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New Extension Interleaved Constructions of Optimal Frequency Hopping Sequence Sets With Low Hit Zone

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ABSTRACT In this paper, a designated interleaved structure of constructing optimal frequency-hopping sequence (FHS) sets with low hit zone (LHZ) is presented based on the Cartesian product. By the general structure, we obtain infinitely many optimal LHZ FHS sets with new and flexible parameters by combining the optimal LHZ FHS sets with one-coincidence sequence sets. Moreover, our constructions remove the constraint requiring that the extension factor is co-prime with the length of the original FHSs. In this paper, most of the extension constructions suffer from this constraint. As a result, our constructions allow great flexibility of choosing parameters of the LHZ FHS sets for a given quasi-synchronization frequency-hopping spread spectrum system.

INDEX TERMS Frequency hopping sequences, low hit zone, optimal Hamming correlation, extension construction.

I. INTRODUCTION

In frequency-hopping multiple access (FHMA) communication systems, the signal of each user hop over the entire transmission bandwidth in a pseudo random fashion. FHMA communication systems are widely adopted in practice [1], [2]. For example, many popular systems, such as military communications [3], ultra wideband communications [4], 5G communication systems [5], HetNets [6], and Bluetooth [7], use FHMA methods. In such systems each user is represented by a sequence of hopping frequencies [8]. Simultaneous transmission by any two users over the same frequency band results in collisions of signals, and hence, it is very desirable that such collisions over the same frequency band are minimized. Thus, the design of a frequency hopping sequence (FHS) set with good property is an important problem.

Different from conventional FHS design, the design of FHSs with low hit zone (LHZ) aims at making Hamming correlation equal to a very low value within a correlation zone [9]. The significance of LHZ FHS set is that, even there are relative delays between the transmitted FHSs, the number of hits will be kept at a very low level between different sequences as long as the relative delay does not exceed certain limit (zone). There have been a number of optimal FHS sets which satisfies Peng-Fan-Lee bound [11]. In 2010, optimal LHZ FHS sets meeting Peng-Fan-Lee bound were constructed firstly by Ma and Sun [12]. In 2012 and 2014, we got some constructions of optimal LHZ FHS sets by interleaving technique [13], [14]. In 2013, Chung et al. introduced some constructions of optimal LHZ FHS sets through Cartesian product [15]. In 2016, Han and Wang constructed sets of optimal LHZ FHSs with different parameters [16], [17]. In 2017, Zhou et al. presented new constructions of LHZ FHS sets via Cartesian product [18], and obtained two constructions of optimal LHZ FHS sets based on the decimated sequences of m-sequence [19]. In 2018, we presented a new of optimal LHZ FHSs with large family size [20].

Throughout this paper, we use $(N, v, \lambda; M)$ to denote an FHS set of M sequences of length N over a frequency slot set of size v, with the maximum Hamming correlation λ, use $(N, v, \lambda; M; Z)$ to denote an FHS set of M sequences of length N over a frequency slot set of size v with the

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TABLE 1. Comparison of parameters for some optimal LHZ FHS sets.

Sequence length	Alphabeta set size	Maximum Hamming correlation	Family size	LHZ	Constrains	Ref.
$s(q^n-1)$	q	$s(q^{n-1}-1)$	m	$w-1$	$q^n-1=wm$, $\gcd(s,q^n-1)=1$.	[12]
$s(p^n-1)$	$e+1$	sf	e	$w-1$	$\gcd(s,p^n-1)=1, w=\frac{p^n-1}{m}$, $m\mid(p^n-1), 1\le m<f$, $e+1>sf$, $sfe^2m<(fe^2-m)(e+1-sf)$.	[17]
sN	v	$s\lambda$	mM	$w-1$	$m=\lceil\frac{N}{w}\rceil, \gcd(s,N)=1$, $s=aw+1$, $a\ge1$, $s<mN$.	[13]
MN	v	$M\lambda$	m	$wM-1$	$m=\lceil\frac{N}{w}\rceil$.	[14]
MN	v	$M\lambda$	mw	$M-2$	$m=\lceil\frac{N}{w}\rceil, w>2M$.	[20]
q^n-1	q^k	q^{n-k}	$q^k(q-1)$	$w-1$	$w=\frac{q^n-1}{q-1}$, $1\le k\le n$.	[16]
$\frac{q^n-1}{s}$	q^k	$\frac{q^{n-k}-1}{s}$	m	$w-1$	$s\mid(q^n-1)$, $q^n-1=wm$, $\gcd(s,n)=1$, $1\le k\le n$.	[19]
$p^2(q-1)$	pq	p	pq	$\min\{p^2-1,q-2\}$	$2p<q-1$, $\gcd(p,q-1)=1$.	[15]
$p^2(p^2-1)$	p^2	$p(p-1)$	p	p^2-2	$\gcd(p^2,p^2-1)=1$	[18]
nN	v_2v	λ	nM	N	$M_C>m(X)$.	Cor. 2
nN	v_2v	λ	nM	Z	$M_C>m(Y)$.	Thm. 2

In Table 1, p is a prime, q is a prime power. The parameters in [13], [14], [20] and Coro. 2 are based on the original $(N,v,\lambda;M)$ FHS set X . Thm. 2 is based on the original $(N,v,\lambda;M;Z)$ LHZ FHS set Y and $(n,v_2;M_C)$ OC set. $m(X)$ denotes the maximum number of appearence of frequency slot in set X.

maximum Hamming correlation λ and low hit zone Z, and use $(n, v_2; M_C)$ to denote an OC sequence set of M sequences of length N over a frequency slot set of size v. We compare the parameters of the LHZ FHS sets in literature as summarized in Table 1.

In this paper, we present an extension interleaved structure of constructing optimal LHZ FHS sets based on Cartesian product. Under the structure, we obtain infinitely many new optimal LHZ FHS sets which increase the length and alphabet size of the original LHZ FHS set by using flexible extension factor. Compared with the existing results, our constructions are with large family size and more flexible parameters. As a result, our constructions allow a great flexibility of choosing parameters of LHZ FHS sets for a given quasi-synchronization FHMA communication system.

The rest of this paper is organized as follows. In Section 2, we give some preliminaries to FHSs. In Section 3, we present a general interleaved structure to construct optimal LHZ FHS sets with flexible parameters based on Cartesian product. In Section 4, we obtain the new classes of optimal LHZ FHS set in which all the sequences are shift distinct. Finally, we conclude the paper in Section 5.

II. PRELIMINARIES

Throughout this paper, the following symbols will be used:

$\mathbb{Z}_n$: the ring of integers modulo n for a positive integer $n > 1$;

$< x >_y$: the least nonnegative residue of x modulo y for an integer x and a positive integer y;

$\lceil z\rceil$: the smallest integer greater than or equal to z.

Let $F = \{f_1, f_2, \ldots, f_v\}$ be a frequency slot set with size $|F| = v$, X be a set of M FHSs of length N. For any two FHSs $\boldsymbol{x}_i = (x_i(0), x_i(1), \ldots, x_i(N-1))$, $\boldsymbol{x}_j = (x_j(0), x_j(1), \ldots, x_j(N-1)) \in X$, $0 \le i, j < M$, the Hamming correlation function $H_{\boldsymbol{x}_i\boldsymbol{x}_j}(\tau)$ of sequences $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$ at time delay τ is defined as follows:

$$H_{\boldsymbol{x}_i\boldsymbol{x}_j}(\tau) = \sum_{t=0}^{N-1} h\left(x_i(t), x_j(\langle t+\tau\rangle_N)\right), \quad 0 \le \tau < N, \quad (1)$$

where $h(a, b) = 1$ if $a = b$, and $h(a, b) = 0$ otherwise, and only positive time shifts are considered.

For any given FHS set X, the maximum Hamming auto-correlation $H_a(X)$, the maximum Hamming crosscorrelation $H_c(X)$ and the maximum Hamming correlation $H_m(X)$ are defined as follows, respectively:

$$H_a(X) = \max\left\{H_{\boldsymbol{x}_i\boldsymbol{x}_i}(\tau) : \boldsymbol{x}_i \in X, 0 < \tau < N\right\},$$
$$H_c(X) = \max\left\{H_{\boldsymbol{x}_i\boldsymbol{x}_j}(\tau) : \boldsymbol{x}_i, \boldsymbol{x}_j \in X, i \neq j, 0 \le \tau < N\right\},$$
$$H_m(X) = \max\{H_a(X), H_c(X)\}.$$

For simplicity, we denote $\lambda = H_m(X)$.

