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Duality of the Evolute-Involute Pair and Its Application

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ABSTRACT We studied the type of cusps on the caustic in the optical model that the coplanar point light source irradiates a singular planar curvilinear mirror in our last paper. In this paper, we give a one-to-one correspondence between the wavefront (orthotomic) and the caustic. According to this one-to-one correspondence, we can derive the property of one of them from that of the other one. To prove it, we formulate the involutes of (n, m)-cusp curves and study their singular properties. Furthermore, we give a duality theorem for evolute-involute pairs. This theorem plays a crucial role in the one-to-one correspondence mentioned earlier.

INDEX TERMS Optical reflection, optical propagation, optical surface waves, optical imaging, geometry.

I. INTRODUCTION

Recently, optics is getting attention increasingly, especially the wide application of the dual-comb spectroscopy (see [5]). Many mathematical theories are used in optics. By selecting appropriate variables, some optical objects could be converted into simple mathematical formulas, such as caustics and wavefronts in the optical system: a point light source irradiates the planar curvilinear mirror or the refringent sphere in the same plane (see [2]–[4], [12]). Some problems about caustics with symmetry could be studied by using the catastrophe theory in geometry and the paraxial approximation (see [11], [13]). The method of differential equations are usually applied to solve problems about Gaussian beams in complex optical systems (see [6], [7]). The development of mathematical theory promotes the research in optics.

In the present paper, what we are most interested in is the property of caustics in the optical system that a point light source irradiates a planar curvilinear mirror in the same plane (see [1], [4]). Caustics in optics are closely related to evolutes in geometry. In the language of differential geometry, the original curve with respect to the evolute is called an involute (see [8], [9]). For regular curves, their evolutes and involutes have been studied comprehensively in the classical differential geometry. But singular curves are unavoidable in practice. As the typical singular maps, fronts have been studied in [14], [16]. More general, Fukui [10] defined the multiplicity for smooth curves and studied the asymptotic behavior of curvature functions. Later we studied curves which allow (n, m)-cusps in 2-dimensional Euclidean space and their evolutes in [17]. Among all plane curves with finite multiplicities, curves which allow (n, m)-cusps are generic. One typical example of the curve which allows (n, m)-cusps is the non-planar spherical helix (see [15]).

In [17], we studied which type of cusps could occur on caustics when the mirror is singular. We convert this problem to the correspondence between the wavefront and the caustic. A natural problem we are facing now is that if this correspondence is a one-to-one correspondence. If this is true, we can directly derive the focal property of the mirror from the analysis of its wavefront. There are rich results about wavefronts in optics and geometry. This is our motivation of the present paper.

The present paper is arranged as follows. We formulate involutes of curves which allow (n, m)-cusps and study their singular properties in Section 2. In Section 3, we research deeper geometric information about involutes of curves which allow (n, m)-cusps. As the main result, we give the duality theorem for the evolute-involute pairs. As an application in optics, we show that there is a one-to-one correspondence between the wavefront and the caustic in the considering optical system in Section 4.

All maps considered in the present paper are of class C^∞.

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TABLE 1. Type of $Inv(\gamma,\ t_0)(t)$ at $t = 0$.

t_0	$t_0 \neq 0$	$t_0 = 0$
Type of $Inv(\gamma,\ t_0)$	$(m-n, 2(m-n))$-cusp	$(m, 2m-n)$-cusp

II. INVOLUTES OF CUSP CURVES

In [17], we say that the curve $\gamma\ :\ (-\epsilon,\ \epsilon) \to \mathbb{R}^2$ has an $(n,\ m)$-cusp at $t = 0$ if $(\gamma^{(n)}(0),\ \gamma^{(m)}(0))$ is a linearly independent vector pair with the lowest order. The curve γ is called an $(n,\ m)$-cusp curve. The two integers

$$ind(\gamma) = m - n$$

and

$$ord(\gamma) = n$$

are called the *index* and the *multiplicity* of γ respectively. The index and the multiplicity are invariants in 2-dimensional Euclidean space. We endow $\gamma(t)$ with an *orientation* which is the direction as the parameter t increases.

