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An Evidential Aggregation Method of Intuitionistic Fuzzy Sets Based on Belief Entropy

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ABSTRACT Intuitionistic fuzzy sets (IFSs) are essential in the multi-criteria decision making (MCDM) under uncertain environment. However, how to reasonably aggregate them with considering the uncertainty contained in the IFSs is still an open issue. In this paper, a new method is proposed to solve such a problem based on the Dempster–Shafer evidence theory, belief entropy, and the weighted ordered weighted averaging (WOWA) operator. One of the advantages of the presented model is that the uncertainty contained in the IFSs is effectively modeled based on belief entropy and the conversion from the IFS to Dempster–Shafer evidence theory. In the framework of evidence theory, the uncertain information contained in the IFSs can be embodied effectively. Then, the belief entropy is calculated to determine the certainty weights for each IFS. With the various definitions of the regular increasing monotone (RIM) quantifier Q function, the preference relationship of a decision maker is considered. A numerical example is shown to illustrate the feasibility and effectiveness of the proposed method.

INDEX TERMS Intuitionistic fuzzy sets, multi-criteria decision making, Dempster-Shafer evidence theory, belief entropy, weighted ordered weighted averaging operator, preference.

I. INTRODUCTION

Multi-criteria decision making (MCDM) problems under uncertain environment have been attracted by many researchers [1]–[4]. Due to its practical features, it has been widely applied in many fields, such as risk assessment [5]–[7], supply selection [8]–[10], failure mode and effects analysis [11]–[13] and so on [14], [15]. In many cases, we need to obtain the evaluation results for each criterion with different alternatives to deal with the MCDM problems. Thus, it's essential to represent the uncertain information by using some technologies such as fuzzy sets to handle the uncertain better. One of the pioneering work is made by Bellman and Zadeh who use fuzzy set methods for dealing with multi-criteria decision problems [16]. In this application, a criterion is represented as a fuzzy set over the space of alternative solutions. With the development of fuzzy theory in recent years, many non-standard fuzzy set, such as intuitionistic fuzzy set (IFS) [17]–[19], interval-valued intuitionistic fuzzy set (IVIFS) [20]–[23] and hesitant fuzzy set (HFS) [24]–[26], have been caused widespread concern and been applied in dealing with MCDM problems.

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Since the process of making relevant decisions need to formulate the overall decision function, the aggregation method of those technologies have been widely discussed [27]–[29]. Numerous operators have been proposed for the aggregation of fuzzy sets [30]. Traditionally, one commonly used method is the ordered weighted averaging (OWA) aggregation operator, which has the ability to model linguistically expressed imperatives for the aggregation of fuzzy sets [31]. Though classical fuzzy set can make the evaluation results more objective, it still has some limitations in expressing uncertainty [32], [33]. Recently, more and more scholars have focused their research on aggregating non-standard fuzzy sets, particularly those which allow for a representation of uncertainty in the membership grade [34]. One common example is the intuitionistic fuzzy set. Since its practical and intuitive feature compared with traditional fuzzy sets, IFS has been paid great attentions in recent years [35]–[37].

In the previous related research, many technologies are introduced to aggregate IFSs such as Choquet integral aggregation (CIA), Sugeno integral aggregation (SIA) operators and so on [38], [39]. However, such methods require the corresponding fuzzy measure, which is not objective enough [40]. Namely, the aggregated results using such methods are mainly depended on the selected fuzzy measure.

Based on it, some other effective aggregation methods are introduced such as ordered weighted averaging (OWA) operator [41]. Although the OWA operator has a certain degree of objectivity, it does not well consider the uncertainty contained in IFSs obtained in specific MCDM problems. Thus, traditional OWA operator is also not effective enough to deal with such an issue. In addition, some researchers utilize the basic operational properties to complete the aggregation process such as intuitionistic aggregation aggregation (IFA) operator [42], arithmetic mean operator [43], intuitionistic fuzzy ordered weighted averaging operator [44] and so on. Nevertheless, IFA operator cannot deal with the situation that the numerical values of IFSs have significant difference. And the mean operator is lack of consideration for uncertain information contained in IFSs. Furthermore, Xu and Yager introduced some operators for intuitionistic fuzzy information, such as the intuitionistic fuzzy weighted geometric (IFWG) operator, the intuitionistic fuzzy ordered weighted geometric (IFOWG) operator, and the intuitionistic fuzzy hybrid geometric (IFHG) operator [45]. Yu proposed the intuitionistic fuzzy geometric Heronian mean (IFGHM) operator and the intuitionistic fuzzy geometric weighed Heronian mean (IFGWHM) operator to deal with such an issue [46]. And Xu proposed the intuitionistic fuzzy power aggregation (IFPA) operators to make use of the advantaged of power aggregation (PA) operator [47].

To address this issue, in this paper, a novel evidential aggregation method of intuitionistic fuzzy sets based on belief entropy is proposed. Due to the importance of uncertain information implicit in IFS, we need to handle it by using some technologies. One of the most effective method is Dempster-Shafer evidence theory, which is widely used in many applications [48]–[51]. Considering the specific meaning of membership, non-membership and hesitancy degree, those variables can be regarded as three mutually exclusive and exhaustive hypotheses. Moreover in the framework of Dempster-Shafer evidence theory, the basic probability assignments (BPAs) can be used to handle the relative uncertain information of IFSs. Then we need to consider the measurement of uncertainty. Based on it, the concept of entropy derived from physics is introduced in this paper. One of the most common technology is Shannon entropy, which is widely adopted to measure the uncertainty of a probability distribution [52]. However, Shannon entropy cannot work effectively to deal with the case of BPAs [53]. Thus, we introduce belief entropy to solve such an issue. After a series of calculation process, the certainty weights can be obtained for each IFS. Moreover, the weighted ordered weighted averaging (WOWA) operator is introduced to complete the polymerization process [54]. With the sundry determination of RIM quantifier Q function, the preference relationship of decision maker is also considered.

The rest of this paper is organized as follows. In Section II, we briefly introduce some basic definitions about the Dempster-Shafer evidence theory, intuitionistic fuzzy set, ordered weighted averaging operator, belief entropy and so on. In Section III, a novel evidential aggregation method of intuitionistic fuzzy sets based on belief entropy is proposed. In Section IV, a numerical example and its computational process are shown to illustrate the effectiveness and practicality of our proposed method. Moreover, the comparisons and discussion have been also mentioned. In Section V, some conclusions of the proposed method are given.

II. PRELIMINARIES

A. DEMPSTER-SHAFER EVIDENCE THEORY

Handling uncertainty is inevitable in real engineering [55]–[58]. Dempster-Shafer evidence theory is also called as evidence theory [59], [60]. Due to its good performance on dealing with uncertain information, it has been widely used in many fields, such as target recognition [61], decision making [62]–[64], conflict management [65], [66], fault diagnosis [67] and information fusion [68], [69]. Here are some of the basic definitions.

Let Ω be a set of N elements called the frame of discernment which denotes a finite nonempty set of mutually exclusive and exhaustive hypotheses that $\Omega = \{H_1, H_2, H_3, \ldots, H_n\}$. The power set of Ω, which was denoted with $P(\Omega)$, contains all the possible subsets of it. And it is composed of 2^N elements of Ω. Each element of 2^N represents a proposition.

Definition 1: A basic probability assignment (BPA) is a function, which is defined by [59], [60]

$$m : P(\Omega) \rightarrow [0, 1], \quad A \mapsto m(A) \tag{1}$$

where

$$\sum_{A \in P(\Omega)} m(A) = 1, \quad m(\emptyset) = 0 \tag{2}$$

BPA is the base of evidence theory with many operations such as negation [70], correlation [71] and divergence [72], [73].

Definition 2: Given a BPA m, for a proposition $A \subseteq \Omega$, the belief function Bel: $2^{\Omega} \rightarrow$ [0,1] is defined as

$$Bel(A) = \sum_{B \subseteq A} m(B) \tag{3}$$

The plausibility function Pl: $2^{\Omega} \rightarrow$ [0,1] is defined as

$$Pl(A) = 1 - Bel(\bar{A}) = \sum_{B \bigcap A \neq \emptyset} m(B) \tag{4}$$

where $\bar{A} = \Omega$ - A. Bel(A) can be seen as a measure of people's belief that the hypothesis A is true and is viewed as a lower limit function on the probability of A. The plausibility Pl(A) can be interpreted as the degree that we absolutely believe in A and is seen as an upper limit function on the probability of A.

B. THE ORDERED WEIGHTED AVERAGING OPERATOR

The OWA operators, which is firstly introduced by Yager, has been paid more and more attention in recent years. Here we briefly introduce some basic concepts.

1) QUANTIFIER-BASED ORDERED WEIGHTED AVERAGING OPERATOR

Definition 3: An OWA operator of dimension n is a mapping $F : R_n \mapsto R$ with an associated group of weights $W = [w_1, w_2, \ldots, w_n]$ which satisfies the condition that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. The OWA operator is determined as

$$F(a_1, a_2, \ldots a_n) = \sum_{j=1}^{n} w_j a_{ind(j)} \quad (5)$$

where $a_{ind(j)} \in [0, 1]$ and $a_{ind(j)}$ is the j^{th} largest of the a_j. And $W = [w_1, w_2, \ldots, w_j, \ldots, w_n]$ is called as the OWA weighting vector.