In 2004, Peng and Fan [10] established the following lower bound of an FHS set.

Lemma 1 (Peng-Fan bound): Let X be a set of M FHSs of length N over a frequency slot set with size v, we have

$$\lambda \geq \left\lceil \frac{(MN-v)N}{(MN-1)v} \right\rceil. \tag{2}$$

For any FHS set X, let integer $\lambda \geq 0$. Then the low hit zone Z, the low hit zone autocorrelation Z_a and the low hit zone crosscorrelation Z_c of X are defined as follows, respectively:

$$\begin{aligned} Z &= \min\{Z_a, Z_c\}, \\ Z_a &= \max\left\{T \,|H_{\boldsymbol{x}_i\boldsymbol{x}_i}(\tau) \leq \lambda : \boldsymbol{x}_i \in X, 0 < \tau \leq T\right\}, \\ Z_c &= \max\left\{T \,|H_{\boldsymbol{x}_i\boldsymbol{x}_j}(\tau) \leq \lambda : \boldsymbol{x}_i, \boldsymbol{x}_j \in X, i \neq j, 0 \leq \tau \leq T\right\}. \end{aligned}$$

Then the set X is said to be an FHS set with low hit zone Z.

In 2006, Peng *et al.* [11] established the following lower bound of an LHZ FHS set.

Lemma 2 (Peng-Fan-Lee bound): Let X be a set of M FHSs of length N over a frequency slot set with size v, and let Z be the low hit zone of X with respect to constants λ. Then for any position integer L_H, $0 \leq L_H \leq Z$, we have

$$\lambda \geq \left\lceil \frac{(ML_H+M-v)N}{(ML_H+M-1)v} \right\rceil. \tag{3}$$

If the maximum Hamming correlation λ is the minimum integer solution of inequality (3), then the corresponding FHS set X is called an optimal $(N, v, \lambda; M; Z)$ LHZ FHS set.

The one-coincidence (OC) sequence set [21] was firstly proposed by Shaar and Davies in 1984.

Definition 1 [21]: A one-coincidence sequence set is a set of nonrepeating sequences, for which the peak of the Hamming crosscorrelation function equals one for any pair of sequences belonging to the set.

Namely, the Hamming autocorrelation and crosscorrelation of the OC sequences are respectively 0 and 1.

Lemma 3: Let $X = \{\boldsymbol{x}_i = (x_i(0), x_i(1), \ldots, x_i(N-1)), 0 \leq i < M\}$ be a sequence set of M FHSs of length N over frequency slot set $F = \{f_1, f_2, \ldots, f_v\}$. For any $f_\alpha \in F$, $1 \leq \alpha \leq v$, the location of occurrences of f_α within $\boldsymbol{x}_i$ and X, denoted by $N_{\boldsymbol{x}_i}(f_\alpha)$ and $N_X(f_\alpha)$, can be written as

$$N_{\boldsymbol{x}_i}(f_\alpha) = \{(i, a) : x_i(a) = f_\alpha, 0 \leq a < N\},$$

$$N_X(f_\alpha) = \bigcup_{i=0}^{M-1} N_{\boldsymbol{x}_i}(f_\alpha).$$

Then, the maximum number of appearance of any frequency slot $f_\alpha \in F$ in set X, denoted by m(X),can be written as $m(X) = \max\{|N_X(f_\alpha)| : f_\alpha \in F, 1 \leq \alpha \leq v\}$.

III. AN EXTENSION INTERLEAVED STRUCTURE OF LHZ FHS SET

In this section, we give an extension interleaved structure of LHZ FHS set based on Cartesian product by combining some known OC sequences with optimal LHZ FHS sets.

Let $F_1 = \{f_{11}, f_{12}, \ldots, f_{1\,v_1}\}$ be a frequency slot set with size $|F_1| = v_1$. Choose a $(N, v_1, H_m(X); M_X; Z)$ LHZ FHS set $X = \{\boldsymbol{x}_i : 0 \leq i < M_X\}$ over F_1 with $\boldsymbol{x}_i = (x_i(0), x_i(1), \ldots, x_i(N-1))$. The maximum number of appearance of any frequency slot $f_\alpha \in F_1$ of set X denote by $m(X)$.

Let $F_2 = \{f_{21}, f_{22}, \ldots, f_{2\,v_2}\}$ be a frequency slot set with size $|F_2| = v_2$. Choose a $(n, v_2, H_m(C); M_C)$ FHS set $C = \{\boldsymbol{c}_i : 0 \leq i < M_C\}$ over F_2 with $\boldsymbol{c}_i = (c_i(0), c_i(1), \ldots, c_i(n-1))$ where $M_C > m(X)$.

For any i, k_1, $0 \leq i < M_X$, $0 \leq k_1 < n$, an $n \times N$ matrix is formed by combining the sequence $\boldsymbol{c}_{\omega_i(t_2)}$ with the FHS $\boldsymbol{x}_i$ as follows, $\boldsymbol{u}_i^{k_1}$, as shown at the bottom of this page, where for any $0 \leq t_2 < N$, $\omega_i(t_2)$ defines as follows:

$$\omega_i(t_2) = \sum_{j=0}^{i-1} |N_{\boldsymbol{x}_j}(x_i(t_2))| + |\{\theta : x_i(\theta) = x_i(t_2), 0 \leq \theta \leq t_2\}|.$$

The above definition can ensure $\omega_i(t_2) \neq \omega_i(t_2 + \tau_2)$ if $x_i(t_2) = x_i(t_2 + \tau_2)$ for any $0 \leq \tau_2 < N$. It is very important property to prove the correlation of extended sequences later.

By reading the elements in $\boldsymbol{u}_i^{k_1}$ row by row, we get an extended FHS $\boldsymbol{u}_i^{k_1} = (u_i^{k_1}(0), u_i^{k_1}(1), \ldots, u_i^{k_1}(nN-1))$ of period nN over $F = F_1 \times F_2$. Thus, we can obtain the extended LHZ FHS set $U = \{\boldsymbol{u}_i^{k_1} : 0 \leq i < M_X, 0 \leq k_1 < n\}$, where the set X is called the original LHZ FHS set.

$$\boldsymbol{u}_i^{k_1} = \begin{pmatrix} (c_{\omega_i(0)}(k_1), x_i(0)) & (c_{\omega_i(1)}(k_1), x_i(1)) & \cdots & (c_{\omega_i(N-1)}(k_1), x_i(N-1)) \\ \vdots & \vdots & \ddots & \vdots \\ (c_{\omega_i(0)}(n-1), x_i(0)) & (c_{\omega_i(1)}(n-1), x_i(1)) & \cdots & (c_{\omega_i(N-1)}(n-1), x_i(N-1)) \\ (c_{\omega_i(0)}(0), x_i(0)) & (c_{\omega_i(1)}(0), x_i(1)) & \cdots & (c_{\omega_i(N-1)}(0), x_i(N-1)) \\ \vdots & \vdots & \ddots & \vdots \\ (c_{\omega_i(0)}(k_1-1), x_i(0)) & (c_{\omega_i(1)}(k_1-1), x_i(1)) & \cdots & (c_{\omega_i(N-1)}(k_1-1), x_i(N-1)) \end{pmatrix}$$

$$= \begin{pmatrix} u_i^{k_1}(0) & u_i^{k_1}(1) & \cdots & u_i^{k_1}(N-1) \\ u_i^{k_1}(N) & u_i^{k_1}(N+1) & \cdots & u_i^{k_1}(2N-1) \\ \vdots & \vdots & \ddots & \vdots \\ u_i^{k_1}((n-1)N) & u_i^{k_1}((n-1)N+1) & \cdots & u_i^{k_1}(nN-1) \end{pmatrix}$$

For short, we write the extended FHS $u_i^{k_1}$ by using the interleaving operator I as

$$u_i^{k_1} = I\Big(\big(L^{(k_1)}(c_{\omega_i(0)}), x_i(0)\big), \ldots, \big(L^{(k_1)}(c_{\omega_i(N-1)}), x_i(N-1)\big)\Big),$$

where L is the (left cyclical) shift operator, i.e. $L^{(2)}(a_i) = (a_i(2), a_i(3), \ldots, a_i(N-1), a_i(0), a_i(1))$. By using the shift operator L and interleaving operator I, the matrix representation of $u_i^{k_1}$ can be abbreviated.

