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The Division Ring Over Conjugate Product

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ABSTRACT In this paper, we investigate the rational fractions in the framework of conjugate product and establish a division ring. Some conjugate properties on the proposed division ring are obtained, and the similarity and consimilarity properties are investigated.

INDEX TERMS Rational fractions, conjugate product, inverse.

I. INTRODUCTION

The rational fraction is a very important concept in the field of mathematics with broad applications. For example, a rational fraction was used to approximate a complicated function in [1], [2]. By using a continued fraction associated with two matrices A and B, a solution to the standard Sylvester matrix equation $AX - XB = C$ was given in [3]. In abstract algebra, the set of rational fractions is often used as an example for constructing a field by extending a commutative ring [4]. In classic control theory, the plant and the stabilizing controller of a single-input-single-output system are mathematically described by a rational fraction [5], [6]. In controller design, the coprime factorization of rational fractions over RH_∞ plays a key role [7]. For example, a parameterization of all stabilizing controllers was given in [8] based on a doubly coprime factorization.

Recently, the concept of conjugate product for polynomial matrices was proposed in [9] and it was used to give a unified approach for solving a class of complex matrix equations. The conjugate product of polynomials was investigated in [10], where some concepts in the framework of ordinary operation are generalized to the case of conjugate product. In [11], some criteria were provided for the coprimeness of two polynomials in the framework of conjugate product in terms of the so-called con-Sylvester matrices. The conjugate product of polynomial matrices was investigated in [12]. Recently, a real representation was provided in [13] for the complex polynomial matrices in the framework of conjugate product.

As the conjugate product is a new concept of mathematical operation and the rational fraction over it has not been investigated so far, in this paper, a division ring in the framework of conjugate product is constructed by extending the polynomial ring in the framework of conjugate product proposed in [9] and [10]. Some interesting properties about this division ring are obtained. In [11], a polynomial description in the framework of conjugate product was given for antilinear systems. With such a result, it is necessary to introduce rational fractions in the framework of conjugate product as tools to investigate antilinear systems. A possible application of rational fractions in the framework of conjugate product is to extend the concept of transfer functions into the context of antilinear systems. We believe this will pave the way for the study of dynamic control systems defined over a conjugate product.

The organization of this paper is as follows. In Section II, some preliminary results on conjugate product of polynomials are given. The construction of rational fraction division ring in the framework of conjugate product is introduced in details in Section III. Some mathematical properties on rational fractions in the framework of conjugate product are provided in Section IV. A series approach is established in Section V for rational fractions in the framework of conjugate product. In Section VI, two concepts, similarity and consimilarity are investigated, and some elementary results are provided. Some concluding remarks are given in Section VII.

Throughout this paper, $\overline{a}$ is used to denote the conjugate of $a \in \mathbb{C}$. For a complex number c and a positive integer k, we define

$$c^{*k} = \overline{c^{*(k-1)}}$$

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with $c^{*0} = c$. By this definition, it is obvious that

$$c^{*k} = \begin{cases} c, & \text{for even } k, \\ \overline{c}, & \text{for odd } k. \end{cases}$$

For an arbitrary real a, the operation $\lfloor a \rfloor$ denotes the greatest integer number not to exceed a, that is, $a = \lfloor a \rfloor + p$ with $0 \leq p < 1$ and $\lfloor a \rfloor \in \mathbb{Z}$.

II. PRELIMINARY RESULTS

The concept of conjugate product for complex polynomial matrices was first proposed in [9] and it was used to establish a unified approach for solving a class of complex matrix equations. Some further properties for conjugate product of polynomials and polynomial matrices were given in [10], [12], respectively. In this section, we first present some basic concepts and properties on conjugate product of polynomials briefly. For detailed information, one can refer to [10].

Definition 1: For two polynomials $a(s) = \sum_{i=0}^{m} a_i s^i \in \mathbb{C}[s]$ and $b(s) = \sum_{j=0}^{n} b_j s^j \in \mathbb{C}[s]$, their conjugate product is defined as

$$a(s) \circledast b(s) = \sum_{i=0}^{m} \sum_{j=0}^{n} a_i b_j^{*i} s^{i+j}.$$

For $f(s) = \sum_{i=0}^{t} a_i s^i \in \mathbb{C}[s]$, with $a_t \neq 0$, t is called the degree of $f(s)$, and we denote the degree t by $\deg f(s) = t$. It is easily checked that

$$\deg f(s) \circledast g(s) = \deg f(s) + \deg g(s)$$

for $0 \neq f(s)$, $0 \neq g(s) \in \mathbb{C}[s]$. For convenience, the degree of the zero polynomial is defined to be $-\infty$, and we denote $\deg 0 = -\infty$. Moreover, for any nonzero polynomial $f(s) \in \mathbb{C}[s]$, there holds $\deg 0 < \deg f(s)$.

Lemma 1: Given $a(s), b(s), c(s) \in \mathbb{C}[s]$, the following relations hold:

(1) Left distributivity: $(a(s) + b(s)) \circledast c(s) = a(s) \circledast c(s) + b(s) \circledast c(s)$;
(2) Right distributivity: $a(s) \circledast (b(s) + c(s)) = a(s) \circledast b(s) + a(s) \circledast c(s)$;
(3) Associativity: $(a(s) \circledast b(s)) \circledast c(s) = a(s) \circledast (b(s) \circledast c(s))$.

Remark 1: Different from the ordinary product, the conjugate product does not obey commutativity. For example, for the following two polynomials

$$f(s) = (1 + 2\mathrm{i})s + 3\mathrm{i},$$
$$g(s) = \mathrm{i}s + 2 - 4\mathrm{i},$$

One can easily check that

$$f(s) \circledast g(s) = (2 - \mathrm{i})s^2 - (9 - 8\mathrm{i})s + 12 + 6\mathrm{i},$$
$$g(s) \circledast f(s) = (2 + \mathrm{i})s^2 + 13s + 12 + 6\mathrm{i}.$$

Obviously, $f(s) \circledast g(s)$ is not equal to $g(s) \circledast f(s)$.

Remark 2: It is well known that an arbitrary polynomial with its degree not less than 1 can be factorized into product of some divisors with degree 1 in the framework of ordinary product in complex domain. However, such a conclusion does not hold in the framework of conjugate product. For example, one can not make factorization for the simple polynomial $f(s) = s^2 + 1$ in the framework of conjugate product.

In order to define rational fractions in the framework of conjugate product, we give the following definitions of right and left divisors.

Definition 2: Given two polynomials $f(s)$, $g(s) \in \mathbb{C}[s]$, a polynomial $\varphi_{\mathrm{R}}(s) \in \mathbb{C}[s]$ is called a common right divisor of them in the framework of conjugate produce if there exist $\tilde{f}(s)$, $\tilde{g}(s) \in \mathbb{C}[s]$ such that

$$f(s) = \tilde{f}(s) \circledast \varphi_{\mathrm{R}}(s), \quad g(s) = \tilde{g}(s) \circledast \varphi_{\mathrm{R}}(s),$$

Further, $\varphi_{\mathrm{R}}(s)$ is called a greatest common right divisor (briefly, gcrd) if an arbitrary common right divisor $\varphi(s)$ of them is a right divisor of $\varphi_{\mathrm{R}}(s)$.

Definition 3: Given two polynomials $f(s)$, $g(s) \in \mathbb{C}[s]$, a polynomial $\varphi_{\mathrm{L}}(s) \in \mathbb{C}[s]$ is called a common left divisor of them in the framework of conjugate produce if $\varphi_{\mathrm{L}}(s)$ satisfies

$$f(s) = \varphi_{\mathrm{L}}(s) \circledast \tilde{f}(s), \quad g(s) = \varphi_{\mathrm{L}}(s) \circledast \tilde{g}(s),$$

where $\tilde{f}(s)$, $\tilde{g}(s) \in \mathbb{C}[s]$. Further, $\varphi_{\mathrm{L}}(s)$ is called a greatest common left divisor (briefly, gcld) if an arbitrary common left divisor $\varphi(s)$ of them is a left divisor of $\varphi_{\mathrm{L}}(s)$.

Next, we give the concept of common right multiples.

Definition 4: For polynomials $f_i(s) \in \mathbb{C}[s]$, $i = 1, 2, \cdots, n$, a polynomial $m(s)$ is called a common right multiple of them if there exist $u_i(s) \in \mathbb{C}[s]$, $i = 1, 2, \cdots, n$, such that

$$m(s) = f_1(s) \circledast u_1(s) = f_2(s) \circledast u_2(s) = \cdots = f_n(s) \circledast u_n(s).$$

As to the existence of a common right multiple of two polynomials, we have the following result.

Lemma 2: For two nonzero polynomials $f(s)$, $g(s) \in \mathbb{C}[s]$, there exist a nonzero common right multiple $m(s) \in \mathbb{C}[s]$. That is, there exist $u(s)$, $v(s) \in \mathbb{C}[s]$ such that

$$f(s) \circledast u(s) = g(s) \circledast v(s). \tag{1}$$

Proof: The conclusion will be proven by using mathematical induction for $\min\{\deg f(s), \deg g(s)\}$.

(1) We consider the case of $\min\{\deg f(s), \deg g(s)\} = 0$. It is assumed that $\deg f(s) \leq \deg g(s)$. In this case, denote

$$f(s) = a, \quad 0 \neq a \in \mathbb{C},$$
$$g(s) = \sum_{i=0}^{n} b_i s^i, \quad 0 \neq b_n \in \mathbb{C}, \ n \geq 0.$$

Then, it is easily found that

$$v(s) = c, \quad u(s) = \sum_{i=0}^{n} \frac{b_i c^{*i}}{a} s^i$$

satisfy (1).

(2) It is assumed that the conclusion holds for $\min\{\deg f(s), \deg g(s)\} \leq m$. We consider the case of $\min\{\deg f(s), \deg g(s)\} = m+1$.

Without loss of generality, let $\deg f(s) \leq \deg g(s)$. In this case, one has $\deg f(s) = m+1$. It follows from the Theorem 3 of [10] that there exist $h(s)$, $r(s) \in \mathbb{C}[s]$ such that

$$g(s) = f(s) \circledast h(s) + r(s) \tag{2}$$

with $r(s) = 0$ or $\deg r(s) < \deg f(s)$. If $r(s) = 0$, then

$$u(s) = h(s), v(s) = 1$$

satisfy (1). If $\deg r(s) < \deg f(s)$, then

$$\min\{\deg f(s), \quad \deg r(s)\} = \deg r(s) < \deg f(s) = m+1,$$

which implies $\min\{\deg f(s), \quad \deg r(s)\} \leq m$. With this, by using the induction assumption, there exist $u_0(s)$, $v_0(s) \in \mathbb{C}[s]$ such that

$$r(s) \circledast u_0(s) = f(s) \circledast v_0(s).$$

By combining this relation with (2), one has

$$f(s) \circledast (v_0(s) + h(s) \circledast u_0(s)) = g(s) \circledast u_0(s),$$

which implies that

$$\begin{cases} u(s) = v_0(s) + h(s) \circledast u_0(s) \\ v(s) = u_0(s) \end{cases}$$

satisfy (1).

