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# SISO Intuitionistic Fuzzy Systems: IF-*t*-Norm, IF-*R*-Implication, and Universal Approximators

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**ABSTRACT** Fuzzy system, as the main method of approximate reasoning, has been widely used to solve practical problems. The existing results on fuzzy systems were based on fuzzy sets, fuzzy *if-then* rule, and fuzzy inference. In this work, we investigate the component of the intuitionistic fuzzy systems, such as the intuitionistic fuzzy triangular norms (IF-*t*-norm), the intuitionistic fuzzy residual implications (IF-*R*-implication), the intuitionistic fuzzy *if-then* rule, the intuitionistic fuzzy inference, and the intuitionistic defuzzification. Accordingly, the single input single output (SISO) intuitionistic fuzzy systems are built. Some examples are paid to illustrate the methods and their application. Moreover, the approximation properties of the intuitionistic Mamdani, the intuitionistic Larsen, the IF-*t*-norm, and the intuitionistic triple-I fuzzy systems are obtained, which show that the SISO intuitionistic fuzzy systems are the universal approximators.

**INDEX TERMS** Intuitionistic fuzzy triangular norm, intuitionistic fuzzy residual implication, SISO intuitionistic fuzzy system, intuitionistic fuzzy inference, universal approximators.

## I. INTRODUCTION

Approximate reasoning means methods and methodologies that enable reasoning with imprecise inputs to acquire meritorious outputs. Zadeh introduced the theory of approximate reasoning in [1]–[5]. This theory offers a powerful framework for reasoning in the face of imprecise and uncertain information. Fuzzy systems, as an approximate reasoning method, are mainly based on fuzzy sets. Let us consider a fuzzy system that is composed of three principle components: fuzzifier, inference mechanism and defuzzifier. Fuzzy inference mechanism needs to be able to model the process of approximate reasoning, through interpolation between the fuzzy *if-then* rules. Of course, good interpolations are also approximations and in this way the approximate reasoning is performed. The study on the approximation theory of fuzzy system is of great importance and necessary. In most applications of the fuzzy systems, the main design objective can be transformed to find designed mapping from a input space to a output space, which may also be denoted as a function.

The existing literature on fuzzy systems were depend on fuzzy sets [6], [8]–[12]. Atanassov's intuitionistic fuzzy sets theory can be regarded as a more powerful and sensitive than fuzzy sets in dealing with imperfect information, especially under imperfectly defined facts and imprecise knowledge and, therefore, to describe in a more adequate way many real problems. It has been widely applied in many fields such as pattern recognition, machine learning, decision making, market prediction, image processing and so on. The working with intuitionistic fuzzy sets instead of fuzzy sets imply the adding of non-membership to membership. Intuitionistic fuzzy set offers a new possibility to represent imperfect knowledge and, therefore, to describe in a more adequate way many real problems. For instance, voting can be taken as a good example of this situation, as human voters may be divided into three groups of those who vote for, who vote against and who abstain. If we take $\langle x_1, 0.6, 0.3\rangle$ as an element of intuitionistic fuzzy set $A$ of voting, we can interpret that "the vote for the candidate $x_1$ is 0.6 in favor to 0.3 against with 0.1 abstentions".

Since there is only one membership function in fuzzy sets, some information will be lost when fuzzifing and defuzzifing in the fuzzy systems. Moreover, note that a fuzzy set is a specific case of an intuitionistic fuzzy set and there are membership functions, non-membership functions and hesitation functions in intuitionistic fuzzy sets. Fuzzy inference in intuitionistic fuzzy systems has to consider the fact that we have the membership as well as the non-membership.

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However, the approximate reasoning based on intuitionistic fuzzy set seems to be not paid enough attention to. So far, there are just a few works that deal with intuitionistic fuzzy systems [7], [13], [14]. The main obstacle is due to that the intuitionistic fuzzy reasoning mechanism is not spontaneous like most of the fuzzy reasoning, especially restrictions on the intuitionistic fuzzy implication operators are much more complicated than that of fuzzy implication operators. Inspired by the above factors, we propose the method of SISO intuitionistic fuzzy systems.

The main goal of this paper is to discuss the intuitionistic fuzzy $t$-norms, the intuitionistic fuzzy $t$-conorms, the intuitionistic fuzzy residual implications and the intuitionistic fuzzy compositional rule on account of intuitionistic fuzzy set theory. Then we provide the intuitionistic fuzzy inference mechanism and the intuitionistic defuzzifier. Next, we establish some novel SISO intuitionistic fuzzy systems based on intuitionistic fuzzy sets. After that, in order to guarantee that this methodology can indeed describe an arbitrary system, it is recommended to check that this methodology is universal.

This paper is arranged as follows: In Section 2, some important concepts and definitions from both fuzzy set theory and fuzzy systems are presented. In Section 3, some operators in intuitionistic fuzzy sets theory, such as the intuitionistic fuzzy numbers, the intuitionistic fuzzy $t$-norm and $t$-conorm, the intuitionistic fuzzy residual principle and the intuitionistic residual implication, are investigated. In Section 4, the intuitionistic fuzzy systems, which being made up of the intuitionistic fuzzy *if-then* rule and rule base, the intuitionistic fuzzy inference and the intuitionistic fuzzy defuzzification, are proposed. The intuitionistic triple-I fuzzy rule and the intuitionistic triple-I inference be built. In Section 5, some approximation properties of the intuitionistic fuzzy systems are obtained, which shows that the SISO intuitionistic fuzzy systems are universal approximators.

## II. FUZZY SETS AND FUZZY SYSTEMS

Here we recall and give some basic concepts and results, which we shall need in the subsequent sections.

### A. FUZZY SETS

A fuzzy set in a universe of discourse $X$ is characterized by a membership function $\mu_A(x)$ that takes values in the unit interval [0, 1]. It may be represented as a set of ordered pairs of a generic element $x$ and its membership value, that is $A = \{(x, \mu_A(x))|x \in X\}$. An important notion in fuzzy set theory is that of triangular norms and conorms, which used to define a generalized intersection and union of fuzzy sets. In general, a triangular norm($t$-norm) $T$ on unit interval [0,1] is defined as an increasing, commutative, associative mapping satisfying $T(1, x) = x$ for all $x \in [0, 1]$. A triangular conorm($t$-conorm) $S$ is defined as an increasing, commutative, associative mapping satisfying $S(0, x) = x$, for all $x \in [0, 1]$.

If a $t$-norm $T$ and a $t$-conorm $S$ satisfies that $S(a, b) = 1 - T(1 - a, 1 - b)$ for any $a, b \in [0, 1]$, the $t$-conorm $S$ is called the dual $t$-conorm of the $t$-norm $T$. Analogously, if a $t$-conorm $S$ and a $t$-norm $T$ satisfies that $T(a, b) = 1 - S(1 - a, 1 - b)$ for any $a, b \in [0, 1]$, the $t$-norm $T$ is called the dual $t$-norm of the $t$-conorm $S$.

A $t$-norm $T$ is called a left-continuous $t$-norm if it satisfies

$$T\big(\bigvee_{i\in I} a_i, b\big) = \bigvee_{i\in I} T(a_i, b),$$

for any $a_i, b \in [0, 1]$.

A $t$-conorm $S$ is called a right-continuous $t$-conorm if it satisfies

$$S(\bigwedge_{i\in I} a_i, b) = \bigwedge_{i\in I} S(a_i, b),$$

for any $a_i, b \in [0, 1]$.

*Example 1 ( [15], [16]):* The following $t$-norms and $t$-conorms are the basic $t$-norms and its dual $t$-conorms given by

(i) *minimum $t$-norm and $t$-conorm.*

$$T_M(a, b) = \min\{a, b\}, \quad S_M(a, b) = \max\{a, b\}.$$

(ii) *product $t$-norm and $t$-conorm.*

$$T_P(a, b) = ab, \quad S_P(a, b) = a + b - ab.$$

(iii) *Łukasiewicz $t$-norm and $t$-conorm.*

$$T_{Lu}(a, b) = \max\{a+b-1, 0\}, \ S_{Lu}(a, b) = \min\{a+b, 1\}.$$

for any $a, b \in [0, 1]$, respectively.

In fuzzy sets theory, a $t$-norm based *Residual Principle* is an important rule that defined as follows:

$$T(a, t) \leq b \text{ iff } t \leq a \to_T b, \quad \text{for all } a, b, t \in [0, 1]. \quad (1)$$

where $\to_T$ is the residual implication induced by $t$-norm $T$. Then, we have

$$a \to_T b = \sup\{t \in [0, 1]|T(a, t) \leq b\} \quad (2)$$

Since the residual implication $\to_T$ is based upon a $t$-norm $T$, it is a natural idea that we can define the *co-residual implication* $\to_S$ based upon the dual $t$-conorm $S$ of a $t$-norm $T$.

*Definition 2:* Let $S$ be the right-continuous dual $t$-conorm of a left-continuous $t$-norm $T$. Then the *co-residual principle*, which is based upon $t$-conorm $S$, is defined as follows:

$$S(a, t) \geq b \text{ iff } t \geq a \to_S b, \quad a, b, t \in [0, 1]. \quad (3)$$

It is clearly that a $t$-norm satisfies the residual principle if and only if the it is left-continuous and that a $t$-conorm satisfies the co-residual principle if and only if it is right-continuous.

*Theorem 3:* Let $S$ be the right-continuous dual $t$-conorm of a left-continuous $t$-norm $T$. Then there exists a binary operation $\to_S$: [0, 1] $\to$ [0, 1] such that $\to_S$ satisfies the *co-residual principle* where $\to_S$ is given by

$$a \to_S b = \inf\{t \in [0, 1]|S(a, t) \geq b\}, \quad a, b \in [0, 1]. \quad (4)$$

*Proof:* By using Eq.(3), if $S(a, t) \geq b$ then $t \geq a \to_S b$. Conversely, if $t \geq a \to_S b$, then

$t \geq \inf\{\delta \in [0,1] | S(a,\delta) \geq b\}$. From the monotonicity and right-continuity of $t$-conorm $S$, we can obtain that

$$\begin{aligned} S(a,t) &\geq S(a, \inf\{\delta | S(a,\delta) \geq b\}) \\ &= \inf\{S(a,\delta) | S(a,\delta) \geq b\} = b. \end{aligned}$$

i.e., $S(a,t) \geq b$. The proof is complete. □

*Theorem 4:* Let $\to_T$ be a residual implication and $\to_S$ be a co-residual implication, which induced by a left-continuous $t$-norm $T$ and a right-continuous dual $t$-conorm $S$, respectively. Then, for any $a, b \in [0,1]$, we have

(i) $a \to_S b = 1 - (1-b) \to_T (1-a)$.