In order to apply the method of moving frames, we introduced two unit orthogonal vectors in [17] as follows:

$$\begin{cases} \mathbf{t}(t) = \dfrac{\gamma'(t)/t^{n-1}}{\|\gamma'(t)/t^{n-1}\|}, \\ \mathbf{n}(t) = \mathbf{J}\mathbf{t}(t), \end{cases}$$

where $\mathbf{J}$ is the counterclockwise rotation by $\frac{\pi}{2}$ on $\mathbb{R}^2$. We have

$$\frac{d}{dt}\begin{pmatrix} \mathbf{t}(t) \\ \mathbf{n}(t) \end{pmatrix} = \begin{pmatrix} 0 & f(t) \\ -f(t) & 0 \end{pmatrix}\begin{pmatrix} \mathbf{t}(t) \\ \mathbf{n}(t) \end{pmatrix}.$$

We say that $f(t)$ is the *associate function*. According to the fundamental theorem, we have that the multiplicity and the associate function determine a cusp curve up to a rigid motion in $\mathbb{R}^2$.

As we all know, the orthogonal trajectories of all the tangent lines to a curve are involutes of the given curve in 2-dimensional Euclidean space. But the classical Frenet-Serret type frame is not well defined for singular curves. It is comparatively convenient to use the modified Frenet-Serret type frame to formulate involutes of curves which admit $(n,\ m)$-cusps and to investigate their properties.

Under above notations, we define the *involute* of $\gamma(t)$ as follows:

$$Inv(\gamma,\ t_0)(t) = \gamma(t) - \left(\int_{t_0}^{t} \gamma'(t)\cdot\mathbf{t}(t)dt\right)\mathbf{t}(t),\ t_0 \in \mathbb{R}.$$

When $n = 1$ and $m = 2$, this is the same as the classical definition of involutes. We can see that involutes form a one-parameter family of parallels depending on the parameter $t_0 \in \mathbb{R}$.

By calculations, we conclude the singular properties of involutes by Table 1. When $t_0 \neq 0$, the $(m-n,\ 2(m-n))$-cusp on $Inv(\gamma,\ t_0)$ is the corresponding point of the $(n,\ m)$-cusp on γ. When $t_0 = 0$, the $(m,\ 2m-n)$-cusp on $Inv(\gamma,\ 0)$ is the corresponding point of the $(n,\ m)$-cusp on γ.

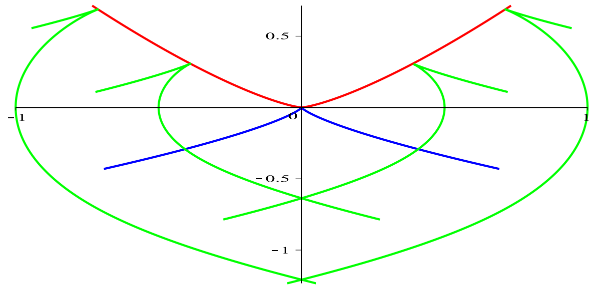

FIGURE 1. The red curve is $\gamma(t) = \left(\frac{t^3}{3},\ \frac{t^4}{4}\right)$, the blue curve is $Inv(\gamma)$ and green curves are $Inv(\gamma,\ t_0)$ with $t_0 \neq 0$.

We mainly study $Inv(\gamma,\ 0)$ in the present paper since properties of other involutes could be induced by analysis of parallels. In what follows, we use $Inv(\gamma)$ to denote $Inv(\gamma,\ 0)$ for simplicity. By calculations, we have that the modified Frenet-Serret type frame of $Inv(\gamma)$ is

$$\{Inv(\gamma)(t);\ -\mathbf{n}(t),\ \mathbf{t}(t)\},$$

the associate function of $Inv(\gamma)$ is also $f(t)$ and

$$\begin{cases} ord(Inv^k(\gamma)) = n + k(m-n), \\ ind(Inv^k(\gamma)) = m - n. \end{cases}$$

Here $Inv^k(\gamma)$ is the k-th involute of γ.

Now we give some explicit examples of involutes in different cases.

Example 1: Let the curve $\gamma\ :\ (-2, 2) \to \mathbb{R}^2$ be parameterized by

$$\gamma(t) = \left(\frac{t^3}{3},\ \frac{t^4}{4}\right).$$

By calculations, we have that $t = 0$ is a $(3,\ 4)$-cusp of γ and

$$Inv(\gamma, t_0)(t) = \left(\frac{t^3}{3} - \frac{\int_{t_0}^{t} t^2(1+t^2)^{1/2}dt}{(1+t^2)^{1/2}},\ \frac{t^4}{4} - \frac{t\int_{t_0}^{t} t^2(1+t^2)^{1/2}dt}{(1+t^2)^{1/2}}\right),$$

where $t_0 \in \mathbb{R}$. Then $t = 0$ is a $(4,\ 5)$-cusp of $Inv(\gamma)$ and $t = 0$ is a $(1,\ 2)$-cusp of $Inv(\gamma,\ t_0)$ when $t_0 \neq 0$ (see Figure 1).

