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Semi Analytical Solutions for Fractional Oldroyd-B Fluid Through Rotating Annulus

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ABSTRACT In this paper, semi analytical solutions for velocity field and tangential stress correspond to fractional Oldroyd-B fluid, in an annulus, are acquired by Laplace transforms and modified Bessel equation. In the beginning, cylinders are stationary, motion is produced after $t = 0$ when both cylinders start rotating about their common axis. The governing equations solved for velocity field and shear stress by using the Laplace transform technique. The inverse Laplace transform is alternately calculated by Stehfest's algorithm using "MATHCAD" numerically. The numerically obtained solutions are in the form of modified Bessel's equations of first and second kind and satisfying all the imposed physical conditions. Finally, there is a comparison between exact and obtained solutions. It is observed that semi analytical technique and exact technique are approximately the same and satisfy imposed boundary conditions. Through graphs, the impact of physical parameters (relaxation time, retardation time kinematic viscosity, and dynamic viscosity) and fractional parameters on both velocity and shear stress is observed.

INDEX TERMS Fractional Oldroyd-B fluid, annulus, integral transformations, modified Bessel equation, velocity field, shear stress, numerical solutions.

I. INTRODUCTION

The complicated correspondence between stress and strain in non-Newtonian fluids and their technological application made study of non-Newtonian fluids valuable. Non-Newtonian fluids, especially in a cylindrical domain have great importance in the field of engineering and mathematics.

In previous era solutions for the flow of non-Newtonian fluids were discussed by many researchers. Mahmood *et al.* established a note on sinusoidal motion of a viscoelastic non-Newtonian fluid [1]. Hayat *et al.* obtained the velocity fields for some simple flows of Oldroyd-B fluids using Fourier transform [2]. Fetecau established exact solutions for some unidirectional flows of the same fluids in unbounded domains which ware geometrically axi-symmetric pipelike [3]. Yin and Zhu studied the oscillating flow of a viscoelastic fluid in a pipe with the fractional Maxwell model [4]. Waters and King studied the start-up Poiseuille flow of an Oldroyd- B fluid in a straight circular tube and found exact solution by using the Laplace transform method [5]. Wood considered the general case of helical flow of an Oldroyd-B fluid, due to the combined action of rotating cylinders (with constant angular velocities) and a constant axial pressure gradient [6].

Now-a-days fractional calculus is beneficial to summarize viscoelastic characteristics. Fractional calculus is advantageous in bio-engineering [7] and bio-rheology [8] (study of flow properties of biological fluids) because many tissue-like materials illustrate power-law when stress or strain are applied on them [9], [10]. Elastic tissues of the aorta are an example for such power-law behavior and fractional order viscoelastic models are used to analyze these types of data [11], [12]. Fractional calculus was born in 1695 during the conversation of L′ Hospital and Leibniz on the possibility of generalizing the operation of differentiation to non-integer orders. Fractional derivatives are actually modified differential equations, where integer order time derivative replaced by Caputo fractional calculus operators [13], [14]. By this abstraction integrals or derivatives of non-integer order can be defined precisely [14]. Fractional calculus is very useful in the field of mathematics as well as in physics [15], [16].

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Bagley and Torvik established a link between molecular theories that predict the macroscopic behavior of certain viscoelastic media and an empirically developed fractional calculus approach to viscoelasticity [17].

Recently many researchers have worked on unsteady flow of fractionalized non-Newtonian fluids (second grade, Maxwell, Oldroyed-B and Burgers') and obtained the solutions by using integral transformations. Jamil and Khan studied the unsteady flows of Burgers' fluid with fractional derivatives model, through a circular cylinder, by means of the Laplace and finite Hankel transforms [18]. Qi and Xu discussaed unsteady flow of viscoelastic fluids with the fractional Maxwell model [19]. Exact solutions for unsteady flow of a fractional Maxwell fluid through moving co-axial circular cylinders obtaianed by Imran *et al.* in (2016) [20]. Ali *et al.* studied fractional Casson fluid with heat generation over the oscillating plate by using fractional Caputo derivative for mathematical formulation [21] and found closed form of the solutions. Jimenez *et al.* discussed relaxation modulus in PMMA and PTFE fitting by fractional Maxwell model [22]. In (2011) Kamran *et al.* investigate the solution for the unsteady linearly accelerating flow of a fractional second grade fluid through a circular cylinder [23]. Tong *et al.* studied some unsteady unidirectional transient flows of fractional Oldroyd-B fluid in an annular pipe and discussed the following four problems: (1) Poiseuille flow due to a constant pressure gradient; (2) axial Couette flow in an annulus; (3) axial Couette flow in an annulus due to a longitudinal constant shear; (4) Poiseuille flow due to a constant pressure gradient and a longitudinal constant shear [24]. In 2017 Qi *et al.* [25] found the exact solutions for fractional Oldroyd-B fluid by using integral transforms technique. Ali *et al.* analysed the effects of magnetohydrodynamics on the blood flow when blood is represented as a Casson fluid, along with magnetic particles in a horizontal cylinder and concluded that the model with fractional order derivatives bring a remarkable change as compared to the ordinary model [26]. Sheikh *et al.* studied generalized Casson fluid model with heat generation and chemical reaction and gives a comparison between the Atangana-Baleanu and Caputo-Fabrizio fractional derivatives for said fluid [27]. Recently Farooq *et al.* [28] studied the viscous fluid flow over an infinite plate by using Caputo-Fabrizio derivative model and present solutions in closed form.

In the present era numerical study of non-Newtonian fluids got more attention of scientists. Recently, many researchers worked on numerical solutions of non-Newtonian fluids with the help of some numerical algorithms, for example Stehfest's, Tzou's and Talbot's algorithm. Villinger solved the inhomogeneous one dimensional heat diffusion equation semi-analytically in a cylindrically layered whole space and calculate the inverse Laplace transform by using a numerical procedure (Gaver Stehfest algorithm) [29]. In (2017) Raza investigate numerical solution of unsteady rotational flow of a second grade fluid with non-integer Caputo time fractional derivative by using Stehfest's algorithm [30]. Tahir *et al.* obtained the solutions for temperature, velocity and shear stress with numerical inversion techniques of Laplace transform namely, Stehfest's and Tzou's algorithms for Maxwell fluid over an oscillating vertical plate [31]. Raza *et al.* obtained numerical solution for fractional Maxwell fluid by using a semi analytical technique [32]. Shah *et al.* [33] studied the effect of magnetic Field and convection on flow of viscous fluid over a plate with by using Caputo-Fabrizio fractional derivative model by using semi analytic technique.

The aspiration of study here is to formulate solutions for fractional Oldroyd-B fluid in cylindrical domain using semi analytical technique. The general solutions are acquired by integral transformation and modified Bessel equation. Laplace inverse transformation has been calculated numerically by using MATHCAD. Furthermore, the influence of motion by variations in material parameters is presented in tabular and graphical illustrations. Also to check the accuracy of obtained solution, there is a comparison with already calculated exact analytical solutions.