For any $j, k_2, 0 \le j < M_X, 0 \le k_2 < n$, another extended FHS $u_j^{k_2}$ can be generated by combining c_{ω_j} with x_j as

$$u_j^{k_2} = I\Big(\big(L^{(k_2)}(c_{\omega_j(0)}), x_j(0)\big), \ldots, \big(L^{(k_2)}(c_{\omega_j(N-1)}), x_j(N-1)\big)\Big).$$

Consider its cyclical shift version $L^{(\tau)}(u_j^{k_2})$, $\tau = N\tau_1 + \tau_2$, $0 \le \tau_1 < n, 0 \le \tau_2 < N$. Obviously, $L^{(\tau)}(u_j^{k_2})$ is just another extended FHS. Namely, we have

$$\begin{aligned} L^{(\tau)}(u_j^{k_2}) &= I\big(L^{(k_2+\tau_1)}(c_{\omega_j(\tau_2)}), x_j(\tau_2)), \ldots, (L^{(k_2+\tau_1)}(c_{\omega_j(N-1)}), \\ &\quad x_j(N-1)), (L^{(k_2+\tau_1+1)}(c_{\omega_j(0)}), x_j(0)), \ldots, \\ &\quad (L^{(k_2+\tau_1+1)}(c_{\omega_j(\tau_2-1)}), x_j(\tau_2-1))\big). \end{aligned}$$

Then, the Hamming correlation function $H_{u_i^{k_1}u_j^{k_2}}(\tau)$ between the extended LHZ FHSs $u_i^{k_1}$ and $u_j^{k_2}$ at time delay τ becomes the following from above formulas, i.e.,

$$\begin{aligned} &H_{u_i^{k_1}u_j^{k_2}}(\tau) \\ &= \sum_{t_2=0}^{N-\tau_2-1}\sum_{t_1=0}^{n-1} h\Big(\big(c_{\omega_i(t_2)}(k_1+t_1), x_i(t_2)\big), \\ &\quad \big(c_{\omega_j(t_2+\tau_2)}(k_2+t_1+\tau_1), x_j(t_2+\tau_2)\big)\Big) \\ &\quad + \sum_{t_2=N-\tau_2}^{N-1}\sum_{t_1=0}^{n-1} h\Big(\big(c_{\omega_i(t_2)}(k_1+t_1), x_i(t_2)\big), \\ &\quad \big(c_{\omega_j(t_2+\tau_2)}(k_2+t_1+\tau_1+1), x_j(t_2+\tau_2)\big)\Big) \\ &= \sum_{t_2=0}^{N-\tau_2-1}\sum_{t_1=0}^{n-1} h\big(c_{\omega_i(t_2)}(k_1+t_1), c_{\omega_j(t_2+\tau_2)}(k_2+t_1+\tau_1)\big) \\ &\quad \cdot h\big(x_i(t_2), x_j(t_2+\tau_2)\big) \\ &\quad + \sum_{t_2=N-\tau_2}^{N-1}\sum_{t_1=0}^{n-1} h\big(c_{\omega_i(t_2)}(k_1+t_1), c_{\omega_j(t_2+\tau_2)}(k_2+t_1+\tau_1+1)\big) \\ &\quad \cdot h\big(x_i(t_2), x_j(t_2+\tau_2)\big) \\ &= \sum_{t_2=0}^{N-\tau_2-1} H_{c_{\omega_i(t_2)}, c_{\omega_j(t_2+\tau_2)}}(k_2+\tau_1-k_1) \cdot h\big(x_i(t_2), x_j(t_2+\tau_2)\big) \\ &\quad + \sum_{t_2=N-\tau_2}^{N-1} H_{c_{\omega_i(t_2)}, c_{\omega_j(t_2+\tau_2)}}(k_2+\tau_1-k_1+1) \\ &\quad \cdot h\big(x_i(t_2), x_j(t_2+\tau_2)\big). \end{aligned} \tag{4}$$

Then, we will discuss the maximum Hamming correlation of the FHS set U. For convenience and simplicity, let $H_m(X)$ be the maximum Hamming correlation of FHS X within LHZ Z, and $H'_m(X)$ denoted the maximum Hamming correlation of FHS X outside the low hit zone. Obviously, $H_m(X) \le H'_m(X)$, and $H'_m(X) \ne N$ where N is the length of FHS X.

Theorem 1: We can construct the LHZ FHS set U by combining the $(N, v_1, H_m(X); M_X; Z)$ LHZ FHS set X with a $(n, v_2, H_m(C); M_C)$ FHS set C. For any LHZ FHS set $u_i^{k_1}$ and $u_j^{k_2} \in U$, $0 \le i, j < M_X$, $0 \le k_1, k_2 < n$, the Hamming correlation between $u_i^{k_1}$ and $u_j^{k_2}$ at time delay τ, $\tau = N\tau_1 + \tau_2$, $0 \le \tau_1 < n$, $0 \le \tau_2 < N$ can be discussed in the following cases:

1) If $0 \le \tau < Z$, that is $\tau_1 = 0$, $0 \le \tau_2 < Z$, then we have

$$H_{u_i^{k_1}u_j^{k_2}}(\tau) \le \begin{cases} nN, & i=j, k_1=k_2, \tau_2=0, \\ NH_a(C), & i=j, k_1\ne k_2, \tau_2=0, \\ H_c(C)\cdot H_m(X), & i=j, 0<\tau<Z, \\ & i\ne j, 0\le\tau<Z. \end{cases}$$

2) If $Z \le \tau < nN$, that is $0 \le \tau_1 < n$, $0 \le \tau_2 < N$, then we have

$$H_{u_i^{k_1}u_j^{k_2}}(\tau) \le \begin{cases} NH_a(C), & i=j, \tau_1 \not\equiv k_1-k_2 \bmod n, \\ & \tau_2=0, \\ nN, & i=j, \tau_1 \equiv k_1-k_2 \bmod n, \\ & \tau_2=0, \\ H_c(C)\cdot H_m(X), & i=j, 1\le\tau_2<Z, \\ & i\ne j, 0\le\tau_2<Z, \\ H_c(C)\cdot H'_m(X), & Z\le\tau_2<N. \end{cases}$$

Proof: Let $0 \le t_1 < n$, $0 \le t_2 < N$, $\tau = N\tau_1 + \tau_2$, where $0 \le \tau_1 < n$, $0 \le \tau_2 < N$.

In order to discuss $H_{u_i^{k_1}u_j^{k_2}}(\tau)$, we divide the problem in following cases:

Case 1). $i = j$, $\tau_2 = 0$.

In this case, it is obviously that the Hamming autocorrelation function of x_i is $h(x_i(t_2), x_i(t_2)) = 1$ for any t_2, $0 \le t_2 < N$. Thus, we can write the Hamming autocorrelation of U from equation (4) as

$$H_{u_i^{k_1}u_i^{k_2}}(\tau) = \sum_{t_2=0}^{N-1} H_{c_{\omega_i(t_2)}, c_{\omega_i(t_2)}}(k_2+\tau_1-k_1)\cdot 1.$$

Case1.1). If $\tau_1 \equiv k_1 - k_2 \mod n$, then we have

$$H_{u_i^{k_1}u_i^{k_2}}(\tau) \le \sum_{t_2=0}^{N-1} H_{c_{\omega_i(t_2)}, c_{\omega_i(t_2)}}(0)\cdot 1 = nN. \tag{5}$$

Case1.2). If $\tau_1 \not\equiv k_1 - k_2 \mod n$, we can get the Hamming correlation between $u_i^{k_1}$ and $u_i^{k_2}$ at shif τ is

$$H_{u_i^{k_1}u_i^{k_2}}(\tau) \le NH_a(C). \tag{6}$$

Case 2). $i = j$, $\tau_2 \ne 0$.

