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H_∞ Piecewise Control for Linear Parabolic Distributed Parameter Systems With Piecewise Observation in Space

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ABSTRACT In this paper, the H_∞ piecewise control problem is discussed for linear parabolic distributed parameter systems (DPSs) with piecewise observation in space. An H_∞ performance constraint is provided in this paper to solve the effect caused by the measurement disturbance. First, a static feedback controller is designed for the collocated piecewise observation in space, then an observer-based dynamic feedback controller is designed for the non-collocated piecewise observation in space. By the Lyapunov technique, it has been proved that if the LMI-based sufficient conditions are fulfilled, the resulting closed-loop systems are exponentially stable under a prescribed H_∞ performance constraint. Finally, the numerical simulation results of the closed-loop system and the H_∞ attenuation level demonstrate the effectiveness of the proposed method.

INDEX TERMS H_∞ control, parabolic distributed parameter systems, non-collocated piecewise observation, measurement disturbance.

I. INTRODUCTION

Parabolic distributed parameter systems (DPSs) are widely used in engineering practice, such as diffusion process, heat exchange and pipeline chemical reaction and so on [1], [2]. Over the past few decades, the research on the control problem of parabolic DPSs [3]–[13] has attracted much attention from many scholars. For example, Dubljevic has studied the finite-dimensional state feedback predictive control for parabolic DPSs in [5], [6]; Wu has studied the finite-dimensional output feedback control for nonlinear parabolic DPSs in [7]; Fridman has studied the robust sampled-data control for semi-linear parabolic DPSs in [8]; Demetriou has studied the consensus control for parabolic DPSs in [9]; Wang has studied the pointwise control based on the Takagi-Sugeno fuzzy PDE model for semi-linear parabolic DPSs in [12]. Generally, the control form of DPSs are divided into boundary control and in-domain control. Boundary control is that the actuators are distributed at the boundary of DPSs, and the research about boundary control for parabolic DPSs has already fruitful achievements [10], [14]–[18]. Although the study of boundary control for DPSs is relatively mature, the in-domain control is also very important in some applications, such as the pointwise heating of a long thin rod [1]. In-domain control is that the actuators are distributed within the spatial domain of DPSs. Some researchers have been carried out for the in-domain control, such as static collocated piecewise control [19]–[21], static collocated pointwise control [12], [22], [23], static non-collocated pointwise control [12], static non-collocated piecewise control [8], [11], observer-based dynamic non-collocated pointwise control [12], [13] and observer-based dynamic non-collocated piecewise control [24].

For the measurement of DPSs, measurement disturbance is usually unavoidable. H_∞ control is a feasible way to reduce the effect of the measurement disturbance. The main idea of H_∞ control is to design an H_∞ feedback controller for the DPSs and ensure that the resulting closed-loop system is exponentially stable under an H_∞ performance constraint. In the past few decades, the H_∞ control has been paid much attention [17], [20], [25]–[29]. In this paper, we will extend the recent results in [13], [24], [30] to discuss the problem of H_∞ control for linear parabolic DPSs. Both the collocated and non-collocated piecewise observation in space are considered. For the collocated piecewise observation case, we propose a static output feedback controller to ensure that

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the closed-loop system satisfies the prescribed H_∞ performance constraint. For the non-collocated piecewise observation case, similar to the previous works [13], [24], [30], an observer-based dynamic output feedback controller is designed and the prescribed H_∞ performance constraint is guaranteed for the resulting closed-loop DPSs.

The main contributions of this paper are summarized as follows:

1) In contrast to the results in [11]–[13], [21], [24] for the in-domain control with non-collocated observation, H_∞ control is considered in this paper.

2) Different from the H_∞ control for DPSs in [25], [27], [29], the original nonlinear PDE systems is represented by an approximate ordinary differential equation (ODE) via the model reduction technique before the controller design. In this paper, the controller is designed based on the original PDE models, and this method can utilize the original system dynamics completely.

3) Although the H_∞ controller is designed based on the original PDE model in [17], it's developed under the boundary control. In this paper, an H_∞ feedback controller based on the original PDE model is designed for the piecewise control form.

The organizational structure of the remaining parts of this paper is arranged as follows. Section II gives problem formulation and preliminaries about the research of this paper. Section III provides an H_∞ static feedback controller design for the piecewise control with collocated piecewise observation in space. Section IV provides an H_∞ observer-based dynamic feedback controller design for the piecewise control with non-collocated piecewise observation in space. Section V demonstrates some numerical simulation results of the closed-loop system with free disturbance and the H_∞ disturbance attenuation level. Finally, Section VI gives the brief conclusions of this paper.

Notation: $\Re$ is the set of real numbers, $\Re^m$ is m-dimensional Euclidean space, $\Re^{m\times m}$ is the set of $m \times m$ matrices. $\boldsymbol{P}$ is a symmetric matrix, $\boldsymbol{P} < 0$ represents $\boldsymbol{P}$ is negative definite. Meanwhile $\boldsymbol{P} > 0$ represents $\boldsymbol{P}$ is negative. $\mathscr{L}^2([0,1])$ is a real Hilbert space of $\epsilon(z)$, in which $|\epsilon(\cdot)|_2 \triangleq \sqrt{\int_0^L \epsilon^2(z)dz}$. $\mathscr{L}^2([0,\infty])$ is a real Hilbert space of $\zeta(\cdot,t)$, in which $|\zeta(\cdot)|_{\mathscr{L}^2} \triangleq \int_0^\infty |\zeta(\cdot,t)|_2 dt$. $\mathscr{H}^1(0,1) \triangleq \mathcal{W}^{1,2}([0,1])$ is a real Sobolev space of absolutely continuous functions $\vartheta(z)$, in which $|\vartheta(\cdot)|_{\mathscr{H}^1} \triangleq \sqrt{\int_0^L \left(\frac{d\vartheta(z)}{dz}\right)^2 dz}$. $y_z(z,t) = dy(z,t)/dx$, $y_t(z,t) = dy(z,t)/dt$ and $y_{zz}(z,t) = d^2y(z,t)/dz^2$ are the partial derivatives of $y(z,t)$. $\mathscr{M} \triangleq \{1,2,\cdots,m\}$ is a natural number set. The superscript 'T' represents the transposition of a vector or a matrix. The symbol '$*$' in a matrix set stands for a symmetric matrix, for example $\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} = \begin{bmatrix} A & B \\ * & C \end{bmatrix}$.