Definition 4: Let $\hat{W} = w_{n-j+1}$ and let $\bar{a}_i$ be the negation, $\bar{a}_i = 1 - a_i$. The OWA aggregation of the $\bar{a}_i$ under $\hat{W}$, $F_{\hat{W}}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n)$. Because of the limit of the indexing, assume that $\hat{a}_i \leq \hat{a}_k$ if $i < k$. Thus

$$F_{\hat{W}}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) = \sum_{j=1}^{n} \widehat{w_j} \bar{a}_{n-j+1} = 1 - \sum_{i=1}^{n} w_i a_i \quad (6)$$

Definition 5: The weighting vector W for all $i = 1, \ldots, n$ is called quantifier-based OWA weights. And $Q : [0, 1] \to [0, 1]$ is called a Regular Increasing Monotone (RIM) quantifier if it satisfies that $Q(0) = 0$, $Q(1) = 1$ and $Q(a) \leq Q(b)$ whenever $a < b$. With a RIM quantifier Q, the weight w_i allocated to the i^{th} variable b_i is defined as [74]

$$w_i = Q(\frac{i}{n}) - Q(\frac{i-1}{n}) \quad (7)$$

where $i = (1, 2, .., n)$ with $w_i \in [0, 1]$, $\sum_{i=1}^{n} w_i = 1$ and n represents the total number of criteria.

It's obviously to see that the specific value of w_i varies with the different RIM quantifier. Many scholars have defined lots of different Q to express the decision preference better in different situations.

2) THE ORNESS OF RIM QUANTIFIER

To better analyze the preference relationship of the RIM quantifier Q, Yager introduced the concept of orness.

Definition 6: Given a RIM quantifier Q, its orness degree is generated as [75]

$$orness(Q) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{n-i}{n-1} \left[Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right) \right] = \lim_{n\to\infty} \frac{1}{n-1} \sum_{i=1}^{n-1} Q\left(\frac{i}{n}\right) = \int_0^1 Q(x)dx. \quad (8)$$

For any RIM quantifier Q, it can be considered to have the ability to express the preference relationship. For example, if any RIM quantifier Q is concave, then generally the larger numerical satisfaction of criterion C_i will carry more numerical weight. Moreover, due to the character of non-decreasing, continuity and defined on the closed unit interval [0,1], such the conditional function Q is Riemann integrable that the Eq. 8 is always effective.

3) WEIGHTED ORDERED WEIGHTED AVERAGING OPERATOR

In many cases, the precise values which needed to be aggregated always have its own weights. Since the quantifier-based OWA operator satisfies is commutativity, it stands for the equal importance of each argument. Thus, the reliability factor of all the information sources is frequently ignored in this way. Based on it, Torra proposed the weighted ordered weighted averaging (WOWA) operator [54] , which combines the advantages of the OWA operator and the weighted mean. Here we simply introduce some of the concepts.

Definition 7: Given a sequence of n variables $\{a_1, a_2, a_3, \ldots, a_n\}$ and let p and w be weighting vectors of dimension n such that $p_j \in [0, 1]$, $\sum_{j=1}^{n} p_j = 1$ and $w_j \in [0, 1]$, $\sum_{j=1}^{n} w_j = 1$ where p_j and w_j individually represents the corresponding weight information obtained by evaluating the reliability of each data source and the weights calculation.

The weighted OWA operator is a mapping $F_{WOWA} : R_n \mapsto R$, which is defined as

$$F_{WOWA}(a_1, a_2, \ldots a_n) = \sum_{j=1}^{n} w_j a_{ind(j)} \quad (9)$$

where $a_{ind(j)} \in [0, 1]$ and $a_{ind(j)}$ is the j^{th} largest of the a_j. And the weight for each argument is defined as

$$w_i = Q(\sum_{k=1}^{i} p_k) - Q(\sum_{k=1}^{i-1} p_k) \quad (10)$$

where Q is a RIM quantifier function, which can embody the preference relationship in many cases.

C. INTUITIONISTIC FUZZY SETS

In classical fuzzy sets theory, for any elements in the domain of discourse, the relationship between each set is only *Belong to* or *Not Belong to* [17]. To express the mathematical model of the uncertain information better, intuitionistic fuzzy sets was introduced by Atanassov [76]. In recent years, IFS has been applied in many applications [77], [78]. Here are some basic definitions.

Definition 8: An IFS A on the space X is defined by two functions, $A =< \mu_A(x), \nu_A(x) >$, $\mu_A(x)$ could be represented by the degree of membership of x in A and $\nu_A(x)$ could be represented by the degree of nonmembership of x in A. What's more, it satisfies the condition that $0 \leq \mu_A(x) + \nu_A(x) \leq 1$, where $\mu_A(x) \in [0, 1]$ and $\nu_A(x) \in [0, 1]$. The degree of hesitancy of x is defined as [76]

$$\pi_A(x) = 1 - (\mu_A(x) + \nu_A(x)). \quad (11)$$

Note that for the standard fuzzy sets, $\nu_A(x) = 1 - \mu_A(x)$.

D. IFS IN THE FRAMEWORK OF EVIDENCE THEORY

The relationship between evidence theory and IFS is discussed in [79].

Definition 9: Assume that there is an IFS $A = \{< x, \mu_A(x), \nu_A(x) > | x \in X\}$. In this case above, the relationship between IFS and evidence theory by mathematical modeling could be found, which could be expressed that [79]

$$\begin{aligned} m(\{Yes\}) &= \mu_A(x) \\ m(\{No\}) &= \nu_A(x) \\ m(\{Yes, No\}) &= \pi_A(x) \end{aligned} \tag{12}$$

Recalling the evidence theory, the IFS A can also be expressed as another form

$$A = \{< x, BI_A(x) > | x \in X\} \tag{13}$$

where

$$BI_A(x) = [Bel_A(x), Pl_A(x)] \tag{14}$$

$$Bel_A(x) = m(\{Yes\}) = \mu_A(x) \tag{15}$$

$$\begin{aligned} Pl_A(x) &= m(\{Yes\}) + m(\{Yes, No\}) \\ &= \mu_A(x) + \pi_A(x) \\ &= 1 - \upsilon_A(x) \end{aligned} \tag{16}$$

Thus, the belief function and plausibility function for any IFS after the conversion process can be obtained. In the framework of evidence theory, the uncertain information contained in IFSs can be effectively modeled. Based on the transformation equation mentioned above, IFSs can be converted into several BPAs, which are also seen as the mass function in evidence theory.

E. BELIEF ENTROPY

As the development of information science, Shannon entropy has played more and more important role in measuring the uncertainty [80]. Here are some of the basic definitions.

Definition 10: Shannon entropy is defined as [52]

$$H = -\sum_{i=1}^{N} p_i \log_b p_i \tag{17}$$

where N is the number of basic states, p_i denotes the probability of state i, and p_i satisfies $\sum_{i=1}^{N} p_i = 1$.

If the unit information is bit, then $b = 2$, Shannon entropy is expressed as

$$H = -\sum_{i=1}^{N} p_i \log_2 p_i \tag{18}$$

Since Dempster-Shafer evidence theory has been widely used in many fields, the method to measure the uncertainty in evidence theory is still an issue worth exploring [81], [82]. Based on Shannon entropy, a belief entropy, named as Deng entropy, is presented to deal with uncertainty measure of BPAs.

Definition 11: In frame of discernment X, the belief entropy is defined as [53]

$$E_d(m) = -\sum_{A \subseteq X} m(A) \log_2 \frac{m(A)}{2^{|A|} - 1} \tag{19}$$

where $|A|$ is the cardinality of the proposition A. Specially, the belief entropy can definitely degenerate to the Shannon entropy, if the belief is only assigned to single elements. In recent years, the belief entropy has attracted more and more people's attention due to the superiority in measuring uncertain information [83], [84].

III. THE PROPOSED METHOD

Since IFS is an effective technology to express uncertainty, the amount of uncertainty contained in it should be modeled effectively in the aggregation process. In this section, an evidential method based on evidence theory, belief entropy and weighted OWA is proposed, which is shown in the following.

A. METHOD ILLUSTRATION

Assume that there are n alternatives with m criteria, which are denoted as $\{A_1, A_2, \ldots, A_n\}$ and $\{C_1, C_2, \ldots, C_m\}$. For any criterion, there are corresponding n IFSs indicating satisfaction based on alternative A_1 to A_n, respectively. The task is to aggregate a collection of IFSs under different criteria for each alternative. Here we note that for the convenience of the following explanation and discussion process, we use x to represent one of the criteria C_k. The details of proposed method are shown as follows.

Step 1: Obtain the three variables of each IFS separately and convert the IFSs into BPAs. To measure the uncertainty contained in IFSs better, we can transform the membership degree, non-membership degree and hesitancy degree of IFSs in the framework of evidence theory. The specific conversion process is shown in Eq. 12.

Step 2: Calculate the certainty weights based on belief entropy. The belief entropy is an effective method to measure the uncertainty. Hence, we then can calculate the belief entropy for each IFS. Here the variables are regarded as three mutually exclusive elements. Based on Eq. 19, the calculation results can be obtained. Take criteria x for instance, the specific calculation is shown in Eq. 20.

Note that Y denotes Yes and N denotes No.