Therefore, the conclusion of this lemma holds for $\min\{\deg f(s), \deg g(s)\} = m+1$.

(3) According to the principle of mathematical induction, it can be concluded from the previous parts that the result of this lemma holds for any two nonzero polynomials $f(s)$ and $g(s)$. ■

The result of Lemma 2 shows the existence of a common right multiple for any two polynomials in the framework of conjugate product. By Lemma 2, the existence for a common right multiple of more than two polynomials in the framework of conjugate product can also be obtained.

Corollary 1: For nonzero polynomials $f_i(s) \in \mathbb{C}[s]$, $i = 1, 2, \cdots, n$, there exist a nonzero common right multiple $m(s) \in \mathbb{C}[s]$. That is, there exist $u_i(s) \in \mathbb{C}[s]$, $i = 1, 2, \cdots, n$, such that

$$m(s) = f_1(s) \circledast u_1(s) = f_2(s) \circledast u_2(s) = \cdots = f_n(s) \circledast u_n(s).$$

Proof: Only the case of $n = 3$ is considered.

It follows from Lemma 2 that there exist nonzero common right multiple $m_1(s)$ for $f_1(s)$ and $f_2(s)$, and $m_2(s)$ for $f_2(s)$ and $f_3(s)$. Correspondingly, there exist $v_1(s)$, $v_2(s)$, $w_2(s)$, $w_3(s) \in \mathbb{C}[s]$ such that

$$m_1(s) = f_1(s) \circledast v_1(s) = f_2(s) \circledast v_2(s),$$
$$m_2(s) = f_2(s) \circledast w_2(s) = f_3(s) \circledast w_3(s).$$

In addition, it can also be known by using Lemma 2 that there exists a nonzero common right multiple $m(s)$ for $m_1(s)$ and $m_2(s)$. Correspondingly, there exist $\gamma_1(s)$, $\gamma_2(s) \in \mathbb{C}[s]$ such that

$$m(s) = m_1(s) \circledast \gamma_1(s) = m_2(s) \circledast \gamma_2(s).$$

From the preceding three relations, one has

$$\begin{aligned} m(s) &= f_1(s) \circledast v_1(s) \circledast \gamma_1(s) = f_2(s) \circledast v_2(s) \circledast \gamma_1(s) \\ &= f_2(s) \circledast w_2(s) \circledast \gamma_2(s) = f_3(s) \circledast w_3(s) \circledast \gamma_2(s). \end{aligned}$$

This implies that the conclusion holds. ■

Lemma 2 and Corollary 1 will play a vital role in the next section.

Remark 3: In the framework of ordinary product, there exists the concept of the least common multiple for polynomials. However, such a concept does not exist in the framework of conjugate product. For example, for

$$f(s) = s + 1 + \mathrm{i}, \quad g(s) = (1 - \mathrm{i})s + 2\mathrm{i},$$

one has

$$\begin{aligned} m_1(s) &= (1-\mathrm{i})\, s^2 + (2+2\mathrm{i})s + (2+2\mathrm{i}) \\ &= f(s) \circledast ((1+\mathrm{i})s + 2) = g(s) \circledast (s + 1 - \mathrm{i}), \end{aligned}$$

and

$$\begin{aligned} m_2(s) &= (1-\mathrm{i})\, s^2 + (1+\mathrm{i})s + 2\mathrm{i} \\ &= f(s) \circledast ((1+\mathrm{i})s + 1 + \mathrm{i}) = g(s) \circledast (s+1). \end{aligned}$$

These relations show that both $m_1(s)$ and $m_2(s)$ are common right multiples of $f(s)$ and $g(s)$. However, it is easily found that $m_1(s)$ is neither a right divisor nor a left divisor of $m_2(s)$, and $m_2(s)$ is neither a right divisor nor a left divisor of $m_1(s)$.

III. THE DIVISION RING OVER $(\mathbb{C}(s), +, \circledast)$

In this section, we aim to establish a rational fraction division ring by extending the polynomial ring in the framework of conjugate product. For this purpose, the quotient set $\mathbb{C}(s)$ under an equivalence relation is first constructed, and then the sum and conjugate product of rational fractions are defined in the set $\mathbb{C}(s)$.

A. THE QUOTIENT SET $\mathbb{C}(s)$

We begin with this section by defining an equivalence relation in the following set

$$\mathbb{C}[s] \times \mathbb{C}[s] \backslash \{0\} = \{(g(s), f(s)) |\ g(s) \in \mathbb{C}[s],\ f(s) \in \mathbb{C}[s] \backslash \{0\}\}.$$

Definition 5: $(g(s), f(s))$, $(\beta(s), \alpha(s)) \in \mathbb{C}[s] \times \mathbb{C}[s] \backslash \{0\}$ have a relation "$\sim$", denoted

$$(g(s), f(s)) \sim (\beta(s), \alpha(s)),$$

if there are two nonzero polynomials $\gamma(s)$, $\pi(s) \in \mathbb{C}[s]$ such that

$$m(s) = f(s) \circledast \gamma(s) = \alpha(s) \circledast \pi(s),$$

implies

$$g(s) \circledast \gamma(s) = \beta(s) \circledast \pi(s).$$

Since common right multiples of $f(s)$ and $\alpha(s)$ are not unique, we need to answer the following question for the rationality of the above definition:

Is the relation "$\sim$" between $(g(s), f(s))$ and $(\beta(s), \alpha(s))$ dependent on the choice of a common right multiple of $f(s)$ and $\alpha(s)$?

The following lemma gives an answer for this question.

Lemma 3: For $(g(s), f(s)), (\beta(s), \alpha(s)) \in \mathbb{C}[s] \times \mathbb{C}[s]\backslash\{0\}$, let $m_1(s)$ and $m_2(s)$ are two nonzero common right multiples of $f(s)$ and $\alpha(s)$. That is, there exist $\gamma_i(s), \pi_i(s)$, $i = 1, 2$, such that

$$m_1(s) = f(s) \circledast \gamma_1(s) = \alpha(s) \circledast \pi_1(s), \tag{3}$$
$$m_2(s) = f(s) \circledast \gamma_2(s) = \alpha(s) \circledast \pi_2(s). \tag{4}$$

Then,

$$g(s) \circledast \gamma_1(s) = \beta(s) \circledast \pi_1(s) \Longleftrightarrow g(s) \circledast \gamma_2(s) = \beta(s) \circledast \pi_2(s).$$

Proof: Let $m(s)$ be a nonzero common right multiple of $m_1(s)$ and $m_2(s)$. Then, there exist nonzero polynomials $x(s)$, $y(s) \in \mathbb{C}[s]$ such that

$$m(s) = m_1(s) \circledast x(s) = m_2(s) \circledast y(s).$$

By combining this relation with (3) and (4), one has

$$\begin{cases} f(s) \circledast \gamma_1(s) \circledast x(s) = f(s) \circledast \gamma_2(s) \circledast y(s) \\ \alpha(s) \circledast \pi_1(s) \circledast x(s) = \alpha(s) \circledast \pi_2(s) \circledast y(s) \end{cases},$$

which gives

$$\begin{cases} \gamma_1(s) \circledast x(s) = \gamma_2(s) \circledast y(s) \\ \pi_1(s) \circledast x(s) = \pi_2(s) \circledast y(s) \end{cases}. \tag{5}$$

Pre-multiplying both sides of the first expression in (5) by $g(s)$, gives

$$g(s) \circledast \gamma_1(s) \circledast x(s) = g(s) \circledast \gamma_2(s) \circledast y(s). \tag{6}$$

Pre-multiplying both sides of the second expression in (5) by $\beta(s)$, gives

$$\beta(s) \circledast \pi_1(s) \circledast x(s) = \beta(s) \circledast \pi_2(s) \circledast y(s). \tag{7}$$

With (6) and (7), the conclusion can be easily obtained. ■

It can be seen from Lemma 3 that the relation defined in Definition 5 is not dependent on the choice of common right multiples. The following result shows that the relation defined in Definition 5 is in fact an equivalence relation.

Theorem 1: The relation "$\sim$" given in Definition 5 is an equivalence relation. That is, the following properties hold.

(1) Reflexivity: $(g(s), f(s)) \sim (g(s), f(s))$;
(2) Symmetry: $(g(s), f(s)) \sim (\beta(s), \alpha(s)) \Longrightarrow (\beta(s), \alpha(s)) \sim (g(s), f(s))$;
(3) Transitivity: if $(g_1(s), f_1(s)) \sim (g_2(s), f_2(s))$, $(g_2(s), f_2(s)) \sim (g_3(s), f_3(s))$, then $(g_1(s), f_1(s)) \sim (g_3(s), f_3(s))$.

Proof: The properties (1) and (2) are obvious according to Definition 5. Now, we give a proof of transitivity.

It follows from Corollary 1 that there exists a common right multiple $m(s)$ of $f_1(s)$, $f_2(s)$ and $f_3(s)$, and correspondingly there exist $b_i(s)$, $i = 1, 2, 3$, such that

$$m(s) = f_1(s) \circledast b_1(s) = f_2(s) \circledast b_2(s) = f_3(s) \circledast b_3(s). \tag{8}$$

With this relation, it follows from $(g_1(s), f_1(s)) \sim (g_2(s), f_2(s))$ and $(g_2(s), f_2(s)) \sim (g_3(s), f_3(s))$ that

$$g_1(s) \circledast b_1(s) = g_2(s) \circledast b_2(s),$$
$$g_2(s) \circledast b_2(s) = g_3(s) \circledast b_3(s),$$

which gives $g_1(s) \circledast b_1(s) = g_3(s) \circledast b_3(s)$. Combining this relation with $m(s) = f_1(s) \circledast b_1(s) = f_3(s) \circledast b_3(s)$ gives $(g_1(s), f_1(s)) \sim (g_3(s), f_3(s))$. The proof is thus completed. ■

Since the relation "$\sim$" given in Definition 5 is an equivalence relation, we can classify the set $\mathbb{C}[s] \times \mathbb{C}[s]\backslash\{0\}$ in terms of the relation "$\sim$". The set of all $(\beta(s), \alpha(s))$ for which $(\beta(s), \alpha(s)) \sim (g(s), g(s))$ holds, will make up an equivalence class in $\mathbb{C}[s] \times \mathbb{C}[s]\backslash\{0\}$ by $\sim$. Let

$$\frac{g(s)}{f(s)} = \{(\beta(s), \alpha(s)) \in \mathbb{C}[s] \times \mathbb{C}[s]\backslash\{0\} | (\beta(s), \alpha(s)) \sim (g(s), g(s))\} \tag{9}$$

denote the equivalence class to which $(g(s), g(s))$ belongs. Then the set of all the elements in $\mathbb{C}[s] \times \mathbb{C}[s]\backslash\{0\}$ being equivalent to each other is one element in the quotient set.