II. PROBLEM FORMULATION AND GOVERNING EQUATIONS

The constitutive equations for an Oldroyd-B fluid are [34], [35]

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}\,;\; \mathbf{S} + \lambda(\overset{\bullet}{\mathbf{S}} - \mathbf{LS} - \mathbf{SL}^T) = \mu[\mathbf{A} + \lambda_r(\overset{\bullet}{\mathbf{A}} - \mathbf{LA} - \mathbf{AL}^T)], \tag{1}$$

where $\mathbf{T}$, p, $\mathbf{I}$, λ, $\mathbf{S}$, μ, $\mathbf{A}$, λ_r, are Cauchy stress tensor, hydrostatic pressure, unit tensor, relaxation time, extra-stress tensor, dynamic viscosity, 1^{st} Rivlin-Ericksen tensor and retardation time respectively. $\frac{\delta}{\delta t}$ denotes upper convected derivative defined as [36],

$$\frac{\delta S}{\delta t} = \left(\frac{d}{dt} - L - L^T\right) S.$$

Consider an incompressible Oldroyd-B fluid between two infinite circular cylinders. For the considered problem flow is only in θ -direction, so we have flow of the form [37]

$$V = V(r,t) = v_\theta(r,t)e_\theta; \quad S = S(r,t), \tag{2}$$

where, e_θ is the unit vector in the θ-direction of the cylindrical coordinate. The governing equations corresponding to Oldroyd-B fluid are [34], [35]

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\tau\,(r,t) = \mu\left(1+\lambda_r\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)v_\theta\,(r,t), \tag{3}$$

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial v_\theta(r,t)}{\partial t} = \nu\left(1+\lambda_r\frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)v_\theta\,(r,t), \tag{4}$$

where, $\tau(r,t) = S_{r\theta}(r,t)$ is the shear stress.

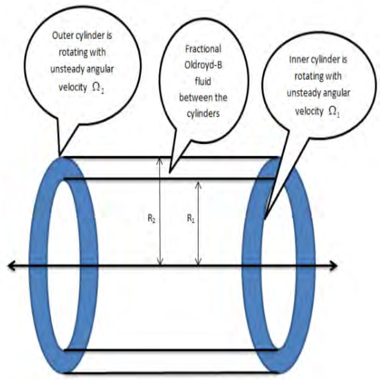

FIGURE 1. Oldrod-B fluid in coaxial cylinders with rotational motion.

The governing equation corresponding to fractional Oldroyd-B fluid obtained from (3) and (4) by replacing the time derivative with the fractional derivative operator

$$\left(1+\lambda^{\alpha} D_t^{\alpha}\right) \tau(r,t) = \mu\left(1+\lambda_r^{\beta} D_t^{\beta}\right)\left(\frac{\partial}{\partial r}-\frac{1}{r}\right) v_\theta(r,t),$$

$$\left(1+\lambda^{\alpha} D_t^{\alpha}\right)\frac{\partial v_\theta(r,t)}{\partial t} \tag{5}$$

$$= \nu\left(1+\lambda_r^{\beta} D_t^{\beta}\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right) v_\theta(r,t). \tag{6}$$

where, the fractional differential operator D_t^{β} is [38]

$$D_t^{\gamma} g(t) = \begin{cases} \dfrac{1}{\Gamma(1-\gamma)}\dfrac{d}{dt}\displaystyle\int_0^t \dfrac{g(\tau)}{(t-\tau)^{\gamma}}d\tau, & 0\le\gamma<1; \\ \dfrac{d}{dt}g(t), & \gamma=1. \end{cases} \tag{7}$$

III. INITIAL AND BOUNDARY CONDITIONS WITH GEOMETRY

The Fractional Oldroyb-B fluid (FOBF) is at rest in the beginning between two infinite coaxial circular cylinders having radii R_1 and R_2. Here R_1 and R_2 are radius of inner and outer cylinders respectively as shown in Fig. 1. The annulus begins to rotate when $t = 0^+$ with unsteady velocities around the axis of the cylinder. Taking this position into account, we set the initial and boundary conditions of the considered problem as follows

$$v_\theta(r,0) = \frac{\partial v_\theta(r,0)}{\partial r} = 0; \quad r\in[R_1,R_2], \tag{8}$$

$$v_\theta(R_1,t) = R_1\Omega_1 t = X(t);$$
$$v_\theta(R_2,t) = R_2\Omega_2 t = Y(t), \quad t>0. \tag{9}$$

IV. SOLUTION OF THE PROBLEM

A. CALCULATION OF THE VELOCITY FIELD

Applying Laplace transformation to Eqs. (5), (9) and using the initial condition (8),

$$\left(q+\lambda^{\alpha}q^{\alpha+1}\right)\bar{v}_\theta(r,q)$$

$$= \nu\left(1+\lambda_r^{\beta}q^{\beta}\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right)\bar{v}_\theta(r,q), \tag{10}$$

$$\bar{v}_\theta(R_1,q) = \frac{R_1\Omega_1}{q^2} = X(q);\ \bar{v}_\theta(R_2,q) = \frac{R_2\Omega_2}{q^2} = Y(q), \tag{11}$$

where, $\overline{v_\theta}(r,q)$ is the Laplace transformation of $v_\theta(r,t)$. Writing the Eq. (10) in another form as,

$$\frac{\partial^2\bar{v}_\theta(r,q)}{\partial r^2}+\frac{1}{r}\frac{\partial\bar{v}_\theta(r,q)}{\partial r}-\frac{\bar{v}_\theta(r,q)}{r^2} \tag{12}$$

$$+\lambda_r^{\beta}q^{\beta}\left(\frac{\partial^2\bar{v}_\theta(r,q)}{\partial r^2}+\frac{1}{r}\frac{\partial\bar{v}_\theta(r,q)}{\partial r}-\frac{\bar{v}_\theta(r,q)}{r^2}\right)$$
$$-\left(\frac{q+\lambda^{\alpha}q^{\alpha+1}}{\nu}\right)\bar{v}_\theta(r,q) = 0, \tag{13}$$

rearranging to have the form

$$[1+a(q)]\frac{\partial^2\bar{v}_\theta(r,q)}{\partial r^2}+[1+a(q)]\frac{1}{r}\frac{\partial\bar{v}_\theta(r,q)}{\partial r}$$
$$-[1+a(q)]\frac{\bar{v}_\theta(r,q)}{r^2}-b(q)\bar{v}_\theta(r,q) = 0, \tag{14}$$

where,

$$\lambda_r^{\beta}q^{\beta} = a(q)\ \ and\ \ \frac{q+\lambda^{\alpha}q^{\alpha+1}}{\nu} = b(q). \tag{15}$$

Using variable transformation $Z = r\sqrt{\frac{b(q)}{1+a(q)}} = rd(q)$ in Eq. (14), we have

$$Z^2\frac{\partial^2\bar{v}_\theta}{\partial Z^2}+Z\frac{\partial\bar{v}_\theta}{\partial Z}-(Z^2-1)\bar{v}_\theta = 0. \tag{16}$$