In this case, we can get the Hamming correlation of U from equation (4) as

$$H_{\boldsymbol{u}_i^{k_1}\boldsymbol{u}_i^{k_2}}(\tau) = \sum_{t_2=0}^{N-\tau_2-1} H_{\boldsymbol{c}_{\omega_i(t_2)},\boldsymbol{c}_{\omega_i(t_2+\tau_2)}}(k_2+\tau_1-k_1)\cdot h\left(x_i(t_2), x_i(t_2+\tau_2)\right) + \sum_{t_2=N-\tau_2}^{N-1} H_{\boldsymbol{c}_{\omega_i(t_2)},\boldsymbol{c}_{\omega_i(t_2+\tau_2)}}(k_2+\tau_1-k_1+1) \cdot h\left(x_i(t_2), x_i(t_2+\tau_2)\right).$$

According to the definition of ω, we have $\omega_i(t_2) \neq \omega_i(t_2+\tau_2)$ if $x_i(t_2) = x_i(t_2+\tau_2)$. Then, we have

Case 2.1). If $0 \le \tau_1 < n$, $1 \le \tau_2 < Z$, we can obtain that

$$H_{\boldsymbol{u}_i^{k_1}\boldsymbol{u}_i^{k_2}}(\tau) \le H_c(C)\cdot H_a(X). \tag{7}$$

Case 2.2).If $0 \le \tau_1 < n$, $Z \le \tau_2 < N$, we can obtain

$$H_{\boldsymbol{u}_i^{k_1}\boldsymbol{u}_i^{k_2}}(\tau) \le H_c(C)\cdot H'_a(X). \tag{8}$$

Case 3). $i \neq j$.

In this case, we can get the Hamming correlation of U from equation (4) as

$$H_{\boldsymbol{u}_i^{k_1}\boldsymbol{u}_j^{k_2}}(\tau) = \sum_{t_2=0}^{N-1} H_c(C)\cdot h\left(x_i(t_2), x_j(t_2+\tau_2)\right). \tag{9}$$

The same as case 2), based on the definition of ω, we have $\omega_i(t_2) \neq \omega_j(t_2+\tau_2)$ if $x_i(t_2) = x_j(t_2+\tau_2)$. Then, we have

Case 3.1). If $0 \le \tau_1 < n$, $0 \le \tau_2 < Z$, we can obtain that

$$H_{\boldsymbol{u}_i^{k_1}\boldsymbol{u}_j^{k_2}}(\tau) \le H_c(C)\cdot H_c(X). \tag{10}$$

Case 3.2). If $0 \le \tau_1 < n$, $Z \le \tau_2 < N$, we can get

$$H_{\boldsymbol{u}_i^{k_1}\boldsymbol{u}_j^{k_2}}(\tau) \le H_c(C)\cdot H'_c(X). \tag{11}$$

Thus, the Hamming correlation of U at time delay τ, $0 \le \tau < nN$ have

$$H_{\boldsymbol{u}_i^{k_1}\boldsymbol{u}_j^{k_2}}(\tau) \le \begin{cases} nN, & i=j, \tau_1 \equiv k_1-k_2 \mod n, \tau_2 = 0,\\ NH_a(C), & i=j, \tau_1 \not\equiv k_1-k_2 \mod n, \tau_2=0,\\ H_c(C)\cdot H_m(X), & i=j, 0\le\tau_1<n, 1\le\tau_2<Z,\\ & i\neq j, 0\le\tau_1<n, 0\le\tau_2<Z,\\ H_c(C)\cdot H'_m(X), & i=j, 0\le\tau_1<n, Z\le\tau_2<N,\\ & i\neq j, 0\le\tau_1<n, Z\le\tau_2<N. \end{cases}$$

Based on the above, we can get the following results:

1) If $0 \le \tau < Z$, that is $\tau_1 = 0$, $0 \le \tau_2 < Z$, then we have

$$H_{\boldsymbol{u}_i^{k_1}\boldsymbol{u}_j^{k_2}}(\tau) \le \begin{cases} nN, & i=j, k_1=k_2, \tau_2=0,\\ NH_a(C), & i=j, k_1\neq k_2, \tau_2=0,\\ H_c(C)\cdot H_m(X), & i=j, 0<\tau<Z,\\ & i\neq j, 0\le\tau<Z. \end{cases}$$

2) If $Z \le \tau < N$, that is $\tau_1 = 0$, $Z \le \tau_2 < N$, then we have

$$H_{\boldsymbol{u}_i^{k_1}\boldsymbol{u}_j^{k_2}}(\tau) \le H_c(C)\cdot H'_m(X).$$

3) If $N \le \tau < nN$, that is $1 \le \tau_1 < n$, $0 \le \tau_2 < N$, then we have

$$H_{\boldsymbol{u}_i^{k_1}\boldsymbol{u}_j^{k_2}}(\tau) \le \begin{cases} NH_a(C), & i=j, \tau_1 \not\equiv k_1-k_2 \mod n, \tau_2=0,\\ nN, & i=j, \tau_1 \equiv k_1-k_2 \mod n, \tau_2=0,\\ H_c(C)\cdot H_m(X), & i=j, 1\le\tau_2<Z,\\ & i\neq j, 0\le\tau_2<Z,\\ H_c(C)\cdot H'_m(X), & Z\le\tau_2<N. \end{cases}$$

In summary, this complete the proof of the Theorem 1. □

Corollary 1: We can construct the LHZ FHS set U by combining the optimal $(N, v_1, H_m(X); M_X; Z)$ *LHZ FHS set X with a* $(n, v_2; M_C)$ *OC FHS set C. The Hamming correlation between* $\boldsymbol{u}_i^{k_1}$ *and* $\boldsymbol{u}_j^{k_2}$ *at time delay* τ *is*

1) If $0 \le \tau < Z$*, that is* $\tau_1 = 0$, $0 \le \tau_2 < Z$*, then we have*

$$H_{\boldsymbol{u}_i^{k_1}\boldsymbol{u}_j^{k_2}}(\tau) \le \begin{cases} nN, & i=j, k_1=k_2, \tau_2=0,\\ 0, & i=j, k_1\neq k_2, \tau_2=0,\\ H_m(X), & i=j, 0<\tau<Z,\\ & i\neq j, 0\le\tau<Z. \end{cases}$$

2) If $Z \le \tau < nN$*, that is* $0 \le \tau_1 < n$, $0 \le \tau_2 < N$*, then we have*

$$H_{\boldsymbol{u}_i^{k_1}\boldsymbol{u}_j^{k_2}}(\tau) \le \begin{cases} 0, & i=j, \tau_1 \not\equiv k_1-k_2 \mod n, \tau_2=0,\\ nN, & i=j, \tau_1 \equiv k_1-k_2 \mod n, \tau_2=0,\\ H_m(X), & i=j, 1\le\tau_2<Z,\\ & i\neq j, 0\le\tau_2<Z,\\ H'_m(X), & Z\le\tau_2<N. \end{cases}$$

The design of FHSs with LHZ aims at making the number of hits equal to a very low value within the LHZ. However, in order to avoid the mutual interference increasing suddenly, the number of hits should also be kept as low as possible when the relative delays outside the LHZ. Thus, in the practical system, it is very desirable that the LHZ FHS set with good Hamming correlation in the LHZ and outside the LHZ. In the above construction, some sequences in the proposed LHZ FHS set U are the cyclical shift of the others. If the sequences are the cyclically shift of each other, the number of hits will equal to the length nN of sequence (full collision) at some time slots outside the LHZ.

The sequence $\boldsymbol{u}_i^{k_1}$ is the cyclical shift of $\boldsymbol{u}_j^{k_2}$ only if $i = j$ and $\tau_1 \equiv k_1 - k_2 \mod n$ with $0 \le i, j < M_X$, $0 \le k_1, k_2 < n$. That is because the any $L^{(k_\delta)}\left(\boldsymbol{c}_{\omega_i}\right)$, $0 \le \delta < n$ column sequences in the interleaved structure use the same shift k_δ. A feasible way is to get a suitable shift values to ensure that the column sequences of U are not the cyclic shift of each other.

The definition of the inequivalent shift sequences [13] is given as follows.