The quotient set of $\mathbb{C}[s] \times \mathbb{C}[s]\backslash\{0\}$ by $\sim$ is the set of all possible equivalence classes of $\mathbb{C}[s] \times \mathbb{C}[s]\backslash\{0\}$ by $\sim$. That is,

$$(\mathbb{C}[s] \times \mathbb{C}[s]\backslash\{0\}) / \sim = \left\{ \frac{g(s)}{f(s)} | (f(s), g(s)) \in \mathbb{C}[s] \times \mathbb{C}[s]\backslash\{0\} \right\}.$$

For convenience, the quotient set $(\mathbb{C}[s] \times \mathbb{C}[s]\backslash\{0\}) / \sim$ with the relation "$\sim$" being defined in Definition 5 is denoted by $\mathbb{C}(s)$ in this paper. Any element $\frac{g(s)}{f(s)}$ in $\mathbb{C}(s)$ is called a rational fraction. Similarly to the case of ordinary rational fractions, $f(s)$ is called the denominator of $\frac{g(s)}{f(s)}$, and $g(s)$ is called the numerator of $\frac{g(s)}{f(s)}$. According to the definition of equivalence class in (9), for any $g(s), f(s), a(s) \in \mathbb{C}[s]$ with $f(s) \neq 0$, $a(s) \neq 0$, there holds

$$\frac{g(s)}{f(s)} = \frac{g(s) \circledast a(s)}{f(s) \circledast a(s)}. \tag{10}$$

B. OPERATIONS IN $\mathbb{C}(s)$

In the previous subsection, the quotient set $\mathbb{C}(s)$ has been established. In this subsection, we define the sum and conjugate product for two rational fractions in $\mathbb{C}(s)$. First, the definition for the sum of two rational fractions in $\mathbb{C}(s)$ is given.

Definition 6: For any $\frac{g(s)}{f(s)}, \frac{\beta(s)}{\alpha(s)} \in \mathbb{C}(s)$, their sum in $\mathbb{C}(s)$ is defined as

$$\frac{g(s)}{f(s)} + \frac{\beta(s)}{\alpha(s)} = \frac{g(s) \circledast a(s) + \beta(s) \circledast b(s)}{m(s)}, \tag{11}$$

where $m(s) \in \mathbb{C}[s]$ is a common right multiple of $f(s)$ and $\alpha(s)$, and $a(s), b(s) \in \mathbb{C}[s]$ satisfy

$$m(s) = f(s) \circledast a(s) = \alpha(s) \circledast b(s). \tag{12}$$

Now, we check the rationality of Definition 6. For such an aim, we need to show first that the sum defined in (11) is independent of the choice of the common right multiple $m(s)$. Let $m_0(s)$ be another common right multiple of $f(s)$ and $\alpha(s)$, and correspondingly there exist $a_0(s), b_0(s) \in \mathbb{C}[s]$ satisfying

$$m_0(s) = f(s) \circledast a_0(s) = \alpha(s) \circledast b_0(s). \tag{13}$$

In addition, let $n(s) \in \mathbb{C}[s]$ be a common right multiple of $m(s)$ and $m_0(s)$, and correspondingly there exist $x(s), y(s) \in \mathbb{C}[s]$ such that

$$n(s) = m(s) \circledast x(s) = m_0(s) \circledast y(s).$$

Combining this relation with (12) and (13), gives

$$\begin{cases} a(s) \circledast x(s) = a_0(s) \circledast y(s) \\ b(s) \circledast x(s) = b_0(s) \circledast y(s) \end{cases}.$$

With the previous relations, by using the property (10) we have

$$\begin{aligned} \frac{g(s)}{f(s)} + \frac{\beta(s)}{\alpha(s)} &= \frac{g(s) \circledast a(s) + \beta(s) \circledast b(s)}{m(s)} \\ &= \frac{(g(s) \circledast a(s) + \beta(s) \circledast b(s)) \circledast x(s)}{m(s) \circledast x(s)} \\ &= \frac{g(s) \circledast a(s) \circledast x(s) + \beta(s) \circledast b(s) \circledast x(s)}{m(s) \circledast x(s)} \\ &= \frac{g(s) \circledast a_0(s) \circledast y(s) + \beta(s) \circledast b_0(s) \circledast y(s)}{m_0(s) \circledast y(s)} \\ &= \frac{(g(s) \circledast a_0(s) + \beta(s) \circledast b_0(s)) \circledast y(s)}{m_0(s) \circledast y(s)} \\ &= \frac{g(s) \circledast a_0(s) + \beta(s) \circledast b_0(s)}{m_0(s)}. \end{aligned}$$

This implies that the sum defined in Definition 6 for two rational fractions in $\mathbb{C}(s)$ does not depends on the choice of the common right multiple of their denominators.

Next, we need to show that the sum in (11) does not depends on the choice of the representative elements of classes $\frac{g(s)}{f(s)}$ and $\frac{\beta(s)}{\alpha(s)}$. Let

$$\frac{g_0(s)}{f_0(s)} = \frac{g(s)}{f(s)} \text{ and } \frac{\beta_0(s)}{\alpha_0(s)} = \frac{\beta(s)}{\alpha(s)}. \tag{14}$$

It follows from Corollary 1 that there exists a common right multiple $n(s) \in \mathbb{C}[s]$ for $f_0(s)$, $f(s)$, $\alpha_0(s)$ and $\alpha(s)$. Correspondingly, there exist $\gamma_0(s)$, $\gamma(s)$, $\pi_0(s)$ and $\pi(s)$ such that

$$\begin{aligned} n(s) &= f_0(s) \circledast \gamma_0(s) = f(s) \circledast \gamma(s) \\ &= \alpha_0(s) \circledast \pi_0(s) = \alpha(s) \circledast \pi(s). \end{aligned}$$

Combining this relation with (14), gives

$$\begin{cases} g_0(s) \circledast \gamma_0(s) = g(s) \circledast \gamma(s) \\ \beta_0(s) \circledast \pi_0(s) = \beta(s) \circledast \pi(s) \end{cases}.$$

With the preceding relations, by using the property (10) we have

$$\begin{aligned} \frac{g_0(s)}{f_0(s)} + \frac{\beta_0(s)}{\alpha_0(s)} &= \frac{g_0(s) \circledast \gamma_0(s) + \beta_0(s) \circledast \pi_0(s)}{n(s)} \\ &= \frac{g(s) \circledast \gamma(s) + \beta(s) \circledast \pi(s)}{n(s)} \\ &= \frac{g(s) \circledast \gamma(s)}{f(s) \circledast \gamma(s)} + \frac{\beta(s) \circledast \pi(s)}{\alpha(s) \circledast \pi(s)} \\ &= \frac{g(s)}{f(s)} + \frac{\beta(s)}{\alpha(s)}. \end{aligned}$$

This implies that the sum defined in Definition 6 for two rational fractions in $\mathbb{C}(s)$ does not depends on the choice of the representative elements of the rational fractions.

In the set $\mathbb{C}(s)$, there exist a class of elements in the form of $\frac{0}{f(s)}$ with $0 \neq f(s) \in \mathbb{C}[s]$. According to the definition of rational fraction in the framework of conjugate product, we can define that $\frac{0}{f(s)} = \frac{0}{g(s)}$ for any two nonzero polynomials $f(s)$, $g(s) \in \mathbb{C}[s]$. Due to this reason, we can denote $0 = \frac{0}{f(s)}$ in $\mathbb{C}(s)$. Before giving the definition of the conjugate product for two rational fractions in $\mathbb{C}(s)$, we first provide an example to illustrate the uniqueness for the sum.

Example 1: Given the following two rational fractions in $\mathbb{C}(s)$,

$$\frac{g(s)}{f(s)} = \frac{(1+\mathrm{i})s + (2+\mathrm{i})}{s+1+\mathrm{i}}, \quad \frac{\beta(s)}{\alpha(s)} = \frac{(1-2\mathrm{i})s + (1+\mathrm{i})}{(1-\mathrm{i})s + 2\mathrm{i}},$$

we aim to calculate their sum in $\mathbb{C}(s)$. First, by simple calculation it can be obtained that

$$\begin{aligned} m_1(s) &= (1-\mathrm{i})\, s^2 + (2+2\mathrm{i})s + (2+2\mathrm{i}) \\ &= f(s) \circledast ((1+\mathrm{i})s + 2) = \alpha(s) \circledast (s+1-\mathrm{i})\,. \end{aligned}$$

With this relation, we have

$$\begin{aligned} \frac{g(s)}{f(s)} + \frac{\beta(s)}{\alpha(s)} &= \frac{g(s) \circledast ((1+\mathrm{i})s + 2) + \beta(s) \circledast (s+1-\mathrm{i})}{m_1(s)} \\ &= \frac{n_1(s)}{m_1(s)}. \end{aligned} \tag{15}$$

with

$$n_1(s) = (3-2\mathrm{i})\, s^2 + (7+5\mathrm{i})s + (6+2\mathrm{i})\,.$$

In addition, it can be checked easily that

$$\begin{aligned} m_2(s) &= (1-\mathrm{i})\, s^2 + (1+\mathrm{i})s + 2\mathrm{i} \\ &= f(s) \circledast ((1+\mathrm{i})s + 1 + \mathrm{i}) = \alpha(s) \circledast (s+1)\,. \end{aligned}$$

With this relation, one has

$$\begin{aligned} \frac{g(s)}{f(s)} + \frac{\beta(s)}{\alpha(s)} &= \frac{g(s) \circledast ((1+\mathrm{i})s + 1 + \mathrm{i}) + \beta(s) \circledast (s+1)}{m_2(s)} \\ &= \frac{n_2(s)}{m_2(s)}, \end{aligned} \tag{16}$$

with

$$n_2(s) = (3-2\mathrm{i})s^2 + (5+2\mathrm{i})s + (2+4\mathrm{i}).$$

It can be seen that the expressions (15) and (16) for the sum look very different. One may doubt whether the results in (15) and (16) are equal. In fact, by simple calculation we have

$$m_1(s) \circledast (s-1) = m_2(s) \circledast (s-1+\mathrm{i}).$$

Moreover, it is also easily known that

$$\begin{aligned} n_1(s) \circledast (s-1) &= n_2(s) \circledast (s-1+\mathrm{i}) \\ &= (3-2\mathrm{i})s^3 + (4+7\mathrm{i})s^2 + (-1-3\mathrm{i})s + (-6-2\mathrm{i}). \end{aligned}$$

By definition, this implies that the expressions in (15) and (16) are the same.