Eq. (16) is the modified Bessel equation when $n = 1$ and its general solution is [39], [40]

$$\bar{v}_\theta(Z,q) = C_1 I_1(Z)+C_2 K_1(Z), \tag{17}$$

where, C_1, C_2 are constants and I_1, K_1 are modified Bessel function of the first and second kind respectively.
Solving Eqs. (11) and (17) we get the values of C_1 and C_2,

$$C_1 = \frac{X(q)}{I_1(R_1d(q))} - \frac{K_1(R_1d(q))}{K_1(R_2d(q))I_1(R_1d(q))-K_1(R_1d(q))I_1(R_2d(q))} \times\left(Y(q)-X(q)\frac{I_1(R_2d(q))}{I_1(R_1d(q))}\right),$$

$$C_2 = \frac{I_1(R_1d(q))}{K_1(R_2d(q))I_1(R_1d(q))-K_1(R_1d)I_1(R_2d(q))} \times\left(Y(q)-X(q)\frac{I_1(R_2d(q))}{I_1(R_1d(q))}\right).$$

Finally, put the values of C_1 and C_2 in Eq. (17) we have

$$\bar{v}_\theta(r,q) = \left[\frac{X(q)}{I_1(R_1d(q))}\right.$$
$$-\frac{K_1(R_1d(q))}{K_1(R_2d(q))I_1(R_1d(q))-K_1(R_1d(q))I_1(R_2d(q))}$$

TABLE 1. Comparison of velocity field corresponding to numerical and exact values for $R_1 = 0.7$, $R_2 = 1$, $\Omega_1 = 0.81$, $\Omega_2 = -0.45$, $\lambda^\alpha = 2$, $\lambda_r^\beta = 1$, $\nu = 8$, $\alpha = 0.0001$, $\beta = 1$, $t = 1$.

r	Numerical Result $v_\theta(r,t)$ Eq. (18)	Exact Result $\omega(r,t)$ Eq. (20) Tahir *at el* [35]	$v_\theta(r,t) - \omega(r,t)$
0.7	0.567	0.567	0
0.72	0.4856	0.4856	0
0.74	0.4069	0.4068	0.0001
0.76	0.3306	0.3304	0.0002
0.78	0.2566	0.2563	0.0003
0.8	0.1846	0.1842	0.0004
0.82	0.1145	0.114	0.0005
0.84	0.0462	0.0456	0.0006
0.86	-0.0204	-0.0211	0.0006
0.88	-0.0856	-0.0862	0.0007
0.9	-0.1493	-0.15	0.0006
0.92	-0.2117	-0.2123	0.0006
0.94	-0.2729	-0.2734	0.0005
0.96	-0.333	-0.3334	0.0004
0.98	-0.392	-0.3922	0.0002
1	-0.45	-0.45	0

$$\times\left(Y(q)-X(q)\frac{I_1(R_2d(q))}{I_1(R_1d(q))}\right)\Bigg]+I_1(rd(q))$$
$$\times\left[\frac{I_1(R_1d(q))}{K_1(R_2d(q))I_1(R_1d(q))-K_1(R_1d(q))I_1(R_2d(q))}\right.$$
$$\left.\times\left(Y(q)-X(q)\frac{I_1(R_2d(q))}{I_1(R_1d(q))}\right)\right]K_1(rd(q). \tag{18}$$

It is very laborious to calculate inverse Laplace transformation of Eq. (18) traditionally. So, inverse Laplace transform calculated numerically by using Gaver-Stehfest algorithm [41]

$$v(r,t)=\frac{\ln(2)}{t}\sum_{j=1}^{2m}d_j\bar{v}\left(r,j\frac{\ln(2)}{t}\right), \tag{19}$$

where m is a positive integer and

$$d_j=(-1)^{j+m}\sum_{i=\left[\frac{j+1}{2}\right]}^{\min(j,m)}\frac{i^m\,(2i)!}{(m-i)!i!\,(i-1)!\,(j-i)!\,(2i-j)!}.$$

In order to check the accuracy of our numerical solution here is a comparison between the numerical and already calculated exact analytical solution of this fluid given in Table 1, in which velocity field is, [35]

$$\omega(r,t)=\frac{\Omega_1R_1^2(R_2^2-r^2)+\Omega_2R_2^2(r^2-R_1^2)}{r(R_2^2-R_1^2)}t^\kappa-\frac{\pi}{\lambda\varsigma}$$
$$\times\sum_{n=1}^{\infty}$$
$$\frac{J_1(R_1r_n)A(r,r_n)\left(\Omega_2R_2J_1(R_1r_n)-\Omega_1R_1J_1(R_2r_n)\right)}{J_1^2(R_1r_n)-J_1^2(R_2r_n)}$$
$$\times\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{k!}{m!(k-m)!}\left(-\frac{\nu r_n^2}{\lambda\varsigma}\right)^k\lambda_r^{\varsigma m}$$
$$\times\left[G_{\varsigma,\varsigma m-k-\kappa-1,k+1}\left(-\lambda^{-1},t\right)\right.$$
$$\left.+\lambda G_{\varsigma,\varsigma+\varsigma m-k-\kappa-1,k+1}\left(-\lambda^{-1},t\right)\right]. \tag{20}$$

In table 1. we presented the equivalence of exact and numerical results.

B. CALCULATION OF SHEAR STRESS

Applying Laplace transformation to Eq. (5) we have

$$\left(1+\lambda^\alpha q^\alpha\right)\bar{\tau}(r,q)=\mu\left(1+\lambda_r^\beta q^\beta\right)\left(\frac{\partial}{\partial r}-\frac{1}{r}\right)\overline{v_\theta}(r,q), \tag{21}$$

writing above equation in another form

$$\bar{\tau}(r,q)=\mu\left(\frac{1+\lambda_r^\beta q^\beta}{1+\lambda^\alpha q^\alpha}\right)\left(\frac{\partial\overline{v_\theta}(r,q)}{\partial r}-\frac{\overline{v_\theta}(r,q)}{r}\right), \tag{22}$$

or

$$\bar{\tau}(r,q)=\kappa(q)\left(\frac{\partial\overline{v_\theta}(r,q)}{\partial r}-\frac{\overline{v_\theta}(r,q)}{r}\right), \tag{23}$$

where,

$$\mu\left(\frac{1+\lambda_r^\beta q^\beta}{1+\lambda^\alpha q^\alpha}\right)=\kappa(q).$$

Since, [39]

$$\frac{\partial I_n}{\partial x}=I_{n+1}(x)-\frac{nI_n(x)}{x}.$$
$$\frac{\partial K_n}{\partial x}=-K_{n+1}(x)-\frac{nK_n(x)}{x}.$$