II. PROBLEM FORMULATION AND PRELIMINARIES

Parabolic DPSs have important applications in industrial production, and it's of great significant to study the parabolic DPSs. In this paper, we consider a class of linear second-order parabolic PDE system, and study the stability of the PDE system using the method of H_∞ control.

The model of linear parabolic PDE system with the Dirichlet-Neumann boundary constraints and initial value is described as

$$\begin{cases} y_t(z,t) = y_{zz}(z,t) + \sigma y(z,t) \\ \qquad\qquad +\boldsymbol{P}^T(z)\boldsymbol{w}(t),\ t > 0,\ z \in (0,1), \\ y(0,t) = 0,\ y_z(z,t)|_{z=1} = 0,\ t > 0, \\ y(z,0) = y_0(z),\ z \in [0,1] \end{cases} \tag{1}$$

where $y(z,t)$ is a state variable of the PDE system related to z and t, in which $z \in [0,1] \subset \Re$ is position variable in space, and $t \geq 0$ is time variable, respectively. $\sigma > 0$ is a known constant of the system, the distribution function of actuators $\boldsymbol{P}(z) \triangleq [p_1(z) \quad \cdots \quad p_m(z)]^T \in \Re^m$ is a m dimension integrable vector, in which $p_i(z)$, $i \in \mathscr{M}$ represents the i actuator's distribution. The control input $\boldsymbol{w}(t) \triangleq [w_1(t) \quad \cdots \quad w_m(t)]^T \in \Re^m$ is a m dimension integrable vector, in which $w_i(t)$, $i \in \mathscr{M}$ represents the i actuator's control input. $y(0,t)$ is the Dirichlet boundary constraint of $y(z,t)$ at the position $z = 0$. $y_z(z,t)|_{z=1}$ is the Neumann boundary constraint, which represents the derivative value of the state $y(z,t)$ at the position $z = 1$. $y_0(\cdot)$ is the given initial value of the state $y(z,t)$. What we need to know is that the system (1) is unstable without the control input ($\boldsymbol{w}(t) = 0$) when the system constant $\sigma > 0.25\pi^{-2}$ [16].

Due to the existence of measurement disturbance in actual measurement unavoidably, so the output with measurement disturbance is

$$\boldsymbol{v}(t) = \int_0^1 \boldsymbol{s}(z)y(z,t)dz + \boldsymbol{d}(t) \tag{2}$$

where the observation output $\boldsymbol{v}(t) \triangleq [v_1(t) \quad v_2(t) \quad \cdots \quad v_m(t)]^T \in \Re^m$ is a m dimension integrable vector, in which $v_i(z)$, $i \in \mathscr{M}$ represents the i sensor's observation output. The distribution function of sensors $\boldsymbol{s}(z) \triangleq [s_1(z) \quad s_2(z) \quad \cdots \quad s_m(z)]^T \in \Re^m$ is a known m dimension integrable vector, in which $s_i(z)$, $i \in \mathscr{M}$ represents the i sensor's distribution, respectively. $\boldsymbol{d}(\cdot) \triangleq [d_1(\cdot) \quad d_2(\cdot) \quad \cdots \quad d_m(\cdot)] \in \mathscr{L}^2([0,\infty]$ is the m-dimensional measurement disturbance vector, that is only related to t.

We consider the piecewise control and piecewise observation in space for the system (1) in this paper. For piecewise control, the distribution function $p_i(z)$, $i \in \mathscr{M}$ of control actuators is chosen as

$$p_i(z) \triangleq \begin{cases} \frac{1}{\bar{z}_i^h - \bar{z}_i^s} & z \in [\bar{z}_i^s, \bar{z}_i^h] \\ 0 & elsewhere \end{cases}, i \in \mathscr{M} \tag{3}$$

The actuators are placed at the discrete piecewise domain $[\bar{z}_i^s, \bar{z}_i^h]$, $i \in \mathscr{M}$ of the spatial domain $[0,1]$, and $0 < \bar{z}_1^s < \bar{z}_1^h < \bar{z}_2^s < \bar{z}_2^h < \cdots < \bar{z}_m^s < \bar{z}_m^h < 1$, respectively.

For piecewise observation, the distribution function $s_i(z)$, $i \in \mathscr{M}$ is chosen as

$$s_i(z) \triangleq \begin{cases} \frac{1}{\bar{z}_i^h - \bar{z}_i^s} & z \in [\bar{z}_i^s, \bar{z}_i^h] \\ 0 & elsewhere \end{cases}, i \in \mathscr{M} \tag{4}$$

The sensors are placed at the discrete piecewise domain $[\bar{z}_i^s, \bar{z}_i^h]$, $i \in \mathscr{M}$ of the spatial domain $[0, 1]$, and $0 < \bar{z}_1^s < \bar{z}_1^h < \bar{z}_2^s < \bar{z}_2^h < \cdots < \bar{z}_m^s < \bar{z}_m^h < 1$, respectively.

Remark 1: The well-posedness analysis of the open-loop system (1) with (3) without control input ($\boldsymbol{w}(t) = 0$) has been addressed in [30]. For more information, please refer to Proposition 1 in [30].

Remark 2: It should be noted that the piecewise control is represented by the distribution of actuators $\boldsymbol{P}(z)$, the piecewise observation is represented by the distribution of sensors $\boldsymbol{s}(z)$. When the distributions of actuators and sensors are same (i.e. $\boldsymbol{P}(z) = \boldsymbol{s}(z)$), it's called collocated piecewise observation. However, when the distributions of actuators and sensors are different (i.e. $\boldsymbol{P}(z) \neq \boldsymbol{s}(z)$), it's called non-collocated piecewise observation

For the development of H_∞ control in this paper, we provide the lemma and definition as follows:

Lemma 1 (Wirtinger's Inequality [24]): Set $y \in \mathscr{H}^1(0, 1)$ be a scalar function, we have

$$\int_0^1 (y(s) - y(0))^2 dz \leq 4\pi^2 \int_0^1 (dy(s)/ds)^2 \, ds, \tag{5}$$

or

$$\int_0^1 (y(s) - y(1))^2 ds \leq 4\pi^2 \int_0^1 (dy(s)/dz)^2 \, ds. \tag{6}$$

Remark 3: Form the inequalities (5) and (6), we can obtain that if $y(0) = 0$ or $y(1) = 0$, then the inequality $\int_0^1 y^2(s)ds \leq 4\pi^2 \int_0^1 (dy(s)/ds)^2 ds$ is fulfilled.