Then the maximum value of belief entropy in this situation E_{dmax} can be calculated. Based on the three variables of IFS in evidence theory's framework, it can be shown in Fig. 1, which is 2.2925. The certainty weights can be calculated as

$$U(A_i(x)) = E_{dmax} - E_d(A_i(x)) \tag{21}$$

Step 3: Order the IFSs based on corresponding numerical value of membership degree and non-membership degree. Here $\mu_{75Aind(i)}(x)$ represents the alternative with the i^{th} largest membership degree and $\nu_{Aind(i)}(x)$ represents the alternative with the i^{th} largest membership degree.

Step 4: Obtain the RIM quantifier Q function. The specific definition of Q function can embody the preference relationship (See in Definition 6). Thus, we need to determine the expression of Q function.

Step 5: Calculate the RIM quantifier $\widehat{Q}$ function. To aggregate the non-membership in the following, here we first calculate $\widehat{Q}$, which is the dual of Q function. Suppose that the

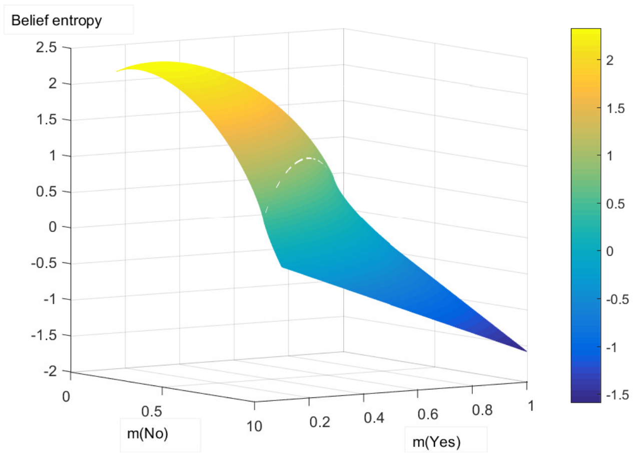

FIGURE 1. The schematic diagram of belief entropy of IFS model in the framework of evidence theory.

Q function with independent variable x is denoted as $Q(t)$, then $\widehat{Q}$ is calculated as

$$\widehat{Q}(t) = 1 - Q(1-t). \tag{22}$$

Step 6: Compute the weighting vectors W based on the order results of belief entropy and the given RIM quantifier Q function. Take the criteria x for example, the calculation expression is defined as

$$w_i(x) = Q\left(\frac{S_i}{S_n}(x)\right) - Q\left(\frac{S_{i-1}}{S_n}(x)\right) \tag{23}$$

where

$$S_i(x) = \sum_{k=1}^{i} U(A_{ind(k)}(x))$$
$$S_n(x) = \sum_{k=1}^{n} U(A_{ind(k)}(x)). \tag{24}$$

It's obviously to see that for any $i \in \{1, 2, \ldots, n\}$, $\frac{S_i}{S_n}(x) \in [0, 1]$ is always established. Moreover, it satisfies the boundary conditions that when $i = 1$, $\frac{S_i}{S_n}(x) = 0$ and $\frac{S_i}{S_n}(x) = 1$ when $i = n$.

Step 7: Obtain the dual of weighting vector $\widehat{W}$ based on the same order results of belief entropy and calculated $\widehat{Q}$. The specific calculation process is expressed as

$$\widehat{w}_i(x) = \widehat{Q}\left(\frac{\widehat{S}_i}{\widehat{S}_n}(x)\right) - \widehat{Q}\left(\frac{\widehat{S}_{i-1}}{\widehat{S}_n}(x)\right). \tag{25}$$

where

$$\widehat{S}_i(x) = \sum_{k=1}^{i} U(A_{\widehat{ind(k)}}(x))$$
$$\widehat{S}_n(x) = \sum_{k=1}^{n} U(A_{\widehat{ind(k)}}(x)). \tag{26}$$

Note that $\widehat{ind(k)}$ is also an index function which is the dual of $ind(k)$. Moreover, $U(A_{\widehat{ind(k)}}(x))$ can be defined as

$$U(A_{\widehat{ind(k)}}(x)) = U(A_{ind(n-k+1)}(x)) \tag{27}$$

Step 8: Respectively calculate the aggregation results for membership and non-membership of IFSs. Based on the calculated weighting vector W and $\widehat{W}$, the aggregated IFS $A(x) =< \mu_A(x), \nu_A(x) >$ for criterion x can be computed as

$$\mu_A(x) = F_W(\mu_{A_{ind(1)}}(x), \mu_{A_{ind(2)}}(x), \ldots, \mu_{A_{ind(n)}}(x))$$
$$= \sum_{k=1}^{n} w_k \mu_{A_{ind(k)}}(x) \tag{28}$$
$$\nu_A(x) = F_{\widehat{W}}(\nu_{A_{ind(k)}}(x), \nu_{A_{ind(k)}}(x), \ldots, \nu_{A_{ind(k)}}(x)$$
$$= \sum_{k=1}^{n} \widehat{w}_k \nu_{A_{ind(k)}}(x). \tag{29}$$

Step 9: Obtain the aggregated IFSs for each criterion. For criterion x, the aggregated result is denoted as $A(x) =< \mu_A(x), \nu_A(x) >$. And for other criteria, the same method shown above can be used to calculate its aggregation result.

B. PROOF

Here we give the specific demonstration to show that the aggregated result is always effective. Namely, the aggregation result $A(x) =< \mu_A(x), \nu_A(x) >$ is always a valid IFS. Moreover, $\frac{S_i}{S_n}(x)$ is expressed as $s_i(x)$ and the corresponding certainty weights are normalized as $u_i(x)$ where $\sum_{k=1}^{n} u_k = 1$.

Note that for simplicity, the membership of any IFS $\mu_{A_i}(x)$ is expressed as a_i. Moreover, we assume that all of them have been indexed that $\mu_1(x) \geq \mu_2(x) \geq \ldots \geq \mu_n(x)$.

Firstly, the aggregation result of membership can be represented as

$$\mu_A(x) = F_W((\mu_1(x), u_1(x)), \ldots, (\mu_n(x), u_n(x)))$$
$$= \sum w_j(x)\mu_j(x)$$

where $w_j = Q(s_i(x)) - Q(s_{j-1}(x))$.

$$E_d(A_i(x)) = -m_{A_i(x)}(\{Y\})\log_2\frac{m_{A_i(x)}(\{Y\})}{2^{|\{Y\}|}-1} - m_{A_i(x)}(N)\log_2\frac{m_{A_i(x)}(\{N\})}{2^{|\{N\}|}-1} - m_{A_i(x)}(\{Y,N\})\log_2\frac{m_{A_i(x)}(\{Y,N\})}{2^{|\{Y,N\}|}-1}$$
$$= -m_{A_i(x)}(\{Y\})\log_2 m_{A_i(x)}(\{Y\}) - m_{A_i(x)}(N)\log_2 m_{A_i(x)}(\{N\}) - m_{A_i(x)}(\{Y,N\})\log_2\frac{m_{A_i(x)}(\{Y,N\})}{3} \tag{20}$$

Based on Eq. 24 and 26, the aggregation result of non-membership can be represented as

$$\nu_A(x) = F_{\widehat{W}}((\nu_1(x), u_1(x)), \ldots, (\nu_n(x), u_n(x)))$$

Since $\mu_i(x) + \nu_i(x) \leq 1$, we must have $\nu_i(x) \leq 1 - \mu_i(x)$. Due to the monotonicity of WOWA operator [54], $\nu_A(x)$ can be embodied as

$$\nu_A(x) \leq F_{\widehat{W}}((1 - \mu_1(x), u_1(x)), \ldots, (1 - \mu_n(x), u_n(x)).$$

Note that $1 - \mu_i(x)$ is denoted as $d_i(x)$ for simplicity that

$$\nu_A(x) \leq F_{\widehat{W}}((d_1(x), u_1(x)), \ldots, (d_n(x), u_n(x))).$$

According to the ordering of $\mu_i(x)$, it's obviously to see that $d_i(x)$ should be ordered inversely where $d_1(x) \leq d_2(x) \leq \ldots \leq d_n(x)$. Thus we have

$$\nu_A(x) \leq \sum_{k=1}^{n} \widehat{w}_k(x) d_{n-j+1}(x)$$

where $\widehat{w}_j(x) = \widehat{Q}(\widehat{s}_j(x)) - \widehat{Q}(\widehat{s}_{j-1}(x))$ and $\widehat{s}_j(x) = \sum_{k=1}^{n} u_{n-j+1}(x)$.