(ii) $a \to_T b = 1 - (1-b) \to_S (1-a)$.

*Proof:* Here we only give the proof of the first case. Since the $t$-norm $T$ and the $t$-conorm $S$ are dual, then we have

$$\begin{aligned} a \to_S b &= \inf\{t | S(a,t) \geq b\} \\ &= \inf\{t | 1 - T(1-a, 1-t) \geq b\} \\ &= 1 - \sup\{1-t | T(1-a, 1-t) \leq 1-b\} \\ &= 1 - (1-a) \to_T (1-b). \end{aligned}$$

The proof of another is similar to it. □

*Theorem 5:* Let $\to_T$ and $\to_S$ be a residual and co-residual implication, which induced by a left-continuous $t$-norm $T$ and a right-continuous dual $t$-conorm $S$, respectively. Then we have (i)

i) $T(a, a \to_T b) \leq b$.

ii) $S(a, a \to_S b) \geq b$.

*Proof:* Here we only proof the second case and the first is similar to it. It follows from Eq.(4) and the monotonicity and the right-continuous of the $t$-conorm $S$ that

$$\begin{aligned} S(a, a \to_S b) &= S(a, \inf\{t \in [0,1] | S(a,t) \geq b\}) \\ &= \inf\{S(a,t) | S(a,t) \geq b, t \in [0,1]\} \\ &\geq b. \end{aligned}$$

□

*Example 6:* The following implications are the three basic residual implications and three co-residual implications, respectively.

$$a \to_{T_M} b = \begin{cases} 1, & \text{if } a \leq b \\ b, & \text{if } a > b \end{cases}, \quad a \to_{S_M} b = \begin{cases} b, & \text{if } a < b \\ 0, & \text{if } a \geq b \end{cases},$$

$$a \to_{T_P} b = \begin{cases} 1, & \text{if } a \leq b \\ \frac{b}{a}, & \text{if } a > b \end{cases}, \quad a \to_{S_P} b = \begin{cases} \frac{b-a}{1-a}, & \text{if } a < b \\ 0, & \text{if } a \geq b \end{cases},$$

$$a \to_{T_{Lu}} b = \min\{1-a+b, 1\}, \quad a \to_{S_{Lu}} b = \max\{b-a, 0\}.$$

### B. FUZZY SYSTEMS

Fuzzy system consists of linguistic variables, fuzzy *if-then* rules and a fuzzy inference mechanism. Linguistic variables allow us to interpret linguistic expressions in terms of fuzzy mathematical quantities. Fuzzy *if-then* rule base is a set of rules that make association between typical input and output data sometimes in a perceptual way, or, on other occasions, in a data driven way.

*Definition 7 ([3]–[5]):* A linguistic variable is characterized by $(X, T, U, M)$, where $X$ is the name of linguistic variable, $T$ is the set of linguistic values that $X$ can take, $U$ is the actual physical domain in which the linguistic variable $X$ takes its set in $U$ and $M$ is a set of semantic rules that relates each linguistic value in $T$ with a fuzzy sets in $U$.

Fuzzy *if-then* rules are able to model expert opinion or commonsense knowledge often expressed in linguistic terms. The intuitive association that exists between given typical input data and typical output data is hard to be described in a mathematically correct way, because of the uncertain, often subjective nature of this information. Fuzzy *if-then* rules are tools that are able to model and use such knowledge. A fuzzy *if-then* rule is a triplet $(A, B, R)$ that consists of an antecedent $A$, a consequence $B$ that are linguistic variables, linked through a fuzzy relation $R$. By using fuzzy sets, a fuzzy *if-then* rule is written as follows:

$$\textit{If } x \textit{ is } A(x) \;\; \textit{then } y \textit{ is } B(y).$$

Since any rule is a natural expression of a relationship between the input and output variable, it is a natural idea to interpret fuzzy *if-then* rules by using fuzzy relations. The fuzzy relations are obtained by a composition of the antecedent and consequence.

For instance,

If "Temperature is High" then "Fan speed is Medium".

Here "Temperature" and "Fan speed" are the linguistic variables. "High" and "Medium" are the linguistic values taken by linguistic variables in a suitable domain.

*Definition 8 ([17]):* The fuzzy *if-then* rule: *If x is* $A(x)$ *then y is* $B(y)$, where $A(x)$ and $B(y)$ are two fuzzy sets, be define as a fuzzy relation $R(x,y)$ as follows:

(i) Mamdani rule: $R_M(x,y) = A(x) \wedge B(y)$.

(ii) Larsen rule: $R_L(x,y) = A(x)B(y)$.

(iii) $t$-norm rule: $R_T(x,y) = T\big(A(x), B(y)\big)$, with $T$ being an arbitrary $t$-norm.

(iv) Residual implication rule:

$$R_R(x,y) = A(x) \to_T B(y),$$

with $\to_T$ being a residual implication with a given $t$-norm $T$.

Generally speaking, a single fuzzy relation is not enough to make an informed decision. In practical applications, we will usually have a fuzzy *if-then* rule base, i.e., a finite collection of fuzzy *if-then* rules.

*Definition 9 ([17]):* Given two non-empty crisp sets $X, Y \subseteq \mathbb{R}$, for a Single-Input-Single-Output(SISO), the fuzzy *if-then* rule base *If x is* $A_i(x)$ *then y is* $B_i(y)$, $i = 1, \ldots, n$ consists of rules of the following forms, as a fuzzy relation as follows:

(i) Mamdani rule base: $R_M(x,y) = \bigvee_{i=1}^{n} A_i(x) \wedge B_i(y)$.

(ii) Larsen rule base: $R_L(x,y) = \bigvee_{i=1}^{n} A_i(x)B_i(y)$.

(iii) $t$-norm rule base: $R_T(x,y) = \bigvee_{i=1}^{n} T\big(A_i(x), B_i(y)\big)$, with $T$ being an arbitrary $t$-norm.

(iv) Residual implication rule base:

$$R_R(x,y) = \bigwedge_{i=1}^{n} A_i(x) \rightarrow_T B_i(y),$$

with $\rightarrow_T$ being a residual implication with a given $t$-norm $T$.

$x \in X, y \in Y$ are the linguistic variables and $A_i(x)$ and $B_i(y)$ are the linguistic values taken by the linguistic variables. These linguistic values are represented by fuzzy sets in their corresponding domains.

Fuzzy inference is the process of obtaining a conclusion for a given input that was possibly never encountered before. The basic rule for fuzzy inference systems is the compositional rule of fuzzy inference(CRI) that proposed by Zadeh. It is based on the classical rule of Modus Ponens.

The compositional rule of inference consists of a

| | | |
|---|---|---|
| *premise* | : | If $x$ is $A_i(x)$ then $y$ is $B_i(y)$, $i = 1, \ldots, n$ |
| *fact* | : | $x$ is $A'(x)$ |
| *conclusion* | : | $y$ is $B'(y)$ |

*Definition 10 ([17]):* Fuzzy inference, which being based on a composition rule, is defined as follows: (i)

(i) Mamdani inference:

$$B'(y) = A' \circ R(x,y) = \bigvee_{x\in X} A'(x) \wedge R(x,y).$$

(ii) Larsen inference:

$$B'(y) = A' \circ_L R(x,y) = \bigvee_{x\in X} A'(x) \cdot R(x,y).$$

(iii) t-norm inference:

$$B'(y) = A' \circ_T R(x,y) = \bigvee_{x\in X} T\big(A'(x), R(x,y)\big).$$

(iv) Residual implication inference:

$$B'(y) = A' \lhd_T R(x,y) = \bigwedge_{x\in X} A'(x) \rightarrow_T R(x,y),$$

with $\rightarrow_T$ being a residual implication with a given $t$-norm $T$.

By the CRI method, the fuzzy logic control has been applied successfully to many fields. However, the CRI method has some imperfections (see [18]–[22]). For example, the composition operation '$\circ$' which being employed in the CRI method lacks clear logical meaning. To improve the previous method, Wang has proposed a new method which comes from a new viewpoint for fuzzy reasoning, i.e., the full implication triple-I method for fuzzy reasoning in [19]:

*Definition 11:* Let $\rightarrow_T$ be a residual implication that induced by a left-continuous $t$-norm $T$, the solution $B'(y)$ of the fuzzy *if-then* rule should be the smallest fuzzy subset of the universe $Y$ satisfying that

$$\big(A(x) \rightarrow_T B(y)\big) \rightarrow_T \big(A'(x) \rightarrow_T B'(y)\big) = 1. \quad (5)$$

The above method is called the full implication triple-I inference rule.

*Theorem 12 ([20]):* Given two non-empty crisp sets $X$, $Y \subseteq \mathbb{R}$. Let $\rightarrow_T$ be a residual implication that induced by a left-continuous $t$-norm $T$. Then the triple-I solution $B'(y)$ of *if-then* rule is given by the following formula:

$$B'(y) = \bigvee_{x\in X} T\big(A'(x), A(x) \rightarrow_T B(y)\big), \quad x \in X, y \in Y. \quad (6)$$

*Definition 13:* Given two non-empty crisp sets $X, Y \subseteq \mathbb{R}$. Let $\rightarrow_T$ be a residual implication that induced by a $t$-norm. The fuzzy *if-then* rule base: *If $x$ is $A_i(x)$ then $y$ is $B_i(y)$*, $i = 1, \ldots, n$ be a fuzzy relation that

$$R_I(x,y) = \bigwedge_{i=1}^{n} A_i(x) \rightarrow_T B_i(y)$$

The triple-I fuzzy inference is defined as follows:

$$\Big(\bigwedge_{i=1}^{n}(A_i(x) \rightarrow_T B_i(y)) \rightarrow_T (A'(x) \rightarrow_T B'(y))\Big) = 1. \quad (7)$$

The solution $B'(y)$ of triple-I fuzzy inference of fuzzy *if-then* rule base should be the smallest fuzzy subset of universe $Y$ satisfying Eq.(7).