Now using equation Eq. (18) and above relations we have

$$\frac{\partial\overline{v_\theta}(r,q)}{\partial r}-\frac{\overline{v_\theta}(r,q)}{r}$$
$$=\left[\frac{X(q)}{I_1(R_1d(q))}\right.$$
$$-\frac{K_1(R_1d(q))}{K_1(R_2d(q))I_1(R_1d(q))-K_1(R_1d(q))I_1(R_2d(q))}$$
$$\left.\times\left(Y(q)-X(q)\frac{I_1(R_2d(q))}{I_1(R_1d(q))}\right)\right]$$
$$\left[dI_0(rd(q))-2\frac{I_1(rd(q))}{r}\right]$$
$$-\left[\frac{I_1(R_1d(q))}{K_1(R_2d(q))I_1(R_1d(q))-K_1(R_1d(q))I_1(R_2d(q))}\right.$$
$$\left.\times\left(Y(q)-X(q)\frac{I_1(R_2d(q))}{I_1(R_1d(q))}\right)\right]$$
$$\times\left[dK_0(rd(q))+2\frac{K_1(rd(q))}{r}\right].$$

Finally Eq. (23) takes the following form by using above equation

$$\bar{\tau}(r,q)=\kappa(q)\left[\frac{X(q)}{I_1(R_1d(q))}\right.$$

TABLE 2. $v_\theta(r,t)$, Eq. (18) for $R_1 = 0.7, R_2 = 1, \Omega_1 = 0.5, \Omega_2 = -0.5, \lambda^\alpha = 2, \lambda_r^\beta = 1, \nu = 0.02, \alpha = 0.9, \beta = 0.8$ and variation in time.

r	$v_\theta(r,t)$ t=1	$v_\theta(r,t)$ t=2	$v_\theta(r,t)$ t=3
0.7	0.34999747	0.700001573	1.049994245
0.72	0.277363418	0.566118667	0.847364116
0.74	0.214440897	0.442330984	0.656705438
0.76	0.159605368	0.326698184	0.475956796
0.78	0.111171036	0.217326453	0.303243487
0.8	0.067376772	0.11248143	0.136793148
0.82	0.026395122	0.010522165	-0.025043406
0.84	-0.013643711	-0.090086409	-0.183809528
0.86	-0.054632129	-0.190760663	-0.340948741
0.88	-0.098451387	-0.29284024	-0.497819867
0.9	-0.146957353	-0.397572193	-0.655707684
0.92	-0.202009431	-0.506098161	-0.815787029
0.94	-0.265574082	-0.619476896	-0.979256157
0.96	-0.34002255	-0.738766055	-1.147157932
0.98	-0.428675658	-0.864793555	-1.320479664
1	-0.537166236	-0.998480005	-1.500309614

TABLE 3. $v_\theta(r,t)$, Eq. (18) for $R_1 = 0.7, R_2 = 1, \Omega_1 = 0.5, \Omega_2 = -0.5, t = 2, \lambda_r^\beta = 0.5, \nu = 0.005, \alpha = 0.02, \beta = 0.9$ and variation in λ^α.

r	$v_\theta(r,t)$ $\lambda^\alpha = 1$	$v_\theta(r,t)$ $\lambda^\alpha = 3$	$v_\theta(r,t)$ $\lambda^\alpha = 5$
0.7	0.700002559	0.700001883	0.700003861
0.72	0.507856038	0.45221678	0.410973213
0.74	0.361780067	0.286320701	0.234798514
0.76	0.250566987	0.177171678	0.130685813
0.78	0.164788666	0.105942506	0.070557056
0.8	0.096550043	0.058871078	0.036044942
0.82	0.039111133	0.025950641	0.015448577
0.84	-0.013461142	-0.000404284	0.001105834
0.86	-0.066685146	-0.026554859	-0.012682185
0.88	-0.126029981	-0.058766483	-0.031434199
0.9	-0.197286292	-0.104323328	-0.062267803
0.92	-0.286880321	-0.172728889	-0.116058624
0.94	-0.402244078	-0.27700395	-0.210418437
0.96	-0.55199781	-0.435187623	-0.374072013
0.98	-0.746186942	-0.671963273	-0.653519217
1	-0.996209809	-1.020064371	-1.123767136

$$
\begin{aligned}
& - \frac{K_1(R_1 d(q))}{K_1(R_2 d(q)) I_1(R_1 d(q)) - K_1(R_1 d(q)) I_1(R_2 d(q))} \\
& \times \left(Y(q) - X(q) \frac{I_1(R_2 d(q))}{I_1(R_1 d(q))} \right) \Bigg] \\
& \left[d I_0(r d(q)) - 2 \frac{I_1(r d(q))}{r} \right] \\
& - \Bigg[\frac{I_1(R_1 d(q))}{K_1(R_2 d(q)) I_1(R_1 d(q)) - K_1(R_1 d) I_1(R_2 d(q))} \\
& \times \left(Y(q) - X(q) \frac{I_1(R_2 d(q))}{I_1(R_1 d(q))} \right) \Bigg] \\
& \times \left[d(q) K_0(r d(q)) + 2 \frac{K_1(r d(q))}{r} \right]. \qquad (24)
\end{aligned}
$$

Now, find the inverse Laplace transform numerically through MATHCAD by using Gaver-Stehfest algorithm [41].

V. DISCUSSION ON RESULTS

In this article, numerical solution of the velocity field and tangential stress is acquired by using the semi analytical technique. The solutions determined by using the integral transformation and modified Bessel equation. The results are in series form of Modified Bessel functions $I_0(.)$, $I_1(.)$, $K_0(.)$ and $K_1(.)$. The general solutions presented by Eq. (18) and Eq. (24).

TABLE 4. $v_\theta(r,t)$, Eq. (18) for $R_1 = 0.7, R_2 = 1, \Omega_1 = 0.5, \Omega_2 = -0.5, \lambda^\alpha = 2, t = 2.5, \nu = 0.005, \alpha = 0.02, \beta = 0.9$ and variation in λ_r.

r	$v_\theta(r,t)$ $\lambda_r^\beta = 1$	$v_\theta(r,t)$ $\lambda_r^\beta = 3$	$v_\theta(r,t)$ $\lambda_r^\beta = 6$
0.7	0.875001914	0.875002806	0.875006241
0.72	0.630387971	0.664907064	0.684483374
0.74	0.446833322	0.494529464	0.520418349
0.76	0.308319764	0.354143721	0.376872146
0.78	0.202096411	0.235659105	0.24876291
0.8	0.117794103	0.132366232	0.131571659
0.82	0.046870126	0.038421392	0.021347195
0.84	-0.018044593	-0.051392354	-0.085598779
0.86	-0.083701542	-0.141939401	-0.192771111
0.88	-0.156800417	-0.238039183	-0.303515102
0.9	-0.244472367	-0.34463559	-0.421292113
0.92	-0.354814409	-0.467067056	-0.549621766
0.94	-0.497426658	-0.611328101	-0.692240224
0.96	-0.683909828	-0.78431205	-0.853352457
0.98	-0.928398997	-0.994135742	-1.037515011
1	-1.248066521	-1.250431281	-1.249939341