Next, we give the definition of conjugate product for two rational fractions in $\mathbb{C}(s)$.

Definition 7: For any $\frac{g(s)}{f(s)}, \frac{\beta(s)}{\alpha(s)} \in \mathbb{C}(s)$, if $\beta(s) = 0$, their conjugate product in $\mathbb{C}(s)$ is defined as

$$\frac{g(s)}{f(s)} \circledast \frac{\beta(s)}{\alpha(s)} = \frac{0}{\alpha(s)}.$$

If $\beta(s) \neq 0$, their conjugate product in $\mathbb{C}(s)$ is defined as

$$\frac{g(s)}{f(s)} \circledast \frac{\beta(s)}{\alpha(s)} = \frac{g(s) \circledast a(s)}{\alpha(s) \circledast b(s)}, \tag{17}$$

where nonzero $a(s), b(s) \in \mathbb{C}[s]$ satisfy

$$m(s) = f(s) \circledast a(s) = \beta(s) \circledast b(s). \tag{18}$$

Similarly to the case of sum, we need to check the rationality of Definition 7 for conjugate product of two rational fractions in $\mathbb{C}(s)$. The case of $\beta(s) = 0$ is very simple, and thus only the case of $\beta(s) \neq 0$ will be considered.

First, we show that the conjugate product defined in (17) does not depend on the choice of the common right multiple of $f(s)$ and $\beta(s)$. It is assumed that $m_0(s)$ is another common right multiple of $f(s)$ and $\beta(s)$, and correspondingly there exist nonzero $a_0(s), b_0(s) \in \mathbb{C}[s]$ satisfying

$$m_0(s) = f(s) \circledast a_0(s) = \beta(s) \circledast b_0(s). \tag{19}$$

In addition, let $n(s) \in \mathbb{C}[s]$ be a common right multiple of $m(s)$ and $m_0(s)$, and correspondingly there exist $x(s), y(s) \in \mathbb{C}[s]$ such that

$$n(s) = m(s) \circledast x(s) = m_0(s) \circledast y(s).$$

With this relation, it follows from (18) and (19) that

$$\begin{cases} a(s) \circledast x(s) = a_0(s) \circledast y(s) \\ b(s) \circledast x(s) = b_0(s) \circledast y(s) \end{cases}.$$

With the previous relations, by using the property (10) it can be derived that

$$\begin{aligned} \frac{g(s)}{f(s)} \circledast \frac{\beta(s)}{\alpha(s)} &= \frac{g(s) \circledast a_0(s)}{\alpha(s) \circledast b_0(s)} \\ &= \frac{(g(s) \circledast a_0(s)) \circledast y(s)}{(\alpha(s) \circledast b_0(s)) \circledast y(s)} \\ &= \frac{g(s) \circledast a(s) \circledast x(s)}{\alpha(s) \circledast b(s) \circledast x(s)} \\ &= \frac{g(s) \circledast a(s)}{\alpha(s) \circledast b(s)}. \end{aligned}$$

Second, we will show that the conjugate product (17) is independent of the choice of the representative elements in the classes $\frac{g(s)}{f(s)}$ and $\frac{\beta(s)}{\alpha(s)}$. Let

$$\frac{g_0(s)}{f_0(s)} = \frac{g(s)}{f(s)}, \quad \text{and} \quad \frac{\beta_0(s)}{\alpha_0(s)} = \frac{\beta(s)}{\alpha(s)}. \tag{20}$$

By using Corollary 1, there exists a common right multiple $n(s) \in \mathbb{C}[s]$ for $f_0(s), f(s), \beta_0(s)$ and $\beta(s)$. Correspondingly, there exist $\gamma_0(s), \gamma(s), \pi_0(s), \pi(s) \in \mathbb{C}[s]$ such that

$$\begin{aligned} n(s) &= f_0(s) \circledast \gamma_0(s) = f(s) \circledast \gamma(s) \\ &= \beta_0(s) \circledast \pi_0(s) = \beta(s) \circledast \pi(s). \end{aligned}$$

Combining this relation with (20), gives

$$\begin{cases} g_0(s) \circledast \gamma_0(s) = g(s) \circledast \gamma(s) \\ \alpha_0(s) \circledast \pi_0(s) = \alpha(s) \circledast \pi(s) \end{cases}.$$

With the preceding relations, by using the property (10) it can be obtained that

$$\begin{aligned} \frac{g_0(s)}{f_0(s)} \circledast \frac{\beta_0(s)}{\alpha_0(s)} &= \frac{g_0(s) \circledast \gamma_0(s)}{f_0(s) \circledast \gamma_0(s)} \circledast \frac{\beta_0(s) \circledast \pi_0(s)}{\alpha_0(s) \circledast \pi_0(s)} \\ &= \frac{g_0(s) \circledast \gamma_0(s)}{n(s)} \circledast \frac{n(s)}{\alpha_0(s) \circledast \pi_0(s)} \\ &= \frac{g(s) \circledast \gamma(s)}{\alpha(s) \circledast \pi(s)}, \end{aligned}$$

and

$$\begin{aligned} \frac{g(s)}{f(s)} \circledast \frac{\beta(s)}{\alpha(s)} &= \frac{g(s) \circledast \gamma(s)}{f(s) \circledast \gamma(s)} \circledast \frac{\beta(s) \circledast \pi(s)}{\alpha(s) \circledast \pi(s)} \\ &= \frac{g(s) \circledast \gamma(s)}{n(s)} \circledast \frac{n(s)}{\alpha(s) \circledast \pi(s)} \\ &= \frac{g(s) \circledast \gamma(s)}{\alpha(s) \circledast \pi(s)}. \end{aligned}$$

The preceding two expressions imply that the defined conjugate product for two rational fractions is independent of the choice of the representative elements. Next we will provide an example for the conjugate product in $\mathbb{C}(s)$.

Example 2: Given the following two rational fractions in $\mathbb{C}(s)$,

$$\frac{g(s)}{f(s)} = \frac{(1+2\mathrm{i})s + 2 - \mathrm{i}}{s+1+\mathrm{i}}, \quad \frac{\beta(s)}{\alpha(s)} = \frac{(1-\mathrm{i})s + 2\mathrm{i}}{(2+\mathrm{i})s + 1 - 2\mathrm{i}},$$

we aim to calculate their conjugate product in $\mathbb{C}(s)$. $f(s)$ in this example is the $f(s)$ in the previous example, and $\beta(s)$ here is the $\alpha(s)$ in the previous example. Thus,

$$\begin{aligned} m_1(s) &= (1-\mathrm{i})\, s^2 + (2+2\mathrm{i})s + (2+2\mathrm{i}) \\ &= f(s) \circledast ((1+\mathrm{i})s + 2) = \beta(s) \circledast (s+1-\mathrm{i})\,. \end{aligned}$$

With this relation, we have

$$\begin{aligned} \frac{g(s)}{f(s)} \circledast \frac{\beta(s)}{\alpha(s)} &= \frac{g(s) \circledast ((1+\mathrm{i})s+2)}{m_1(s)} \circledast \frac{m_1(s)}{\alpha(s) \circledast (s+1-\mathrm{i})} \\ &= \frac{\delta_1(s)}{\theta_1(s)}. \end{aligned} \tag{21}$$

where

$$\delta_1(s) = g(s) \circledast ((1+\mathrm{i})s+2) = (3+\mathrm{i})s^2 + (5+5\mathrm{i})s + (4-2\mathrm{i}),$$

$$\theta_1(s) = \alpha(s) \circledast (s+1-\mathrm{i}) = (2+\mathrm{i})s^2 + (2+\mathrm{i})s - 1 - 3\mathrm{i}.$$

In addition, there holds

$$m_2(s) = (1-\mathrm{i})\, s^2 + (1+\mathrm{i})s + 2\mathrm{i} = f(s) \circledast ((1+\mathrm{i})s + 1 + \mathrm{i}) = \beta(s) \circledast (s+1).$$

With this relation, it can be derived that

$$\frac{g(s)}{f(s)} \circledast \frac{\beta(s)}{\alpha(s)} = \frac{g(s) \circledast ((1+\mathrm{i})s+1+\mathrm{i})}{m_2(s)} \circledast \frac{m_2(s)}{\alpha(s) \circledast (s+1)} = \frac{\delta_2(s)}{\theta_2(s)}, \tag{22}$$

where

$$\delta_2(s) = g(s) \circledast ((1+\mathrm{i})s+1+\mathrm{i}) = (3+\mathrm{i})s^2 + (6+2\mathrm{i})s + (3+\mathrm{i}),$$

$$\theta_2(s) = \alpha(s) \circledast (s+1) = (2+\mathrm{i})s^2 + (3-\mathrm{i})s + 1 - 2\mathrm{i}.$$

By simple calculation, it can be immediately obtained that

$$\theta_1(s) \circledast c_1(s) = \theta_2(s) \circledast c_2(s) = m(s)$$

with

$$c_1(s) = \mathrm{i}s + \mathrm{i}, c_2(s) = \mathrm{i}s + 1 + \mathrm{i},$$
$$m(s) = (-1+2\mathrm{i})s^3 + (4-3\mathrm{i})s + 3 - \mathrm{i}.$$

In addition, it can be checked that

$$\delta_1(s) \circledast c_1(s) = \delta_2(s) \circledast c_2(s) = (-1+3\mathrm{i})s^3 + (4-2\mathrm{i})s^2 + (7-\mathrm{i})s + 2 + 4\mathrm{i}.$$

This shows that $\frac{\delta_1(s)}{\theta_1(s)}$ in (21) and $\frac{\delta_2(s)}{\theta_2(s)}$ in (22) are equal.

In fact, for both the sum and conjugate product defined in this section, it is easy to prove their uniqueness. This paves the way for main results in next section.

C. THE DIVISION RING ($\mathbb{C}(s)$, +, $\circledast$)

In this subsection, we show that the quotient set defined in Subsection III-A with two operations "+" and "$\circledast$" respectively given in Definitions 6 and 7 is a division ring. First, we show that ($\mathbb{C}(s)$, +) with the operation "+" defined in Definition 6 is an Abelian group.

Theorem 2: Given $\frac{g(s)}{f(s)}, \frac{g_1(s)}{f_1(s)}, \frac{g_2(s)}{f_2(s)}, \frac{g_3(s)}{f_3(s)} \in \mathbb{C}(s)$, for the operation "+" defined in Definition 6 the following relations hold.