*Theorem 14:* Given two non-empty crisp sets $X, Y \subseteq \mathbb{R}$. Suppose that $\rightarrow_T$ be a residual implication which induced by a left-continuous $t$-norm $T$. Then for any $x \in X, y \in Y$, the triple-I solution $B'(y)$ of the fuzzy rule base is given by the following

$$B'(y) = \bigvee_{x\in X} T\Big(A'(x), \quad \bigwedge_{i=1}^{n} A_i(x) \rightarrow_T B_i(y)\Big). \quad (8)$$

*Proof:* For any $x \in X, y \in Y$, it follows from Eq.(8) that

$$T\Big(A'(x), \bigwedge_{i=1}^{n} A_i(x) \rightarrow_T B_i(y)\Big) \le B'(y). \quad (9)$$

Since the $t$-norm $T$ is commutative, then it follows from Eq.(1) that

$$\bigwedge_{i=1}^{n} A_i(x) \rightarrow_T B_i(y) \le A'(x) \rightarrow_T B'(y). \quad (10)$$

Hence, we can obtain that Eq.(7) holds by using Eq.(10). Moreover, for any $x \in X, y \in Y$, suppose that $C(y)$ is a fuzzy subset of $Y$ such that

$$\Big((\bigwedge_{i=1}^{n} A_i(x) \rightarrow_T B_i(y)) \rightarrow_T (A'(x) \rightarrow_T C(y))\Big) = 1. \quad (11)$$

Then, from Eq.(1) we have that

$$T\Big(A'(x), \bigwedge_{i=1}^{n} A_i(x) \rightarrow_T B_i(y)\Big) \le C(y). \quad (12)$$

Thus, it means that $C(y)$ is an upper bound of the set

$$\{T\Big(A'(x), \bigwedge_{i=1}^{n} A_i(x) \rightarrow_T B_i(y)|x \in X\Big)\}.$$

Hence, $B'(y) \leq C(y)$. Therefore, $B'(y)$ is the triple-I solution of the fuzzy *if-then* inference. □

### C. DEFUZZIFICATION

Defuzzification is the last step in a fuzzy control procedure. Based on the output of a fuzzy controller one has to give an estimate of the crisp quantity (a representative crisp element) for the output value of the SISO fuzzy system. In this case one has to use a defuzzification. There are many different defuzzification methods and based on the given application that we are working on, we can select the center of gravity(COG) defuzzifier. It is simple and elegant-the center of gravity(COG) defuzzification. The value selected is the center of gravity of the fuzzy set $\mu_A(x)$. More formally, we have

$$COG(A) = \frac{\int_X x \cdot A(x)\mathrm{d}x}{\int_X A(x)\mathrm{d}x}. \tag{13}$$

where $\int_X$ is the conventional integral.

## III. INTUITIONISTIC FUZZY T-NORM(IF-*T*-NORM) AND INTUITIONISTIC FUZZY RESIDUAL IMPLICATION(IF-R-IMPLICATION)

### A. INTUITIONISTIC FUZZY SET (IFS) AND INTUITIONISTIC FUZZY NUMBER (IFN)

We briefly recall the basic definitions of an intuitionistic fuzzy set, which were introduced by Atanassov in 1986 in [23] and is defined as follows.

*Definition 15 ( [23], [24]):* Let $X$ be a nonempty set. An intuitionistic fuzzy set(IFS) $A$ in $X$ is defined by $A = \langle x, \mu_A(x), \nu_A(x)\rangle$, where $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ such that $0 \leq \mu_A(x)+\nu_A(x) \leq 1, \forall x \in X$. The numbers $\mu_A(x), \nu_A(x) \in [0, 1]$ denote the degree of membership and non-membership of $x \in A$ respectively. For each intuitionistic fuzzy set $A$ in $X$, $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ stands for the degree of hesitation or uncertainty for the object $x$ to belong to $A$.

Since intuitionistic fuzzy sets being a generalization of fuzzy sets which gives us an additional possibility to represent imperfect knowledge, they can make it possible to describe many real problems in a more adequate way.

Consider the set $\mathcal{L}$ and a partial order $\leq_{\mathcal{L}}$ on $\mathcal{L}$ is defined by

$$\mathcal{L} = \{(x_1, x_2)|x_1 + x_2 \leq 1, (x_1, x_2) \in [0, 1]^2\}$$
$$x \leq_{\mathcal{L}} y \text{ iff } x_1 \leq y_1 \text{ and } x_2 \geq y_2,$$

$\forall x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{L}$. We denote $0_{\mathcal{L}} = (0, 1)$ and $1_{\mathcal{L}} = (1, 0)$. Then, it is easy to verify that $(\mathcal{L}, \leq_{\mathcal{L}})$ is a complete lattice. This lattice, which is useful in the later of this paper, can also be defined as an algebraic structure $(\mathcal{L}, \wedge, \vee)$ where the meet operator $\wedge$ and the join operator $\vee$ are defined as follows, for any $x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{L}$:

$$x \wedge y = \big(\min\{x_1, y_1\}, \max\{x_2, y_2\}\big). \tag{14}$$
$$x \vee y = \big(\max\{x_1, y_1\}, \min\{x_2, y_2\}\big). \tag{15}$$

Intuitionistic fuzzy set was extended naturally from fuzzy set by adding an additional non-membership function, which can be viewed as a tool that may help better model imperfect information, especially under imperfectly defined facts and imprecise knowledge. It is easily to work with intuitionistic fuzzy numbers.

Accordingly, the concept of real valued intuitionistic fuzzy number was introduced by Burillo and Bustince [25], also Xu and Yager [26]. The definition of an intuitionistic fuzzy number is as follows:

*Definition 16 ([26])*: An intuitionistic fuzzy number (IFN) $A = (\mu_A, \nu_A)$ in set of real numbers $\mathbb{R}$, is defined as follows:

$$\mu_A(x) = \begin{cases} f_A(x), & a \leq x < b_1, \\ 1, & b_1 \leq x < b_2, \\ g_A(x), & b_2 \leq x \leq c, \\ 0, & \text{otherwise}. \end{cases} \tag{16}$$

$$\nu_A(x) = \begin{cases} h_A(x), & e \leq x < f_1, \\ 0, & f_1 \leq x < f_2, \\ k_A(x), & f_2 \leq x \leq g, \\ 1, & \text{otherwise}. \end{cases} \tag{17}$$

where $0 \leq \mu_A(x)+\nu_A(x) \leq 1$ and $a, b_1, b_2, c, e, f_1, f_2, g \in \mathbb{R}$ such that $e \leq a, f_1 \leq b_1 \leq b_2 \leq f_2, c \leq g$, and four functions $f_A, g_A, h_A, k_A : \mathbb{R} \to [0, 1]$ are called the legs of membership function $\mu_A$ and non-membership function $\nu_A$. The functions $f_A$ and $k_A$ are non-decreasing continuous functions and the functions $h_A$ and $g_A$ are non-increasing continuous functions.

The triangular fuzzy number(TFN) was introduced firstly by Dubois and Prade [10]. The concept of a triangular intuitionistic fuzzy number(TIFN) is defined in a similar way as follows:

*Definition 17 ([28])* : A TIFN $A(x) = (\mu_A(x), \nu_A(x))$ is a special IFS on a real number set $\mathbb{R}$, whose membership function $\mu(x)$ and non-membership function $\nu(x)$ are defined respectively as follows:

$$\mu(x) = \begin{cases} 0, & x < a; \\ \frac{w}{b-a}(x-a), & a \leq x < b; \\ \frac{w}{c-b}(c-x), & b \leq x \leq c; \\ 0, & x > c. \end{cases} \tag{18}$$

$$\nu(x) = \begin{cases} 1, & x < a; \\ \frac{1-u}{b-a}(b-x)+u, & a \leq x < b; \\ \frac{1-u}{c-b}(c-x)+u, & b \leq x \leq c; \\ 1, & x > c. \end{cases} \tag{19}$$

In this paper, we denote a TIFN by $A = \langle(a, b, c); w, u\rangle$, where $w$ and $u$ represent the maximum degree of membership and the minimum degree of non-membership, respectively, such that satisfied the conditions $0 \leq w, u \leq 1$ and $0 \leq w + u \leq 1$.

### B. INTUITIONISTIC FUZZY NEGATION(IF-NEGATION) AND INTUITIONISTIC FUZZY T-NORM(IF-t-NORM)

*Definition 18 ([29])*: An intuitionistic fuzzy negation (IF-negation) is any decreasing mapping $\mathcal{N} : \mathcal{L} \to \mathcal{L}$ satisfying $\mathcal{N}(0_{\mathcal{L}}) = 1_{\mathcal{L}}$ and $\mathcal{N}(1_{\mathcal{L}}) = 0_{\mathcal{L}}$. If $\mathcal{N}(\mathcal{N}(x)) = x$,

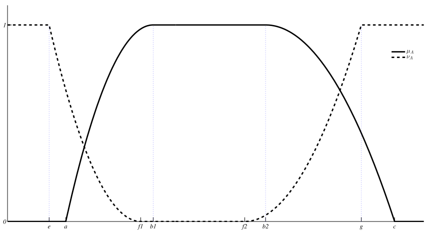

**FIGURE 1. An intuitionistic fuzzy number (IFN).**

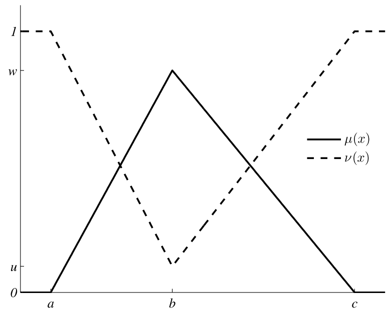


**FIGURE 2. A triangular intuitionistic fuzzy number (TIFN) $A = \langle(a, b, c); w, u\rangle$.**

for all $x \in \mathcal{L}$, then $\mathcal{N}$ is called an involutive intuitionistic fuzzy negation.

The mapping $\mathcal{N}_s(x_1, x_2) = (x_2, x_1)$ for all $x = (x_1, x_2) \in \mathcal{L}$, be called the standard IF-negation.

In the similarly way of the $t$-norm $T$ and $t$-conorm $S$ for fuzzy sets, the following definitions can be straightforwardly extended to the intuitionistic fuzzy case.

*Definition 19 ([29]):* An intuitionistic fuzzy triangular norm(IF-$t$-norm) is any monotonous, commutative, associative $\mathcal{L}^2 \to \mathcal{L}$ mapping $\mathcal{T}$ satisfying $\mathcal{T}(1_{\mathcal{L}}, x) = x$, for all $x \in \mathcal{L}$. An intuitionistic fuzzy triangular conorm (IF-$t$-conorm) is any monotonous, commutative, associative $\mathcal{L}^2 \to \mathcal{L}$ mapping $\mathcal{S}$ satisfying $\mathcal{S}(0_{\mathcal{L}}, x) = x$, for all $x \in \mathcal{L}$.

*Definition 20 ([27], [29]):* Given a $t$-norm $T$ and a $t$-conorm $S$ satisfying $T(a, b) \leq 1 - S(1 - a, 1 - b)$ for any $a, b \in [0, 1]$, the mappings $\mathcal{T}$ and $\mathcal{S}$ defined as follows, for $x = (x_1, x_2), y = (y_1, y_2)$ in $\mathcal{L}$:

$$\mathcal{T}(x, y) = \big(T(x_1, y_1), S(x_2, y_2)\big). \tag{20}$$

and

$$\mathcal{S}(x, y) = \big(S(x_1, y_1), T(x_2, y_2)\big). \tag{21}$$

are called an IF-$t$-norm and an IF-$t$-conorm, respectively.