Similar to the method of exponential stability analysis described in the previous works [11]–[13], [24], [28], [30], the exponential stability of the system with free disturbance is defined as follows:

Definition 1 (Exponential Stability in the Sense of $|\cdot|_2$): If there are two constants $\rho_1 \geq 1$ and $\rho_2 > 0$ satisfying the expression $|y(\cdot, t)|_2 \leq \rho_1 |y_0(\cdot)|_2 \exp(-\rho_2\, t)$ for any $t \geq 0$, then the linear prabolic PDE system (1) with free disturbance ($\boldsymbol{w}(t) = 0$) is locally exponentially stable in the sense of $|\cdot|_2$.

An H_∞ performance is provided for the H_∞ control problem:

$$\int_0^\infty |y(\cdot, t)|_2^2 dt \leq \tau^2 \int_0^\infty \|\boldsymbol{d}(t)\|^2 dt \tag{7}$$

where $\tau > 0$ is a prescribed disturbance attenuation level.

In this paper, due to the existence of measurement disturbance in the output $\boldsymbol{v}(t)$, it brings some effects to the stability of the system. To deal with the influences of measurement disturbance, the H_∞ performance constraint is proposed. The goal of this paper is to design an H_∞ feedback controller to ensure that the resulting closed-loop system is exponentially stable under a prescribed H_∞ performance constraint. Next, we will analyze the H_∞ control problems with two different observation methods respectively.

III. H_∞ CONTROL WITH COLLOCATED PIECEWISE OBSERVATION

We first consider the H_∞ control problem with collocated piecewise observation in space (i.e. $\boldsymbol{P}(z) = \boldsymbol{s}(z)$). Based on the theory of feedback control, the following static feedback controller is designed:

$$\boldsymbol{w}(t) = -\boldsymbol{K}\boldsymbol{v}(t) \tag{8}$$

where the control gain $0 < \boldsymbol{K} \triangleq \text{diag}\{k_1, k_2, \cdots, k_m\} \in \Re^{m\times m}$ is $m \times m$ diagonal matrix.

Substituting the designed feedback controller (8) and the output with measurement disturbance (2) into the system (1), we can get the following closed-loop system for collocated observation case:

$$\begin{cases} y_t(z, t) = y_{zz}(z, t) + \sigma y(z, t) \\ \qquad -\boldsymbol{P}^T(z)\boldsymbol{K}\left(\int_0^1 \boldsymbol{P}(z)y(z, t)dz + \boldsymbol{d}(t)\right), \\ \qquad t > 0,\ z \in (0, 1), \\ y(0, t) = 0,\ y_z(z, t)|_{z=1} = 0,\ t > 0, \\ y(z, 0) = y_0(z),\ z \in [0, 1] \end{cases} \tag{9}$$

The object of this section is to seek an appropriate controller gain k_i, $i \in \mathscr{M}$ to ensure the closed-loop system (9) is exponentially stable under a prescribed H_∞ performance constraint. The following definition gives the *exponential stability analysis method at a prescribed H_∞ attenuation level τ*:

Definition 2: If the closed-loop PDE system (9) with free-disturbance ($\boldsymbol{d}(t) = 0$) is exponentially stable, meanwhile the closed-loop system (9) satisfies the H_∞ performance constraint (7) when the initial value of $y(z, t)$ is zero ($y_0(z) = 0$) and all $\boldsymbol{d} \in \mathscr{L}_2(0, \infty)$, then the closed-loop system (9) is exponentially stable at a prescribed H_∞ attenuation level τ.

The following theorem provides a sufficient condition for exponentially stability of the closed-loop system (9) at a prescribed H_∞ attenuation level τ based on the LMI constraint:

Theorem 1: For a class of linear parabolic PDE system described in (1)-(2) with the collocated piecewise observation case (i.e. $\boldsymbol{P}(z) = \boldsymbol{s}(z)$), given a disturbance attenuation level $\tau > 0$, if there exist controller gains k_i, $i \in \mathscr{M}$ satisfying the following LMIs:

$$\Phi_i \triangleq \begin{bmatrix} \sigma + 1 - \frac{k_i}{\Delta z_i} & \frac{k_i}{\Delta z_i} & 0 \\ \frac{k_i}{\Delta z_i} & -\frac{\pi^2}{4\phi_i} - \frac{k_i}{\Delta z_i} & -\frac{k_i}{2\Delta z_i} \\ 0 & -\frac{k_i}{2\Delta z_i} & -\frac{\tau^2}{\Delta z_i} \end{bmatrix},\ i \in \mathscr{M} \tag{10}$$

in which

$$\Delta z_i \triangleq z_{i+1} - z_i,\ i \in \mathscr{M}$$
$$\phi_i \triangleq \max\{(z_{i+1} - \bar{z}_i^s)^2, (\bar{z}_i^h - z_i)^2\},\ i \in \mathscr{M}$$

then there existing a static feedback controller (8) such that the closed-loop system (9) is exponentially stable at a prescribed H_∞ attenuation level τ in (7).

Proof: Applying the Lyapunov stability method, we consider a Lyapunov function of $y(z, t)$:

$$V_y(t) = 0.5 \int_0^1 y^2(z, t)dz \tag{11}$$

According to Definition 2, we can get that the proof of Theorem 1 consists of two parts: (1) H_∞ performance analysis that the H_∞ performance constraint is guaranteed for the closed-loop system (9) when the initial value of $y(z,t)$ is zero ($y_0(\cdot)=0$) and all $\boldsymbol{d} \in \mathscr{L}_2(0,\infty)$ (2) exponential stability analysis that the closed-loop system (9) with free disturbance ($\boldsymbol{d}(t)=0$) is exponentially stable . Next we will prove the two parts in turn:

(1) H_∞ Performance Analysis:

According to the Lyapunov stability method, the time derivative of Lyapunov function $V_y(t)$ is

$$\begin{aligned}\dot{V}_y(t) &= \int_0^1 y(z,t)y_t(z,t)dz \\ &= \int_0^1 y(z,t)y_{zz}(z,t)dz + \sigma\int_0^1 y^2(z,t)dz \\ &\quad -\int_0^1 y(z,t)\boldsymbol{P}^T(z)dz\boldsymbol{K}\int_0^1 \boldsymbol{P}(z)y(z,t)dz \\ &\quad -\int_0^1 y(z,t)\boldsymbol{P}^T(z)\boldsymbol{K}dz\boldsymbol{d}(t) \qquad (12)\end{aligned}$$

Using the technique of integration by parts to deal with $y_{zz}(z,t)$, and considering the Dirichlet boundary constrain $y(0,t)$ and the Neumann boundary constraint $y_z(z,t)|_{z=1}$ in (9), we have

$$\begin{aligned}\int_0^1 y(z,t)y_{zz}(z,t)dz &= y(z,t)y_z(z,t)|_{z=0}^{z=1} - \int_0^1 y_z^2(z,t)dz \\ &= -\int_0^1 y_z^2(z,t)dz \qquad (13)\end{aligned}$$

Eliminating the integral symbols by the first mean value theorem for definite integrals [31], we can get

$$\int_{\bar{z}_i^s}^{\bar{z}_i^h} y(z,t)dz = (\bar{z}_i^h - \bar{z}_i^s)y(\bar{z}_{it},t) \qquad (14)$$

where $y(\bar{z}_{it},t)$ is the *mean value* of $y(z,t)$ on $[\bar{z}_i^s,\bar{z}_i^h]$ with respect to t. The values of variable $\bar{z}_{it}$ is varying on $[\bar{z}_i^s,\bar{z}_i^h]$ along with time t.