(1) Commutativity: $\frac{g_1(s)}{f_1(s)} + \frac{g_2(s)}{f_2(s)} = \frac{g_2(s)}{f_2(s)} + \frac{g_1(s)}{f_1(s)}$;

(2) Associativity:

$$\left(\frac{g_1(s)}{f_1(s)} + \frac{g_2(s)}{f_2(s)}\right) + \frac{g_3(s)}{f_3(s)} = \frac{g_1(s)}{f_1(s)} + \left(\frac{g_2(s)}{f_2(s)} + \frac{g_3(s)}{f_3(s)}\right);$$

(3) Zero element: $0 + \frac{g(s)}{f(s)} = \frac{g(s)}{f(s)}$;

(4) Negative element: $\frac{g(s)}{f(s)} + \frac{-g(s)}{f(s)} = 0$.

Proof: Only the proof of Item 2 is given here due to limitation of space. It follows from Corollary 1 that there exists a common right multiple $m(s)$ of $f_1(s)$, $f_2(s)$ and $f_3(s)$, correspondingly there exist $a_i(s) \in \mathbb{C}[s]$, $i = 1, 2, 3$, such that

$$m(s) = f_1(s) \circledast a_1(s) = f_2(s) \circledast a_2(s) = f_3(s) \circledast a_3(s).$$

With this relation, we have

$$\begin{aligned}
&\left(\frac{g_1(s)}{f_1(s)} + \frac{g_2(s)}{f_2(s)}\right) + \frac{g_3(s)}{f_3(s)} \\
&= \left(\frac{g_1(s) \circledast a_1(s)}{f_1(s) \circledast a_1(s)} + \frac{g_2(s) \circledast a_2(s)}{f_2(s) \circledast a_2(s)}\right) + \frac{g_3(s) \circledast a_3(s)}{f_3(s) \circledast a_3(s)} \\
&= \frac{g_1(s) \circledast a_1(s) + g_2(s) \circledast a_2(s)}{m(s)} + \frac{g_3(s) \circledast a_3(s)}{m(s)} \\
&= \frac{g_1(s) \circledast a_1(s) + g_2(s) \circledast a_2(s) + g_3(s) \circledast a_3(s)}{m(s)},
\end{aligned}$$

and

$$\begin{aligned}
&\frac{g_1(s)}{f_1(s)} + \left(\frac{g_2(s)}{f_2(s)} + \frac{g_3(s)}{f_3(s)}\right) \\
&= \frac{g_1(s) \circledast a_1(s)}{f_1(s) \circledast a_1(s)} + \left(\frac{g_2(s) \circledast a_2(s)}{f_2(s) \circledast a_2(s)} + \frac{g_3(s) \circledast a_3(s)}{f_3(s) \circledast a_3(s)}\right) \\
&= \frac{g_1(s) \circledast a_1(s)}{m(s)} + \frac{g_2(s) \circledast a_2(s) + g_3(s) \circledast a_3(s)}{m(s)} \\
&= \frac{g_1(s) \circledast a_1(s) + g_2(s) \circledast a_2(s) + g_3(s) \circledast a_3(s)}{m(s)}.
\end{aligned}$$

The preceding two relations imply the conclusion of Item 2. ■

The following theorem shows ($\mathbb{C}(s)$, $\circledast$) with the operation "$\circledast$" defined in Definition 7 is a noncommutative group.

Theorem 3: Given $\frac{g(s)}{f(s)}, \frac{g_1(s)}{f_1(s)}, \frac{g_2(s)}{f_2(s)}, \frac{g_3(s)}{f_3(s)} \in \mathbb{C}(s)$, for the operation "$\circledast$" defined in Definition 7 the following relations hold.

(1) Associativity:

$$\left(\frac{g_1(s)}{f_1(s)} \circledast \frac{g_2(s)}{f_2(s)}\right) \circledast \frac{g_3(s)}{f_3(s)} = \frac{g_1(s)}{f_1(s)} \circledast \left(\frac{g_2(s)}{f_2(s)} \circledast \frac{g_3(s)}{f_3(s)}\right);$$

(2) Identity element: $\frac{1}{1} \circledast \frac{g(s)}{f(s)} = \frac{g(s)}{f(s)} \circledast \frac{1}{1} = \frac{g(s)}{f(s)}$;

(3) Inverse element: there exists a $\rho(s) \in \mathbb{C}(s)$ such that

$$\rho(s) \circledast \frac{g(s)}{f(s)} = \frac{g(s)}{f(s)} \circledast \rho(s) = \frac{1}{1}$$

for any $0 \neq \frac{g(s)}{f(s)} \in \mathbb{C}(s)$.

Proof: (1) Let $a(s)$, $b(s)$ be two nonzero polynomials in $\mathbb{C}[s]$ such that

$$m_1(s) = f_1(s) \circledast a(s) = g_2(s) \circledast b(s).$$

Further, let $c(s)$, $d(s)$ be two nonzero polynomials in $\mathbb{C}[s]$ such that

$$m_2(s) = f_2(s) \circledast b(s) \circledast c(s) = g_3(s) \circledast d(s).$$

With these relations, we have

$$\begin{aligned}
&\left(\frac{g_1(s)}{f_1(s)} \circledast \frac{g_2(s)}{f_2(s)}\right) \circledast \frac{g_3(s)}{f_3(s)} \\
&= \left(\frac{g_1(s) \circledast a(s)}{m_1(s)} \circledast \frac{m_1(s)}{f_2(s) \circledast b(s)}\right) \circledast \frac{g_3(s)}{f_3(s)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{g_1(s) \circledast a(s)}{f_2(s) \circledast b(s)} \circledast \frac{g_3(s)}{f_3(s)} \\
&= \frac{g_1(s) \circledast a(s) \circledast c(s)}{f_2(s) \circledast b(s) \circledast c(s)} \circledast \frac{g_3(s) \circledast d(s)}{f_3(s) \circledast d(s)} \\
&= \frac{g_1(s) \circledast a(s) \circledast c(s)}{f_3(s) \circledast d(s)},
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{g_1(s)}{f_1(s)} \circledast \left(\frac{g_2(s)}{f_2(s)} \circledast \frac{g_3(s)}{f_3(s)} \right) \\
&= \frac{g_1(s)}{f_1(s)} \circledast \left(\frac{g_2(s) \circledast b(s) \circledast c(s)}{f_2(s) \circledast b(s) \circledast c(s)} \circledast \frac{g_3(s) \circledast d(s)}{f_3(s) \circledast d(s)} \right) \\
&= \frac{g_1(s) \circledast a(s) \circledast c(s)}{f_1(s) \circledast a(s) \circledast c(s)} \circledast \left(\frac{m_1(s) \circledast c(s)}{m_2(s)} \circledast \frac{m_2(s)}{f_3(s) \circledast d(s)} \right) \\
&= \frac{g_1(s) \circledast a(s) \circledast c(s)}{m_1(s) \circledast c(s)} \circledast \frac{m_1(s) \circledast c(s)}{f_3(s) \circledast d(s)} \\
&= \frac{g_1(s) \circledast a(s) \circledast c(s)}{f_3(s) \circledast d(s)}.
\end{aligned}
$$

The preceding two relations imply the result of Item 1.

(2) The proof is very simple, and thus is omitted.

(3) $\rho(s) = \frac{f(s)}{g(s)}$ satisfies the condition. ■

In fact, in the definition of an inverse element in Theorem 3, only one condition of $\rho(s) \circledast \frac{g(s)}{f(s)} = \frac{1}{1}$ or $\frac{g(s)}{f(s)} \circledast \rho(s) = \frac{1}{1}$ is sufficient.

Lemma 4: For $\rho(s), r(s) \in \mathbb{C}(s)$, there holds

$$
\rho(s) \circledast r(s) = \frac{1}{1} \Longleftrightarrow r(s) \circledast \rho(s) = \frac{1}{1}.
$$

Proof: Only the part of "$\Longrightarrow$" is proven.

By using definition of conjugate product for two rational fractions and the associativity in the Item 1 of Theorem 3, we have

$$
\begin{aligned}
r(s) \circledast \rho(s) &= r(s) \circledast \frac{1}{1} \circledast \rho(s) \\
&= r(s) \circledast (\rho(s) \circledast r(s)) \circledast \rho(s) \\
&= (r(s) \circledast \rho(s)) \circledast (r(s) \circledast \rho(s)),
\end{aligned}
$$

which gives

$$
(r(s) \circledast \rho(s)) \circledast \left(\frac{1}{1} - r(s) \circledast \rho(s) \right) = 0. \tag{23}
$$

Since $\rho(s) \circledast r(s) = \frac{1}{1}$, there hold $\rho(s) \neq 0$, and $r(s) \neq 0$. Thus, $r(s) \circledast \rho(s) \neq 0$. With this, it follows from (23) that $\frac{1}{1} - r(s) \circledast \rho(s) = 0$. This is the conclusion. The proof is thus completed. ■

For $r(s) \in \mathbb{C}(s)$, if $\rho(s) \circledast r(s) = r(s) \circledast \rho(s) = 1$, then $\rho(s)$ is called the inverse of $r(s)$, and is denoted by $\rho(s) = r^{-1}(s) = (r(s))^{-1}$.

Theorem 4: Given $\frac{g(s)}{f(s)}, \frac{g_1(s)}{f_1(s)}, \frac{g_2(s)}{f_2(s)}, \frac{\beta(s)}{\alpha(s)}, \frac{\beta_1(s)}{\alpha_1(s)}, \frac{\beta_2(s)}{\alpha_2(s)} \in \mathbb{C}(s)$, for the operation "+" defined in Definition 6 and the operation "$\circledast$" defined in Definition 7, the following relations hold.

(1) Left distributivity: $\left(\frac{g_1(s)}{f_1(s)} + \frac{g_2(s)}{f_2(s)}\right) \circledast \frac{\beta(s)}{\alpha(s)} = \frac{g_1(s)}{f_1(s)} \circledast \frac{\beta(s)}{\alpha(s)} + \frac{g_2(s)}{f_2(s)} \circledast \frac{\beta(s)}{\alpha(s)}$;

(2) Right distributivity: $\frac{g(s)}{f(s)} \circledast \left(\frac{\beta_1(s)}{\alpha_1(s)} + \frac{\beta_2(s)}{\alpha_2(s)}\right) = \frac{g(s)}{f(s)} \circledast \frac{\beta_1(s)}{\alpha_1(s)} + \frac{g(s)}{f(s)} \circledast \frac{\beta_2(s)}{\alpha_2(s)}$.

