity.

s-convexity by using maximum number of iterations 50, $s = 0.4$ and varying parameters are following:

- Fig. 13: $\lambda_1 = 0.2, \lambda_2 = 0.3, \lambda_3 = 0.2$ and $A = [-1.6, 1] \times [-1.5, 1.5]$
- Fig. 14: $\lambda_3 = 0.5, \lambda_2 = 0.3, \lambda_1 = 0.4$ and $A = [-1.6, 1] \times [-1.5, 1.5]$
- Fig. 15: $\lambda_3 = 0.4, \lambda_2 = 0.2, \lambda_1 = 0.5$ and $A = [-1.5, 1] \times [-1.5, 1.5]$
- Fig. 16: $\lambda_3 = 0.4, \lambda_2 = 0.8, \lambda_1 = 0.6$ and $A = [-1.5, 1] \times [-1.4, 1.4]$
- Fig. 17: $\lambda_3 = 0.2, \lambda_2 = 0.3, \lambda_1 = 0.7$ and $A = [-1.5, 1] \times [-1.5, 1.5]$

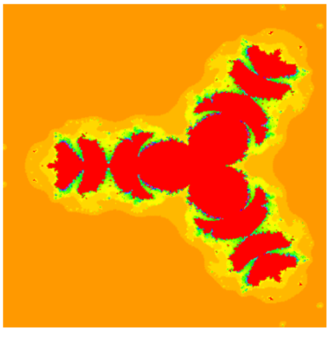

FIGURE 13. Biquadratic Mandelbrot set generated via CR iteration with s-convexity.

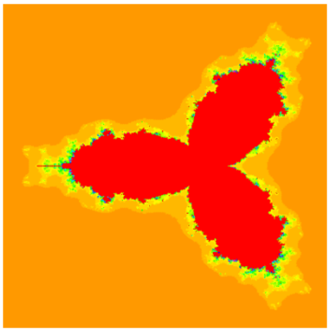

FIGURE 14. Biquadratic Mandelbrot set generated via CR iteration with s-convexity.

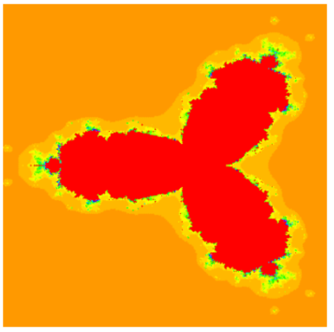

FIGURE 15. Biquadratic Mandelbrot set generated via CR iteration with s-convexity.

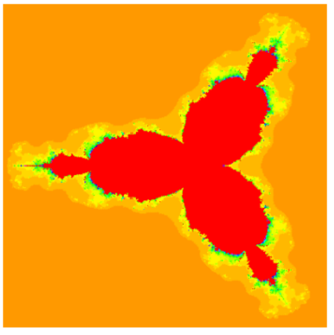

FIGURE 16. Biquadratic Mandelbrot set generated via CR iteration with s-convexity.

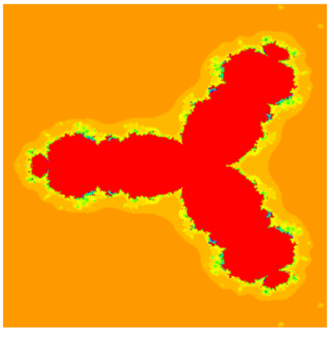

FIGURE 17. Biquadratic Mandelbrot set generated via CR iteration with s-convexity.

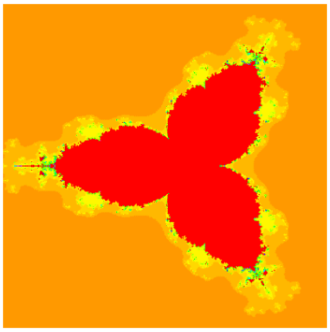

FIGURE 18. Biquadratic Mandelbrot set generated via CR iteration with s-convexity.

- Fig. 18: $\lambda_3 = \lambda_2 = \lambda_1 = 0.8$ and $A = [-1.3, 1] \times [-1.3, 1.3]$.

D. JULIA SETS FOR THE QUADRATIC FUNCTION $Q_C(z) = z^2 + c$

Julia sets for the function $Q_c(z) = z^2 + c$ are presented in CR orbit with s-convexity in Figs. 19–24. The usual parameters to generate the images are the following: $K = 50$, $\lambda_3 = 0.6$, $\lambda_2 = 0.5$, $\lambda_1 = 0.7$ and $c = -1.45$. Whereas, the varying parameters are the following:

- Fig. 19: $A = [-2.1, 1.4] \times [-1.3, 1.3]$, $s = 0.1$,
- Fig. 20: $A = [-2.1, 1.4] \times [-1.8, 1.8]$, $s = 0.2$,

FIGURE 19. Quadratic Julia set generated via CR iteration with s-convexity.