Definition 2: Any two shift sequences $\boldsymbol{a} = (a(0), a(1), \ldots, a(l-1))$, $\boldsymbol{b} = (b(0), b(1), \ldots, b(l-1))$ *over* $\mathbb{Z}_n$ *are said to be* *inequavalent if $b(i)-a(i)=b(j)-a(j)$, for all $0\le i\ne j<l$ does not hold.*

Without loss of generality, we can use shift sequences $\boldsymbol{e}=(e(0),e(1),\ldots,e(N-1))$ and $\boldsymbol{g}=(g(0),g(1),\ldots,g(N-1))$ to express the shift value. Therefore, we can obtain the cyclically distinct LHZ FHS set U by using the inequivalent shift sequences.

IV. CONSTRUCTION OF CYCLICALLY DISTINCT LHZ FHS SETS

In this section, we present optimal LHZ FHS set based on the extension interleaved structure in Section 3. Moreover, all the sequences in our new set are cyclically distinct.

Let $F_1=\{f_{11},f_{12},\ldots,f_{1\,v_1}\}$ be a frequency slot set with size $|F_1|=v_1$, and $F_2=\{f_{21},f_{22},\ldots,f_{2\,v_2}\}$ be a frequency slot set with size $|F_2|=v_2$. Our procedure of the extension construction is described as follows.

Construction 1: Construction of Cyclically Distinct LHZ FHS Sets with length nN.

Step 1: *Select an optimal $(N,v_1,H_m(X);M_X;Z)$ LHZ FHS set X over F_1, we have*

$$X=\{\boldsymbol{x}_i=(x_i(0),x_i(1),\ldots,x_i(N-1)):0\le i<M_X\}.$$

The maximum number of appearance of frequency slot in X is $m(X)$.

Step 2: *Select a $(n,v_2;M_C)$ OC sequence set C over F_2, satisfy $M_C>m(X)$. For any $0\le t_2<N$, we have*

$$\omega_i(t_2)=\sum_{j=0}^{i-1}|N_{\boldsymbol{x}_j}(x_i(t_2))| +|\{\theta:x_i(\theta)=x_i(t_2),0\le\theta\le t_2\}|.$$

Step 3: *For $0\le i<M_X$, generate a set $G_i=\{\boldsymbol{g}_i^k:0\le k<n\}$ with*

$$\boldsymbol{g}_i^k=\left(g_i^k(0),g_i^k(1),\ldots,g_i^k(l_i-1)\right)=\left(P_0(k),P_1(k),\ldots,P_{l_i-1}(k)\right)$$

where $\{P_0,P_1,\ldots,P_{l_i-1}\}$ are permutations over $\mathbb{Z}_n$, such that all the sequences in G_i are pairwise inequivalent in literature [13], and l_i can be written as

$$l_i=\max\{\omega_i(t_2):0\le t_2<N\}-\min\{\omega_i(t_2):0\le t_2<N\}+1.$$

Step 4: *For $0\le i<M_X$, we can obtain a shift sequence set $E_i=\{\boldsymbol{e}_i^k:0\le k<n\}$ over $\mathbb{Z}_n$ with*

$$\boldsymbol{e}_i^k=\left(e_i^k(0),e_i^k(1),\ldots,e_i^k(N-1)\right)=\left(g_i^k(\eta_i(0)),g_i^k(\eta_i(1)),\ldots,g_i^k(\eta_i(N-1))\right)$$

where $\eta_i(0)=0$, and for any j, $1\le j<N$, the function $\eta_i(j)$ denotes as follows:

$$\eta_i(j)=\begin{cases}\eta_i(j-\theta), & \omega_i(j)=\omega_i(j-\theta),0\le\theta<j,\\ \max\{\eta_i(\theta)\}+1, & \omega_i(j)\ne\omega_i(j-\theta),0\le\theta<j.\end{cases}$$

Step 5: *We can construct the desired LHZ FHS set $U=\{\boldsymbol{u}_i^k:0\le i<M_X,0\le k<n\}$. Then, we have*

$$\boldsymbol{u}_i^k=I\left(\left(L^{(e_i^k(0))}(\boldsymbol{c}_{\omega_i(0)}),x_i(0)\right),\ldots,\left(L^{(e_i^k(N-1))}(\boldsymbol{c}_{\omega_i(N-1)}),x_i(N-1)\right)\right).$$

Theorem 2: The proposed LHZ FHS set U constructed by Construction 1 is an optimal $(nN,v_1v_2,H_m(X);nM_X;Z)$ LHZ FHS set if $\left\lceil\frac{(M_XZ+M_X-v_1)N}{(M_XZ+M_X-1)v_1}\right\rceil=\left\lceil\frac{(nM_XZ+nM_X-v_1v_2)nN}{(nM_XZ+nM_X-1)v_1v_2}\right\rceil$.

Proof: From the Corollary 1, it is easily checked that the proposed FHS set U in Construction 1 is a LHZ FHS set with sequence length nN, frequency slot set size v_1v_2, and the maximum Hamming correlation $H_m(X)$ within the LHZ Z. Then we will proof the Hamming correlation outside the LHZ is less than nN.

Let $0\le t_1<n,0\le t_2<N,0\le k_1,k_2<n,\tau=N\tau_1+\tau_2$, where $0\le\tau_1<n,0\le\tau_2<N$. The case 1.1) in the proof of Theorem 1 evolved into the following cases:

Case1.1). $i=j,\tau_2=0,\tau_1\ne0$, and $\tau_1\equiv k_1-k_2\mod n$.

For any ε, $0<\varepsilon<$ N, ε denotes the number of columns in which the full collision occurs between the result sequence matrices. Then, we can get the Hamming correlation function of U as

$$H_{\boldsymbol{u}_i^{k_1}\boldsymbol{u}_i^{k_2}}(\tau)\le H_{\boldsymbol{u}_i^{k_1}\boldsymbol{u}_i^{k_2}}(\tau)\le\sum_{\sigma=0}^{\varepsilon-1}H_{\boldsymbol{c}_{\omega_i(t_2)}\boldsymbol{c}_{\omega_i(t_2)}}(0)\cdot1 +\sum_{\sigma=0}^{N-\varepsilon-1}H_{\boldsymbol{c}_{\omega_i(t_2)}\boldsymbol{c}_{\omega_i(t_2)}}(e_i^{k_2}(t_2)-e_i^{k_1}(t_2)+\tau_1)\cdot1.$$

According to the inequivalent nature of the shift sequence $\boldsymbol{e}_i^{k_1}$ and the shift sequence $\boldsymbol{e}_i^{k_2}$, we can get $e_i^{k_2}(t_2)-e_i^{k_1}(t_2)+\tau_1\ne0\mod n$. Therefore, the Hamming correlation function of U is

$$H_{\boldsymbol{u}_i^{k_1}\boldsymbol{u}_i^{k_2}}(\tau)\le\varepsilon n+H_a(C)\cdot(N-\varepsilon)=\varepsilon n. \tag{12}$$

Case1.2). $i=j,\tau_2=0,\tau_1\not\equiv k_1-k_2\mod n$.

We can get the Hamming correlation between $\boldsymbol{u}_i^{k_1}$ and $\boldsymbol{u}_i^{k_2}$ at shif τ is the same as case 1.2) in the proof of Theorem 1.

According to the above situation and combining with the proof of Theorem 1, we can get the Hamming correlation value of the LHZ FHS set U at time delay $Z\le\tau<nN$, $\tau=N\tau_1+\tau_2,0\le\tau_1<n,0\le\tau_2<Z$ is

$$H_{\boldsymbol{u}_i^{k_1}\boldsymbol{u}_i^{k_2}}(\tau)\le\begin{cases}\varepsilon n, & i=j,\tau_1\equiv k_1-k_2\mod n,\tau_2=0,\\ 0, & i=j,\tau_1\not\equiv k_1-k_2\mod n,\tau_2=0,\\ H_m(X), & i=j,0<\tau<Z,\\ & i\ne j,0\le\tau<Z.\end{cases}$$

Thus, we can obtain the proposed LHZ FHS set U with size nM_X by using the inequivalent shift sequence set G_i with size n in [13]. The maximum Hamming correlation within low hit zone Z is $H_m(U)=H_m(X)$, and the FHSs of the set U will not full hit outside the LHZ.