*Definition 21 ([29]):* An IF-$t$-norm $\mathcal{T}$ is called $t$-representable if and only if there exist a $t$-norm $T$ and a $t$-conorm $S$ on [0,1] such that

$$\mathcal{T}(x, y) = \big(T(x_1, y_1), S(x_2, y_2)\big). \tag{22}$$

for all $x, y \in \mathcal{L}$. An IF-$t$-conorm $\mathcal{S}$ is called $s$-representable iff there exist a $t$-norm $T$ and a $t$-conorm $S$ on [0,1] such that

$$\mathcal{S}(x, y) = \big(S(x_1, y_1), T(x_2, y_2)\big), \tag{23}$$

for any $x, y \in \mathcal{L}$.

*Example 22:* The following $\mathcal{T}_M$, $\mathcal{T}_P$ and $\mathcal{T}_{Lu}$ are the usual $t$-representable IF-$t$-norms and $s$-representable IF-$t$-conorm, for all $x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{L}$,

(1) $\mathcal{T}_M(x, y) = \big(\min\{x_1, y_1\}, \max\{x_2, y_2\}\big)$,
$\mathcal{S}_M(x, y) = \big(\max\{x_1, y_1\}, \min\{x_2, y_2\}\big)$.

(2) $\mathcal{T}_P(x, y) = \big(x_1 y_1, x_2 + y_2 - x_2 y_2\big)$,
$\mathcal{S}_P(x, y) = \big(x_1 + y_1 - x_1 y_1, x_2 y_2\big)$.

(3) $\mathcal{T}_{Lu}(x, y) = \big(\max\{x_1 + y_1 - 1, 0\}, \min\{x_2 + y_2, 1\}\big)$,
$\mathcal{S}_{Lu}(x, y) = \big(\min\{x_1 + y_1, 1\}, \max\{x_2 + y_2 - 1, 0\}\big)$.

*Definition 23:* Let $\mathcal{T}$ be a $t$-representable IF-$t$-norm, $\mathcal{S}$ be a $s$-representable IF-$t$-conorm and $\mathcal{N}$ be a IF-negation. If $\mathcal{T}(x, y) = \mathcal{N}\big(\mathcal{S}(\mathcal{N}(x), \mathcal{N}(y))\big)$, then $\mathcal{T}$ and $\mathcal{S}$ are called dual.

*Remark 24:* Suppose $\mathcal{N} = \mathcal{N}_s$, i.e. $\mathcal{N}_s(x_1, x_2) = (x_2, x_1)$. Then $\mathcal{T}_M$ and $\mathcal{S}_M$ are dual. $\mathcal{T}_P$ and $\mathcal{S}_P$ are dual. $\mathcal{T}_L$ and $\mathcal{S}_L$ are dual.

### C. INTUITIONISTIC FUZZY RESIDUAL IMPLICATION(IF-R-IMPLICATION)

In a similarly way of the residual principle in fuzzy set theory, we give the intuitionistic residual principle in an intuitionistic fuzzy theory.

*Theorem 25 ([31]):* Let $\mathcal{T}$ be an IF-$t$-norm that satisfies the intuitionistic residual principle, i.e.,

$$\mathcal{T}(x, z) \leq_{\mathcal{L}} y \text{ iff } z \leq_{\mathcal{L}} x \to_{\mathcal{L}} y, \quad x, y, z \in \mathcal{L}. \tag{24}$$

where $\to_{\mathcal{L}}$ denotes the intuitionistic residual implication induced by an IF-$t$-norm $\mathcal{T}$, is given by

$$x \to_{\mathcal{L}} y = \sup\{z \in \mathcal{L} | \mathcal{T}(x, z) \leq_{\mathcal{L}} y\}. \tag{25}$$

*Theorem 26:* Let $\to_T$ be a residual implication that induced by a $t$-norm, $\to_S$ be a co-residual implication that induced by a $t$-conorm and $\to_{\mathcal{L}}$ be an intuitionistic residual implication. Then for any $x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{L}$ we have that

$$\begin{aligned} x \to_{\mathcal{L}} y = \big(&x_1 \to_T y_1 \wedge (1 - x_2 \to_S y_2), \\ &x_2 \to_S y_2 \vee (1 - x_1 \to_T y_1)\big). \end{aligned} \tag{26}$$

*Proof:* On the one hand, Let $x \to_{\mathcal{L}} y = (\mu, \nu)$ and $(t_1, t_2) \in \mathcal{L}$. From Eq.(25), we can obtain that

$$\begin{aligned} x \to_{\mathcal{L}} y &= \sup\{(t_1, t_2) | \mathcal{T}((t_1, t_2), \quad (x_1, x_2)) \leq_{\mathcal{L}} (y_1, y_2)\} \\ &= \sup\{(t_1, t_2) | T(t_1, x_1) \leq y_1, \\ &\qquad S(t_2, x_2) \geq y_2, \quad t_1 + t_2 \leq 1.\}. \end{aligned}$$

For the first argument $\mu$ of $(\mu, \nu)$, we have

$$\begin{aligned} \mu &= \sup\{t_1 | T(t_1, x_1) \leq y_1, S(t_2, x_2) \geq y_2, t_1 + t_2 \leq 1.\} \\ &\leq \sup\{t_1 | T(t_1, x_1) \leq y_1\} \\ &\qquad \wedge (1 - \inf\{t_2 | S(t_2, x_2) \geq y_2\}) \\ &= (x_1 \to_T y_1) \wedge (1 - x_2 \to_S y_2). \end{aligned}$$

For the second argument $\nu$ of $(\mu, \nu)$, by the similarly way, we have

$$\nu = \inf\{t_2 | S(t_2, x_2) \geq y_2, T(t_1, x_1) \leq y_1, t_1 + t_2 \leq 1.\}$$

$$\geq \inf\{t_2|S(t_2,x_2)\geq y_2.\} \vee (1-\sup\{t_1|T(t_1,x_1)\leq y_1.\}) = (x_2\rightarrow_S y_2)\vee\ (1-x_1\rightarrow_T y_1).$$

So, $(\mu,\nu)\leq \big(x_1\rightarrow_T y_1\wedge(1-x_2\rightarrow_S y_2), x_2\rightarrow_S y_2\vee(1-x_1\rightarrow_T y_1)\big)$.

On the other hand,

$$\mathcal{T}(x, x\rightarrow_{\mathcal{L}} y)=\mathcal{T}\big((x_1,x_2),(u,v)\big).$$

According to Theorem 5, we obtain that

$$T(x_1,u)\leq T(x_1,(x_1\rightarrow_T y_1)\wedge(1-x_2\rightarrow_S y_2))\leq y_1,$$

and

$$S(x_2,\nu)\leq S(x_2,(x_2\rightarrow_S y_2)\vee(1-x_1\rightarrow_T y_1))\geq y_2.$$

Hence, $\mathcal{T}((\mu,\nu),(x_1,x_2))\ \leq\ (y_1,y_2)$. Therefore, we have that

$$x\rightarrow_{\mathcal{L}} y=\big((x_1\rightarrow_T y_1)\wedge(1-x_2\rightarrow_S y_2), (x_2\rightarrow_S y_2)\vee(1-x_1\rightarrow_T y_1)\big).$$

□

*Example 27:* The following are the usual intuitionistic fuzzy residual implications $\rightarrow_{\mathcal{L}_M}, \rightarrow_{\mathcal{L}_P}$ and $\rightarrow_{\mathcal{L}_{Lu}}$, for any $x=(x_1,x_2), y=(y_1,y_2)\in\mathcal{L}$,

(1) *Intuitionistic minimum residual implication*. If the $t$-norm is minimum $t$-norm, i.e., $T_M(a,b)=\min\{a,b\}$, for any $a,b\ \in\ [0,1]$. According to Example 1 and Example 6, we obtain

$$x\rightarrow_{\mathcal{L}_M} y = \begin{cases}(1-y_2,y_2), & x_1\leq y_1, x_2\leq y_2\\ (1,0), & x_1\leq y_1, x_2>y_2\\ \big(y_1\wedge(1-y_2), y_2\vee(1-y_1)\big), & x_1>y_1, x_2\leq y_2\\ (y_1,1-y_1), & x_1>y_1, x_2>y_2.\end{cases} \tag{27}$$

(2) *Intuitionistic product residual implication*. If the $t$-norm is product $t$-norm, i.e., $T_P(a,b)=ab$, for any $a,b\in[0,1]$, then

$$x\rightarrow_{\mathcal{L}_P} y = \begin{cases}(\frac{1-y_2}{1-x_2}, \frac{y_2-x_2}{1-x_2}), & x_1\leq y_1, x_2<y_2;\\ (1,0), & x_1\leq y_1, x_2\geq y_2;\\ (\frac{y_1}{x_1}\wedge\frac{1-y_2}{1-x_2}, \frac{y_2-x_2}{1-x_2}\vee\frac{x_1-y_1}{x_1}), & x_1>y_1, x_2<y_2;\\ (\frac{y_1}{x_1}, \frac{x_1-y_1}{x_1}), & x_1>y_1, x_2\geq y_2.\end{cases} \tag{28}$$

(3) *Intuitionistic Łukasiewicz residual implication*. If the $t$-norm is Łukasiewicz $t$-norm, i.e., $T_{Lu}(a,b)=\max\{a+b-1,0\}$, for any $a,b\in[0,1]$, then

$$x\rightarrow_{\mathcal{L}_{Lu}} y = \begin{cases}\big(1+x_2-y_2, y_2-x_2\big), & x_1\leq y_1, x_2\leq y_2;\\ (1,0), & x_1\leq y_1, x_2>y_2;\\ \big((1-x_1+y_1)\wedge(1-y_2+x_2), (y_2-x_2)\vee(x_1-y_1)\big), & x_1>y_1, x_2\leq y_2;\\ \big(1-x_1+y_1, x_1-y_1\big), & x_1>y_1, x_2>y_2.\end{cases} \tag{29}$$

## IV. INTUITIONISTIC FUZZY SYSTEMS

### A. *INTUITIONISTIC FUZZY* IF-THEN *RULE*

*Definition 28:* The intuitionistic fuzzy *if-then* rule:

*If x is A(x) then y is B(y).*

is defined as an intuitionistic fuzzy relation $\mathcal{R}(x,y)$ as follows:

(i) The intuitionistic Mamdani rule:

$$\mathcal{R}_M(x,y)=A(x)\wedge B(y) = \big(\min\{\mu_A(x),\mu_B(y)\}, \max\{\nu_A(x),\nu_B(y)\}\big).$$

(ii) The intuitionistic Larsen rule:

$$\mathcal{R}_L(x,y)=A(x)\cdot B(y) = \big(\mu_A(x)\mu_B(y), \nu_A(x)+\nu_B(y)-\nu_A(x)\nu_B(y)\big).$$

(iii) IF-$t$-norm rule:

$$\mathcal{R}_{\mathcal{T}}(x,y)=\mathcal{T}(A(x),B(y)) = \big(T(\mu_A(x),\mu_B(y)), S(\nu_A(x),\nu_B(y))\big).$$

with $\mathcal{T}$ being a $t$-representable IF-$t$-norm correspond to a $t$-norm $T$ and a $t$-conorm $S$;

(iv) The intuitionistic residual implication rule and the intuitionistic triple-I implication rule:

$$\mathcal{R}_R(x,y)=A(x)\rightarrow_{\mathcal{T}} B(y),$$

with $\rightarrow_{\mathcal{T}}$ being intuitionistic fuzzy residual implications with a given $t$-norm $T$ and a $t$-conorm $S$.

where $A(x)\ =\ (\mu_A(x),\nu_A(x))$ and $B(y)\ =\ (\mu_B(y),\nu_B(y))$ are two IFSs being corresponding to the intuitionistic fuzzy *if-then* rule.