Substituting the equations (13) and (14) into (12), we obtain

$$\begin{aligned}\dot{V}_y(t) &= -\int_0^1 y_z^2(z,t)dz + \sigma\int_0^1 y^2(z,t)dz \\ &\quad -\sum_{i=1}^m k_i y^2(\bar{z}_{it},t) - \sum_{i=1}^m k_i y(\bar{z}_{it},t)d_i(t) \qquad (15)\end{aligned}$$

Define $y_{di}(z,t) \triangleq y(z,t) - y(\bar{z}_{it},t),\ i\in\mathscr{M}$, then

$$y_{di,z}(z,t) = y_z(z,t),\ i\in\mathscr{M} \qquad (16)$$

According to the Wirtinger's inequality described in Lemma 1, the following inequalities is obtained for any $i\in\mathscr{M}$

$$\int_{\bar{z}_{it}}^{z_{i+1}} y_z^2(z,t)dz \geq \frac{\pi^2}{4(z_{i+1}-\bar{z}_{it})^2}\int_{\bar{z}_{it}}^{z_{i+1}} y_{di}^2(z,t)dz, \qquad (17)$$

and

$$\int_{z_i}^{\bar{z}_{it}} y_z^2(z,t)dz \geq \frac{\pi^2}{4(\bar{z}_{it}-z_i)^2}\int_{z_i}^{\bar{z}_{it}} y_{di}^2(z,t)dz. \qquad (18)$$

As $\bar{z}_{it}\in[\bar{z}_i^s,\bar{z}_i^h],\ i\in\mathscr{M}$, we obtain the following inequalities

$$\bar{z}_{it} - z_i \leq \bar{z}_i^h - z_i \text{ and } z_{i+1} - \bar{z}_{it} \leq z_{i+1} - \bar{z}_i^s. \qquad (19)$$

From the inequalities (17)-(19), we have

$$\begin{aligned}\int_{z_i}^{z_{i+1}} y_z^2(z,t)dz &= \int_{\bar{z}_{it}}^{z_{i+1}} y_z^2(z,t)dz + \int_{z_i}^{\bar{z}_{it}} y_z^2(z,t)dz \\ &\geq \frac{\pi^2}{4(z_{i+1}-\bar{z}_{it})^2}\int_{\bar{z}_{it}}^{z_{i+1}} y_{di,z}^2(z,t)dz \\ &\quad + \frac{\pi^2}{4(\bar{z}_{it}-z_i)^2}\int_{z_i}^{\bar{z}_{it}} y_{di}^2(z,t)dz \\ &\geq \frac{\pi^2}{4\phi_i}\int_{z_i}^{z_{i+1}} y_{di}^2(z,t)dz \qquad (20)\end{aligned}$$

where $\phi_i \triangleq \max\{(z_{i+1}-\bar{z}_i^s)^2, (\bar{z}_i^h - z_i)^2\},\ i\in\mathscr{M}$.

Substituting the inequalities (20) into (15), the time derivative of $V_y(t)$ is rewritten as

$$\begin{aligned}\dot{V}_y(t) &\leq -\sum_{i=1}^m \frac{\pi^2}{4\phi_i}\int_{z_i}^{z_{i+1}} (y(z,t)-y(\bar{z}_{it},t))^2dz \\ &\quad +\sigma\sum_{i=1}^m\int_{z_i}^{z_{i+1}} y^2(z,t)dz - \sum_{i=1}^m \frac{k_i}{\Delta z_i}\int_{z_i}^{z_{i+1}} y^2(\bar{z}_{it},t)dz \\ &\quad -\sum_{i=1}^m \frac{k_i}{\Delta z_i}\int_{z_i}^{z_{i+1}} y(\bar{z}_{it},t)d_i(t)dz \\ &= \sum_{i=1}^m\int_{z_i}^{z_{i+1}} \boldsymbol{y}_i^T(z,t)\Xi_i\boldsymbol{y}_i(z,t)dz \\ &\quad -\sum_{i=1}^m \frac{k_i}{\Delta z_i}\int_{z_i}^{z_{i+1}} y(\bar{z}_{it},t)d_i(t)dz \qquad (21)\end{aligned}$$

where $\Delta z_i \triangleq z_{i+1}-z_i$, $\boldsymbol{y}_i(z,t) \triangleq [y(z,t) \quad y(\bar{z}_{it},t)]^T,\ i\in\mathscr{M}$, and

$$\Xi_i \triangleq \begin{bmatrix} \sigma - \frac{k_i}{\Delta z_i} & \frac{k_i}{\Delta z_i} \\ \frac{k_i}{\Delta z_i} & -\frac{\pi^2}{4\phi_i} - \frac{k_i}{\Delta z_i}\end{bmatrix},\ i\in\mathscr{M}$$

From the H_∞ performance (7), we get the following inequality

$$\begin{aligned}&\dot{V}_y(t) + |y(\cdot,t)|_2^2 - \tau^2\|\boldsymbol{d}(t)\|^2 \\ &\leq \sum_{i=1}^m\int_{z_i}^{z_{i+1}} \boldsymbol{y}_i^T(z,t)\Xi_i\boldsymbol{y}_i(z,t)dz \\ &\quad -\sum_{i=1}^m \frac{k_i}{\Delta z_i}\int_{z_i}^{z_{i+1}} y(\bar{z}_{it},t)d_i(t)dz \\ &\quad +\sum_{i=1}^m\int_{z_i}^{z_{i+1}} y^2(z,t)dz - \sum_{i=1}^m\int_{z_i}^{z_{i+1}} \frac{\tau^2}{\Delta z_i}d_i^2(t)dz \\ &= \sum_{i=1}^m\int_{z_i}^{z_{i+1}} \boldsymbol{\xi}_i^T(z,t)\Phi_i\boldsymbol{\xi}_i(z,t)dz \qquad (22)\end{aligned}$$

where $\boldsymbol{\xi}_i(z,t) \triangleq [\boldsymbol{y}_i^T(z,t) \quad d_i(t)]^T$.