And we have $\sum_{k=1}^{n} \widehat{w}_k(x) = 1$ and $d_k(x) = 1 - \mu_k(x)$, the following inequality can be obtained that

$$\nu_A(x) \leq 1 - \sum_{j=1}^{n} \widehat{w}_j(x) \mu_{n-j+1}(x).$$

Based on Eq. 22, we can find that

$$\begin{aligned}\widehat{w}_j(x) &= \widehat{Q}(\widehat{s}_j(x)) - \widehat{Q}(\widehat{s}_{j-1}(x)) \\ &= Q(1 - s_{j-1}(x)) - Q(1 - s_j(x)).\end{aligned}$$

According to Eq. 26 and 27, we can obtain that

$$\begin{aligned}1 - \widehat{s}_j(x) &= \sum_{i=j+1}^{n} u_{n-i+1}(x) \\ &= u_1(x) + u_2(x) + \ldots + u_{n-j}(x) = \sum_{i=1}^{n-j} u_i(x).\end{aligned}$$

Since $1 - \widehat{s}_j(x) = s_j(x)$ and $1 - \widehat{s}_{j-1}(x) = s_{n-j+1}(x)$, then

$$\widehat{w}_j(x) = Q(s_{n-j+1}(x)) - Q(s_{n-j}(x)) = w_{n-j+1}(x).$$

and

$$\begin{aligned}\sum_{j=1}^{n} \widehat{w}_j(x) \mu_{n-j+1}(x) &= \sum_{j=1}^{n} w_{n-j+1}(x) \mu_{n-j+1}(x) \\ &= \sum_{j=1}^{n} w_j(x) \mu_j(x) = \mu_A.\end{aligned}$$

Thus we can observe that

$$\nu_A(x) \leq 1 - \sum_{j=1}^{n} \widehat{w}_j(x) \mu_{n-j+1}(x) \leq 1 - \mu_A(x).$$

Thus $\mu_A(x) + \nu_A(x) \leq 1$ is always established. Namely, $A(x)$ is always a valid IFS.

IV. NUMERICAL EXAMPLE

In this section, a numerical example is performed to show the whole procedures of our proposed method. Note that to show the whole process intuitively, there are two criteria and four alternatives.

A. THE IMPLEMENTATION OF THE PROPOSED APPROACH

Let $X = \{x, y\}$, assume that there are four IFSs

$$\begin{aligned}A_1(x) &= <0.6, 0.2> \quad A_1(y) = <1.0, 0.0> \\ A_2(x) &= <0.5, 0.3> \quad A_2(y) = <0.4, 0.4> \\ A_3(x) &= <0.9, 0.0> \quad A_3(y) = <0.3, 0.5> \\ A_4(x) &= <0.2, 0.7> \quad A_4(y) = <0.8, 0.1>\end{aligned}$$

The aim is to consider the aggregation result for criteria x and y with four alternatives A_1, A_2, A_3 and A_4. Here we note that the aggregation result is respectively denoted as

$$\begin{aligned}A(x) &= <\mu_A(x), \nu_A(x)> \\ A(y) &= <\mu_A(y), \nu_A(y)>\end{aligned}$$

Step 1: Obtain the three variables of each IFS respectively and convert the IFSs into BPAs. It's simple to find that

$$\begin{aligned}&\mu_{A_1}(x) = 0.6, \quad \nu_{A_1}(x) = 0.2, \ \pi_{A_1}(x) = 0.2 \\ &\mu_{A_2}(x) = 0.5, \quad \nu_{A_2}(x) = 0.3, \ \pi_{A_2}(x) = 0.2 \\ &\mu_{A_3}(x) = 0.9, \quad \nu_{A_3}(x) = 0.0, \ \pi_{A_3}(x) = 0.1 \\ &\mu_{A_4}(x) = 0.2, \quad \nu_{A_4}(x) = 0.7, \ \pi_{A_4}(x) = 0.1\end{aligned}$$

Similarly,

$$\begin{aligned}&\mu_{A_1}(y) = 1.0, \quad \nu_{A_1}(y) = 0.0, \ \pi_{A_1}(y) = 0.0 \\ &\mu_{A_2}(y) = 0.4, \quad \nu_{A_2}(y) = 0.4, \ \pi_{A_2}(y) = 0.2 \\ &\mu_{A_3}(y) = 0.3, \quad \nu_{A_3}(y) = 0.5, \ \pi_{A_3}(y) = 0.2 \\ &\mu_{A_4}(y) = 0.8, \quad \nu_{A_4}(y) = 0.1, \ \pi_{A_4}(y) = 0.1\end{aligned}$$

The result of the criteria x and y as shown in Table 1.

Step 2: Calculate the certainty weights based on belief entropy. Based on Eq. 20, the belief entropy for each alternative can be shown in Table 2.

Step 3: Order the IFSs based on corresponding numerical value of membership degree and non-membership degree. Take criterion x for an instance, here we use $\mu_{A_{ind(k)}}(x)$ to express the alternative k^{th} largest membership degree of IFSs with criterion x. After the process of comparison, the order result of criteria x and y is shown in Table 3.

Step 4: Obtain the *RIM* quantifier Q function. Here we assume that Q function is defined as $Q(x) = 1 - (1-x)^2$ for all criteria, where $orness(Q) = \int_0^1 Q(x)dx = \int_0^1 1 - (1-x)^2 dx = \frac{2}{3}$.

Step 5: Calculate the *RIM* quantifier $\widehat{Q}$ function. Based on the determined Q before, the $\widehat{Q}$ can be calculated as

$$\widehat{Q}(x) = 1 - [1 - (1 - (1-x)^2)] = x^2.$$

Step 6: Compute the weighting vectors W for criteria x and y respectively. Based on the calculated Q function and

TABLE 1. The result of conversion process from IFS to BPA for the criteria *x* and *y*.

BPA	The degree of IFS variable	Value
$m_{A_1(x)}(Y)$	$\mu_{A_1}(x)$	0.6
$m_{A_1(x)}(N)$	$\nu_{A_1}(x)$	0.2
$m_{A_1(x)}(Y,N)$	$\pi_{A_1}(x)$	0.2
$m_{A_2(x)}(Y)$	$\mu_{A_2}(x)$	0.5
$m_{A_2(x)}(N)$	$\nu_{A_2}(x)$	0.3
$m_{A_2(x)}(Y,N)$	$\pi_{A_2}(x)$	0.2
$m_{A_3(x)}(Y)$	$\mu_{A_3}(x)$	0.9
$m_{A_3(x)}(N)$	$\nu_{A_3}(x)$	0.0
$m_{A_3(x)}(Y,N)$	$\pi_{A_3}(x)$	0.1
$m_{A_4(x)}(Y)$	$\mu_{A_4}(x)$	0.2
$m_{A_4(x)}(N)$	$\nu_{A_4}(x)$	0.7
$m_{A_4(x)}(Y,N)$	$\pi_{A_4}(x)$	0.1
$m_{A_1(y)}(Y)$	$\mu_{A_1}(y)$	1.0
$m_{A_1(y)}(N)$	$\nu_{A_1}(y)$	0.0
$m_{A_1(y)}(Y,N)$	$\pi_{A_1}(y)$	0.0
$m_{A_2(y)}(Y)$	$\mu_{A_2}(y)$	0.4
$m_{A_2(y)}(N)$	$\nu_{A_2}(y)$	0.4
$m_{A_2(y)}(Y,N)$	$\pi_{A_2}(y)$	0.2
$m_{A_3(y)}(Y)$	$\mu_{A_3}(y)$	0.3
$m_{A_3(y)}(N)$	$\nu_{A_3}(y)$	0.5
$m_{A_3(y)}(Y,N)$	$\pi_{A_3}(y)$	0.2
$m_{A_4(y)}(Y)$	$\mu_{A_4}(y)$	0.8
$m_{A_4(y)}(N)$	$\nu_{A_4}(y)$	0.1
$m_{A_4(y)}(Y,N)$	$\pi_{A_4}(y)$	0.1

TABLE 2. The result of belief entropy and associated certainty weights for each alternative.

Belief entropy	Value	Certainty weights	Value
$E_d(A_1(x))$	1.3710	$U(A_1(x))$	0.9215
$E_d(A_2(x))$	1.4855	$U(A_2(x))$	0.8070
$E_d(A_3(x))$	0.4690	$U(A_3(x))$	1.8235
$E_d(A_4(x))$	1.1568	$U(A_4(x))$	1.1357
$E_d(A_1(y))$	0.0000	$U(A_1(y))$	2.2925
$E_d(A_2(y))$	1.5219	$U(A_2(y))$	0.7726
$E_d(A_3(y))$	1.4855	$U(A_3(y))$	0.8070
$E_d(A_4(y))$	0.9219	$U(A_4(y))$	1.3706

TABLE 3. The order result of criterion *x* and *y*.