Example 2: Let the curve $\gamma\ :\ (-2, 2) \to \mathbb{R}^2$ be parameterized by

$$\gamma(t) = \left(\frac{t^2}{2},\ \frac{t^5}{5}\right).$$

By calculations, we have that $t = 0$ is a $(2,\ 5)$-cusp of γ and

$$Inv(\gamma,\ t_0)(t) = \left(\frac{t^2}{2} - \frac{\int_{t_0}^{t} t(1+t^6)^{1/2}dt}{(1+t^6)^{1/2}},\ \frac{t^5}{5} - \frac{t^3\int_{t_0}^{t} t(1+t^6)^{1/2}dt}{(1+t^6)^{1/2}}\right),$$

where $t_0 \in \mathbb{R}$. Then $t = 0$ is a $(5,\ 8)$-cusp of $Inv(\gamma)$ and $t = 0$ is a $(3,\ 6)$-cusp of $Inv(\gamma,\ t_0)$ when $t_0 \neq 0$ (see Figure 2).

In order to have an intuitive understanding of the evolute-involute pairs, we give some typical examples of evolutes.

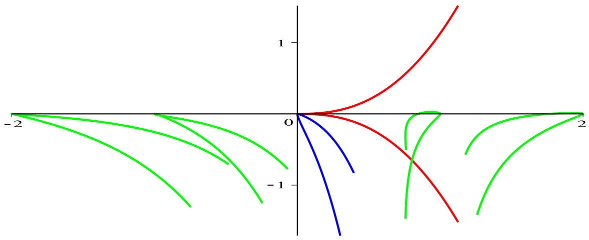

FIGURE 2. The red curve is $\gamma(t) = \left(\frac{t^2}{2}, \frac{t^5}{5}\right)$, the blue curve is $Inv(\gamma)$ and green curves are $Inv(\gamma,\ t_0)$ with $t_0 \neq 0$.

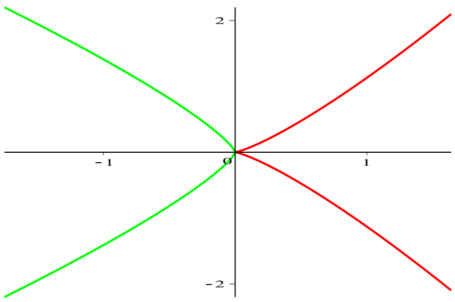

FIGURE 3. The red curve is $\left(\frac{t^4}{4}, \frac{t^5}{5}\right)$ and $Ev(\gamma)$ is green.

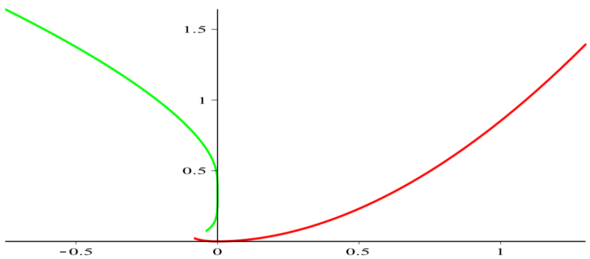

FIGURE 4. The red curve is $\left(\frac{t^3}{3} + \frac{t^4}{4}, \frac{t^6}{6} + \frac{t^7}{7}\right)$ and $Ev(\gamma)$ is green.

Example 3: Let the curve $\gamma : (-2, 2) \to \mathbb{R}^2$ be parameterized by

$$\gamma(t) = \left(\frac{t^4}{4}, \frac{t^5}{5}\right).$$

By calculations, we have $t = 0$ is a $(3,\ 4)$-cusp of $Ev(\gamma)$ (See Figure 3).

Example 4: Let the curve $\gamma : (-2, 2) \to \mathbb{R}^2$ be parameterized by

$$\gamma(t) = \left(\frac{t^3}{3} + \frac{t^4}{4}, \frac{t^6}{6} + \frac{t^7}{7}\right).$$

By calculations, we have $t = 0$ is a $(1,\ 4)$-cusp of $Ev(\gamma)$ (See Figure 4).