Assume the LMIs (10) are satisfied, it's easily obtained from (22) that

$$\dot{V}_y(t) + |y(\cdot,t)|_2^2 - \tau^2\|\boldsymbol{d}(t)\|^2 \le 0,\ \forall t \ge 0 \tag{23}$$

Intergrating (23) from 0 to ∞ and considering $y_0(\cdot) = 0$, we get expression (7), that is the H_∞ performance constraint (7) is guaranteed.

(2) Exponential Stability Analysis:

When there is no measurement disturbance in closed-loop system (9) ($\boldsymbol{d}(t) = 0$), the derivative of Lyapunov function $V_y(t)$ with respect to t is

$$\dot{V}_y(t) \le \alpha \sum_{i=1}^{m} \int_{z_i}^{z_{i+1}} \boldsymbol{y}_i^T(z,t)\Xi_i \boldsymbol{y}_i(z,t)dz \tag{24}$$

The following inequality can be derived from the LMI (10)

$$\Xi_i < 0,\ i \in \mathscr{M} \tag{25}$$

Thus there exists an appropriate constant $\beta > 0$ satisfy the following inequality

$$\Xi_i + 0.5\beta I \le 0,\ i \in \mathscr{M} \tag{26}$$

Substituting the inequality (26) to (24), we obtain

$$\begin{aligned}\dot{V}_y(t) &\le -0.5\beta \sum_{i=1}^{m} \int_{z_i}^{z_{i+1}} \boldsymbol{y}_i^T(z,t)\boldsymbol{y}_i(z,t)dz \\ &\le -0.5\beta \int_0^1 y^2(z,t)dz = -\beta V_y(t)\end{aligned} \tag{27}$$

By integrating from 0 to t for the inequality (27), we can get $V_y(t) \le V_y(0)\exp(-\beta t),\ t \ge 0$, which implies $|y(\cdot,t)|_2 \le |y_0(\cdot)|_2 \exp(-0.5\beta t),\ t \ge 0$. In other words, the closed-loop PDE system (9) with $\boldsymbol{d}(t) = 0$ is exponentially stable in the sense of $|\cdot|_2$.

From the proof of H_∞ performance analysis, the H_∞ performance constraint is guaranteed for the closed-loop system (9) when the initial value of the $y(z,t)$ is zero ($y_0(\cdot) = 0$) and all $\boldsymbol{d} \in \mathscr{L}_2(0,\infty)$. Moreover from the proof of exponential stability analysis, the closed-loop system (9) with $\boldsymbol{d}(t) = 0$ is exponentially stable respectively. According to the description of Definition 2, we can get the closed-loop system (9) is exponentially stable under a prescribed H_∞ performance constraint from the proof of the H_∞ performance analysis and exponential stability analysis. The proof is complete. ■

IV. H_∞ OBSERVER-BASED CONTROL WITH NON-COLLOCATED PIECEWISE OBSERVATION

In this section, we consider the non-collocated piecewise observation in space (i.e. $\boldsymbol{P}(z) \ne \boldsymbol{s}(z)$) for the PDE system (1) and the output with measurement disturbance (2). We first propose a Luenberger observer to observe the collocated state of the system (1), and then design an H_∞ observer-based feedback controller to ensure the resulting closed-loop coupled systems are exponentially stable under a prescribed H_∞ performance constraint.

The form of the Luenberger observer is

$$\begin{cases}\hat{y}_t(z,t) = \hat{y}_{zz}(z,t) + \sigma\hat{y}(z,t) \\ \qquad\qquad + \boldsymbol{s}^T(z)\hat{\boldsymbol{L}}\left(\boldsymbol{v}(t) - \hat{\boldsymbol{v}}(t)\right),\ t > 0,\ z \in (0,1), \\ \hat{y}(0,t) = 0,\ \hat{y}_z(z,t)|_{z=1} = 0,\ t > 0, \\ \hat{y}(z,0) = \hat{y}_0(z),\ z \in [0,1]\end{cases} \tag{28}$$

where $\hat{y}(z,t)$ is the observation state from the observer, $0 < \hat{\boldsymbol{L}} \triangleq \mathrm{diag}\{l_1, l_2, \cdots, l_m\} \in \Re^{m\times m}$ is the observer gain to be determined in (34) and it's a $m \times m$ diagonal matrix.

The observation output obtained from the observer is

$$\hat{\boldsymbol{v}}(t) = \int_0^1 \boldsymbol{s}(z)\hat{y}(z,t)dz,\ t \ge 0, \tag{29}$$

Based on the observer (28) and observation output (29), a dynamic feedback controller is constructed:

$$\boldsymbol{w}(t) = -\boldsymbol{K}\int_0^1 \boldsymbol{P}(z)\hat{y}(z,t)dz \tag{30}$$

where $\boldsymbol{K} \triangleq \mathrm{diag}\{k_1, k_2, \cdots, k_m\} \in \Re^{m\times m}$ is the observer-based control gain.

Define the estimation error function as

$$e(z,t) \triangleq y(z,t) - \hat{y}(z,t). \tag{31}$$

According to the definition of estimation error, the estimation error system is obtained:

$$\begin{cases}e_t(z,t) = e_{zz}(z,t) + \sigma e(z,t) \\ \qquad\qquad -\boldsymbol{s}^T(z)\hat{\boldsymbol{L}}\int_0^1 \boldsymbol{c}(z)e(z,t)dz \\ \qquad\qquad -\boldsymbol{s}^T(z)\hat{\boldsymbol{L}}\boldsymbol{d}(t),\ t > 0,\ z \in (0,1), \\ e(0,t) = 0,\ e_z(z,t)|_{z=1} = 0,\ t > 0, \\ e(z,0) = e_0(z),\ z \in [0,1]\end{cases} \tag{32}$$

where $e_0(z) \triangleq y_0(z) - \hat{y}_0(z)$.