Membership		Non-membership	
Alternative	Order	Alternative	Order
$A_1(x)$	2	$A_1(x)$	3
$A_2(x)$	3	$A_2(x)$	2
$A_3(x)$	1	$A_3(x)$	4
$A_4(x)$	4	$A_4(x)$	1
$A_1(y)$	1	$A_1(y)$	4
$A_2(y)$	3	$A_2(y)$	2
$A_3(y)$	4	$A_3(y)$	1
$A_4(y)$	2	$A_4(y)$	3

entropy-based order S_k for $k \in \{1, 2, 3, 4\}$ can be calculated, which is shown as follows

$$\frac{S_1}{S_n}(x) = \frac{\sum_{i=1}^{1} U(A_{ind(i)}(x))}{\sum_{k=1}^{4} U(A_{ind(k)}(x))} = \frac{1.8235}{4.6877} = 0.3890$$

$$\frac{S_2}{S_n}(x) = \frac{\sum_{i=1}^{2} U(A_{ind(i)}(x))}{\sum_{k=1}^{4} U(A_{ind(k)}(x))} = \frac{2.7450}{4.6877} = 0.5856$$

$$\frac{S_3}{S_n}(x) = \frac{\sum_{i=1}^{3} U(A_{ind(i)}(x))}{\sum_{k=1}^{4} U(A_{ind(k)}(x))} = \frac{3.5520}{4.6877} = 0.7577$$

$$\frac{S_4}{S_n}(x) = \frac{\sum_{i=1}^{4} U(A_{ind(i)}(x))}{\sum_{k=1}^{4} U(A_{ind(k)}(x))} = \frac{4.6877}{4.6877} = 1.0000$$

$$\frac{S_1}{S_n}(y) = \frac{\sum_{i=1}^{1} U(A_{ind(i)}(y))}{\sum_{k=1}^{4} U(A_{ind(k)}(y))} = \frac{2.2925}{5.2427} = 0.4373$$

$$\frac{S_2}{S_n}(y) = \frac{\sum_{i=1}^{2} U(A_{ind(i)}(y))}{\sum_{k=1}^{4} U(A_{ind(k)}(y))} = \frac{3.6631}{5.2427} = 0.6987$$

$$\frac{S_3}{S_n}(y) = \frac{\sum_{i=1}^{3} U(A_{ind(i)}(y))}{\sum_{k=1}^{4} U(A_{ind(k)}(y))} = \frac{4.4357}{5.2427} = 0.8461$$

$$\frac{S_4}{S_n}(y) = \frac{\sum_{i=1}^{4} U(A_{ind(i)}(y))}{\sum_{k=1}^{4} U(A_{ind(k)}(y))} = \frac{5.2427}{5.2427} = 1.0000.$$

where $\frac{S_0}{S_n}(x) = 0$ and $\frac{S_0}{S_n}(y) = 0$.

Then the weighting vector W for criteria x and y can be computed as follows.

Step 7 : Obtain the dual of weighting vector $\widehat{W}$. Based on the calculated Q function and entropy-based order, $\widehat{S}_k$ for $k \in \{1, 2, 3, 4\}$ can be calculated, which is shown as

$$\frac{\widehat{S}_1}{\widehat{S}_n}(x) = \frac{\sum_{i=1}^{1} U(A_{\widehat{ind(i)}}(x))}{\sum_{k=1}^{4} U(A_{\widehat{ind(k)}}(x))} = \frac{1.1357}{4.6877} = 0.2423$$

$$\frac{\widehat{S}_2}{\widehat{S}_n}(x) = \frac{\sum_{i=1}^{2} U(A_{\widehat{ind(i)}}(x))}{\sum_{k=1}^{4} U(A_{\widehat{ind(k)}}(x))} = \frac{1.9427}{4.6877} = 0.4144$$

$$\frac{\widehat{S}_3}{\widehat{S}_n}(x) = \frac{\sum_{i=1}^{3} U(A_{\widehat{ind(i)}}(x))}{\sum_{k=1}^{4} U(A_{\widehat{ind(k)}}(x))} = \frac{2.8642}{4.6877} = 0.6110$$

$$w_1(x) = Q\left(\frac{S_1}{S_n}(x)\right) - Q\left(\frac{S_0}{S_n}(x)\right) = Q\,(0.3890) - Q\,(0.0000) = 0.6267 - 0.0000 = 0.6267$$
$$w_2(x) = Q\left(\frac{S_2}{S_n}(x)\right) - Q\left(\frac{S_1}{S_n}(x)\right) = Q\,(0.5856) - Q\,(0.3890) = 0.8283 - 0.6267 = 0.2015$$
$$w_3(x) = Q\left(\frac{S_3}{S_n}(x)\right) - Q\left(\frac{S_2}{S_n}(x)\right) = Q\,(0.7577) - Q\,(0.5856) = 0.9413 - 0.8283 = 0.1131$$
$$w_4(x) = Q\left(\frac{S_4}{S_n}(x)\right) - Q\left(\frac{S_3}{S_n}(x)\right) = Q\,(1.0000) - Q\,(0.7577) = 1.0000 - 0.9413 = 0.0587$$

$$w_1(y) = Q\left(\frac{S_1}{S_n}(y)\right) - Q\left(\frac{S_0}{S_n}(y)\right) = Q\,(0.4373) - Q\,(0.0000) = 0.6833 - 0.0000 = 0.6833$$
$$w_2(y) = Q\left(\frac{S_2}{S_n}(y)\right) - Q\left(\frac{S_1}{S_n}(y)\right) = Q\,(0.6987) - Q\,(0.4373) = 0.9092 - 0.6833 = 0.2259$$
$$w_3(y) = Q\left(\frac{S_3}{S_n}(y)\right) - Q\left(\frac{S_2}{S_n}(y)\right) = Q\,(0.8461) - Q\,(0.6987) = 0.9763 - 0.9092 = 0.0671$$
$$w_4(y) = Q\left(\frac{S_4}{S_n}(y)\right) - Q\left(\frac{S_3}{S_n}(y)\right) = Q\,(1.0000) - Q\,(0.8461) = 1.0000 - 0.9763 = 0.0237.$$

$$\frac{\widehat{S}_4}{\widehat{S}_n}(x) = \frac{\sum_{i=1}^{4} U(A_{\widehat{ind(i)}}(x))}{\sum_{k=1}^{4} U(A_{\widehat{ind(k)}}(x))} = \frac{4.6877}{4.6877} = 1.0000$$

$$\frac{\widehat{S}_1}{\widehat{S}_n}(y) = \frac{\sum_{i=1}^{1} U(A_{\widehat{ind(i)}}(y))}{\sum_{k=1}^{4} U(A_{\widehat{ind(k)}}(y))} = \frac{0.8070}{5.2427} = 0.1539$$

$$\frac{\widehat{S}_2}{\widehat{S}_n}(y) = \frac{\sum_{i=1}^{2} U(A_{\widehat{ind(i)}}(y))}{\sum_{k=1}^{4} U(A_{\widehat{ind(k)}}(y))} = \frac{1.5796}{5.2427} = 0.3012$$

$$\frac{\widehat{S}_3}{\widehat{S}_n}(y) = \frac{\sum_{i=1}^{3} U(A_{\widehat{ind(i)}}(y))}{\sum_{k=1}^{4} U(A_{\widehat{ind(k)}}(y))} = \frac{2.9502}{5.2427} = 0.5627$$

$$\frac{\widehat{S}_4}{\widehat{S}_n}(y) = \frac{\sum_{i=1}^{4} U(A_{\widehat{ind(i)}}(y))}{\sum_{k=1}^{4} U(A_{\widehat{ind(k)}}(y))} = \frac{5.2427}{5.2427} = 1.0000.$$

where $\frac{\widehat{S}_0}{\widehat{S}_n}(x) = 0$ and $\frac{\widehat{S}_0}{\widehat{S}_n}(y) = 0$.

Similarly, the weighting vector $\widehat{W}$ can be computed as follows.

Step 8: Respectively calculate the aggregation results for membership and non-membership of IFSs. Based on the calculated weighting vector W and $\widehat{W}$, the two aggregated IFSs $A(x) =< \mu_A(x), \nu_A(x) >$ and $A(y) =< \mu_A(y), \nu_A(y) >$ can be computed as

$$\begin{aligned}
&\mu_A(x)\\
&= F_W(\mu_{A_{ind(1)}}(x), \mu_{A_{ind(2)}}(x), \ldots, \mu_{A_{ind(n)}}(x)\\
&= \sum_{k=1}^{n} w_k(x) * \mu_{A_{ind(k)}}(x)\\
&= 0.6267 * 0.9 + 0.2016 * 0.6 + 0.1131 * 0.5 + 0.0587 * 0.2\\
&= 0.7533\\
&\nu_A(x)\\
&= F_{\widehat{W}}(\nu_{A_{\widehat{ind(1)}}}(x), \nu_{A_{\widehat{ind(2)}}}(x), \ldots, \nu_{A_{\widehat{ind(n)}}}(x)\\
&= \sum_{k=1}^{n} \widehat{w}_k(x) * \nu_{A_{\widehat{ind(k)}}}(x)\\
&= 0.0587 * 0.7 + 0.1130 * 0.3 + 0.2016 * 0.2 + 0.6267 * 0.0\\
&= 0.1153\\
&\mu_A(y)\\
&= F_W(\mu_{A_{ind(1)}}(x), \mu_{A_{ind(2)}}(x), \ldots, \mu_{A_{ind(n)}}(x)\\
&= \sum_{k=1}^{n} w_k(x) * \mu_{A_{ind(k)}}(y)\\
&= 0.6833 * 1.0 + 0.2259 * 0.8 + 0.0671 * 0.4 + 0.0237 * 0.3\\
&= 0.8978\\
&\nu_A(y)\\
&= F_{\widehat{W}}(\nu_{A_{\widehat{ind(1)}}}(y), \nu_{A_{\widehat{ind(2)}}}(y), \ldots, \nu_{A_{\widehat{ind(n)}}}(y)\\
&= \sum_{k=1}^{n} \widehat{w}_k(y) * \nu_{A_{\widehat{ind(k)}}}(y)
\end{aligned}$$