TABLE 5. $v_\theta(r,t)$, Eq. (18) for $R_1 = 0.7, R_2 = 1, \Omega_1 = 0.5, \Omega_2 = -0.5, \lambda^\alpha = 2, \lambda_r^\beta = 0.5, t = 1, \alpha = 0.01, \beta = 0.9$ and variation in ν.

r	$v_\theta(r,t)$ $\nu = 0.01$	$v_\theta(r,t)$ $\nu = 0.015$	$v_\theta(r,t)$ $\nu = 0.03$
0.7	0.349999454	0.350000342	0.350001653
0.72	0.245584524	0.258635336	0.272691
0.74	0.169868778	0.187726644	0.206889903
0.76	0.114761174	0.132126399	0.150075651
0.78	0.074067973	0.087640164	0.100021102
0.8	0.042987995	0.050783018	0.054751669
0.82	0.0177156	0.018599075	0.012477022
0.84	-0.004925425	-0.011519607	-0.028458288
0.86	-0.027784151	-0.041998159	-0.069620326
0.88	-0.053679917	-0.075235967	-0.112559558
0.9	-0.085681297	-0.113746291	-0.158813626
0.92	-0.127392234	-0.160312332	-0.209990076
0.94	-0.183261736	-0.218135804	-0.267770987
0.96	-0.258923473	-0.290966242	-0.333986818
0.98	-0.361542696	-0.383279304	-0.410609224
1	-0.500207075	-0.500423769	-0.499827811

TABLE 6. $\tau(r,t)$, Eq. (24) for $R_1 = 0.5, R_2 = 1, \Omega_1 = 1, \Omega_2 = -1, \lambda^\alpha = 6, \lambda_r^\beta = 3, \nu = 5, \alpha = 0.01, \beta = 1, \mu = 2.196$ and variation in time.

r	$\tau(r,t)$ t=0.6	$\tau(r,t)$ t=0.7	$\tau(r,t)$ t=0.8
0.5	-3.981149495	-4.655110841	-5.307065097
0.53	-3.549948669	-4.14247502	-4.716717216
0.56	-3.174594974	-3.706268258	-4.218337712
0.59	-2.862880932	-3.330685168	-3.808977538
0.62	-2.582281886	-3.020225522	-3.447943275
0.65	-2.353886006	-2.755798108	-3.141894041
0.68	-2.144383106	-2.505507907	-2.856285714
0.71	-1.972439206	-2.304194385	-2.631234956
0.74	-1.823035005	-2.129817163	-2.424919216
0.77	-1.692170127	-1.969096958	-2.240967586
0.8	-1.560580589	-1.833441375	-2.086103221
0.83	-1.460680046	-1.709819964	-1.946384033
0.86	-1.360597993	-1.58453699	-1.81430395
0.89	-1.272594733	-1.497254181	-1.71026876
0.92	-1.209093623	-1.402404627	-1.594003979
0.95	-1.140952862	-1.330383502	-1.507282993
0.98	-1.069022708	-1.260486902	-1.429302563

TABLE 7. $\tau(r,t)$, Eq. (24) for $R_1 = 0.5, R_2 = 1, \Omega_1 = 1, \Omega_2 = -1, t = 1, \lambda_r^\beta = 1, \nu = 1, \alpha = 0.8, \beta = 0.5, \mu = 2.196$ and variation in λ^α.

r	$\tau(r,t)$ $\lambda^\alpha = 6$	$\tau(r,t)$ $\lambda^\alpha = 7$	$\tau(r,t)$ $\lambda^\alpha = 8$
0.5	-3.153065561	-2.745233042	-2.437955437
0.53	-2.773346931	-2.41850245	-2.138952028
0.56	-2.4604601	-2.145547852	-1.895276397
0.59	-2.203514818	-1.917667361	-1.691937222
0.62	-1.987855903	-1.735272142	-1.532046831
0.65	-1.80687679	-1.575038342	-1.391992497
0.68	-1.663632789	-1.441849204	-1.27839219
0.71	-1.534684863	-1.336255373	-1.186132377
0.74	-1.430629264	-1.247925671	-1.111609086
0.77	-1.341467317	-1.17436171	-1.05103524
0.8	-1.269444728	-1.114534917	-1.004950903
0.83	-1.208552509	-1.067880272	-0.968815499
0.86	-1.162445533	-1.033511323	-0.946800169
0.89	-1.124182747	-1.007292626	-0.927886588
0.92	-1.092555779	-0.99012947	-0.924844828
0.95	-1.079967988	-0.983079844	-0.920039752
0.98	-1.070864852	-0.974701205	-0.92961586

TABLE 8. $\tau(r,t)$, Eq. (24) for $R_1 = 0.5, R_2 = 1, \Omega_1 = 1, \Omega_2 = -1, \lambda^\alpha = 6, t = 0.8, \nu = 1, \alpha = 0.1, \beta = 0.1, \mu = 2.196$ and variation in λ_r.

r	$\tau(r,t)$ $\lambda_r^\beta = 3$	$\tau(r,t)$ $\lambda_r^\beta = 4$	$\tau(r,t)$ $\lambda_r^\beta = 5$
0.5	-1.955261756	-2.158100236	-2.334366489
0.53	-1.721492337	-1.905179965	-2.063605975
0.56	-1.52537159	-1.692966332	-1.837688174
0.59	-1.366409418	-1.517053579	-1.647063223
0.62	-1.232550294	-1.36999172	-1.489615445
0.65	-1.121843643	-1.247201603	-1.354736595
0.68	-1.02907585	-1.143677813	-1.242020212
0.71	-0.953005067	-1.057686188	-1.145755922
0.74	-0.88999624	-0.985165895	-1.064868454
0.77	-0.838928292	-0.925048923	-0.997227826
0.8	-0.797410102	-0.875621559	-0.9404958
0.83	-0.764618088	-0.835783332	-0.892807583
0.86	-0.739453532	-0.803660023	-0.852901781
0.89	-0.720663772	-0.77874612	-0.821409529
0.92	-0.710743522	-0.76116208	-0.797399593
0.95	-0.702735786	-0.749440092	-0.777884086
0.98	-0.697796189	-0.743971046	-0.766966963

TABLE 9. $\tau(r,t)$, Eq. (24) for $R_1 = 0.5, R_2 = 1, \Omega_1 = 1, \Omega_2 = -1, \lambda^\alpha = 6, \lambda_r^\beta = 1, \nu = 3, \alpha = 0.5, \beta = 0.5, t = 1$ and variation in μ.

r	$\tau(r,t)$ $\mu = 2.196$	$\tau(r,t)$ $\mu = 2.496$	$\tau(r,t)$ $\mu = 2.896$
0.5	-2.616240496	-2.973806961	-3.450651707
0.53	-2.320803545	-2.639205075	-3.060529352
0.56	-2.072058919	-2.354275798	-2.733771658
0.59	-1.859465727	-2.114329735	-2.451767468
0.62	-1.68041494	-1.910012557	-2.216550627
0.65	-1.529656522	-1.738833312	-2.018068915
0.68	-1.401109019	-1.592773212	-1.848196374
0.71	-1.288419124	-1.464450957	-1.699930395
0.74	-1.19426635	-1.357933575	-1.575300608
0.77	-1.111600162	-1.263984235	-1.46631822
0.8	-1.039377032	-1.18138729	-1.371271311
0.83	-0.978467461	-1.111857967	-1.290201279
0.86	-0.928614307	-1.055378581	-1.224052993
0.89	-0.881432154	-1.001274217	-1.161638076
0.92	-0.842513868	-0.957330648	-1.111551989
0.95	-0.811562644	-0.922294879	-1.070194726
0.98	-0.782028683	-0.888200095	-1.031270904

Now, in order to reveal some relevant physical aspects of the obtained results, the figures of the velocity as well as those of the shear stress are depicted against r for different values of time t and of the physical parameters. Also, to check the accuracy of our solution, there is a comparison graph between numerical and exact solutions.