Substituting the observer-based dynamic feedback controller (30) into (1), and considering the definition of estimation error (32), we obtain the following closed-loop system for non-collocated observation case:

$$\begin{cases}y_t(z,t) = y_{zz}(z,t) + \sigma y(z,t) \\ \qquad\qquad -\boldsymbol{P}^T(z)\boldsymbol{K}\int_0^1 \boldsymbol{P}(z)\left(y(z,t) - e(z,t)\right)dz, \\ \qquad\qquad t > 0,\ z \in (0,1), \\ y(0,t) = 0,\ y_z(z,t)|_{z=1} = 0,\ t > 0, \\ y(z,0) = y_0(z),\ z \in [0,1]\end{cases} \tag{33}$$

Hence, the closed-loop coupled PDE systems are represented by the estimation error system (32) and the closed-loop system (33) with the distribution functions of actuators and sensors (3)-(4).

The object of this section is to seek appropriate controller gain $k_i,\ i \in \mathscr{M}$ and observer gain $l_i,\ i \in \mathscr{M}$ to ensure the closed-loop coupled PDE systems (32)- (33) and (3)-(4) is exponentially stable under a prescribed H_∞ performance constraint. The following definition gives the *exponential stability analysis method at a prescribed* H_∞ *attenuation level* τ :

Definition 3: If the closed-loop coupled PDE systems (32)- (33) and (3)-(4) with free-disturbance ($\boldsymbol{d}(t) = 0$)

$$\Gamma_i \triangleq \begin{bmatrix} \sigma+1-\frac{k_i}{\Delta z_i} & \frac{k_i}{\Delta z_i} & 0 & 0 & 0 & 0 \\ * & -\frac{\pi^2}{4\phi_i}-\frac{k_i}{\Delta z_i} & 0 & \frac{k_i}{2\Delta z_i} & 0 & 0 \\ * & * & \alpha\omega_i & \frac{\alpha\pi^2 r_i}{4\phi_i} & \frac{\alpha\pi^2(1-r_i)}{4\varphi_i} & 0 \\ * & * & * & -\frac{\alpha\pi^2 r_i}{4\phi_i} & 0 & -\frac{\hat{l}_i}{2\Delta z_i} \\ * & * & * & * & \kappa_i & 0 \\ * & * & * & * & * & -\frac{\tau^2}{\Delta z_i} \end{bmatrix} < 0, \quad i \in \mathscr{M} \tag{34}$$

is exponentially stable, meanwhile the closed-loop coupled PDE systems satisfy the H_∞ performance constraint (7) when the initial value of the state $y(z,t)$ is zero ($y_0(z)=0$) and all $\boldsymbol{d} \in \mathscr{L}_2(0,\infty)$, then the closed-loop coupled PDE systems (32)- (33) and (3)-(4) are exponentially stable at a prescribed H_∞ attenuation level τ.

The following theorem provides a sufficient condition for exponentially stability of the closed-loop coupled PDE systems (32)-(33) and (3)-(4) at a prescribed H_∞ attenuation level τ based on the LMI constraint:

Theorem 2: For a class of linear parabolic PDE system described in (1)-(2) with non-collocated piecewise observation in space (i.e. $\boldsymbol{P}(z) \geq \boldsymbol{s}(z)$), given constants $\tau > 0$ and $0 < r_i < 1$, $i \in \mathscr{M}$, if there exist controller gains k_i, $i \in \mathscr{M}$ and scalars $\hat{l}_i$, $i \in \mathscr{M}$ satisfying the following LMIs: see (34), as shown at the top of this page, in which

$$\varphi_i \triangleq \max\{(z_{i+1}-\tilde{z}_i^s)^2, (\tilde{z}_i^h - z_i)^2\},\ i \in \mathscr{M}$$
$$\omega_i \triangleq \sigma - \frac{\pi^2 r_i}{4\phi_i} - \frac{\pi^2(1-r_i)}{4\varphi_i},\ i \in \mathscr{M}$$
$$\kappa_i \triangleq -\frac{\hat{l}_i}{\Delta z_i} - \frac{\alpha\pi^2(1-r_i)}{4\varphi_i},\ i \in \mathscr{M}$$

then there exists an observer-based dynamic feedback controller (30) such that the resulting closed-loop coupled PDE systems (32)-(33) and (3)-(4) is exponentially stable at a prescribed H_∞ attenuation level τ in (7), where the observer gains l_i, $i \in \mathscr{M}$ are

$$l_i = \hat{l}_i/\alpha,\ i \in \mathscr{M}. \tag{35}$$

Proof: For the closed-loop coupled PDE systems (32)-(33) and (3)-(4), we consider the coupled Lyapunov function $V_{ye}(t)$ composed of $V_y(t)$ and $V_e(t)$:

$$V_{ye}(t) = V_y(t) + V_e(t) \tag{36}$$

where $V_e(t) = 0.5\alpha \int_0^1 e^2(z,t)dz$ and $\alpha > 0$ is a Lyapunov parameter.

Similar to the inequality (21), we can get

$$\dot{V}_y(t) \leq \sum_{i=1}^m \int_{z_i}^{z_{i+1}} \boldsymbol{y}_i^T(z,t)\Xi_i \boldsymbol{y}_i(z,t)dz + \sum_{i=1}^m k_i y(\bar{z}_{it},t)e(\bar{z}_{it},t) \tag{37}$$

Similar to the equation (13), considering the boundary conditions in (32), we obtain

$$\int_0^1 e(z,t)e_{zz}(z,t)dz = -\int_0^1 e_z^2(z,t)dz \tag{38}$$

From (14), we can obtain that for each $i \in \mathscr{M}$ and any $t \geq 0$,

$$\int_{\tilde{z}_i^s}^{\tilde{z}_i^h} e(z,t)dz = (\tilde{z}_i^h - \tilde{z}_i^s)e(\tilde{z}_{it},t) \tag{39}$$

where $y(\tilde{z}_{it},t)$ is the *mean value* of the state variable $y(z,t)$ on $[\tilde{z}_i^s, \tilde{z}_i^h]$ with respect to t. The values of variable $\tilde{z}_{it}$ is varying on $[\tilde{z}_i^s, \tilde{z}_i^h]$ along with time t.