$$\widehat{w}_1(x) = \widehat{Q}\left(\frac{\widehat{S}_1}{\widehat{S}_n}(x)\right) - \widehat{Q}\left(\frac{\widehat{S}_0}{\widehat{S}_n}(x)\right) = \widehat{Q}\,(0.2423) - \widehat{Q}\,(0.0000) = 0.0587 - 0.0000 = 0.0587$$
$$\widehat{w}_2(x) = \widehat{Q}\left(\frac{\widehat{S}_2}{\widehat{S}_n}(x)\right) - \widehat{Q}\left(\frac{\widehat{S}_1}{\widehat{S}_n}(x)\right) = \widehat{Q}\,(0.4144) - \widehat{Q}\,(0.2423) = 0.1717 - 0.0587 = 0.1130$$
$$\widehat{w}_3(x) = \widehat{Q}\left(\frac{\widehat{S}_3}{\widehat{S}_n}(x)\right) - \widehat{Q}\left(\frac{\widehat{S}_2}{\widehat{S}_n}(x)\right) = \widehat{Q}\,(0.6110) - \widehat{Q}\,(0.4144) = 0.3733 - 0.1717 = 0.2016$$
$$\widehat{w}_4(x) = \widehat{Q}\left(\frac{\widehat{S}_4}{\widehat{S}_n}(x)\right) - \widehat{Q}\left(\frac{\widehat{S}_3}{\widehat{S}_n}(x)\right) = \widehat{Q}\,(1.0000) - \widehat{Q}\,(0.6110) = 1.0000 - 0.3733 = 0.6267$$

$$\widehat{w}_1(y) = \widehat{Q}\left(\frac{\widehat{S}_1}{\widehat{S}_n}(y)\right) - \widehat{Q}\left(\frac{\widehat{S}_0}{\widehat{S}_n}(y)\right) = \widehat{Q}\,(0.1539) - \widehat{Q}\,(0.0000) = 0.0237 - 0.0000 = 0.0237$$
$$\widehat{w}_2(y) = \widehat{Q}\left(\frac{\widehat{S}_2}{\widehat{S}_n}(y)\right) - \widehat{Q}\left(\frac{\widehat{S}_1}{\widehat{S}_n}(y)\right) = \widehat{Q}\,(0.3013) - \widehat{Q}\,(0.1549) = 0.0907 - 0.0237 = 0.0671$$
$$\widehat{w}_3(y) = \widehat{Q}\left(\frac{\widehat{S}_3}{\widehat{S}_n}(y)\right) - \widehat{Q}\left(\frac{\widehat{S}_2}{\widehat{S}_n}(y)\right) = \widehat{Q}\,(0.5627) - \widehat{Q}\,(0.3013) = 0.3167 - 0.0907 = 0.2259$$
$$\widehat{w}_4(y) = \widehat{Q}\left(\frac{\widehat{S}_4}{\widehat{S}_n}(y)\right) - \widehat{Q}\left(\frac{\widehat{S}_3}{\widehat{S}_n}(y)\right) = \widehat{Q}\,(1.0000) - \widehat{Q}\,(0.5627) = 1.0000 - 0.3167 = 0.6833$$

$$= 0.0237 * 0.5 + 0.0671 * 0.4 + 0.2259 * 0.1 + 0.6833 * 0.0$$
$$= 0.0613$$

Step 9: Obtain the aggregated IFSs for each criterion. Based on the calculation result shown above, the aggregated IFS $A(x)$ and $A(y)$ can be obtained, which are

$$A(x) = <\mu_A(x), \nu_A(x)> = <0.7533, 0.1153>$$
$$A(y) = <\mu_A(y), \nu_A(y)> = <0.8978, 0.0613>$$

B. DISCUSSION

In the previous related research, the aggregation of IFS has caused extensive discussion among many scholars. Many technologies are introduced to solve such a question such as Choquet integral aggregation (CIA), Sugeno integral aggregation (SIA) operators, OWA operator and so on. However, such methods have some limitations to be applied in real decision-making environment. For example, Choquet integral aggregation (CIA) and Sugeno integral aggregation (SIA) operators require the corresponding fuzzy measure, which are not objective enough. Moreover, although the ordered weighted averaging (OWA) operator has a certain degree of objectivity, it does not consider the uncertain information contained in the IFSs.

In some specific applications such as MCDM problems, the amount of uncertainty contained in IFSs given by different experts are different. For instance, there may exist the situation that some of the experts are not familiar with the target objects, which makes them give the evaluation results contained many uncertain information. Hence, if we want to aggregate the IFSs to make a synthesized assessment, the IFS which have less uncertainty should be allocated more weights compared with others. Moreover, decision makers are more willing to obtain the certain result than the uncertain one because we can get more accurate information from it.

According to the previous articles, entropy is particularly effective in measuring uncertain information. If entropy is introduced to define weights, the results can be obtained more scientifically and effectively. Thus, in the proposed method, we introduce the belief entropy to measure and analyze the uncertainty contained in IFSs which need to be aggregated. Since IFS is an effective technique to express uncertainty, the amount of uncertain information should be modeled effectively. Moreover, considering the ingenious connection of intuitionistic fuzzy sets and evidence theory, the membership degree, non-membership degree and hesitancy degree is converted to BPAs. Then we introduce belief entropy to measure the uncertainty, which allows decision makers to weight the uncertainty factor. Note that the obtained uncertainty weights based on the calculation results of belief entropy can well embody the uncertainty contained in associated IFSs.

Furthermore, the preference relationship of decision makers is also modeled by using RIM quantifier Q function. For example, in Section IV-A, the Q function for membership degree is determined as $Q(x) = 1 - (1 - x)^2$. After the calculation process, the $\widehat{Q}$ for non-membership degree is calculated as $Q(x) = x^2$, which is regard as the dual of Q function. The schematic diagram of Q and $\widehat{Q}$ function is shown in Fig. 2.

Here we note that the red line represents $Q(x) = x$, which means decision makers do not consider the impact of personal preferences. Furthermore, the blue line represents the selected Q function in Section IV and the yellow line represents the dual of Q function $\widehat{Q}$. As shown in Fig. 2, we take two coordinates (0.4, 0.64) and (0.4, 0.16) for instance. With the same independent variable, the values obtained after the Q

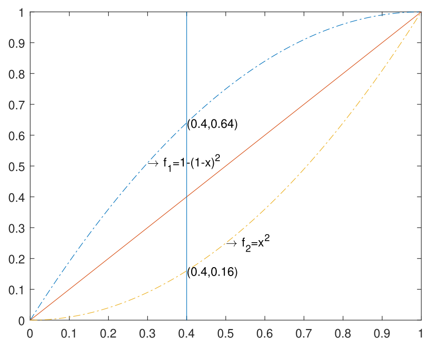

FIGURE 2. The schematic diagram of Q and $\widehat{Q}$ function.

and $\widehat{Q}$ function operation are different. Moreover, the weights allocated to the preceding arguments are not the same due to the difference in the nature of the function. For the Q function, To be specific, the higher the weight of the sorted values, the higher the weight allocated to the arguments. And the case of $\widehat{Q}$ is opposite. Thus, the selection of Q function can effectively embody the preference relationship for decision makers.

As shown in the results, it expresses the decision maker prefers the more certain evaluation. Note that the specific definition of Q function is sundry such as linguistic method, etc. And the aggregated IFS has been proved to be valid at all times. Thus, it's obviously to see the practical significance of the new method.

The calculation results for some of other relative approaches are shown in Table 4. Note that the mean operator and IFA operator are completely data driven which do not need decision makers to offer subjectively weights. Moreover, for IFOWA operator, we assume that the RIM Q function for membership degree is $Q(x) = 1 - (1 - x)^2$, which is consistent with the Q function used in Section IV.

TABLE 4. The calculation results for several relative approaches.

Aggregation method	Result for criterion X	Result for criterion Y
IFA method [42]	$< 0.2400, 0.0000 >$	$< 0.2500, 0.0000 >$
Arithmetic mean operator [43]	$< 0.5500, 0.3000 >$	$< 0.6250, 0.2500 >$
IFOWA method [44]	$< 0.6875, 0.0625 >$	$< 0.7812, 0.1375 >$
Proposed method *	$< 0.7533, 0.1153 >$	$< 0.8978, 0.0613 >$

It's obviously to see that the aggregation result for IFA method is counterintuitive, which is not effective to deal with the cases that the values of IFSs have significant difference [42]. As for the arithmetic mean operator [43], it cannot respond well to the willingness of decision makers' preference relationship and does not take into account the uncertainty of experts, which is plain and not flexible enough. For the IFOWA operator [44], it mainly consider the size relationship of numerical values and do not make use of the amount of uncertain information contained in IFSs. Therefore, even if the aggregated results of such approaches can express the difference between the criterion x and y, they still have some limitations compared with the proposed method.

V. CONCLUSION

In this paper, a new method to aggregate IFSs is proposed to handle MCDM problems. The main contribution is to consider the uncertain information contained in associated IFSs to make the aggregation more considerate and effective. The belief entropy and evidence theory are introduced to measure the uncertainty. Furthermore, the preference relationship is also premeditated based on various definitions of quantifier Q function in WOWA operator.

It should be pointed that there are some limitations of the proposed method. For example, when the data sets have their own induced value, the proposed method is not efficient enough, which is one of our ongoing works.

REFERENCES

[1] G. Kabir, R. Sadiq, and S. Tesfamariam, "A review of multi-criteria decision-making methods for infrastructure management," *Struct. Infrastruct. Eng.*, vol. 10, no. 9, pp. 1176–1210, 2014.