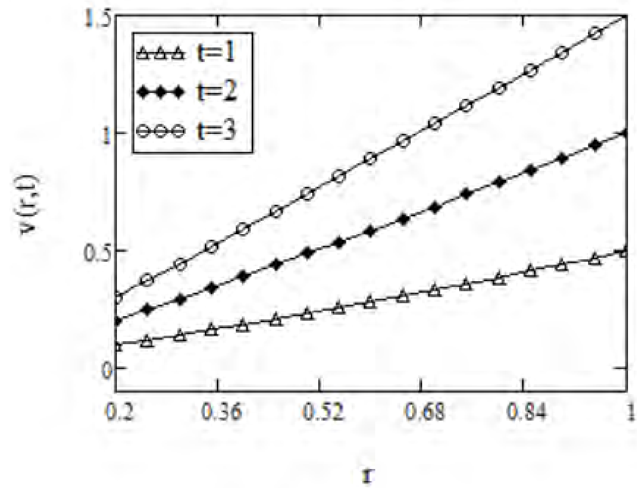

FIGURE 2. $v_\theta(r,t)$, Eq. (18) for $R_1 = 0.2, R_2 = 1, \Omega_1 = 0.5, \Omega_2 = 0.5, \lambda^\alpha = 3, \lambda_r^\beta = 1, \nu = 2, \alpha = 0.9, \beta = 0.8$ and variation in time.

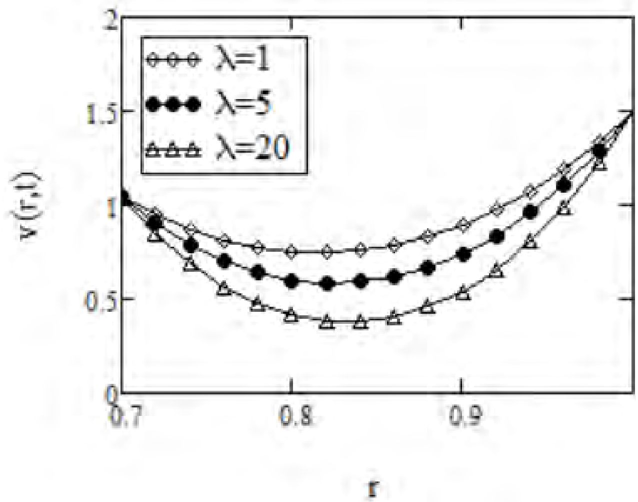

FIGURE 3. $v_\theta(r,t)$, Eq. (18) for $R_1 = 0.7, R_2 = 1, \Omega_1 = 0.5, \Omega_2 = 0.5, t = 3, \lambda_r^\beta = 4, \nu = 0.006, \alpha = 0.5, \beta = 0.7$ and variation in λ^α.

The velocity field $v_\theta(r,t)$ and shear stress $\tau(r,t)$ given in Eqs. (18) and (24) have been drawn against r. The velocity profiles illustrate in figures (2-7) for different values of physical parameters also there are tables for respective parameter. Fig. 2 for the effect of t, it is clearly observed that as we increase the value of time the fluid flow develop and velocity of the fluid increases. So one can say that velocity of fluid is an increasing function with respect to time. Similarly increase in velocity can also be observed in figure 2 with respect to radial component r while fixing the other dependent parameters and our solution satisfy the applied boundary conditions. It can also be observed that velocity has linear relation for r with respect to t. Fig. 3, 4 depicted for the effect of λ and λ_r respectively, the velocity is decreasing functions with

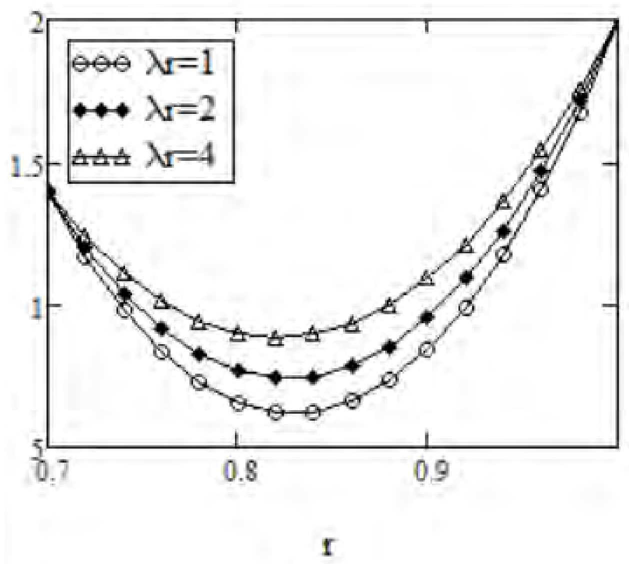

FIGURE 4. $v_\theta(r,t)$, Eq. (18) for $R_1 = 0.7, R_2 = 1, \Omega_1 = 0.5, \Omega_2 = 0.5, \lambda^\alpha = 6, t = 4, \nu = 0.006, \alpha = 0.5, \beta = 0.7$ and variation in λ_r^β.

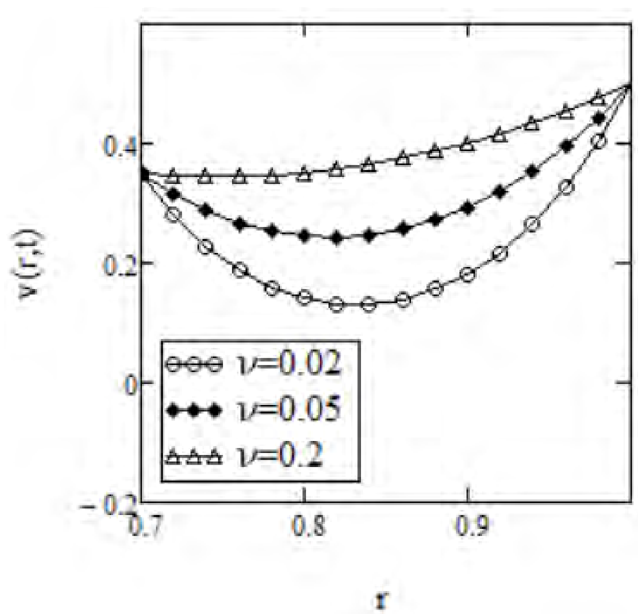

FIGURE 5. $v_\theta(r,t)$, Eq. (18) for $R_1 = 0.7, R_2 = 1, \Omega_1 = 0.5, \Omega_2 = 0.5, \lambda^\alpha = 2, \lambda_r^\beta = 0.5, t = 1, \alpha = 0.01, \beta = 0.9$ and variation in ν.