Example 5: Let the curve $\gamma : (-2, 2) \to \mathbb{R}^2$ be parameterized by

$$\gamma(t) = \left(\frac{t^3}{3}, \frac{t^8}{8}\right).$$

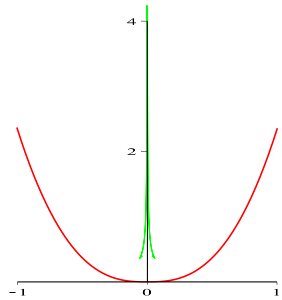

FIGURE 5. The red curve is $\left(\frac{t^3}{3}, \frac{t^8}{8}\right)$ and $Ev(\gamma)$ is green.

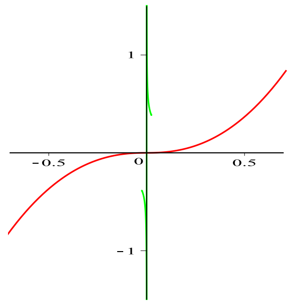

FIGURE 6. The red curve is $\left(\frac{t^3}{3}, \frac{t^7}{7}\right)$ and $Ev(\gamma)$ is green.

By calculations, we have that the normal line of γ is the asymptote of the two parts of $Ev(\gamma)$. The two parts have the same asymptotic direction (See Figure 5).

Example 6: Let the curve $\gamma : (-2, 2) \to \mathbb{R}^2$ be parameterized by

$$\gamma(t) = \left(\frac{t^3}{3}, \frac{t^7}{7}\right).$$

By calculations, we have that the normal line of γ is the asymptote of the two parts of $Ev(\gamma)$. The two parts have the opposite asymptotic direction (See Figure 6).

III. DUALITY THEOREM FOR THE EVOLUTE-INVOLUTE PAIRS

We will give the main theorem in the present section.

Let $\gamma : (-\epsilon,\ \epsilon) \to \mathbb{R}^2$ be a curve with an $(n,\ m)$-cusp at $t = 0$. Suppose $Inv(\gamma)$ is an $(n_i,\ m_i)$-cusp curve at $t = 0$. Then $m_i < 2n_i$. Therefore we can always define the evolute of $Inv(\gamma)$. We easily have

$$Ev(Inv(\gamma))(t) = \gamma(t).$$

Therefore every $(n,\ m)$-cusp curve can be regard as the evolute of it involute.

Now we study the correspondence of their cusps. On the one hand, if $t = 0$ is an $(n,\ m)$-cusp of $\gamma(t)$. Then we have that $t = 0$ is an $(m,\ 2m - n)$-cusp of $Inv(\gamma)(t)$. On the other hand, if $t = 0$ is an $(m,\ 2m - n)$-cusp of $Inv(\gamma)(t)$. Since

$$Ev(Inv(\gamma))(t) = \gamma(t).$$

Then according to [17], we have that $t = 0$ on $\gamma(t)$ is just an (n, m)-cusp. In conclusion, we obtain the following duality theorem for the evolute-involute pairs.

Theorem 1 (Duality theorem): Let $\gamma : (-\epsilon, \epsilon) \to \mathbb{R}^2$ be a smooth curve. Then $t = 0$ is an (n, m)-cusp of $\gamma(t)$ if and only if $t = 0$ is an $(m, 2m - n)$-cusp of its involute $Inv(\gamma)(t)$.

IV. APPLICATION IN OPTICS

In [17], we know that the wavefront (orthotomic) is an $(n, n + 1)$-cusp curve for some integer n. According to Table 1, we have

Case 1: If wavefront is an involute of the caustic C in the form of $Inv(C, t_0)$ $(t_0 \neq 0)$. We have $n+1 = 2n$, then $n = 1$. It is a regular point on the wavefront.

Case 2: If wavefront is an involute of the caustic C in the form of $Inv(C, 0)$. We have $n + 1 < 2n$, then $n > 1$. It is a singular point on the wavefront.

What we are more interested in is the property of singular points in the present paper. Therefore we only consider the case $n > 1$, that is, the case

$$W = Inv(C) = Inv(C, 0).$$

Therefore we obtain a one-to-one correspondence between singular wavefronts and caustics.

V. CONCLUSION

In conclusion, we obtain a one-to-one correspondence between singularities on the wavefront (orthotomic) and cusps on the caustic in the optical system: a point light source irradiates the planar curvilinear mirror placed in the same plane. In explicit, the $(n - 1, n)$-cusp on the caustic one-to-one corresponds with the $(n, n+1)$-cusp on the wavefront (orthotomic) when $n > 1$. However, there also exist some other issues to be studied for improving our work. Vertices on the wavefront might induce singularities on caustics. We have not verified this correspondence is one-to-one or not up to now. Because this proof involves very complicated calculations. In future work, we will take vertices on the wavefront into consideration in the correspondence between caustics and wavefronts.

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