Using the equations (38) and (39), the time derivative of $V_e(t)$ is

$$\dot{V}_e(t) = -\alpha \int_0^1 e_z^2(z,t)dz + \alpha\sigma \int_0^1 e^2(z,t)dz - \alpha \sum_{i=1}^m l_i e^2(\tilde{z}_{it},t) - \alpha \sum_{i=1}^m l_i e(\tilde{z}_{it},t)d_i(t) \tag{40}$$

Define $e_{di}(z,t) \triangleq e(z,t) - e(\bar{z}_{it},t)$, $i \in \mathscr{M}$ and $e_{si}(z,t) \triangleq e(z,t) - e(\tilde{z}_{it},t)$, $i \in \mathscr{M}$, according to the Wirtinger's inequality in Lemma 1, similar to the inequality (20), we get for any $i \in \mathscr{M}$

$$\int_{z_i}^{z_{i+1}} e_z^2(z,t)dz \geq \frac{\pi^2}{4\phi_i}\int_{z_i}^{z_{i+1}} e_{di}^2(z,t)dz \tag{41}$$

and

$$\int_{z_i}^{z_{i+1}} e_z^2(z,t)dz \geq \frac{\pi^2}{4\varphi_i}\int_{z_i}^{z_{i+1}} e_{si}^2(z,t)dz \tag{42}$$

where $\varphi_i \triangleq \max\{(z_{i+1}-\tilde{z}_i^s)^2, (\tilde{z}_i^h - z_i)^2\},\ i \in \mathscr{M}$.

Set

$$\hat{l}_i = \alpha l_i,\ i \in \mathscr{M} \tag{43}$$

Utilizing (41)-(42) and considering $0 < r_i < 1$, $i \in \mathscr{M}$, the derivative of Lyapunov function $V_e(t)$ with respect to t is

$$\dot{V}_e(t) \leq -\alpha \sum_{i=1}^m \frac{\pi^2}{4\phi_i} r_i \int_{z_i}^{z_{i+1}} e_{di}^2(z,t)dz - \alpha \sum_{i=1}^m \frac{\pi^2}{4\varphi_i}(1-r_i)\int_{z_i}^{z_{i+1}} e_{si}^2(z,t)dz + \alpha\sigma \sum_{i=1}^m \int_{z_i}^{z_{i+1}} e^2(z,t)dz$$

$$
\begin{aligned}
&-\sum_{i=1}^{m}\frac{\hat{l}_i}{\Delta z_i}\int_{z_i}^{z_{i+1}} e^2(\tilde{z}_{it},t)dz-\sum_{i=1}^{m}\hat{l}_i e(\tilde{z}_{it},t)d_i(t)\\
&=\sum_{i=1}^{m}\int_{z_i}^{z_{i+1}}\boldsymbol{e}_i^T(z,t)\Psi_i\boldsymbol{e}_i(z,t)dz\\
&\quad-\sum_{i=1}^{m}\hat{l}_i e(\tilde{z}_{it},t)d_i(t) \qquad (44)
\end{aligned}
$$

where $\boldsymbol{e}_i(z,t)\triangleq[e(z,t)\quad e(\bar{z}_{it},t)\quad e(\tilde{z}_{it},t)]$, $i\in\mathscr{M}$, and

$$
\Psi_i\triangleq\begin{bmatrix}\alpha\omega_i & \frac{\alpha\pi^2 r_i}{4\phi_i} & \frac{\alpha\pi^2(1-r_i)}{4\varphi_i}\\ \frac{\alpha\pi^2 r_i}{4\phi_i} & -\frac{\alpha\pi^2 r_i}{4\phi_i} & 0\\ \frac{\alpha\pi^2(1-r_i)}{4\varphi_i} & 0 & \kappa_i\end{bmatrix},\quad i\in\mathscr{M},
$$

in which $\omega_i\triangleq\sigma-\frac{\pi^2 r_i}{4\phi_i}-\frac{\pi^2(1-r_i)}{4\varphi_i}$ and $\kappa_i\triangleq-\frac{\hat{l}_i}{\Delta z_i}-\frac{\alpha\pi^2(1-r_i)}{4\varphi_i}$.

From (37) and (44), the time derivative of Lyapunov function $V_{ye}(t)$ is

$$
\begin{aligned}
\dot{V}_{ye}(t)&=\dot{V}_y(t)+\dot{V}_e(t)\\
&\le\sum_{i=1}^{m}\int_{z_i}^{z_{i+1}}\boldsymbol{y}_i^T(z,t)\Xi_i\boldsymbol{y}_i(z,t)dz\\
&\quad+\sum_{i=1}^{m}k_i y(\bar{z}_{it},t)e(\bar{z}_{it},t)\\
&\quad+\sum_{i=1}^{m}\int_{z_i}^{z_{i+1}}\boldsymbol{e}_i^T(z,t)\Psi_i\boldsymbol{e}_i(z,t)dz\\
&\quad-\sum_{i=1}^{m}\hat{l}_i e(\tilde{z}_{it},t)d_i(t)\\
&=\sum_{i=1}^{m}\int_{z_i}^{z_{i+1}}\boldsymbol{\chi}_i^T(z,t)\Omega_i\boldsymbol{\chi}_i(z,t)dz\\
&\quad-\sum_{i=1}^{m}\hat{l}_i e(\tilde{z}_{it},t)d_i(t) \qquad (45)
\end{aligned}
$$

where $\boldsymbol{\chi}_i(z,t)\triangleq[\boldsymbol{y}_i^T(z,t)\quad \boldsymbol{e}_i^T(z,t)]^T$, and

$$
\Omega_i\triangleq\begin{bmatrix}\sigma-\frac{k_i}{\Delta z_i} & \frac{k_i}{\Delta z_i} & 0 & 0 & 0\\ * & -\frac{\pi^2}{4\phi_i}-\frac{k_i}{\Delta z_i} & 0 & \frac{k_i}{2\Delta z_i} & 0\\ * & * & \alpha\omega_i & \frac{\alpha\pi^2 r_i}{4\phi_i} & \frac{\alpha\pi^2(1-r_i)}{4\varphi_i}\\ * & * & * & -\frac{\alpha\pi^2 r_i}{4\phi_i} & 0\\ * & * & * & * & \kappa_i\end{bmatrix},\quad i\in\mathscr{M}
$$

From (45) and consider the H_∞ performance in (7),we get

$$
\begin{aligned}
&\dot{V}_{ye}(t)+|y(\cdot,t)|_2^2-\tau^2\|\boldsymbol{d}(t)\|^2\\
&\le\sum_{i=1}^{m}\int_{z_i}^{z_{i+1}}\boldsymbol{\chi}_i^T(z,t)\Omega_i\boldsymbol{\chi}_i(z,t)dz\\
&\quad-\sum_{i=1}^{m}\hat{l}_i e(\tilde{z}_{it},t)d_i(t)+\sum_{i=1}^{m}\int_{z_i}^{z_{i+1}}y^2(z,t)dz\\
&\quad-\sum_{i=1}^{m}\int_{z_i}^{z_{i+1}}\frac{\tau^2}{\Delta z_i}d_i^2(t)dz\\
&=\sum_{i=1}^{m}\int_{z_i}^{z_{i+1}}\boldsymbol{\varpi}_i^T(z,t)\Gamma_i\boldsymbol{\varpi}_i(z,t)dz \qquad (46)
\end{aligned}
$$

where $\boldsymbol{\varpi}_i(z,t)\triangleq[\boldsymbol{\chi}_i^T(z,t)\quad d_i(t)]^T$.