Proof: (1) Let $m(s)$ be a common right multiple of $f_1(s)$, $f_2(s)$ and $\beta(s)$, and correspondingly there exist $a_1(s)$, $a_2(s)$, $b(s)$ such that

$$
m(s) = f_1(s) \circledast a_1(s) = f_2(s) \circledast a_2(s) = \beta(s) \circledast b(s).
$$

With this, by using the property (10) we have

$$
\begin{aligned}
&\left(\frac{g_1(s)}{f_1(s)} + \frac{g_2(s)}{f_2(s)} \right) \circledast \frac{\beta(s)}{\alpha(s)} \\
&= \left(\frac{g_1(s) \circledast a_1(s)}{f_1(s) \circledast a_1(s)} + \frac{g_2(s) \circledast a_2(s)}{f_2(s) \circledast a_2(s)} \right) \circledast \frac{\beta(s) \circledast b(s)}{\alpha(s) \circledast b(s)} \\
&= \frac{g_1(s) \circledast a_1(s) + g_2(s) \circledast a_2(s)}{m(s)} \circledast \frac{m(s)}{\alpha(s) \circledast b(s)} \\
&= \frac{g_1(s) \circledast a_1(s) + g_2(s) \circledast a_2(s)}{\alpha(s) \circledast b(s)},
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{g_1(s)}{f_1(s)} \circledast \frac{\beta(s)}{\alpha(s)} + \frac{g_2(s)}{f_2(s)} \circledast \frac{\beta(s)}{\alpha(s)} \\
&= \frac{g_1(s) \circledast a_1(s)}{f_1(s) \circledast a_1(s)} \circledast \frac{\beta(s) \circledast b(s)}{\alpha(s) \circledast b(s)} + \frac{g_2(s) \circledast a_2(s)}{f_2(s) \circledast a_2(s)} \\
&\quad \circledast \frac{\beta(s) \circledast b(s)}{\alpha(s) \circledast b(s)} \\
&= \frac{g_1(s) \circledast a_1(s)}{\alpha(s) \circledast b(s)} + \frac{g_2(s) \circledast a_2(s)}{\alpha(s) \circledast b(s)} \\
&= \frac{g_1(s) \circledast a_1(s) + g_2(s) \circledast a_2(s)}{\alpha(s) \circledast b(s)},
\end{aligned}
$$

which imply the conclusion of Item 1.

(2) It can be proven similarly to the case Item 1. ■

From Theorems 2, 3, and 4, we have the following main result of this paper.

Theorem 5: The set $\mathbb{C}(s)$ with two operations "+" and "$\circledast$" respectively defined in Definitions 6 and 7 is a division ring. It is denoted by $(\mathbb{C}(s), +, \circledast)$.

For convenience, we denote $\frac{a(s)}{1} = a(s)$ for $a(s) \in \mathbb{C}[s]$ in the division ring $(\mathbb{C}(s), +, \circledast)$. With this notation, for any $0 \neq f(s) \in \mathbb{C}[s]$, there holds

$$
\frac{f(s)}{f(s)} = 1.
$$

In addition, it is easily known that

$$
a^{-1}(s) = \left(\frac{a(s)}{1} \right)^{-1} = \frac{1}{a(s)}.
$$

Thus, for a rational fraction $r(s) = \frac{g(s)}{f(s)}$ with $f(s), g(s) \in \mathbb{C}[s]$, one has

$$
r(s) = \frac{g(s)}{1} \circledast \frac{1}{f(s)} = g(s) \circledast f^{-1}(s).
$$

By now, we establish the division ring $(\mathbb{C}(s), +, \circledast)$ by extending the ring $(\mathbb{C}[s], +, \circledast)$. With the notation $\frac{a(s)}{1} = a(s)$, we can say that $\mathbb{C}[s] \subset \mathbb{C}(s)$.

The inverse of rational fractions has the following property.

Lemma 5: For nonzero $r_1(s), r_2(s) \in \mathbb{C}(s)$, there holds

$$(r_1(s) \circledast r_2(s))^{-1} = r_2^{-1}(s) \circledast r_1^{-1}(s).$$

Proof: By using the associativity of conjugate product for rational fractions, it can be derived that

$$\begin{aligned}(r_1(s) \circledast r_2(s)) \circledast \left(r_2^{-1}(s) \circledast r_1^{-1}(s)\right) &= r_1(s) \circledast \left(r_2(s) \circledast \left(r_2^{-1}(s) \circledast r_1^{-1}(s)\right)\right) \\ &= r_1(s) \circledast \left(\left(r_2(s) \circledast r_2^{-1}(s)\right) \circledast r_1^{-1}(s)\right) \\ &= r_1(s) \circledast \left(1 \circledast r_1^{-1}(s)\right) \\ &= r_1(s) \circledast r_1^{-1}(s) \\ &= 1,\end{aligned}$$

which implies the conclusion. ■

The following corollary is obvious.

Lemma 6: For nonzero $r_i(s) \in \mathbb{C}(s)$, $i = 1, 2, \cdots, n$, there holds

$$(r_1(s)\circledast r_2(s)\circledast \cdots \circledast r_n(s))^{-1} = r_n^{-1}(s)\circledast \cdots \circledast r_2^{-1}(s)\circledast r_1^{-1}(s).$$

At the end of this section, we investigate a special rational fraction. For $c \in \mathbb{C}$ and an integer k, the operation $c^{\overleftarrow{k}}$ is defined as

$$c^{\overleftarrow{k}} = c^{k-2\left\lfloor \frac{k}{2} \right\rfloor} (\overline{c}c)^{\left\lfloor \frac{k}{2} \right\rfloor}.$$

Given $f(s) = \sum_{i=0}^{n} a_i s^i \in \mathbb{C}[s]$, and $c \in \mathbb{C}$, let

$$g(s) = \sum_{k=0}^{n-1} \left(\sum_{j=k+1}^{n} a_j \left(c^{*(k+1)}\right)^{\overleftarrow{j-(k+1)}} \right) s^k.$$

Then, we have

$$\frac{f(s)}{x-c} = g(s) + \frac{\sum_{j=0}^{n} a_j c^{\overleftarrow{j}}}{x-c}.$$

This relation implies that $x - c$ is a right divisor of $f(s)$ if and only if $\sum_{j=0}^{n} a_j c^{\overleftarrow{j}} = 0$. Due to this reason, in the framework of conjugate product we define for $f(s) = \sum_{i=0}^{n} a_i s^i \in \mathbb{C}[s]$ and $c \in \mathbb{C}$

$$f(c) = \sum_{j=0}^{n} a_j c^{\overleftarrow{j}}.$$

It is easily checked that a polynomial has a right divisor with degree 1 if and only if it has a monic right divisor with degree 1. Combining this fact with the preceding reason, the following conclusion can be obtained.

Lemma 7: A nonzero polynomial $f(s) \in \mathbb{C}[s]$ has a right divisor with degree 1 if and only if there exists a $c \in \mathbb{C}$ such that $f(c) = 0$.

In Remark 2, it is pointed out that $s^2 + 1$ can not been factorized into the conjugate product of two divisors with degree 1. Such a fact can be easily intepreted by using Lemma 7. Obviously, if $s^2 + 1$ can be factorized into the conjugate product of two divisors with degree 1, then it has a monic right divisor with degree 1. By applying Lemma 7, there exists a $c \in \mathbb{C}$ such that $c^{\overleftarrow{2}} + 1 = 0$. However,

$$c^{\overleftarrow{2}} + 1 = \overline{c}c + 1 = |c|^2 + 1 > 0.$$

Such a contradiction implies that $s^2 + 1$ can not been factorized into the conjugate product of two divisors with degree 1.

IV. CONJUGATE PROPERTIES FOR RATIONAL FRACTIONS IN $\mathbb{C}(s)$

In this section, we will investigate some mathematical properties for rational fractions in $\mathbb{C}(s)$. Most of these properties are related to the conjugate operation. First, we give the definition of conjugate for a polynomial in the framework of conjugate product.

Definition 8: For a polynomial $f(s) = \sum_{i=0}^{n} a_i s^i \in \mathbb{C}[s]$, its conjugate is defined as

$$\overline{f}(s) = \sum_{i=0}^{n} \overline{a_i} s^i.$$

The conjugate of polynomials in the framework of conjugate product has some interesting properties.

Lemma 8: Given $f(s), g(s) \in \mathbb{C}[s]$, the following relations hold.

(1) $\overline{\overline{f}}(s) = f(s)$;
(2) If $p(s) = f(s) + g(s)$. Then $\overline{p}(s) = \overline{f}(s) + \overline{g}(s)$;
(3) If $h(s) = f(s) \circledast g(s)$. Then $\overline{h}(s) = \overline{f}(s) \circledast \overline{g}(s)$.

Proof: (1) The conclusions of Items 1 and 2 are obvious.

(3) Denote $f(s) = \sum_{i=0}^{n} a_i s^i$, $g(s) = \sum_{j=0}^{m} b_j s^j$. Then, we have

$$\begin{aligned}\overline{f}(s) \circledast \overline{g}(s) &= \left(\sum_{i=0}^{n} \overline{a_i} s^i\right) \circledast \left(\sum_{j=0}^{n} \overline{b_j} s^j\right) \\ &= \sum_{i=0}^{n}\sum_{j=0}^{n} \overline{a_i}\left(\overline{b_j}\right)^{*i} s^{i+j} \\ &= \sum_{i=0}^{n}\sum_{j=0}^{n} \overline{a_i b_j^{*i}} s^{i+j} \\ &= \overline{h}(s).\end{aligned}$$

The proof is completed. ■

For $f(s) = \sum_{i=0}^{n} a_i s^i \in \mathbb{C}[s]$ and $c \in \mathbb{C}$, one has

$$\overline{f}(\overline{c}) = \sum_{i=0}^{n} \overline{a_i c^{\overleftarrow{i}}} = \overline{\sum_{i=0}^{n} a_i c^{\overleftarrow{i}}} = \overline{f(c)},$$

which implies that $\overline{f}(\overline{c}) = 0$ if and only $f(c) = 0$. From this fact, it can be seen that if $s - c$ is a right divisor of $f(s)$, then $s - \overline{c}$ is a right divisor of $\overline{f}(s)$.

With the definition of conjugate for polynomials, the definition of conjugate for rational fractions can be given as follows.

Definition 9: For a rational fraction $r(s) = \frac{g(s)}{f(s)} \in \mathbb{C}(s)$ with $f(s), g(s) \in \mathbb{C}[s]$, its conjugate is defined as

$$\overline{r}(s) = \frac{\overline{g}(s)}{\overline{f}(s)}.$$

The conjugate of rational fractions in the framework of conjugate product has some interesting properties.

Lemma 9: Given $r(s), \rho(s) \in \mathbb{C}(s)$, the following relations hold.