### B. *INTUITIONISTIC FUZZY* IF-THEN *RULE BASE*

*Definition 29:* The fuzzy *if-then* rule base: *If x is $A_i(x)$ then y is $B_i(y)$* is defined as an intuitionistic fuzzy relation $\mathcal{R}(x,y)$ as follows:

(i) The intuitionistic Mamdani rule base:

$$\mathcal{R}_M(x,y) = \bigvee_{i=1}^{n} A_i(x)\wedge B_i(y) = \big(\bigvee_{i=1}^{n}\min\{\mu_{A_i}(x),\mu_{B_i}(y)\}, \bigwedge_{i=1}^{n}\max\{\nu_{A_i}(x),\nu_{B_i}(y)\}\big),$$

(ii) The intuitionistic Larsen rule base:

$$\mathcal{R}_L(x,y) = \bigvee_{i=1}^{n} A_i(x)\cdot B_i(y) = \Big(\bigvee_{i=1}^{n}(\mu_{A_i}(x)\mu_{B_i}(y)), \bigwedge_{i=1}^{n}(\nu_{A_i}(x)+\nu_{B_i}(y)-\nu_{A_i}(x)\nu_{B_i}(y))\Big),$$

(iii) IF-$t$-norm rule base:

$$\mathcal{R}_{\mathcal{T}}(x,y) = \bigvee_{i=1}^{n}\mathcal{T}(A_i(x),B_i(y)) = \Big(\bigvee_{i=1}^{n}T(\mu_{A_i}(x),\mu_{B_i}(y)), \bigwedge_{i=1}^{n}S(\nu_{A_i}(x),\nu_{B_i}(y))\Big),$$

with $T$ being an arbitrary $t$-norm and $S$ being a $t$-conorm correspond to $t$-norm $T$;

(iv) Intuitionistic residual implication rule base:

$$\mathcal{R}_R(x,y) = \bigwedge_{i=1}^{n}(A_i(x)\rightarrow_{\mathcal{T}} B_i(y)),$$

with $\rightarrow_{\mathcal{T}}$ being an intuitionistic residual implication.

### C. INTUITIONISTIC FUZZY INFERENCE(IF-INFERENCE)

*Definition 30:* An intuitionistic fuzzy inference, which based on a composition law, is defined as follows:

(i) The intuitionistic Mamdani inference:

$$B'(y) = A'(x)\circ_M \mathcal{R}_M(x,y) = \bigvee_{x\in X} A'(x)\wedge \mathcal{R}_M(x,y)$$

(ii) The intuitionistic Larsen inference:

$$B'(y) = A'(x)\circ_L \mathcal{R}_L(x,y) = \bigvee_{x\in X} A'(x)\cdot \mathcal{R}_L(x,y).$$

(iii) IF-$t$-norm inference:

$$B'(y) = A'(x)\circ_T \mathcal{R}_{\mathcal{T}}(x,y) = \bigvee_{x\in X}\mathcal{T}(A'(x),\mathcal{R}_{\mathcal{T}}(x,y)),$$

(iv) Intuitionistic residual implication inference:

$$B'(y) = A'(x)\triangleleft_{\mathcal{T}} \mathcal{R}_{\mathcal{T}}(x,y) = \bigwedge_{x\in X}\big(A'(x)\rightarrow_{\mathcal{T}} \mathcal{R}_{\mathcal{T}}(x,y)\big),$$

with $\rightarrow_{\mathcal{T}}$ being an intuitionistic residual implication with a given $t$-representable IF-$t$-norm $\mathcal{T}$.

(v) The intuitionistic triple-I residual implication inference: $B'(y)$ should be the smallest intuitionistic fuzzy set on $Y$ satisfying

$$\big(A(x)\rightarrow_{\mathcal{T}} B(y)\big)\rightarrow_{\mathcal{T}} (A'(x)\rightarrow_{\mathcal{T}} (B'(y))) = 1.$$

with $\rightarrow_{\mathcal{T}}$ being an intuitionistic residual implication.

*Theorem 31:* Let $\rightarrow_{\mathcal{T}}$ be an intuitionistic residual implication which induced by a $t$-representable IF-$t$-norm $\mathcal{T}$. Then, the intuitionistic triple-I solution $B'(y)$ of the intuitionistic fuzzy *if-then* rule, is given by the following

$$B'(y) = \bigvee_{x\in X}\mathcal{T}(A'(x),(A(x)\rightarrow_{\mathcal{T}} B(y))), \quad y\in Y. \tag{30}$$

*Proof:* It is similar to the proof of Theorem 14. □

*Example 32:* Suppose that $X = \{x_1,x_2,x_3\}$, $Y = \{y_1,y_2,y_3\}$ and

$$A = \{\langle x_1,0.5,0.4\rangle,\langle x_2,0.7,0.3\rangle,\langle x_3,0.8,0.1\rangle\}$$
$$B = \{\langle y_1,0.6,0.3\rangle,\langle y_2,0.8,0.2\rangle,\langle y_3,0.7,0.2\rangle\}$$
$$A' = \{\langle x_1,0.5,0.3\rangle,\langle x_2,0.6,0.4\rangle,\langle x_3,0.8,0.2\rangle\}.$$

By employing the intuitionistic triple-I fuzzy system, suppose the $t$-representable IF-$t$-norm be the intuitionistic Łukasiewicz $t$-norm, i.e., for any $\alpha = (a_1,a_2), \beta = (b_1,b_2)\in\mathcal{L}$,

$$\mathcal{T}(\alpha,\beta) = (T_{Lu}(a_1,b_1),S_{Lu}(a_2,b_2)) = \big(\max\{a_1+b_1-1,0\},\min\{a_2+b_2,1\}\big),$$

Here the intuitionistic residual implication is given as follows:

$$\alpha\rightarrow_{\mathcal{T}_{Lu}}\beta = \big(a_1\rightarrow_{T_{Lu}} b_1\wedge(1-a_2\rightarrow_{S_{Lu}} b_2), a_2\rightarrow_{S_{Lu}} b_2\vee(1-a_1\rightarrow_{T_{Lu}} b_1)\big).$$

Further, from Eqs.(29) and (30), we obtain that

$$B'(y_1) = \bigvee_{x\in X}\mathcal{T}\big(A'(x),A(x)\rightarrow_{\mathcal{T}_{Lu}} B(y_1)\big) = (0.5,0.3)\vee(0.5,0.5)\vee(0.6,0.4) = (0.6,0.3),$$
$$B'(y_2) = \bigvee_{x\in X}\mathcal{T}\big(A'(x),A(x)\rightarrow_{\mathcal{T}_{Lu}} B(y_2)\big) = (0.5,0.3)\vee(0.6,0.4)\vee(0.7,0.3) = (0.7,0.3),$$
$$B'(y_3) = \bigvee_{x\in X}\mathcal{T}\big(A'(x),A(x)\rightarrow_{\mathcal{T}_{Lu}} B(y_3)\big) = (0.5,0.3)\vee(0.6,0.4)\vee(0.7,0.3) = (0.7,0.3).$$

Therefore, the intuitionistic triple-I solution of $B'(y)$ of the intuitionistic fuzzy *if-then* rule base is obtained as follows:

$$B'(y) = \{\langle y_1,0.6,0.3\rangle,\langle y_2,0.7,0.3\rangle,\langle y_3,0.7,0.3\rangle\}.$$

### D. INTUITIONISTIC DEFUZZIFICATION

By using the similar method of defuzzification for a fuzzy inference, we propose the method of the intuitionistic defuzzification for an intuitionistic fuzzy inference.

*Definition 33:* Let $A(x) = (\mu_A(x),\nu_A(x))$ be an IFS on $X$, $\lambda\in[0,1]$.

$$ICOG(A)^{\lambda} = \lambda\frac{\int_X x\cdot\mu_A(x)\mathrm{d}x}{\int_X \mu_A(x)\mathrm{d}x} + (1-\lambda)\frac{\int_X x\cdot\nu_A(x)\mathrm{d}x}{\int_X \nu_A(x)\mathrm{d}x} \tag{31}$$

is called the $\lambda$-hybrid of an Intuitionistic Center of Gravity(ICOG) output. Here $\lambda$ stands for the firing strength of the membership and $1-\lambda$ of the non-membership.

When $\lambda > \frac{1}{2}$, the membership function plays a crucial role compared with the non-membership function. When $\lambda < \frac{1}{2}$, the non-membership function plays a crucial role compared with the membership function.

*Theorem 34:* If the IFS $A'(x)$ is a crisp input, i.e., $A'(x_0) = (1, 0)$, of an intuitionistic fuzzy inference system with a given intuitionistic rule base $\mathcal{R}(x, y)$, then the output of the intuitionistic Mamdani inference, the intuitionistic Larsen inference, the IF-$t$-norm inference and the intuitionistic triple-I inference systems coincide.