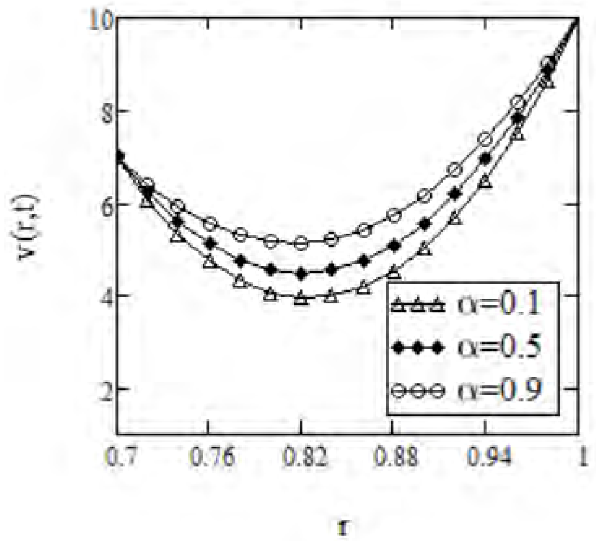

FIGURE 6. $v_\theta(r,t)$, Eq. (18) for $R_1 = 0.7, R_2 = 1, \Omega_1 = 0.5, \Omega_2 = 0.5, \lambda^\alpha = 5, \lambda_r^\beta = 1, t = 20, \nu = 0.001, \beta = 0.1$ and variation in α.

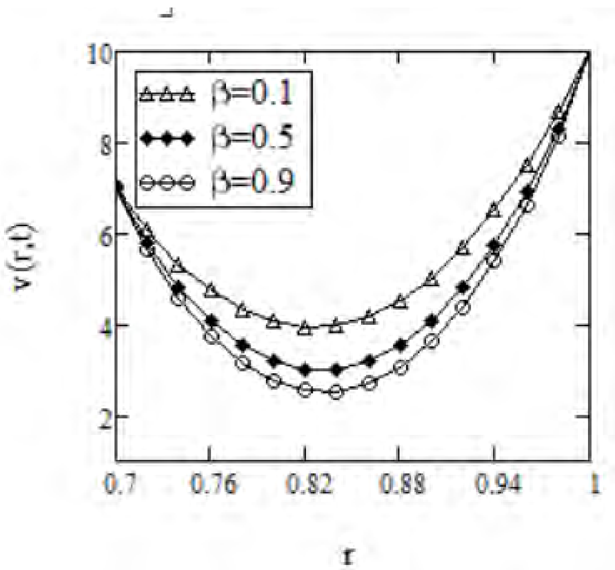

FIGURE 7. $v_\theta(r,t)$, Eq. (18) for $R_1 = 0.7, R_2 = 1, \Omega_1 = 0.5, \Omega_2 = 0.5, \lambda^\alpha = 5, \lambda_r^\beta = 1, t = 20, \nu = 0.001, \alpha = 0.1$ and variation in β.

respect to λ and increasing functions with respect to λ_r. The relaxation time parameter also called the memory time of the fluid associated with the microstructure of the material. When elastic fluid is deformed after the memory time or the relaxation time of the material has elapsed, the stress would reach a steady value and has no effect on fluid motion after that time. So velocity is a decreasing function of relaxation time as larger value of relaxation time offer more elasticity stress to fluid which slow down its velocity. Retardation time is the delayed response to an applied force or stress and can be described as delay of the elasticity and its effect on velocity is opposite to relaxation time. Fig. 5 shows the effect of ν, velocity is a decreasing function of kinematic viscosity which validates our results as increase in viscosity offer resistance to fluid flow which deceases the fluid velocity. Figs. 6 and 7 drawn to observe the effects of fractional parameter on fluid velocity. As expected the effects of fractional parameter α and β is similar to relaxation time and retardation time respectively. From tables (2 - 5) we can observe that in all cases the velocity of the fluid is zero near the inner cylinder when both the cylinders are rotating in the opposite direction with the same speed.

The tangential shear stress illustrates in figures (8 - 11) for different values of physical parameters also there are tables (6 - 9) for respective parameters. These figures

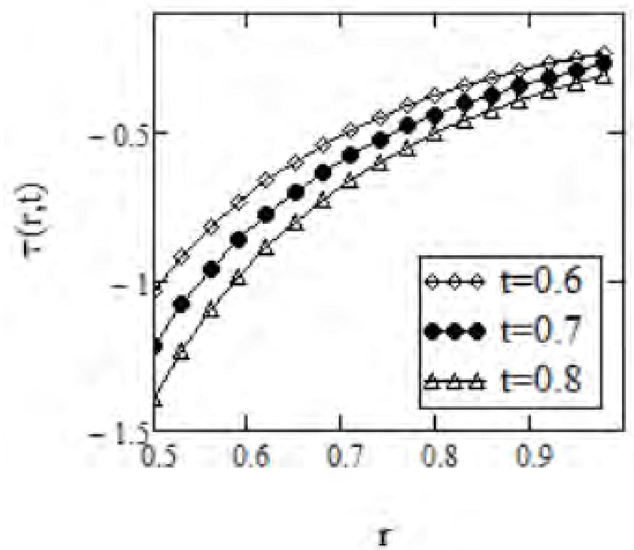

FIGURE 8. $\tau(r,t)$, Eq. (24) for $R_1 = 0.5, R_2 = 1, \Omega_1 = 1, \Omega_2 = 0.5, \lambda^{\alpha} = 6, \lambda_r^{\gamma} = 3, \nu = 5, \alpha = 0.01, \gamma = 1, \mu = 2.196$ and variation in time.