[2] F. Xiao, "A hybrid fuzzy soft sets decision making method in medical diagnosis," *IEEE Access*, vol. 6, pp. 25300–25312, 2018.

[3] L. Fabisiak, "Web service usability analysis based on user preferences," *J. Organizational End User Comput. (JOEUC)*, vol. 30, no. 4, pp. 1–13, 2018.

[4] C. Fu, D.-L. Xu, and M. Xue, "Determining attribute weights for multiple attribute decision analysis with discriminating power in belief distributions," *Knowl.-Based Syst.*, vol. 143, no. 1, pp. 127–141, 2018.

[5] K. Chatterjee, E. K. Zavadskas, J. Tamošaitiene, K. Adhikary, and S. Kar, "A hybrid MCDM technique for risk management in construction projects," *Symmetry*, vol. 10, no. 2, p. 46, 2018.

[6] Y. Han and Y. Deng, "A hybrid intelligent model for assessment of critical success factors in high-risk emergency system," *J. Ambient Intell. Humanized Comput.*, vol. 9, no. 6, pp. 1933–1953, 2018.

[7] H. Seiti, A. Hafezalkotob, S. E. Najafi, and M. Khalaj, "A risk-based fuzzy evidential framework for FMEA analysis under uncertainty: An interval-valued DS approach," *J. Intell. Fuzzy Syst.*, vol. 35, no. 2, pp. 1419–1430, 2018

[8] K. Chatterjee, D. Pamucar, and E. K. Zavadskas, "Evaluating the performance of suppliers based on using the R'AMATEL-MAIRCA method for green supply chain implementation in electronics industry," *J. Cleaner Prod.*, vol. 184, pp. 101–129, May 2018.

[9] H. Seiti, A. Hafezalkotob, and R. Fattahi, "Extending a pessimistic–optimistic fuzzy information axiom based approach considering acceptable risk: Application in the selection of maintenance strategy," *Appl. Soft Comput.*, vol. 67, pp. 895–909, Jun. 2018.

[10] L. Fei, Y. Deng, and Y. Hu, "DS-VIKOR: A new multi-criteria decision-making method for supplier selection," *Int. J. Fuzzy Syst.*, vol. 21, no. 1, pp. 157–175, 2019.

[11] W. Jiang, C. Xie, M. Zhuang, and Y. Tang, "Failure mode and effects analysis based on a novel fuzzy evidential method," *Appl. Soft Comput.*, vol. 57, pp. 672–683, Aug. 2017.

[12] L. Chen and Y. Deng, "A new failure mode and effects analysis model using Dempster–Shafer evidence theory and grey relational projection method," *Eng. Appl. Artif. Intell.*, vol. 76, pp. 13–20, Nov. 2018.

[13] Z. Liu and F. Xiao, "An intuitionistic evidential method for weight determination in FMEA based on belief entropy," *Entropy*, vol. 21, no. 2, p. 211, 2019.

[14] F. Xiao, "A multiple-criteria decision-making method based on D numbers and belief entropy," *Int. J. Fuzzy Syst.*, vol. 21, no. 4, pp. 1144–1153, 2019.

[15] X. Deng, W. Jiang, and Z. Wang, "Zero-sum polymatrix games with link uncertainty: A Dempster–Shafer theory solution," *Appl. Math. Comput.*, vol. 340, pp. 101–112, Jan. 2019.

[16] R. E. Bellman and L. A. Zadeh, "Decision-making in a fuzzy environment," *Manage. Sci.*, vol. 17, no. 4, pp. 141–164, 1970.

[17] L. A. Zadeh, "Fuzzy sets," *Inf. Control*, vol. 8, no. 3, pp. 338–353, Jun. 1965.

[18] G. Wei, "Maximizing deviation method for multiple attribute decision making in intuitionistic fuzzy setting," *Knowl.-Based Syst.*, vol. 21, no. 8, pp. 833–836, 2008.