Then, we will prove the optimality of the LHZ FHS set U. The Hamming correlation of LHZ FHS set U within LHZ is

TABLE 2. The Extended LHZ FHS Set from an Original $(N, v, \lambda; M; Z)$-LHZ FHS Set X by Construction 1.

Extended LHZ FHS sets	Based on OC sets	Constrains	Ref.
$((q-1)N, qv, \lambda; (q-1)M; Z)$	$(q-1, q; q)$ [21]	$q > m(X)$.	Cor.3
$(pN, pv, \lambda; pM; Z)$	$(p, p; p-1)$ [21], [25]	$p > m(X)+1$.	Cor.4
$((n-2d-1)N, nv, \lambda; (n-2d-1)M; Z)$	$(n-2d-1, n; n)$ [22]	$2 \nmid n,\ n > m(X),\ 0 < d \le \frac{\sqrt{4n+1}-1}{2}$.	Cor.5
$(lN, qv, \lambda; lM; Z)$	$\left(l, q; q^{\frac{q-1}{l}}\right)$ [23]	$q = el+1, l > 2,\ e > 0, q^{\frac{q-1}{l}} > m(X)$.	Cor.6
$((n-1)n_1N, nn_1v, \lambda; ((n-1)n_1)M; Z)$	$((n-1)n_1, nn_1; n)$ [24]	$n_1 - 1 > n > m(X), \gcd(n-1, n_1) = 1$.	Cor.7
$(pq_1N, pq_1v, \lambda; pq_1M; Z)$	$(pq_1, pq_1; p-1)$ [26]	$p_1 > p > m(X)+1$.	Cor.8

In Table 2, q is the power of odd pirme p, q_1 is the power of odd pirme p_1. $m(X)$ denotes the maximum number of appearance of frequency slot in original set X.

the same with the original LHZ FHS set X. As the original set X is an optimal $(N, v_1, H_m(X); M; Z)$ LHZ FHS set that satisfy Peng-Fan-Lee bound, we can obtain that the Hamming correlation of purposed FHS set U within LHZ Z is

$$H_m(U) = H_m(X) = \left\lceil \frac{(M_X Z + M_X - v_1)N}{(M_X Z + M_X - 1)v_1} \right\rceil.$$

According to the Peng-Fan-Lee bound, the optimal Hamming correlation of FHS set U should be

$$H_{mo}(U) = \left\lceil \frac{(nM_X Z + nM_X - v_1 v_2)nN}{(nM_X Z + nM_X - 1)v_1 v_2} \right\rceil.$$

Therefore, the Hamming correlation $H_m(U) = H_m(X) = H_{mo}(U)$ of LHZ FHS set U within LHZ Z is optimal if

$$\left\lceil \frac{(nM_X Z + nM_X - v_1 v_2)nN}{(nM_X Z + nM_X - 1)v_1 v_2} \right\rceil = \left\lceil \frac{(M_X Z + M_X - v_1)N}{(M_X Z + M_X - 1)v_1} \right\rceil.$$

In summary, the extended LHZ FHS set U is an optimal $(nN, v_1 v_2, H_m(X); nM_X; Z)$ LHZ FHS set. □

If the original set X is a $(N, v_1, H_m(X); M_X)$ FHS set, we can obtain the following corollary.

Corollary 2: Choose the $(N, v_1, H_m(X); M_X)$ FHS set as original set in Construction 1, we can get an optimal $(nN, v_1 v_2, H_m(X); nM_X; N)$ LHZ FHS set U.

Remark 1: It should be noted that our above construction remove the constraint requiring that the extension factor is co-prime with the length of original FHSs. The extension constructions in [12], [13], [15], [17], [18] all suffer from this constraint. Moreover, the LHZ FHS set constructed by Construction 1 is optimal and cyclically distinct.

By using the inequivalent shift set, LHZ FHS set can keep excellent properties both in and outside the LHZ. The relaxation of conditions provides more flexibility for the choice of shift sequences and further leads to more parameters of LHZ FHS sets.

Based on the above construction, we can obtain infinitely many optimal LHZ FHS set with large family size by choosing any optimal $(N, v, \lambda; M; Z)$ LHZ FHS set X and OC sequence sets C with different parameters. Let the maximum number of appearance of any frequency slot in set X is $m(X)$. Then we can obtain the following corollaries by choosing some OC sequence sets in literature [16]–[22]. In the following corollaries, let p be an odd prime, and q be the power of p.

Corollary 3: Choose a $(q-1, q; q)$ OC sequence set C_1 in [21] where $q > m(X)$. We can obtain an optimal $((q-1)N, qv, \lambda; (q-1)M; Z)$ LHZ FHS set over $\mathbb{Z}_q \times F_1$ if $\left\lceil \frac{((q-1)MZ+(q-1)M-qv)(q-1)N}{((q-1)MZ+(q-1)M-1)qv} \right\rceil = \left\lceil \frac{(MZ+M-v)N}{(MZ+M-1)v} \right\rceil$.

Corollary 4: Choose a $(p, p; p-1)$ OC sequence set C_2 in [21], [25] where $p > m(X)+1$. We can obtain an optimal $(pN, pv, \lambda; pM; Z)$ LHZ FHS set over $\mathbb{Z}_p \times F_1$, if $\left\lceil \frac{(MZ+M-v)pN}{(pMZ+pM-1)v} \right\rceil = \left\lceil \frac{(MZ+M-v)N}{(MZ+M-1)v} \right\rceil$.

Corollary 5: Choose a $(n-2d-1, n; n)$ OC sequence set C_3 in [22] where n, d are positive integers with $2 \nmid n$, $n > m(X)$ and $d \le \frac{\sqrt{4n+1}-1}{2}$. We can obtain an optimal $((n-2d-1)N, nv, \lambda; (n-2d-1)M; Z)$ LHZ FHS set over $\mathbb{Z}_n \times F_1$, if $\left\lceil \frac{((n-2d-1)MZ+(n-2d-1)M-nv)(n-2d-1)N}{((n-2d-1)MZ+(n-2d-1)M-1)nv} \right\rceil = \left\lceil \frac{(MZ+M-v)N}{(MZ+M-1)v} \right\rceil$.

Corollary 6: Choose a $(l, q; q^{\frac{q-1}{l}})$ OC sequence set C_4 in [23] where $q^{\frac{q-1}{l}} > m(X)$, $q = el+1$ for any $0 < e$, $2 < l$. We can obtain an optimal $(lN, qv, \lambda; lM; Z)$ LHZ FHS set over $\mathbb{Z}_q \times F_1$, if $\left\lceil \frac{(lMZ+lM-qv)lN}{(lMZ+lM-1)qv} \right\rceil = \left\lceil \frac{(MZ+M-v)N}{(MZ+M-1)v} \right\rceil$.

Corollary 7: Choose a $((n-1)n_1, nn_1; n)$ OC sequence set C_5 in [24] where $n_1 - 1 > n > m(X)$ for positive integers n and n_1 with $\gcd(n-1, n_1) = 1$. We can obtain an optimal $((n-1)n_1N, nn_1v, \lambda; (n-1)n_1M; Z)$ LHZ FHS set over the alphabet of size nn_1v, if $\left\lceil \frac{((n-1)MZ+(n-1)M-nv)(n-1)n_1N}{((n-1)n_1MZ+(n-1)n_1M-1)nv} \right\rceil = \left\lceil \frac{(MZ+M-v)N}{(MZ+M-1)v} \right\rceil$.

Corollary 8: Choose a $(pq_1, pq_1; p-1)$ OC sequence set C_6 in [26] where $p > m(X)+1$, p_1 is odd prime with $p_1 > p$, q_1 is the power of p_1. We can obtain an optimal $(pq_1N, pq_1v, \lambda; pq_1M; Z)$ LHZ FHS set over the alphabet of size pq_1v, if $\left\lceil \frac{(MZ+M-v)pq_1N}{(pq_1MZ+pq_1M-1)v} \right\rceil = \left\lceil \frac{(MZ+M-v)N}{(MZ+M-1)v} \right\rceil$.