(1) $\overline{\overline{r}}(s) = r(s)$;

(2) If $\omega(s) = r(s) + \rho(s)$. Then $\overline{\omega}(s) = \overline{r}(s) + \overline{\rho}(s)$;

(3) If $\eta(s) = r(s) \circledast \rho(s)$. Then $\overline{\eta}(s) = \overline{r}(s) \circledast \overline{\rho}(s)$.

Proof: (1) The conclusion is obvious.

(2) Denote $r(s) = \frac{g(s)}{f(s)}$, $\rho(s) = \frac{\beta(s)}{\alpha(s)}$ with $g(s), f(s), \alpha(s), \beta(s) \in \mathbb{C}[s]$. Let $m_1(s)$ be a common right multiple of $f(s)$ and $g(s)$, and correspondingly there exist nonzeros $a(s), b(s) \in \mathbb{C}[s]$ such that

$$m_1(s) = f(s) \circledast a(s) = \alpha(s) \circledast b(s).$$

By using Lemma 8, one has

$$\overline{m_1}(s) = \overline{f}(s) \circledast \overline{a}(s) = \overline{\alpha}(s) \circledast \overline{b}(s).$$

With this relation, by using Lemma 8 it can be obtained that

$$\begin{aligned}\overline{r}(s) + \overline{\rho}(s) &= \frac{\overline{g}(s)}{\overline{f}(s)} + \frac{\overline{\beta}(s)}{\overline{\alpha}(s)} \\ &= \frac{\overline{g}(s) \circledast \overline{a}(s)}{\overline{f}(s) \circledast \overline{a}(s)} + \frac{\overline{\beta}(s) \circledast \overline{b}(s)}{\overline{\alpha}(s) \circledast \overline{b}(s)} \\ &= \frac{\overline{g}(s) \circledast \overline{a}(s) + \overline{\beta}(s) \circledast \overline{b}(s)}{\overline{m_1}(s)}.\end{aligned}$$

In addition, it is known that

$$\omega(s) = r(s) + \rho(s) = \frac{g(s) \circledast a(s) + \beta(s) \circledast b(s)}{m_1(s)}.$$

According to Definition 9, the preceding two relations imply the conclusion.

(3) Define $r(s)$ and $\rho(s)$ as before, and let $m_2(s)$ be a common right multiple of $f(s)$ and $\beta(s)$. Correspondingly, there exist nonzeros $c(s), d(s) \in \mathbb{C}[s]$ such that

$$m_2(s) = f(s) \circledast c(s) = \beta(s) \circledast d(s).$$

By using Lemma 8, we have

$$\overline{m_2}(s) = \overline{f}(s) \circledast \overline{c}(s) = \overline{\beta}(s) \circledast \overline{d}(s).$$

With this relation, by using Lemma 8 it can be obtained that

$$\begin{aligned}\overline{r}(s) \circledast \overline{\rho}(s) &= \frac{\overline{g}(s)}{\overline{f}(s)} \circledast \frac{\overline{\beta}(s)}{\overline{\alpha}(s)} \\ &= \frac{\overline{g}(s) \circledast \overline{c}(s)}{\overline{f}(s) \circledast \overline{c}(s)} \circledast \frac{\overline{\beta}(s) \circledast \overline{d}(s)}{\overline{\alpha}(s) \circledast \overline{d}(s)} \\ &= \frac{\overline{g}(s) \circledast \overline{c}(s)}{\overline{\alpha}(s) \circledast \overline{d}(s)}.\end{aligned}$$

In addition, it is known that

$$\eta(s) = r(s) \circledast \rho(s) = \frac{g(s) \circledast c(s)}{\alpha(s) \circledast d(s)}.$$

According to Definition 9, the preceding two relations imply the conclusion. ■

For a rational fraction $r(s) \in \mathbb{C}(s)$, if $h(s) \circledast \overline{r}(s) = 1$, we denote $h(s) = \overline{r}^{-1}(s)$. In addition, according to the preceding definitions it is easily known that $\overline{r^{-1}}(s)$ denotes the conjugate of the inverse $r^{-1}(s)$ of $r(s)$. With these notations, the following conclusion is obtained.

Lemma 10: For a nonzero $r(s) \in \mathbb{C}(s)$, there holds $\overline{r}^{-1}(s) = \overline{r^{-1}}(s)$.

Further, by using Lemmas 6 and 9 we can obtain the following result.

Lemma 11: Given nonzero $r_i(s) \in \mathbb{C}(s)$, $i = 1, 2, \cdots, n$, denote $\omega(s) = r_1(s) \circledast r_2(s) \circledast \cdots \circledast r_n(s)$. Then

$$\overline{\omega}^{-1}(s) = \overline{r_n^{-1}}(s) \circledast \overline{r_{n-1}^{-1}}(s) \circledast \cdots \circledast \overline{r_2^{-1}}(s) \circledast \overline{r_1^{-1}}(s).$$

V. A SERIES APPROACH IN $(\mathbb{C}(s), +, \circledast)$

In previous sections, we investigate the rational fractions in $\mathbb{C}(s)$ in general. In this section, we investigate it from a different perspective. We begin with this section by investigating a class of special rational fractions in $(\mathbb{C}(s), +, \circledast)$. Consider a rational fraction $r(s) = \frac{a}{bs^i}$, with $a, b \in \mathbb{C}$, $b \neq 0$. One has

$$\begin{aligned}r(s) &= \frac{a}{bs^i} = a \circledast \frac{1}{bs^i} = a \circledast \frac{1}{s^i \circledast b^{*i}} \\ &= a \circledast \frac{\left(b^{-1}\right)^{*i}}{s^i \circledast b^{*i} \circledast \left(b^{-1}\right)^{*i}} \\ &= a\left(b^{-1}\right)^{*i} \circledast \frac{1}{s^i}.\end{aligned}$$

If we denote

$$c_{-i} = a\left(b^{-1}\right)^{*i}, \quad s^{-i} = \frac{1}{s^i},$$

then, $r(s) = \frac{a}{bs^i} = c_{-i} \circledast s^{-i}$. With these notations, for $f(s) = a_j s^j \in \mathbb{C}[s]$ and $r(s) = c_{-i} \circledast s^{-i} \in \mathbb{C}(s)$, $i \geq 1$, one has

$$f(s) \circledast r(s) = a_j c_{-i}^{*j} \circledast s^{j-i}.$$

In addition, by using Definition 7 and the property (10) we have

$$\begin{aligned}r(s) \circledast f(s) &= c_{-i} \circledast \frac{1}{s^i} \circledast a_j s^j \\ &= c_{-i} \circledast \left(\frac{1}{s^i} \circledast a_j\right) \circledast s^j \\ &= c_{-i} \circledast \left(\frac{a_j^{*(-i)}}{s^i \circledast a_j^{*(-i)}} \circledast \frac{a_j s^i}{s^i}\right) \circledast s^j \\ &= c_{-i} \circledast \left(\frac{a_j^{*(-i)}}{a_j s^i} \circledast \frac{a_j s^i}{s^i}\right) \circledast s^j \\ &= c_{-i} \circledast a_j^{*(-i)} \circledast \frac{1}{s^i} \circledast s^j \\ &= c_{-i} a_j^{*(-i)} \circledast s^{j-i}.\end{aligned}$$

For $\rho(s)=d_{-k}\circledast s^{-k}\in\mathbb{C}(s)$, $k\geq 1$, and $r(s)=c_{-i}\circledast s^{-i}\in\mathbb{C}(s)$, $i\geq 1$, by Definition 7 and the property (10) it can be derived that

$$\begin{aligned}
r(s)\circledast d(s) &= c_{-i}\circledast\frac{1}{s^i}\circledast d_{-k}\circledast\frac{1}{s^k}\\
&= c_{-i}\circledast\left(\frac{d_{-k}^{*(-i)}}{s^i\circledast d_{-k}^{*(-i)}}\circledast\frac{d_{-k}s^i}{s^i}\right)\circledast\frac{1}{s^k}\\
&= c_{-i}\circledast\left(\frac{d_{-k}^{*(-i)}}{d_{-k}s^i}\circledast\frac{d_{-k}s^i}{s^i}\right)\circledast\frac{1}{s^k}\\
&= c_{-i}\circledast d_{-k}^{*(-i)}\circledast\frac{1}{s^i}\circledast\frac{1}{s^k}\\
&= c_{-i}d_{-k}^{*(-i)}\circledast s^{-(i+k)}.
\end{aligned}$$

The preceding relations imply that, if we denote $r(s)=c_{-i}\circledast s^{-i}=c_{-i}s^{-i}$, for $i\geq 1$, then there holds

$$a_is^i\circledast b_js^j=a_ib_j^{*i}s^{i+j},\quad \text{for any integers } i \text{ and } j.$$

With this observation, the following conclusion can be easily derived.

Theorem 6: For $r(s)=\sum_{i=-m}^{n}a_is^i\in\mathbb{C}(s)$, $\rho(s)=\sum_{i=-t}^{l}b_js^j\in\mathbb{C}(s)$ with m, n, t and l being integers not less than 0, there holds

$$r(s)\circledast\rho(s)=\sum_{i=-m}^{n}\sum_{j=-t}^{l}a_ib_j^{*i}s^{i+j}.$$

This theorem states that the conjugate product of two rational fractions can be defined similarly to the case of polynomials in $\mathbb{C}[s]$ when they are represented in the form of series. Such a fact also shows that the definition of rational fractions given in Subsection III-A and the operations "+" and "$\circledast$" for rational fractions in Subsection III-B can be unified.

VI. SIMILARITIES OF RATIONAL FRACTIONS OVER $\mathbb{C}(s)$

There are several similarity concepts for conjugate product [10] and here we fist investigate the normal similarity concept defined below.

A. SIMILARITY

We first give the so-called similarity for two rational fractions over $\mathbb{C}(s)$.

Definition 10: Two rational fractions $r(s)$, $\rho(s)\in\mathbb{C}(s)$ are called similar in the framework of conjugate product, denoted

$$r(s)\approx\rho(s)$$

if there exists a nonzero rational fraction $p(s)\in\mathbb{C}(s)$ such that

$$r(s)=p(s)\circledast\rho(s)\circledast p^{-1}(s).$$

We first prove that the similarity defined above is an equivalence relation as stated below.

Theorem 7: The similarity of rational fractions in Definition 10 is an equivalence relation. That is, the following properties hold.