[19] K. Kumar and H. Garg, "Connection number of set pair analysis based TOPSIS method on intuitionistic fuzzy sets and their application to decision making," *Appl. Intell.*, vol. 48, no. 8, pp. 2112–2119, Aug. 2018.
[20] G.-W. Wei, "Gray relational analysis method for intuitionistic fuzzy multiple attribute decision making," *Expert Syst. Appl.*, vol. 38, no. 9, pp. 11671–11677, 2011.
[21] H. Garg and R. Arora, "A nonlinear-programming methodology for multi-attribute decision-making problem with interval-valued intuitionistic fuzzy soft sets information," *Appl. Intell.*, vol. 48, no. 8, pp. 2031–2046, Aug. 2018.
[22] J.-Q. Wang, P. Wang, J. Wang, H.-Y. Zhang, and X.-H. Chen, "Atanassov's interval-valued intuitionistic linguistic multicriteria group decision-making method based on the trapezium cloud model," *IEEE Trans. Fuzzy Syst.*, vol. 23, no. 3, pp. 542–554, Jun. 2015.
[23] P. Liu, "Some Hamacher aggregation operators based on the interval-valued intuitionistic fuzzy numbers and their application to group decision making," *IEEE Trans. Fuzzy Syst.*, vol. 22, no. 1, pp. 83–97, Feb. 2014.
[24] V. Torra, "Hesitant fuzzy sets," *Int. J. Intell. Syst.*, vol. 25, no. 6, pp. 529–539, 2010.
[25] H. Garg and R. Arora, "Dual hesitant fuzzy soft aggregation operators and their application in decision-making," *Cogn. Comput.*, vol. 10, no. 5, pp. 769–789, Oct. 2018.
[26] X.-K. Wang, H.-G. Peng, and J.-Q. Wang, "Hesitant linguistic intuitionistic fuzzy sets and their application in multicriteria decision-making problems," *Int. J. Uncertainty Quantification*, vol. 8, no. 4, pp. 321–341, 2018.
[27] R. R. Yager, "OWA aggregation over a continuous interval argument with applications to decision making," *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 34, no. 5, pp. 1952–1963, Oct. 2004.
[28] P. Liu and P. Wang, "Some q-rung orthopair fuzzy aggregation operators and their applications to multiple-attribute decision making," *Int. J. Intell. Syst.*, vol. 33, no. 2, pp. 259–280, Feb. 2018.
[29] P. Liu and S.-M. Chen, "Group decision making based on Heronian aggregation operators of intuitionistic fuzzy numbers," *IEEE Trans. Cybern.*, vol. 47, no. 9, pp. 2514–2530, Sep. 2017.
[30] W. Pedrycz and F. Gomide, *Fuzzy Systems Engineering: Toward Human-Centric Computing*. Hoboken, NJ, USA: Wiley, 2007.
[31] R. R. Yager, "On ordered weighted averaging aggregation operators in multicriteria decisionmaking," *IEEE Trans. Syst., Man, Cybern.*, vol. 18, no. 1, pp. 183–190, Jan./Feb. 1988.
[32] F. Xiao, "A novel multi-criteria decision making method for assessing health-care waste treatment technologies based on D numbers," *Eng. Appl. Artif. Intell.*, vol. 71, pp. 216–225, May 2018.
[33] F. Xiao and W. Ding, "Divergence measure of Pythagorean fuzzy sets and its application in medical diagnosis," *Appl. Soft Comput.*, vol. 79, pp. 254–267, Jun. 2019.
[34] P. Ji, J.-Q. Wang, and H.-Y. Zhang, "Frank prioritized Bonferroni mean operator with single-valued neutrosophic sets and its application in selecting third-party logistics providers," *Neural Comput. Appl.*, vol. 30, no. 3, pp. 799–823, Aug. 2018.
[35] W. Jiang, B. Wei, X. Liu, X. Li, and H. Zheng, "Intuitionistic fuzzy power aggregation operator based on entropy and its application in decision making," *Int. J. Intell. Syst.*, vol. 33, no. 1, pp. 49–67, 2018.
[36] Y. Song, X. Wang, W. Wu, L. Lei, and W. Quan, "Uncertainty measure for Atanassov's intuitionistic fuzzy sets," *Appl. Intell.*, vol. 46, no. 4, pp. 757–774, 2017.
[37] C. Fu, D.-L. Xu, and S.-L. Yang, "Distributed preference relations for multiple attribute decision analysis," *J. Oper. Res. Soc.*, vol. 67, no. 3, pp. 457–473, 2016.
[38] A. I. Ban, "Sugeno integral with respect to intuitionistic fuzzy-valued fuzzy measures," *Notes Intuitionistic Fuzzy Sets*, vol. 11, no. 1, pp. 47–61, 2005.
[39] M. Grabisch, T. Murofushi, and M. Sugeno, *Fuzzy Measures and Integrals: Theory and Applications*. Berlin, Germany: Physica-Verlag, 2000.
[40] Z. Xu, "Choquet integrals of weighted intuitionistic fuzzy information," *Inf. Sci.*, vol. 180, no. 5, pp. 726–736, 2010.
[41] R. R. Yager, "Families of OWA operators," *Fuzzy Sets Syst.*, vol. 59, no. 2, pp. 125–148, 1993.
[42] Z. Xu, "Intuitionistic fuzzy aggregation operators," *IEEE Trans. Fuzzy Syst.*, vol. 15, no. 6, pp. 1179–1187, Dec. 2007.
[43] X. Wang, "Fuzzy number intuitionistic fuzzy arithmetic aggregation operators," *Int. J. Fuzzy Syst.*, vol. 10, no. 2, pp. 104–111, Jun. 2008.
[44] H. Mitchell, "An intuitionistic OWA operator," *Int. J. Uncertainty Fuzziness Knowl.-Based Syst.*, vol. 12, no. 6, pp. 843–860, Dec. 2004.
[45] Z. Xu and R. R. Yager, "Some geometric aggregation operators based on intuitionistic fuzzy sets," *Int. J. Gen. Syst.*, vol. 35, no. 4, pp. 417–433, Aug. 2006.
[46] D. Yu, "Intuitionistic fuzzy geometric Heronian mean aggregation operators," *Appl. Soft Comput.*, vol. 13, no. 2, pp. 1235–1246, 2013.
[47] Z. Xu, "Approaches to multiple attribute group decision making based on intuitionistic fuzzy power aggregation operators, " *Knowl.-Based Syst.*, vol. 24, no. 6, pp. 749–760, 2011.
[48] X. Deng, "Analyzing the monotonicity of belief interval based uncertainty measures in belief function theory," *Int. J. Intell. Syst.*, vol. 33, no. 9, pp. 1869–1879, Sep. 2018. doi: 10.1002/int.21999.
[49] X. Gao and Y. Deng, "The generalization negation of probability distribution and its application in target recognition based on sensor fusion," *Int. J. Distrib. Sensor Netw.*, 2019. doi: 10.1177/1550147719849381.
[50] C. Fu and D.-L. Xu, "Determining attribute weights to improve solution reliability and its application to selecting leading industries," *Ann. Oper. Res.*, vol. 245, pp. 401–426, Oct. 2016.
[51] R. Sun and Y. Deng, "A new method to identify incomplete frame of discernment in evidence theory," *IEEE Access*, vol. 7, pp. 15547–15555, 2019.
[52] C. E. Shannon, "A mathematical theory of communication," *Bell Syst. Tech. J.*, vol. 27, no. 3, pp. 379–423, Jul./Oct. 1948.
[53] Y. Deng, "Deng entropy," *Chaos, Solitons Fractals*, vol. 91, pp. 549–553, Oct. 2016.
[54] V. Torra, "The weighted OWA operator," *Int. J. Intell. Syst.*, vol. 12, no. 2, pp. 153–166, Feb. 1997.
[55] I. Dzitac, F. G. Filip, and M.-J. Manolescu, "Fuzzy logic is not fuzzy: World-renowned computer scientist lotfi a. zadeh," *Int. J. Comput. Commun. Control*, vol. 12, no. 6, pp. 748–789, 2017.
[56] D. Meng, M. Liu, S. Yang, H. Zhang, and R. Ding, "A fluid–structure analysis approach and its application in the uncertainty-based multidisciplinary design and optimization for blades," *Adv. Mech. Eng.*, vol. 10, no. 6, pp. 1–7, 2018, Art. no. 1687814018783410.
[57] H. Yang, Y. Deng, and J. Jones, "Network division method based on cellular growth and physarum-inspired network adaptation," *Int. J. Unconventional Comput.*, vol. 13, no. 6, pp. 477–491, 2018.
[58] D. Meng, S. Yang, Y. Zhang, and S.-P. Zhu, "Structural reliability analysis and uncertainties-based collaborative design and optimization of turbine blades using surrogate model," *Fatigue Fract. Eng. Mater. Struct.*, vol. 42, no. 6, pp. 1219–1227, 2019.
[59] A. P. Dempster, "Upper and lower probabilities induced by a multivalued mapping," *Ann. Math. Statist.*, vol. 38, no. 2, pp. 325–339, 1967.
[60] G. Shafer, *A Mathematical Theory of Evidence*. Princeton, NJ, USA: Princeton Univ. Press, 1976.
[61] Y. Han and Y. Deng, "An evidential fractal AHP target recognition method," *Defence Sci. J.*, vol. Vol., 68, no. 4, Jul. 2018, pp. 367–373, 2018.
[62] L. Chen and X. Deng, "A modified method for evaluating sustainable transport solutions based on AHP and Dempster–Shafer evidence theory," *Appl. Sci.*, vol. 8, no. 4, p. 563, 2018.
[63] B. Kang, P. Zhang, Z. Gao, G. Chhipi-Shrestha, K. Hewage, and R. Sadiq, "Environmental assessment under uncertainty using Dempster–Shafer theory and Z-numbers," *J. Ambient Intell. Humanized Comput.*, 2019. doi: 10.1007/s12652-019-01228-y
[64] X. Deng and W. Jiang, "An evidential axiomatic design approach for decision making using the evaluation of belief structure satisfaction to uncertain target values," *Int. J. Intell. Syst.*, vol. 33, no. 1, pp. 15–32, 2018.
[65] J. Wang, K. Qiao, and Z. Zhang, "An improvement for combination rule in evidence theory," *Future Gener. Comput. Syst.*, vol. 91, pp. 1–9, Feb. 2019.
[66] R. Sun and Y. Deng, "A new method to determine generalized basic probability assignment in the open world," *IEEE Access*, vol. 7, no. 1, pp. 52827–52835, 2019.
[67] H. Zhang and Y. Deng, "Engine fault diagnosis based on sensor data fusion considering information quality and evidence theory," *Adv. Mech. Eng.*, vol. 10, no. 11, pp. 1–10, 2018. doi: 10.1177/1687814018809184.
[68] H. Xu and Y. Deng, "Dependent evidence combination based on DEMATEL method," *Int. J. Intell. Syst.*, vol. 34, no. 7, pp. 1555–1571, 2019. doi: 10.1002/int.22107.
[69] W. Jiang and W. Hu, "An improved soft likelihood function for Dempster–Shafer belief structures," *Int. J. Intell. Syst.*, vol. 33, no. 6, pp. 1264–1282, 2018. doi: 10.1002/int.21980.
[70] X. Gao and Y. Deng, "The negation of basic probability assignment," *IEEE Access*, no. 1, 2019. doi: 10.1109/ACCESS.2019.2901932.
[71] W. Jiang, "A correlation coefficient for belief functions," *Int. J. Approx. Reasoning*, vol. 103, pp. 94–106, Dec. 2018.

[72] F. Xiao, "Multi-sensor data fusion based on the belief divergence measure of evidences and the belief entropy," *Inf. Fusion*, vol. 46, pp. 23–32, Mar. 2019.

[73] Y. Song and Y. Deng, "A new method to measure the divergence in evidential sensor data fusion," *Int. J. Distrib. Sensor Netw.*, vol. 15, no. 4, pp. 1–8, 2019, Art. no. 1550147719841295. doi: 10.1177/1550147719841295.

[74] D. Filev and R. R. Yager, "On the issue of obtaining OWA operator weights," *Fuzzy Sets Syst.*, vol. 94, no. 2, pp. 157–169, 1998.

[75] R. R. Yager, "Quantifier guided aggregation using OWA operators," *Int. J. Intell. Syst.*, vol. 11, no. 1, pp. 49–73, 1996.

[76] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets Syst.*, vol. 20, pp. 87–96, Aug. 1986.

[77] W. Ding, C.-T. Lin, and Z. Cao, "Deep neuro-cognitive co-evolution for fuzzy attribute reduction by quantum leaping PSO with nearest-neighbor memeplexes," *IEEE Trans. Cybern.*, vol. 49, no. 7, pp. 2744–2757, Jul. 2019.

[78] Y. Song and X. Wang, "A new similarity measure between intuitionistic fuzzy sets and the positive definiteness of the similarity matrix," *Pattern Anal. Appl.*, vol. 20, no. 1, pp. 215–226, 2017.

[79] L. Dymova and P. Sevastjanov, "An interpretation of intuitionistic fuzzy sets in terms of evidence theory: Decision making aspect," *Knowl.-Based Syst.*, vol. 23, no. 8, pp. 772–782, Dec. 2010.

[80] I. Tuğal and A. Karcı, "Comparisons of Karcı and Shannon entropies and their effects on centrality of social networks," *Phys. A, Stat. Mech. Appl.*, vol. 523, pp. 352–363, Jun. 2019.

[81] L. Pan and Y. Deng, "A new belief entropy to measure uncertainty of basic probability assignments based on belief function and plausibility function," *Entropy*, vol. 20, no. 11, p. 842, 2018.

[82] Y. Li and Y. Deng, "Generalized ordered propositions fusion based on belief entropy," *Int. J. Comput. Commun. Control*, vol. 13, no. 5, pp. 792–807, 2018.

[83] H. Cui, Q. Liu, J. Zhang, and B. Kang, "An improved deng entropy and its application in pattern recognition," *IEEE Access*, vol. 7, pp. 18284–18292, 2019.

[84] Y. Dong, J. Zhang, Z. Li, Y. Hu, and Y. Deng, "Combination of evidential sensor reports with distance function and belief entropy in fault diagnosis," *Int. J. Comput. Commun. Control*, vol. 14, no. 3, pp. 293–307, 2019.

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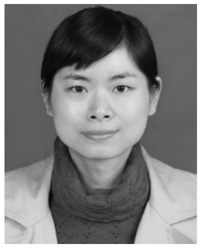

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