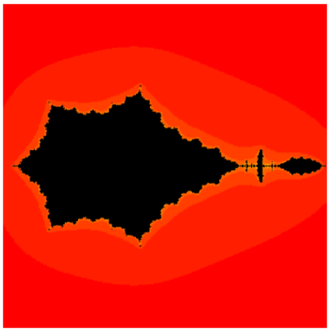

FIGURE 20. Quadratic Julia set generated via CR iteration with s-convexity.

FIGURE 21. Quadratic Julia set generated via CR iteration with s-convexity.

FIGURE 22. Quadratic Julia set generated via CR iteration with s-convexity.

FIGURE 23. Quadratic Julia set generated via CR iteration with s-convexity.

- Fig. 21: $A = [-2.3, 1.7] \times [-2, 2]$, $s = 0.3$,
- Fig. 22: $A = [-2.3, 1.7] \times [-2, 2]$, $s = 0.4$,
- Fig. 23: $A = [-2.3, 1.7] \times [-2, 2]$, $s = 0.5$,
- Fig. 24: $A = [-2.3, 1.7] \times [-2, 2]$, $s = 0.6$.

E. JULIA SETS FOR THE CUBIC FUNCTION $Q_C(z) = z^3 + c$

Julia sets for the function $Q_c(z) = z^3 + c$ are presented in CR orbit with s-convexity in Figs. 25–30. The usual parameters to generate the images are the following: $K = 50$, $\lambda_3 = 0.4$, $\lambda_2 = 0.7$, $\lambda_1 = 0.3$ and $c = -0.02 + 1.0i$. Whereas, the varying parameters are the following:

- Fig. 25: $A = [-1.1, 1.1] \times [-1.6, 1.8]$, $s = 0.4$,

FIGURE 24. Quadratic Julia set generated via CR iteration with s-convexity.

FIGURE 25. Cubic Julia set generated via CR iteration with s-convexity.

FIGURE 26. Cubic Julia set generated via CR iteration with s-convexity.

FIGURE 27. Cubic Julia set generated via CR iteration with s-convexity.

FIGURE 28. Cubic Julia set generated via CR iteration with s-convexity.

FIGURE 29. Cubic Julia set generated via CR iteration with s-convexity.

FIGURE 30. Cubic Julia set generated via CR iteration with s-convexity.

- Fig. 26: $A = [-1.2, 1.2] \times [-1.7, 1.9]$, $s = 0.5$,
- Fig. 27: $A = [-1.3, 1.3] \times [-1.8, 2]$, $s = 0.6$,
- Fig. 28: $A = [-1.4, 1.4] \times [-1.9, 2.1]$, $s = 0.7$,
- Fig. 29: $A = [-1.5, 1.5] \times [-2, 2.2]$, $s = 0.8$,
- Fig. 30: $A = [-1.5, 1.5] \times [-2, 2.2]$, $s = 0.9$.

V. CONCLUSIONS

Approximate fractals found in nature display self-similarity over extended, but finite, scale ranges. The connection between fractals and leaves, for instance, is currently being used to determine how much carbon is contained in trees. Recently, fractal analysis has been used to achieve a 93% success rate in distinguishing real from imitation Pollocks.

Cognitive neuroscientists have shown that Pollock's fractals induce the same stress-reduction in observers as computer-generated fractals and Nature's fractals. Fractals are used everywhere in technology, for example: Fractal antennas, Fractal transistor, Fractal heat exchangers, Digital imaging Architecture' Urban growth, Enzymes (Michaelis-Menten kinetics), Generation of new music. Signal and image compression. Creation of digital photographic enlargements. Fractal in soil mechanics, Computer and video game design, Computer Graphics, Organic environments, Procedural generation, Fractography and fracture mechanics, Small angle scattering theory of fractally rough systems, T-shirts and other fashion, Generation of patterns for camouflage, such as MARPAT, Digital sundial, Technical analysis of price series, Fractals in networks, Medicine, Neuroscience, Diagnostic Imaging, Pathology, Geology, Geography, Archaeology, Soil mechanics, Seismology, Search and rescue, Technical analysis,and many more. In this paper we have presented escape criterion for quadratic, cubic and $(k+1)$th degree polynomials to generate Mandelbrot sets and Julia sets via CR iteration scheme with s-convexity. We obtained new fractals for complex functions in CR orbit with s-convexity that are totally different from those introduced in [24], [25]. Presented results are applications of s-convexity. By using different values of λ_1, λ_2, λ_3 and s we obtained interesting Mandelbrot sets and Julia sets. A few examples of Mandelbrot sets have been presented for complex quadratic, cubic and $(k+1)$th degree polynomials. Very fascinating changing can be seen in Mandelbrot sets for different values of λ_1, λ_2 and λ_3. Also, Julia sets have been explored for quadratic and cubic polynomials. Very interesting changes in Julia sets can be seen when s varies from lowest to higher values.

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