(1) Reflexivity: $r(s)\approx r(s)$ for $r(s)\in\mathbb{C}(s)$;

(2) Symmetry: $r(s)\approx\rho(s)\Longrightarrow\rho(s)\approx r(s)$ for $r(s)$, $\rho(s)\in\mathbb{C}(s)$;

(3) Transitivity: if $r_1(s)\approx r_2(s)$, $r_2(s)\approx r_3(s)$, then $r_1(s)\approx r_3(s)$, for $r_1(s)$, $r_2(s)$, $r_3(s)\in\mathbb{C}(s)$.

Proof: The properties (1) and (2) are obvious. Only the transitivity is proven.

Since $r_1(s)\approx r_2(s)$, there exists $p_2(s)\in\mathbb{C}(s)$ satisfying

$$r_1(s)=p_2(s)\circledast r_2(s)\circledast p_2^{-1}(s). \tag{24}$$

Since $r_2(s)\approx r_3(s)$, there exists $p_3(s)\in\mathbb{C}(s)$ satisfying

$$r_2(s)=p_3(s)\circledast r_3(s)\circledast p_3^{-1}(s).$$

Substituting this relation into (24), gives

$$\begin{aligned}
r_1(s) &= p_2(s)\circledast\left(p_3(s)\circledast r_3(s)\circledast p_3^{-1}(s)\right)\circledast p_2^{-1}(s)\\
&= (p_2(s)\circledast p_3(s))\circledast r_3(s)\circledast\left(p_3^{-1}(s)\circledast p_2^{-1}(s)\right)\\
&= (p_2(s)\circledast p_3(s))\circledast r_3(s)\circledast(p_2(s)\circledast p_3(s))^{-1}.
\end{aligned}$$

This relation implies $r_1(s)\approx r_3(s)$. ■

In the field of ordinary rational fractions, it is not necessary to define the concept of similarity since the ordinary product of two rational fractions obeys commutativity law. Now, it is routine to ask whether we can give necessary and sufficient conditions for similarity of rational fractions in the framework of conjugate product. Further, is there a canonical form under the similarity transformation? These questions are not trivial and we can not answer them currently. For a perceptual intuition understanding to similarity, we give the following example to show its complexity.

Example 3: Given the following polynomial

$$r(s)=(1+\mathrm{i})s+1,$$

by calculation we can find that $r(s)$ is similar to

$$\rho(s)=\frac{(1-\mathrm{i})a}{\bar{a}}s+1,$$

and the corresponding $p(s)$ satisfying $r(s)=p(s)\circledast\rho(s)\circledast p^{-1}(s)$ is given by

$$p(s)=as+\frac{(1+\mathrm{i})c}{a}$$

with c being a real number. If $a=1$, it is obtained that $r(s)$ is similar to $\rho(s)=(1-\mathrm{i})s+1$. If $a=2+3\mathrm{i}$, it is obtained that $r(s)$ is similar to $\rho(s)=\frac{7+17\mathrm{i}}{13}s+1$.

It can be seen from this example that there exist infinitely many similar fractions for a given fraction even if the transformation fraction $p(s)$ is restricted to be a polynomial. Next we consider another similarity concept.

B. CONSIMILARITY

In the area of numerical matrices, there exists the concept of consimilarity for two complex matrices [14], [15]. Two square matrices $A, B \in \mathbb{C}^{n\times n}$ are called consimilar if there exists a nonsingular matrix P such that $A = PB\overline{P}^{-1}$. By generalizing this idea, the concept of consimilarity for two rational fractions is proposed in the framework of conjugate product.

Definition 11: Two rational fractions $r(s), \rho(s) \in \mathbb{C}(s)$ are called consimilar in the framework of conjugate product, denoted

$$r(s) \approxeq \rho(s),$$

if there exists a nonzero rational fraction $p(s) \in \mathbb{C}(s)$ such that

$$r(s) = p(s) \circledast \rho(s) \circledast \overline{p}^{-1}(s).$$

The following theorem shows that consimilarity over $\mathbb{C}(s)$ is also an equivalence relation.

Theorem 8: The consimilarity of rational fractions in Definition 11 is an equivalence relation. That is, the following properties hold.

(1) Reflexivity: $r(s) \approxeq r(s)$ for $r(s) \in \mathbb{C}(s)$;
(2) Symmetry: $r(s) \approxeq \rho(s) \Longrightarrow \rho(s) \approxeq r(s)$ for $r(s), \rho(s) \in \mathbb{C}(s)$;
(3) Transitivity: if $r_1(s) \approxeq r_2(s)$, $r_2(s) \approxeq r_3(s)$, then $r_1(s) \approxeq r_3(s)$, for $r_1(s), r_2(s), r_3(s) \in \mathbb{C}(s)$.

Proof: The properties (1) and (2) are obvious. Only the transitivity is proven here.

Since $r_1(s) \approxeq r_2(s)$, there exists $p_2(s) \in \mathbb{C}(s)$ satisfying

$$r_1(s) = p_2(s) \circledast r_2(s) \circledast \overline{p_2}^{-1}(s). \tag{25}$$

Since $r_2(s) \approxeq r_3(s)$, there exists $p_3(s) \in \mathbb{C}(s)$ satisfying

$$r_2(s) = p_3(s) \circledast r_3(s) \circledast \overline{p_3}^{-1}(s).$$

Let $p(s) = p_2(s) \circledast p_3(s)$. Then, by using Lemmas 6 and 8, substituting this relation into (25), gives

$$\begin{aligned} r_1(s) &= p_2(s) \circledast \left(p_3(s) \circledast r_3(s) \circledast \overline{p_3}^{-1}(s)\right) \circledast \overline{p_2}^{-1}(s) \\ &= (p_2(s) \circledast p_3(s)) \circledast r_3(s) \circledast \left(\overline{p_3}^{-1}(s) \circledast \overline{p_2}^{-1}(s)\right) \\ &= (p_2(s) \circledast p_3(s)) \circledast r_3(s) \circledast (\overline{p_2}(s) \circledast \overline{p_3}(s))^{-1} \\ &= p(s) \circledast r_3(s) \circledast \overline{p}^{-1}(s). \end{aligned}$$

This relation implies $r_1(s) \approxeq r_3(s)$. ■

For a perceptual intuition to understand consimilarity, the following example is given.

Example 4: Given the following polynomial

$$r(s) = (1 - \mathrm{i})s + 2,$$

by calculation it can be found that $r(s)$ is consimilar to

$$\rho(s) = (1 + \mathrm{i})s + \frac{2\overline{b}}{b},$$

and the corresponding $p(s)$ satisfying $r(s) = p(s) \circledast \rho(s) \circledast \overline{p}^{-1}(s)$ is given by

$$p(s) = as + b$$

where a and b satisfy

$$\overline{ab} = ab + \mathrm{i}b\overline{b}.$$

If $a = 1$, $b = -2\mathrm{i}$, it can be obtained that $r(s)$ is similar to $\rho(s) = (1 + \mathrm{i})s - 2$. If $a = 2 + 3\mathrm{i}$, $b = -1 + \mathrm{i}$, it can be obtained that $r(s)$ is similar to $\rho(s) = (1 + \mathrm{i})s + 2\mathrm{i}$.

It can be seen from this example that there exist infinitely many consimilar fractions for a given fraction even if the transformation fraction $p(s)$ is restricted to be a polynomial.

VII. CONCLUSION

In this paper, the division ring of rational fractions in the framework of conjugate product has been constructed by extending the ring of polynomials in the framework of conjugate product. Some interesting properties of rational fractions are also provided. It should be noted that the concept of common right multiples for polynomials in the framework of conjugate product plays a key role in the construction of rational fractions division ring. Unlike the polynomials in the framework of ordinary product, the concept of least common multiples can not be defined in the framework of conjugate product. Consequently, the rational fractions in the framework of conjugate product have some interesting properties. For example, two rational fractions may be equal to each other even if they have very different expressions.

In the proposed rational fraction division ring, the inverse of a nonzero rational fraction exists. In addition, the conjugate product of two rational fractions does not obey commutative law. Similarly to the case of numerical matrices, the concepts of similarity and consimilarity are proposed for rational fractions in the framework of conjugate product. As a future work, it is suggested that similarity and consimilarity of rational fractions in the framework of conjugate product deserve further investigation.

REFERENCES

[1] L. Wuytack, "An algorithm for rational interpolation similar to the *qd*-algorithm," *Numer. Math.*, vol. 20, no. 5, pp. 418–424, 1973.
[2] F. Wielonsky, "Rational approximation to the exponential function with complex conjugate interpolation points," *J. Approximation Theory*, vol. 111, no. 2, pp. 344–368, 2001.
[3] W. X. Gao, "Continued-fraction solution of matrix equation $AX-XB=C$," *Sci. China A*, vol. 32, no. 9, pp. 1025–1035, 1989.
[4] S. Roman, *Field Theory*, 2nd ed. Berlin, Germany: Springer, 2011.
[5] T. Kailath, *Linear Systems*. Englewood Cliffs, NJ, USA: Prentice-Hall, 1980.
[6] K. M. Zhou, J. C. Doyle, and K. Glover, *Robust and Optimal Control*. Upper Saddle River, NJ, USA: Prentice-Hall, 1996.
[7] K. Zhou, "On the parameterization of H_∞ controllers," *IEEE Trans. Autom. Control*, vol. 37, no. 9, pp. 1442–1446, Sep. 1992.
[8] J. Y. Ishihara and R. M. Sales, "Doubly coprime factorizations related to any stabilizing controllers in state space," *Automatica*, vol. 35, no. 9, pp. 1573–1577, 1999.
[9] A. G. Wu, G. Feng, W. Liu, and G.-R. Duan, "The complete solution to the Sylvester-polynomial-conjugate matrix equations," *Math. Comput. Model.*, vol. 53, pp. 2044–2056, May 2011.

[10] A. G. Wu, G. R. Duan, G. Feng, and W. Q. Liu, "On conjugate product of complex polynomials," *Appl. Math. Lett.*, vol. 24, pp. 735–741, May 2011.
[11] A.-G. Wu and Y.-R. Xu, "On coprimeness of two polynomials in the framework of conjugate product," *IET Control Theory Appl.*, vol. 11, no. 10, pp. 1522–1529, 2017.
[12] A.-G. Wu, W. Liu, and G.-R. Duan, "On the conjugate product of complex polynomial matrices," *Math. Comput. Model.*, vol. 53, nos. 9–10, pp. 2031–2043, 2011.
[13] A.-G. Wu and H.-Z. Wang, "A real representation of complex polynomial matrices in the framework of conjugate product," *Int. J. Comput. Math.*, vol. 96, no. 8, pp. 1567–1575, 2019.
[14] Y. P. Hong and R. A. Horn, "A canonical form for matrices under consimilarity," *Linear Algebra Appl.*, vol. 102, pp. 143–168, Apr. 1988.
[15] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge, U.K.: Cambridge Univ. Press, 1990.

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