Remark 2: Suppose the original LHZ FHS set X with parameters $(N, v, \lambda; M; Z)$, we can obtain infinitely optimal extended LHZ FHS sets with new parameters by using

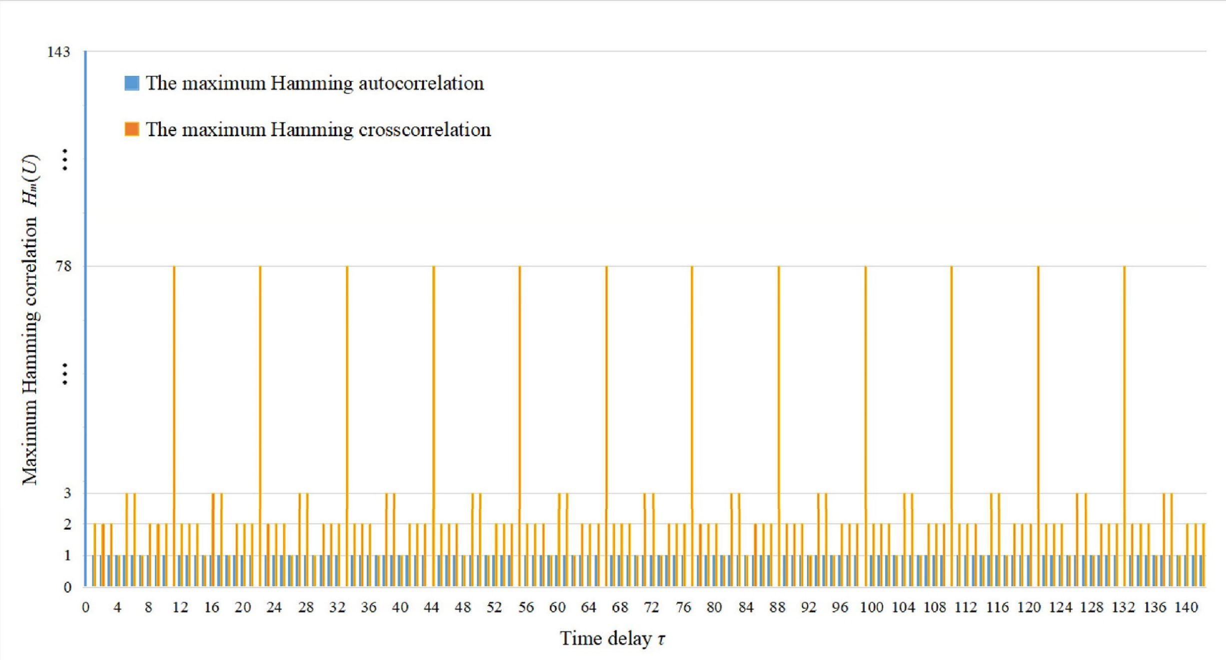

FIGURE 1. The maximum Hamming correlation $H_m(U) = 2$ of U within low hit zone $Z = 6$.

different OC sequence set. We list some parameters of the new LHZ FHS sets in Table 2. By choosing the optimal $(N, v, \lambda; M; Z)$ *LHZ FHS set* X *with specific parameters, we can obtain infinitely many new optimal LHZ FHS sets with flexible parameters, we omit it here.*

We now illustrate the Construction 1 by the following example.

Example 1:

Step 1: We select an optimal $(11, 6, 2; 3; 6)$ *LHZ FHS set* $X = \{\boldsymbol{x}_i : 0 \le i < 3\}$ *over* $\mathbb{Z}_6$*, such that*

$$\boldsymbol{x}_0 = (5, 4, 1, 3, 2, 0, 0, 2, 3, 1, 4),$$
$$\boldsymbol{x}_1 = (3, 2, 5, 1, 0, 4, 4, 0, 1, 5, 2),$$
$$\boldsymbol{x}_2 = (1, 0, 3, 5, 4, 2, 2, 4, 5, 3, 0).$$

So, we have $m(X) = 6$.

Step 2: Select a $(13, 13; 12)$ *OC sequence set of* $C = \{\boldsymbol{c}_j : 0 \le j < 12\}$ *over* $\mathbb{Z}_{13}$*, where*

$$\boldsymbol{c}_0 = (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12),$$
$$\boldsymbol{c}_1 = (0, 2, 4, 6, 8, 10, 12, 1, 3, 5, 7, 9, 11),$$
$$\vdots$$
$$\boldsymbol{c}_{11} = (0, 7, 1, 8, 2, 9, 3, 10, 4, 11, 5, 12, 6).$$

For any $0 \le i < 3, 0 \le t_2 < 11$*, a special expression of* $\omega = \{\omega_i : 0 \le i < 3\}$ *can be given by*

$$\omega_0 = (1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2),$$
$$\omega_1 = (3, 3, 2, 3, 3, 3, 4, 4, 4, 3, 4),$$
$$\omega_2 = (5, 5, 4, 4, 5, 5, 6, 6, 5, 5, 6).$$

And $l_0 = 2$, $l_1 = 3$, $l_2 = 3$.

Step 3: Choose a sequence set of $G_0 = \{\boldsymbol{g}_0^k = (g_0^k(0), g_0^k(1)) : 0 \le k < 13\}$*, we have*

$$\boldsymbol{g}_0^0 = (0, 0), \quad \boldsymbol{g}_0^1 = (1, 2), \quad \ldots, \quad \boldsymbol{g}_0^{12} = (12, 11).$$

For any $1 \le i < 3$*, choose a set of* $G_i = \{\boldsymbol{g}_i^k = (g_i^k(0), g_i^k(1), g_i^k(2) : 0 \le k < 13\}$ *as follows:*

$$\boldsymbol{g}_i^0 = (0, 0, 0), \quad \boldsymbol{g}_i^1 = (1, 2, 3), \quad \ldots, \quad \boldsymbol{g}_i^{12} = (12, 11, 10).$$

Step 4: For any $0 \le i < 3$*, a shift sequence set of* $E_i = \{\boldsymbol{e}_i^k = (e_i^k(0), e_i^k(1), \ldots, e_i^k(10)) : 0 \le k < 13\}$ *can be given by*

$$\boldsymbol{e}_0^0 = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0),$$
$$\boldsymbol{e}_0^1 = (1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2),$$
$$\vdots$$
$$\boldsymbol{e}_0^{12} = (12, 12, 12, 12, 12, 12, 11, 11, 11, 11, 11),$$
$$\boldsymbol{e}_1^0 = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0),$$
$$\vdots$$
$$\boldsymbol{e}_1^{12} = (12, 12, 11, 12, 12, 12, 10, 10, 10, 12, 10),$$
$$\boldsymbol{e}_2^0 = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0),$$
$$\vdots$$
$$\boldsymbol{e}_2^{12} = (12, 12, 11, 11, 12, 12, 10, 10, 12, 12, 10).$$

Step 5: We can construct the LHZ FHS set $U = \{\boldsymbol{u}_i^k : 0 \le i < 3, 0 \le k < 13\}$ *by the Construction 1.*

$$\boldsymbol{u}_0^0 = ((0, 5), (0, 4), (0, 1), (0, 3), (0, 2), (0, 0), (0, 0), (0, 2), (0, 3), \ldots,$$

$$(12,3),(12,2),(12,0),(11,0),(11,2),(11,3),$$
$$(11,1),(11,4)),$$
$$\vdots$$
$$\boldsymbol{u}_0^{12} = ((12,5),(12,4),(12,1),(12,3),(12,2),(12,0),$$
$$(9,0),(9,2),\ldots,$$
$$(11,4),(11,1),(11,3),(11,2),(11,0),(7,0),$$
$$(7,2),(7,3),(7,1)),$$
$$\boldsymbol{u}_1^{0} = ((0,3),(0,2),(0,5),(0,1),(0,0),(0,4),(0,4),$$
$$(0,0),(0,1),\ldots,$$
$$(9,2),(11,5),(9,1),(9,0),(9,4),(5,4),(5,0),$$
$$(5,1),(9,5),(5,2)),$$
$$\vdots$$
$$\boldsymbol{u}_1^{12} = ((9,3),(9,2),(9,5),(9,1),(9,0),(9,4),(2,4),$$
$$(2,0),(2,1),\ldots,$$
$$(5,2),(7,5),(5,1),(5,0),(5,4),(7,4),(7,0),$$
$$(7,1),(5,5),(7,2)),$$
$$\boldsymbol{u}_2^{0} = ((0,1),(0,0),(0,3),(0,5),(0,4),(0,2),(0,2),$$
$$(0,4),(0,5),\ldots,$$
$$(5,3),(5,5),(10,4),(10,2),(7,2),(7,4),$$
$$(10,5),(10,3),(7,0)),$$
$$\vdots$$
$$\boldsymbol{u}_2^{12} = ((10,1),(10,0),(10,3),(10,5),(10,4),$$
$$(10,2),(8,2),(8,4),\ldots,$$
$$(7,0),(2,3),(2,5),(7,4),(7,2),(2,2),(2,4),$$
$$(7,5),(7,3),(2,0)).$$

Then, the maximum Hamming autocorrelation and cross-correlation of U can be seen in Figure 1.