*Proof:* Since $A'(x_0) = (1, 0)$, i.e., $\mu_{A'}(x_0) = 1$ and $\nu_{A'}(x_0) = 0$, then for the intuitionistic Mamdani inference we have

$$\begin{aligned} B'(y) &= \bigvee_{x \in X} A'(x) \wedge \mathcal{R}_M(x, y) \\ &= A'(x_0) \wedge \big(A(x_0) \wedge B(y)\big) \\ &= \big(\min\{\mu_A(x_0), \mu_B(y)\}, \max\{\nu_A(x_0), \nu_B(y)\}\big) \\ &= \mathcal{R}_M(x_0, y). \end{aligned}$$

For the intuitionistic Larsen inference, we have that

$$\begin{aligned} B'(y) &= \bigvee_{x \in X} A'(x_0) \wedge \mathcal{R}_L(x_0, y) \\ &= A'(x_0) \wedge \big(A(x_0) \cdot B(y)\big) \\ &= (\mu_A(x_0)\mu_B(y), \nu_A(x_0)+\nu_B(y)-\nu_A(x_0)\nu_B(y)) \\ &= \mathcal{R}_L(x_0, y). \end{aligned}$$

For the IF-$t$-norm inference, we have that

$$\begin{aligned} B'(y) &= \bigvee_{x \in X} \mathcal{T}\big(A'(x_0), \mathcal{R}_{\mathcal{T}}(x_0, y)\big) \\ &= \mathcal{T}\big((1, 0), \mathcal{R}_{\mathcal{T}}(x_0, y)\big) \\ &= \mathcal{R}_{\mathcal{T}}(x_0, y). \end{aligned}$$

For the intuitionistic triple-I inference, we have that

$$\begin{aligned} B'(y) &= \bigvee_{x \in X} \mathcal{T}\big(A'(x_0), \mathcal{R}_R(x_0, y)\big) \\ &= \mathcal{T}\big((1, 0), \mathcal{R}_R(x_0, y)\big) \\ &= \mathcal{R}_R(x_0, y). \end{aligned}$$

It is clear that the conclusion holds. □

*Example 35:* Let us consider a simple SISO intuitionistic fuzzy system with the intuitionistic *if-then* rule: *If x is $A_i(x)$ then y is $B_i(y)$*, where $A(x) = (\mu_A(x), \nu_A(x))$ and $B(y) = (\mu_B(y), \nu_B(y))$ are two intuitionistic fuzzy sets. Let $A_1 = \langle(1, 2, 3); 0.90, 0.05\rangle$, $A_2 = \langle(2, 3, 4); 0.80, 0.15\rangle)$, $B_1 = \langle(2, 4, 6); 0.95, 0\rangle$ and $B_2 = \langle(4, 6, 8); 0.90, 0.10\rangle$ are TIFNs.

Input the crisp value $x_0 = 2.5$. Let the firing strength of the membership be $\lambda = 0.7$ and the firing strength of the non-membership be 0.3.

(1) According to Theorem 34, by employing the Mamdani intuitionistic rule base and intuitionistic Mamdani fuzzy inference, for a crisp input $x_0 = 2.5$, we have

$$B'(y) = A'(x_0) \circ_M \mathcal{R}_M(x_0, y)$$

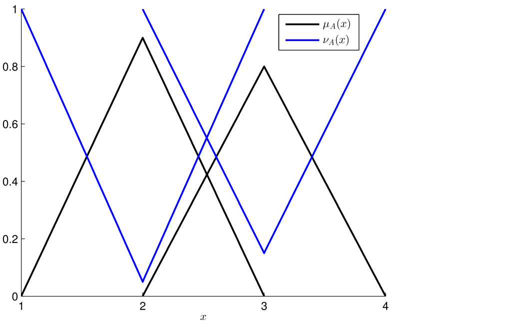


**FIGURE 3.** $A_1 = \langle(1, 2, 3); 0.90, 0.05\rangle$ and $A_2 = \langle(2, 3, 4); 0.80, 0.15\rangle)$ are the input of intuitionistic fuzzy inference rule.

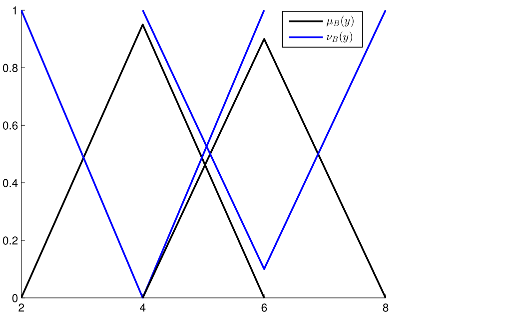


**FIGURE 4.** $B_1 = \langle(2, 4, 6); 0.95, 0\rangle$ and $B_2 = \langle(4, 6, 8); 0.90, 0.10\rangle$ are the output of intuitionistic fuzzy inference rule.

$$\begin{aligned} &= A'(x_0) \wedge \Big(\bigvee_{i=1}^{n} \min\{\mu_{A_i}(x), \mu_{B_i}(y)\}, \\ &\quad \bigwedge_{i=1}^{n} \max\{\nu_{A_i}(x), \nu_{B_i}(y)\}\Big) \\ &= (\mu_{A'}(x_0), \nu_{A'}(x_0)) \bigvee \big(\min\{\mu_A(x_0), \mu_B(y)\}, \\ &\quad \max\{\nu_A(x_0), \nu_B(y)\}\big) \\ &= \big(\min\{\mu_A(x_0), \mu_B(y)\}, \max\{\nu_A(x_0), \nu_B(y)\}\big). \end{aligned}$$

Hence,

$$\begin{aligned} B'(y) &= \big(\max\{\min\{\mu_{A_1}(2.5), \mu_{B_1}(y)\}, \\ &\quad \min\{\mu_{A_2}(2.5), \mu_{B_2}(y)\}\}, \\ &\quad \min\{\max\{\nu_{A_1}(2.5), \nu_{B_1}(y)\}\}, \\ &\quad \max\{\nu_{A_2}(2.5), \nu_{B_2}(y)\}\}\big) \\ &= \big(\max\{\min\{0.45, \mu_{B_1}(y)\}, \min\{0.40, \mu_{B_2}(y)\}\}, \\ &\quad \min\{\max\{0.525, \nu_{B_1}(y)\}, \\ &\quad \max\{\min\{0.575, \nu_{B_2}(y)\}\}\}\big). \end{aligned}$$

Then the output of the intuitionistic fuzzy system $B'(y) = \big(\mu_{B'}(y), \nu_{B'}(y)\big)$ is

$$\mu_{B'}(y) = \begin{cases} \dfrac{0.95}{2}(y-2), & y \in [2, 3.05); \\ 0.45, & y \in [3.05, 5.05); \\ \dfrac{0.95}{2}(6-y), & y \in [5.05, 5.16); \\ 0.425, & y \in [5.16, 7.11) \\ \dfrac{0.90}{2}(8-y), & y \in [7.11, 8] \end{cases}$$

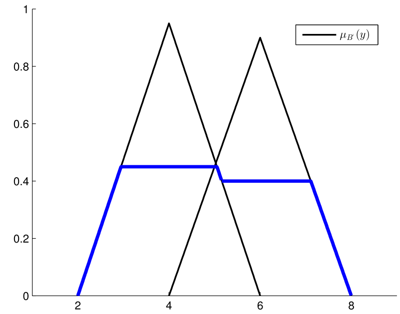


**FIGURE 5.** $\mu_{B'}(y)$ **is the membership function of the output of intuitionistic fuzzy system which employed with Mamdani intuitionistic rule base and Mamdani intuitionistic fuzzy inference.**

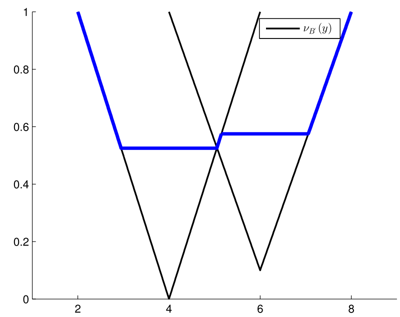


**FIGURE 6.** $\nu_{B'}(y)$ **is the non-membership function of the output of intuitionistic fuzzy system which employed with Mamdani intuitionistic rule base and Mamdani intuitionistic fuzzy inference.**

$$\nu_{B'}(x)=\begin{cases}\frac{1}{2}(4-y), & y\in[2,2.95);\\ 0.525, & y\in[2.95,5.05)];\\ \frac{1}{2}(y-4), & y\in[5.05,5.15);\\ 0.575, & y\in[5.15,7.06)\\ \frac{0.95}{2}(y-6)+0.10, & y\in[7.06,8]\end{cases}$$

After intuitionistic defuzzifying the result, the output value $y_0$ is obtained

$$y_0^{0.7}=0.7\cdot\frac{\int_2^8 y\cdot\mu_{B'}(y)\mathrm{d}y}{\int_2^8\mu_{B'}(y)\mathrm{d}y}+0.3\cdot\frac{\int_2^8 y\cdot\nu_{B'}(y)\mathrm{d}y}{\int_2^8\nu_{B'}(y)\mathrm{d}y}=4.97.$$

Intuitionistic Mamdani inference has simple expression on par with great computational and intuitive properties. Also, these were historically the systems used in the intuitionistic fuzzy systems.

(2) According to Theorem 34, by employing IF-$t$-norm rule base and intuitionistic fuzzy residual inference, for a crisp input $x_0=2.5$, we have

$$\begin{aligned}B'(y)&=A'(x_0)\triangleleft_{\mathcal{T}}\mathcal{R}_{\mathcal{T}}(x_0,y)\\&=\bigwedge_{x\in X}\left(A'(x_0)\rightarrow_{\mathcal{T}_{Lu}}\mathcal{R}_{\mathcal{T}}(x_0,y)\right)\\&=\big(\max\{T(\mu_{A_1}(x_0),\mu_{B_1}(y)),T(\mu_{A_2}(x_0),\mu_{B_2}(y))\},\\&\quad\min\{S(\nu_{A_1}(x_0),\nu_{B_1}(y)),S(\nu_{A_2}(x_0),\nu_{B_2}(y))\}\big),\end{aligned}$$

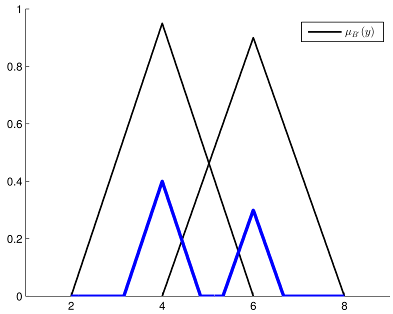


**FIGURE 7.** $\mu_{B'}(y)$ **is the membership function of the output of intuitionistic fuzzy system which employed with IF-*t*-norm rule base and intuitionistic fuzzy residual inference.**

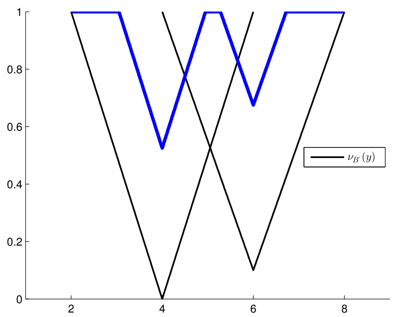


**FIGURE 8.** $\nu_{B'}(y)$ **is the non-membership function of the output of intuitionistic fuzzy system which employed with IF-*t*-norm rule base and intuitionistic fuzzy residual inference.**