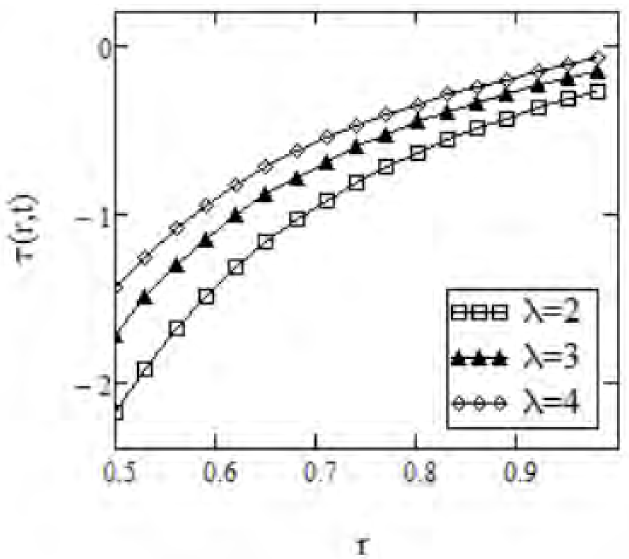

FIGURE 9. $\tau(r,t)$, Eq. (24) for $R_1 = 0.5, R_2 = 1, \Omega_1 = 1, \Omega_2 = 0.5, t = 1, \lambda_r^{\beta} = 1, \nu = 1, \alpha = 0.8, \beta = 0.5, \mu = 2.196$ and variation in λ^{α}.

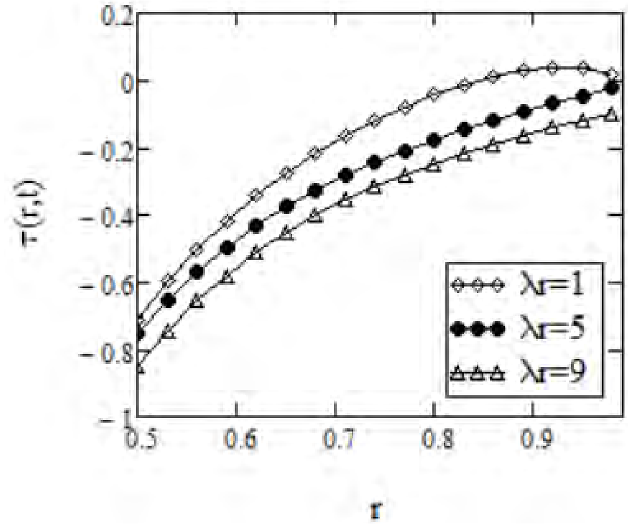

FIGURE 10. $\tau(r,t)$, Eq. (24) for $R_1 = 0.5, R_2 = 1, \Omega_1 = 1, \Omega_2 = 0.5, \lambda^{\alpha} = 6, t = 0.8, \nu = 1, \alpha = 0.8, \beta = 0.5, \mu = 2.196$ and variation in λ_r^{β}.

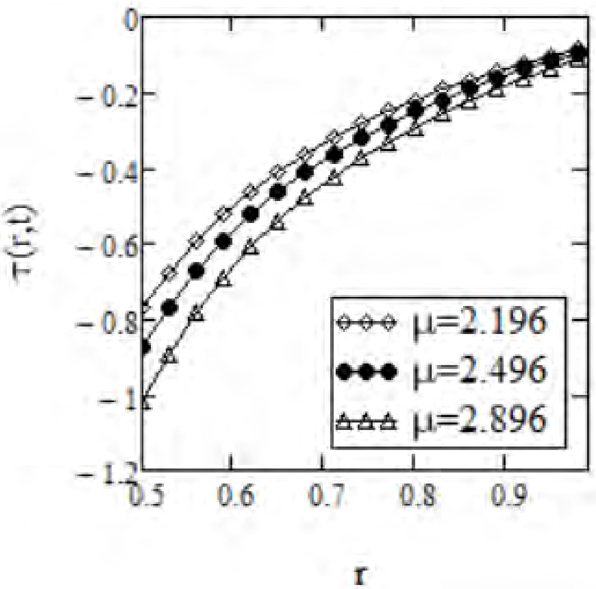

FIGURE 11. $\tau(r,t)$, Eq. (24) for $R_1 = 0.5, R_2 = 1, \Omega_1 = 1, \Omega_2 = -1, \lambda^{\alpha} = 6, \lambda_r^{\beta} = 1, \nu = 3, \alpha = 0.5, \beta = 0.5, t = 1$ and variation in μ.

showing that the shear stress is influenced by parameters r, t, λ, λ_r and μ. One can say that the relaxation parameters λ have decreasing behaviour for stress. This fact can be observed in figure 9. Whereas, stress is increasing function (in absolute values)for the retardation parameter λ_r, viscosity μ and t as shown in figures 10, 11 and 8. These figures $2-11$ make it possible to check the point to point variations, increment or decrement (among which the graphs are made) for shear stress profile.

Finally a comparison between obtained and already existing exact solution for the same values of physical and fractional parameters given in Fig. 12 and table 1 to check the accuracy of our solution. Fig. 12 and table 1 shows that numerical and exact solutions for fractional Oldoyd-B fluid are approximately same.

VI. CONCLUSIONS

In this article semi analytical solutions for fractional Oldoyd-B fluid through an annulus are calculated by using Laplace transformation and modified Bessel function, numerical scheme is used to obtain final results. The main conclusions are,

- Obtained results satisfy all imposed physical conditions.
- The velocity and shear stress are increasing functions of time. It has been observed that the velocity of the fluid is zero nearer to the inner cylinder when both the

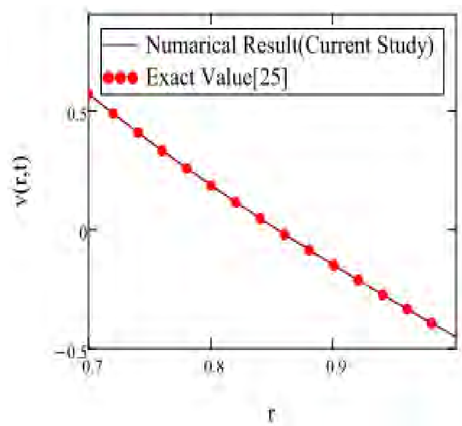

FIGURE 12. $v_\theta(r,t)$ **corresponding to numerical and exact values for** $R_1 = 0.7, R_2 = 1, \Omega_1 = 0.81, \Omega_2 = -0.45, \lambda^\alpha = 2, \lambda_r^\beta = 1, \nu = 8, \alpha = 0.0001, \beta = 1, t = 1.$

cylinders are rotating in the opposite direction with the same speed.

- As expected, that the behaviors of relaxation and retardation time are opposite in direction, both functions, velocity and shear stress are decreasing with respect to relaxation time 'λ' and increasing with respect to retardation time 'λ_r'.
- The effects of fractional parameter α and β on velocity is similar to λ and λ_r respectively.
- Kinematic viscosity ν has a forceful domination on the velocity, as value of ν increases velocity field is increasing or vice versa.
- Shear stress is increased by an increase in dynamic viscosity.
- In case of opposite rotation of cylinders, velocity is zero nearer to the inner cylinder because the radius of inner cylinder is less than outer cylinder.
- Numerical solution Eq. (18) approximately same as exact solution Eq. (20).
- In future authors will try to generalize the study by considering the heat and magnetic field effects on fluid motion.

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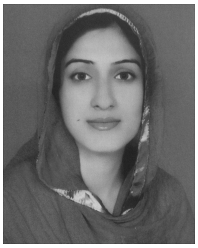

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