It can be verified that the maximum Hamming correlation $H_m(U) = 2$ for the time delay τ in the low hit zone $Z = 6$. Thus, the set U is an optimal $(143, 78, 2; 39; 6)$ *LHZ FHS set, and all the FHSs in U are cyclically distinct.*

V. CONCLUSIONS

In this paper, we present a general interleaved structure of constructing optimal LHZ FHS sets based on Cartesian product. Under the structure, we obtain infinitely many new optimal LHZ FHS sets by combining optimal LHZ FHS sets with some OC FHS sets. Compared with previous extension methods, our constructions remove the constraint requiring that the extension factor is co-prime with the length of original FHSs and get new flexible parameters(see Table 1 and Table 2). By using different know optimal $(N, v, \lambda; M; Z)$ LHZ FHS set and OC sequence set, we can increase the length and alphabet size of the original LHZ FHS set by using flexible extension factor, but preserve the maximum Hamming correlation. We just list some of this results in this paper. As a result, our constructions allow a great flexibility of choosing parameters of LHZ FHS sets for a given quasi-synchronization frequency-hopping spread spectrum system.

REFERENCES

[1] P. Z. Fan and M. Darnell, *Sequence Design for Communications Applications*. London, U.K.: Wiley, 1996.
[2] S. W. Golomb and G. Gong, *Signal Design for Good Correlation: For Wireless Communication, Cryptography, and Radar*. Cambridge, U.K.: Cambridge Univ. Press, 2005.
[3] B. Ning, L. Guan, and H. Huang, "A novel frequency-hopping sequence for covert communication," *IEEE Access*, vol. 5, pp. 20157–20163, 2017.
[4] L. Yu, Y. H. Xu, and Q. H. Wu, "Self-organizing hit avoidance in distributed frequency hopping multiple access networks," *IEEE Access*, vol. 5, pp. 26614–26622, 2017.
[5] S. Baig, H. Muhammad, S. Mumtaz, M. Shafiq, J.-G. Choi, and T. Umer, "High data rate discrete wavelet transform-based PLC-VLC design for 5G communication systems," *IEEE Access*, vol. 6, pp. 52490–52499, 2018.
[6] Q. Zeng, H. Nasser, and G. Gradoni, "A secure waveform format for interference mitigation in heterogeneous uplink networks," *IEEE Access*, vol. 6, pp. 41688–41696, 2018.
[7] A. Alghaihab, Y. Shi, H.-S. Kim, B. H. Calhoun, D. D. Wentzloff, and J. Breiholz, "Enhanced interference rejection Bluetooth low-energy back-channel receiver with LO frequency hopping," *IEEE J. Solid-State Circuits*, to be published.
[8] A. Lempel and H. Greenberger, "Families of sequences with optimal Hamming-correlation properties," *IEEE Trans. Inf. Theory*, vol. IT-20, no. 1, pp. 90–94, Jan. 1974.
[9] X. Wang and P. Fan, "A class of frequency hopping sequences with no hit zone," in *Proc. IEEE Int. Conf. Parallel Distrib. Comput.*, Aug. 2003, pp. 896–898.
[10] D. Peng and P. Fan, "Lower bounds on the Hamming auto- and cross correlations of frequency-hopping sequences," *IEEE Trans. Inf. Theory*, vol. 50, no. 9, pp. 2149–2154, Sep. 2004.
[11] P. Daiyuan, F. Pingzhi , and M. H. Lee, "Lower bounds on the periodic Hamming correlations of frequency hopping sequences with low hit zone," *Sci. China F, Inf. Sci.*, vol. 49, no. 2, pp. 208–218, Jun. 2006.
[12] W. Ma and S. H. Sun, "New designs of frequency hopping sequences with low hit zone," *Des. Codes Cryptogr.*, vol. 60, no. 2, pp. 145–153, Jul. 2010.
[13] X. Niu, D. Y. Peng, and Z. C. Zhou, "New classes of optimal low hit zone frequency hopping sequences with new parameters by interleaving technique," *IEICE Trans. Fundam.*, vol. E95-A, no. 11, pp. 1835–1842, Nov. 2012.
[14] X. Niu, D. Peng, and Z. Zhou, "Frequency/time hopping sequence sets with optimal partial Hamming correlation properties," *IEICE Trans. Fundam.*, vol. E97-A, no. 12, pp. 2567–2571, Dec. 2014.
[15] J.-H. Chung and K. Yang, "New classes of optimal low-hit-zone frequency-hopping sequence sets by cartesian product," *IEEE Trans. Inf. Theory*, vol. 59, no. 1, pp. 726–732, Jan. 2013.
[16] H. Han, D. Peng, and U. Parampalli, "New sets of optimal low-hit-zone frequency-hopping sequences based on *m*-sequences," *Cryptogr. Commun.*, vol. 9, no. 4, pp. 511–522, Jul. 2016.
[17] C. Wang, D. Peng, and L. Zhou, "New constructions of optimal frequency-hopping sequence sets with low-hit-zone," *Int. J. Found. Comput. Sci.*, vol. 27, no. 1, pp. 53–66, Jan. 2016.
[18] L. Zhou, D. Peng, C. Wang, H. Han, and H. Liang, "Generalized methods to construct low-hit-zone frequency-hopping sequence sets and optimal constructions," *Cryptogr. Commun.*, vol. 9, no. 6, pp. 707–728, Nov. 2017.
[19] L. Zhou, D. Peng, C. Wang, Z. Ma, and H. Liang, "Constructions of optimal low-hit-zone frequency hopping sequence sets," *Des. Codes Cryptogr.*, vol. 85, no. 2, pp. 219–232, Nov. 2017.
[20] L. Ling, X. Niu, and X. Liu, "New classes of optimal low hit zone frequency hopping sequence set with large family size," *IEICE Trans. Fundam.*, vol. E101-A, no. 12, pp. 2213–2216, Dec. 2018.
[21] A. A. Shaar and P. A. Davies, "A survey of one-coincidence sequences for frequency-hopped spread-spectrum systems," *IEE Proc. F, Commun., Radar Signal Process.*, vol. 131, no. 7, pp. 719–724, Dec. 1984.
[22] L. Bin, "One-coincidence sequences with specified distance between adjacent symbols for frequency-hopping multiple access," *IEEE Trans. Commun.*, vol. 45, no. 4, pp. 408–410, Apr. 1997.
[23] Z. Cao, G. Ge, and Y. Miao, "Combinatorial characterizations of one-coincidence frequency-hopping sequences," *Des. Codes Cryptogr.*, vol. 41, no. 2, pp. 177–184, Nov. 2006.

[24] H. Wang and P. Huang, "Construction of a one-coincidence frequency-hopping sequence set with optimal performance," in *Proc. 2nd Int. Conf. Mechatronics Autom. Control*. Cham, Switzerland: Springer, 2015, pp. 915–923.

[25] L. Fukshansky and A. A. Shaar, "A new family of one-coincidence sets of sequences with dispersed elements for frequency hopping CDMA systems," Jan. 2017, *arXiv:1701.05209*. [Online]. Available: https://arxiv.org/abs/1701.05209

[26] W. Ren, F.-W. Fu, J. Gao, and F. Wang, "A class of optimal one-coincidence frequency-hopping sequence sets with composite length," *IEICE Trans. Fundam. Electron. Commun. Comput. Sci.*, vol. E100-A, no. 11, pp. 2528–2533, Nov. 2017.

[27] X. Niu and C. Xing, "New extension constructions of optimal frequency hopping sequence sets," 2018, *arXiv:1806.09869*. [Online]. Available: https://arxiv.org/abs/1806.09869

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