Here we employ Łukasiewicz IF-$t$-norm $\mathcal{T}_{Lu}$ that given in Example 22 and Łukasiewicz intuitionistic fuzzy residual implication $\rightarrow_{\mathcal{T}_{Lu}}$ that given in Example 27. Thus, we obtain that

$$\begin{aligned}B'(y)&=\big(\max\{\max\{\mu_{A_1}(2.5)+\mu_{B_1}(y)-1,0\},\\&\quad\max\{\mu_{A_2}(2.5)+\mu_{B_2}(y)-1,0\}\},\\&\quad\min\{\min\{\nu_{A_1}(2.5)+\nu_{B_1}(y),1\}\},\\&\quad\min\{\nu_{A_2}(2.5)+\nu_{B_2}(y),1\}\}\big)\\&=\big(\max\{\max\{0.45+\mu_{B_1}(y)-1,0\},\\&\quad\max\{0.40+\mu_{B_2}(y)-1,0\}\},\\&\quad\min\{\min\{0.525+\nu_{B_1}(y),1\}\},\\&\quad\min\{0.575+\nu_{B_2}(y),1\}\}\big)\end{aligned}$$

Then the output of the system $B'(y)=\big(\mu_{B'}(y),\nu_{B'}(y)\big)$ is

$$\mu_{B'}(y)=\begin{cases}0, & y\in[2,3.10);\\ 0.45+\frac{0.95}{2}(y-2)-1, & y\in[3.10,4);\\ 0.45+\frac{0.95}{2}(6-y)-1, & y\in[4,4.9);\\ 0, & y\in[4.9,5.33);\\ 0.40+\frac{0.90}{2}(y-4)-1, & y\in[5.33,6.22);\\ 0.40+\frac{0.90}{2}(8-y)-1, & y\in[6.22,7.11);\\ 0, & y\in[7.11,8].\end{cases}$$

$$\nu_{B'}(x)=\begin{cases}0, & y\in[2,3.05);\\ 0.525+\frac{4-y}{2}-1, & y\in[3.05,4);\\ 0.525+\frac{y-4}{2}-1, & y\in[4,4.95);\\ 0, & y\in[4.95,5.35);\\ 0.575+\frac{6-y}{2}+0.10-1, & y\in[5.35,6);\\ 0.575+\frac{y-6}{2}+0.10-1, & y\in[6,6.65);\\ 0, & y\in[6.65,8].\end{cases}$$

After intuitionistic defuzzifying the result, the output value $y_0$ be obtained

$$y_0^{0.7}=0.7\cdot\frac{\int_2^8 y\cdot\mu_{B'}(y)\mathrm{d}y}{\int_2^8\mu_{B'}(y)\mathrm{d}y}+0.3\cdot\frac{\int_2^8 y\cdot\nu_{B'}(y)\mathrm{d}y}{\int_2^8\nu_{B'}(y)\mathrm{d}y}=4.62.$$

## V. APPROXIMATION PROPERTIES OF THE SISO INTUITIONISTIC MAMDANI, LARSEN, IF-*T*-NORM AND TRIPLE-I FUZZY SYSTEMS

Let us consider the problem of approximating an unknown continuous function $f:[a,b]\to[c,d]$, with $[c,d]\subseteq\mathbb{R}$ by a SISO intuitionistic fuzzy system. We consider $[c,d]$ to be the range of the function $f$, i.e., $f([a,b])=[c,d]$.

*Definition 36 ([6])*: Let $(X,d)$ be a compact metric space and $([0,\infty),|\cdot|)$ the metric space of positive reals endowed with the usual Euclidean distance. Let $f:X\to[0,\infty]$ be bounded. Then the function $w(f,\cdot):[0,\infty)\to[0,\infty)$, defined by $\omega(f,\delta)=\max\{|f(x)-f(y)|:x,y\in X,d(x,y)\le\delta\}$ is called the modulus of continuity of $f$.

*Lemma 37 ([6])*: The following properties of the modulus of continuity of function $f$ hold true

(i) $|f(x)-f(y)|\le\omega(f,d(x,y))$ for any $x,y\in X$;

(ii) $\omega(f,0)=0$;

(iii) $\omega(f,\delta)$ is non decreasing in $\delta$;

Now we consider a SISO intuitionistic fuzzy system with intuitionistic Mamdani rule base and the defuzzification of Intuitionistic Center of Gravity (ICOG). Let $x\in[a,b]$ be a crisp input. The intuitionistic fuzzy output $B'(y)=(\mu_{B'}(y),\nu_{B'}(y))$, is calculated as follows:

$$B'(y)=\big(\mu_{B'}(y),\nu_{B'}(y)\big)=\bigvee_{i=1}^{n}A_i(x)\wedge B_i(y)=\Big(\bigvee_{i=1}^{n}\min\{\mu_{A_i}(x),\mu_{B_i}(y)\},\bigwedge_{i=1}^{n}\max\{\nu_{A_i}(x),\nu_{B_i}(y)\}\Big),\tag{32}$$

which is subjected to the intuitionistic defuzzification and the result is

$$ICOG(B')^{\lambda}=\lambda\cdot\frac{\int_c^d y\cdot\mu_{B'}(y)\mathrm{d}y}{\int_c^d\mu_{B'}(y)\mathrm{d}y}+(1-\lambda)\cdot\frac{\int_c^d y\cdot\nu_{B'}(y)\mathrm{d}y}{\int_c^d\nu_{B'}(y)\mathrm{d}y}.\tag{33}$$

Combining these two relations, Eqs.(32) and (33), we can write the SISO intuitionistic Mamdani fuzzy system with intuitionistic Mamdani rule base in Definition 29.

$$F(f,x)=\lambda\frac{\int_c^d y\big(\bigvee_{i=1}^{n}\min\{\mu_{A_i}(x),\mu_{B_i}(y)\}\big)\mathrm{d}y}{\int_c^d\big(\bigvee_{i=1}^{n}\min\{\mu_{A_i}(x),\mu_{B_i}(y)\}\big)\mathrm{d}y}+(1-\lambda)\frac{\int_c^d y\big(\bigwedge_{i=1}^{n}\max\{\nu_{A_i}(x),\nu_{B_i}(y)\}\big)\mathrm{d}y}{\int_c^d\big(\bigwedge_{i=1}^{n}\max\{\nu_{A_i}(x),\nu_{B_i}(y)\}\big)\mathrm{d}y}.\tag{34}$$

Since $\mu_{A_i}(x)$ and $\nu_{A_i}(x)$ are continuous and $\mu_{B_i}(y)$ and $\nu_{B_i}(y)$ are integrable, $i=1,\ldots,n$, we have $F(f,x)$ well defined and continuous. Then, the following approximation theorem can be obtained.

*Theorem 38 (Approximation Property of the SISO Intuitionistic Mamdani Fuzzy System)*: Any continuous function $f:[a,b]\to\mathbb{R}$ can be uniformly approximated by the SISO intuitionistic Mamdani fuzzy system $F(f,x)$ with intuitionistic fuzzy sets for the antecedents $A_i(x)=(\mu_{A_i}(x),\nu_{A_i}(x))$ and the consequences $B_i(y)=(\mu_{B_i}(y),\nu_{B_i}(y))$, $i=1,\ldots,n$, satisfying

(i) $\mu_{A_i}(x)$ and $\nu_{A_i}(x)$ continuous with $(A_i)_0=[x_{i-1},x_{i+1}]$, $i=1,\ldots,n$;

(ii) $\mu_{B_i}(y)$ and $\nu_{B_i}(y)$ integral with

$$(B_i)_0=[\min\{y_{i-1},y_i,y_{i+1}\},\max\{y_{i-1},y_i,y_{i+1}\}]$$

for $i=1,\ldots,n$, where $y_i=f(x_i)$, $i=0,\ldots,n+1$

Moreover the following error estimate holds true $\|F(f,x)-f(x)\|\le 3\omega(f,\delta)$, with $\delta=\max_{i=1,\ldots,n}\{x_i-x_{i-1}\}$.

*Proof:* It is easy to see that

$$f(x)=\lambda\frac{\int_c^d\big[\bigvee_{i=1}^{n}\min\{\mu_{A_i}(x),\mu_{B_i}(y)\}\big]f(x)\mathrm{d}y}{\int_c^d\big[\bigvee_{i=1}^{n}\min\{\mu_{A_i}(x),\mu_{B_i}(y)\}\big]\mathrm{d}y}+(1-\lambda)\frac{\int_c^d\big[\bigwedge_{i=1}^{n}\max\{\nu_{A_i}(x),\nu_{B_i}(y)\}\big]f(x)\mathrm{d}y}{\int_c^d\big[\bigwedge_{i=1}^{n}\max\{\nu_{A_i}(x),\nu_{B_i}(y)\}\big]\mathrm{d}y}.$$

and we have that Eq.(35), as shown at the top of the next page, holds.

The membership function $\mu_{A_i}(x)$ and the non-membership functions $\nu_{A_i}(x)$ are both null outside of their support. That is to say any fixed point $x\in[a,b]$ belongs to the support of at most two intuitionisitc fuzzy sets, suppose that $(A_j)_0\cup(A_{j+1})_0$. In addition, consider that $(B_i)_0=[\min\{y_{i-1},y_i,y_{i+1}\},\max\{y_{i-1},y_i,y_{i+1}\}]$, we can restrict the integrals to the union of the supports of $B_j$ and $B_{j+1}$, which is a subset of

$$[\min\{y_{j-1},y_j,y_{j+1},y_{j+2}\},\max\{y_{j-1},y_j,y_{j+1},y_{j+2}\}].$$

$$|F(f,x)-f(x)| = \left|\lambda \frac{\int_c^d \left[\bigvee_{i=1}^n \min\{\mu_{A_i}(x), \mu_{B_i}(y)\}\right]\left[y-f(x)\right]dy}{\int_c^d \left[\bigvee_{i=1}^n \min\{\mu_{A_i}(x), \mu_{B_i}(y)\}\right]dy} + (1-\lambda)\frac{\int_c^d \left[\bigwedge_{i=1}^n \max\{\nu_{A_i}(x), \nu_{B_i}(y)\}\right]\left[y-f(x)\right]dy}{\int_c^d \left[\bigwedge_{i=1}^n \max\{\nu_{A_i}(x), \nu_{B_i}(y)\}\right]dy}\right|$$
$$\leq \lambda \frac{\int_c^d \left[\bigvee_{i=1}^n \min\{\mu_{A_i}(x), \mu_{B_i}(y)\}\right]|y-f(x)|dy}{\int_c^d \left[\bigvee_{i=1}^n \min\{\mu_{A_i}(x), \mu_{B_i}(y)\}\right]dy} + (1-\lambda)\frac{\int_c^d \left[\bigwedge_{i=1}^n \max\{\nu_{A_i}(x), \nu_{B_i}(y)\}\right]|y-f(x)|dy}{\int_c^d \left[\bigwedge_{i=1}^n \max\{\nu_{A_i}(x), \nu_{B_i}(y)\}\right]dy}. \quad (35)$$

$$|F(f,x)-f(x)| = \left|\lambda \frac{\int_{(B_j)_0\cup(B_{j+1})_0} \max\left[\min\{\mu_{A_j}(x), \mu_{B_j}(y)\}, \min\{\mu_{A_{j+1}}(x), \mu_{B_{j+1}}(y)\}\right]\left(y-f(x)\right)dy}{\int_{(B_j)_0\cup(B_{j+1})_0} \max\left[\min\{\mu_{A_j}(x), \mu_{B_j}(y)\}, \min\{\mu_{A_{j+1}}(x), \mu_{B_{j+1}}(y)\}\right]dy}\right.$$
$$\left. + (1-\lambda)\frac{\int_{(B_j)_0\cup(B_{j+1})_0} \min\left[\max\{\nu_{A_j}(x), \nu_{B_j}(y)\}, \max\{\nu_{A_{j+1}}(x), \nu_{B_{j+1}}(y)\}\right]\left(y-f(x)\right)dy}{\int_{(B_j)_0\cup(B_{j+1})_0} \min\left[\max\{\nu_{A_j}(x), \nu_{B_j}(y)\}, \max\{\nu_{A_{j+1}}(x), \nu_{B_{j+1}}(y)\}\right]dy}\right|$$
$$\leq \lambda \frac{\int_{(B_j)_0\cup(B_{j+1})_0} \max\left[\min\{\mu_{A_j}(x), \mu_{B_j}(y)\}, \min\{\mu_{A_{j+1}}(x), \mu_{B_{j+1}}(y)\}\right]\left|y-f(x)\right|dy}{\int_{(B_j)_0\cup(B_{j+1})_0} \max\left[\min\{\mu_{A_j}(x), \mu_{B_j}(y)\}, \min\{\mu_{A_{j+1}}(x), \mu_{B_{j+1}}(y)\}\right]dy}$$
$$+ (1-\lambda)\frac{\int_{(B_j)_0\cup(B_{j+1})_0} \min\left[\max\{\nu_{A_j}(x), \nu_{B_j}(y)\}, \max\{\nu_{A_{j+1}}(x), \nu_{B_{j+1}}(y)\}\right]\left|y-f(x)\right|dy}{\int_{(B_j)_0\cup(B_{j+1})_0} \min\left[\max\{\nu_{A_j}(x), \nu_{B_j}(y)\}, \max\{\nu_{A_{j+1}}(x), \nu_{B_{j+1}}(y)\}\right]dy}. \quad (36)$$

We have that Eq.(36), as shown at the top of this page, holds.

By the definition of $B_j$, $B_{j+1}$, and by the intermediate value theorem applied to the continuous function $f$, given $y \in (B_j)_0 \cup (B_{j+1})_0$ there exists $z \in [x_{j-1}, x_{j+2}]$ such that $f(z) = y$. Then using the properties of the modulus of continuity in Definition 36 and Lemma 37, we obtain

$$|y - f(z)| = |f(z) - f(x)| \leq \omega(f, |x_{j+2} - x_{j-1}|)$$
$$\leq \omega(f, 3\delta) \leq 3\omega(f, \delta),$$

with $\delta = \max_{i=1,\ldots,n} \{x_i - x_{i-1}\}$. Hence, we have that $|F(f,x) - f(x)| \leq 3\omega(f, \delta)$. □

*Theorem 39:* The SISO intuitionistic Mamdani fuzzy system $F(f,x)$ with Mamdani rule base is able to approximate any continuous function $f(x)$ with arbitrary accuracy.

*Proof:* From Lemma 37, we obtain that if $\delta \to 0$ then $F(f,x) \to f(x)$ and the result is immediate. □

By employing the intuitionistic Mamdani fuzzy inference with intuitionistic Larsen rule base and the defuzzification of ICOG, we can write the SISO intuitionistic Larsen fuzzy system as follows:

*Theorem 40: (Approximation Property of the SISO Intuitionistic Larsen Fuzzy System):* Let $A_i(x) = (\mu_{A_i}(x), \nu_{A_i}(x))$ and $B_i(y) = (\mu_{B_i}(y), \nu_{B_i}(y))$ be IFSs which being corresponding to the intuitionistic *if-then* rule base. Then, we have that any continuous function $f : [a, b] \to \mathbb{R}$ can be uniformly approximated by the SISO intuitionistic Mamdani fuzzy system $F(f,x)$ with the intuitionistic Larsen rule base as follows:

$$F(f,x) = \lambda \frac{\int_c^d y\left(\bigvee_{i=1}^n \mu_{A_i}(x)\mu_{B_i}(y)\right)dy}{\int_c^d \left(\bigvee_{i=1}^n \mu_{A_i}(x)\mu_{B_i}(y)\right)dy} + (1-\lambda)\frac{\int_c^d y \cdot \left(\bigwedge_{i=1}^n (\nu_{A_i}(x)+\nu_{B_i}(y)-\nu_{A_i}(x)\nu_{B_i}(y))\right)dy}{\int_c^d \left(\bigwedge_{i=1}^n (\nu_{A_i}(x)+\nu_{B_i}(y)-\nu_{A_i}(x)\nu_{B_i}(y))\right)dy}. \quad (37)$$

*Proof:* The proof is similarly to that of Theorem 38. □

*Theorem 41:* The SISO intuitionistic Larsen fuzzy system $F(f,x)$ is able to approximate any continuous function $f(x)$ with arbitrary accuracy.

*Proof:* The proof is similar to that of Theorem 39. □

By employing the intuitionistic Mamdani fuzzy inference, the IF-$t$-norm rule base and the defuzzification of ICOG,

$$F(f,x)=\lambda\frac{\int_c^d y\cdot\big[\bigvee_{i=1}^{n}\mu_{A_i}(x)\to_T\mu_{B_i}(y)\wedge(1-\nu_{A_i}(x)\to_S\nu_{B_i}(y))\big]\mathrm{d}y}{\int_c^d\big[\bigvee_{i=1}^{n}(\mu_{A_i}(x)\to_T\mu_{B_i}(y)\wedge(1-\nu_{A_i}(x)\to_S\nu_{B_i}(y)))\big]\mathrm{d}y}+(1-\lambda)\frac{\int_c^d y\cdot\big[\bigwedge_{i=1}^{n}\nu_{A_i}(x)\to_S\nu_{B_i}(y)\vee(1-\mu_{A_i}(x)\to_T\mu_{B_i}(y))\big]\mathrm{d}y}{\int_c^d\big[\bigwedge_{i=1}^{n}\nu_{A_i}(x)\to_S\nu_{B_i}(y)\vee(1-\mu_{A_i}(x)\to_T\mu_{B_i}(y))\big]\mathrm{d}y}. \quad (39)$$

we can write the SISO intuitionistic Mamdani fuzzy system with the intuitionistic Larsen rule base as follows:

*Theorem 42 (Approximation Property of the SISO IF-t-Norm Fuzzy System):* Let $A_i(x)=(\mu_{A_i}(x),\nu_{A_i}(x))$ and $B_i(y)=(\mu_{B_i}(y),\nu_{B_i}(y))$ be IFSs which being corresponding to the intuitionistic *if-then* rule base. Then, we have that any continuous function $f:[a,b]\to\mathbb{R}$ can be uniformly approximated by the SISO IF-$t$-norm fuzzy system $F(f,x)$ with IF-$t$-norm intuitionistic rule base as follows:

$$F(f,x)=\lambda\frac{\int_c^d y\big[\bigvee_{i=1}^{n}T(\mu_{A_i}(x),\mu_{B_i}(y))\big]\mathrm{d}y}{\int_c^d\big[\bigvee_{i=1}^{n}T(\mu_{A_i}(x),\mu_{B_i}(y))\big]\mathrm{d}y}+(1-\lambda)\frac{\int_c^d y\cdot\big[\bigwedge_{i=1}^{n}S(\nu_{A_i}(x),\nu_{B_i}(y))\big]\mathrm{d}y}{\int_c^d\big[\bigwedge_{i=1}^{n}S(\nu_{A_i}(x),\nu_{B_i}(y))\big]\mathrm{d}y}. \quad (38)$$

where $\mathcal{T}(x,y)=(T(x_1,y_1),S(x_2,y_2))$ is a $t$-representable IF-$t$-norm.

*Proof:* The proof is similarly to that of Theorem 38. □

*Theorem 43 (Approximation Property of the SISO Intuitionistic Triple-I Fuzzy System):* Let $A_i(x)=(\mu_{A_i}(x),\nu_{A_i}(x))$ and $B_i(y)=(\mu_{B_i}(y),\nu_{B_i}(y))$ be IFSs which being corresponding to the intuitionistic *if-then* rule base. Suppose $\mathcal{T}$ be a $t$-representable IF-$t$-norm and $A_i(x)\to_{\mathcal{T}}B_i(y)=(\mu_{A_i}(x)\to_T\mu_{B_i}(y)\wedge(1-\nu_{A_i}(x)\to_S\nu_{B_i}(y)),\nu_{A_i}(x)\to_S\nu_{B_i}(y)\vee(1-\mu_{A_i}(x)\to_T\mu_{B_i}(y)))$ Then, we have that any continuous function $f:[a,b]\to\mathbb{R}$ can be uniformly approximated by the SISO intuitionistic triple-I fuzzy system $F(f,x)$ as follows Eq.(39), as shown at the top of this page.

*Proof:* The proof is similarly with that of Theorem 38. □

## VI. CONCLUSIONS

On the one hand, in this paper, we investigate the $t$-representable intuitionistic fuzzy $t$-norm $\mathcal{T}$ and the intuitionistic fuzzy residual implication $\to_{\mathcal{T}}$ further. Then, we introduce the SISO intuitionistic fuzzy systems, which involved the intuitionistic *if-then* rule, the intuitionistic *if-then* rule base, the intuitionistic fuzzy inference and the intuitionistic defuzzification. Moreover, as a general conclusion, we obtain that the intuitionistic fuzzy systems are more power than fuzzy systems not only because of the membership and the non-membership function of intuitionistic fuzzy sets but also because of its simplicity and ease of its calculations. Furthermore, some simple examples are given to illustrate these SISO intuitionistic fuzzy systems. On the other hand, after having been instructed the methods of SISO intuitionistic fuzzy systems, we would like to extend our work to MISO intuitionistic fuzzy systems in the future. Finally, we think that intuitionistic fuzzy systems can be applied to lots of problems like in intelligent control, image processing, data mining, time series prediction, and so on.

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