Assume the LMIs (34) in Theorem 2 are fulfilled, it is easily obtained from (46) that

$$
\dot{V}_{ye}(t)+|y(\cdot,t)|_2^2-\tau^2\|\boldsymbol{d}(t)\|^2\le 0,\ \forall t\ge 0 \qquad (47)
$$

Intergrating (47) from 0 to ∞ and considering $e_0(\cdot)=0$, we get expression (7), i.e. the H_∞ performance constraint (7) is guaranteed for any $t\ge 0$.

When $\boldsymbol{d}(t)=0$, according to the expression (45), the time derivative of Lyapunov function $V_{ye}(t)$ is rewritten as

$$
\dot{V}_{ye}(t)\le\sum_{i=1}^{m}\int_{z_i}^{z_{i+1}}\boldsymbol{\chi}_i^T(z,t)\Omega_i\boldsymbol{\chi}_i(z,t)dz \qquad (48)
$$

It is easily derived from (34) that

$$
\Omega_i<0,\ i\in\mathscr{M} \qquad (49)
$$

Similar to the proof of exponential stability in section III, from the inequalities (25)-(27), it can be derived that the closed-loop coupled PDE systems (32)-(33) and (3)-(4) with free disturbance ($\boldsymbol{d}(t)=0$) are exponentially stable in the sense of $|\cdot|_2$.

From the proof of H_∞ performance analysis, the H_∞ performance is guaranteed for the closed-loop coupled PDE systems (32)-(33) and (3)-(4) with the zero initial value $y_0(\cdot)=0$. Moreover from the proof of exponential stability analysis, the closed-loop coupled PDE systems (32)-(33) and (3)-(4) with $\boldsymbol{d}(t)=0$ is exponential stable, respectively.

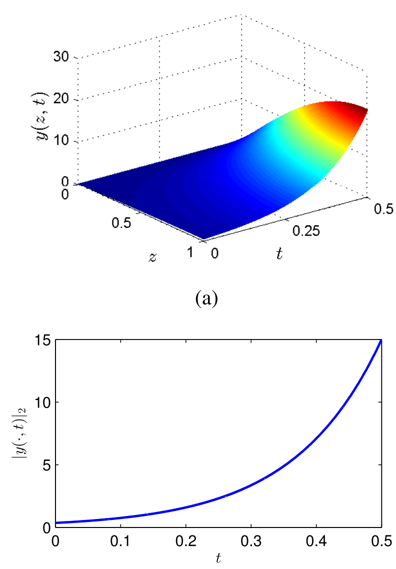

FIGURE 1. Open-loop simulation results: (a) Evolutionary profile of $y(z,t)$ (b) Trajectory of $|y(\cdot,t)|_2$.

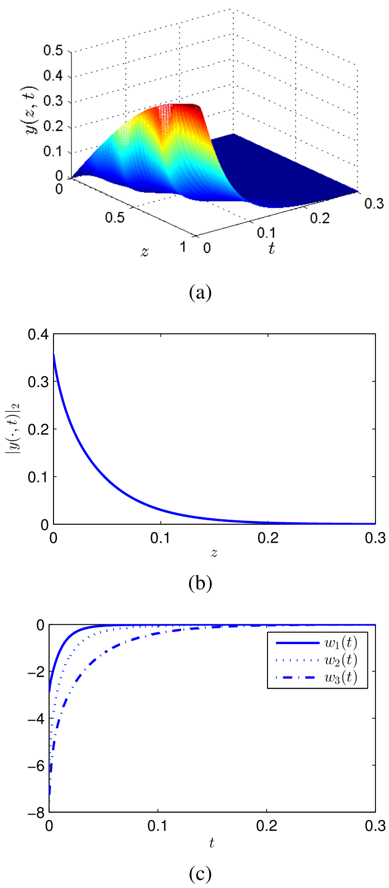

FIGURE 2. Closed-loop simulation results of the closed-loop PDE system (9) with $P(z) = s(z)$ and $d(t) = 0$: (a) Evolutionary profile of $y(z, t)$ (b) Trajectory of $|y(\cdot, t)|_2$ (c)Trajectory of the static output feedback controller (8).

According to the description of Definition 3, we can get the closed-loop coupled PDE systems (32)-(33) and (3)-(4) is exponentially stable at a prescribed H_∞ attenuation level τ. The proof is complete. ∎

V. SIMULATIONS AND EXPERIMENTS

In this section, we will provide some numerical simulations to verify that the proposed H_∞ control method can stabilize the system (1) . Given the constant $\sigma = 10$ and the initial value $y_0(z) = 0.5\sin(0.5\pi z)$, $z \in [0, 1]$. The simulation results of the open-loop system (1) with $\boldsymbol{w}(t) = 0$ is first presented in Figure 1, which describes the evolutionary profile of the system state $y(z, t)$ and trajectory of $|y(\cdot, t)|_2$. In Figure 1, the state of the system $y(z, t)$ and the value of $|y(\cdot, t)|_2$ is increasing along with t. That is, the linear parabolic PDE system (1) with $\boldsymbol{w}(t) = 0$ is unstable. Then we provide the simulation results of closed-loop system for collocated observation case and non-collocated observation case respectively. Assume $m = 3$, that is there are three actuators and three sensors, and the space region can be divided into three parts by the actuators and sensors, i.e., $z_1 = 0$, $z_2 = \frac{1}{3}$, $z_3 = \frac{2}{3}$, $z_4 = 1$, thus $\Delta z_1 = \Delta z_2 = \Delta z_3 = \frac{1}{3}$.

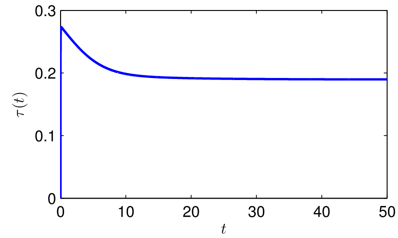

FIGURE 3. Closed-loop PDE system (9) simulation results with $y_0(z) = 0$: the trajectory of $\tau(t)$.

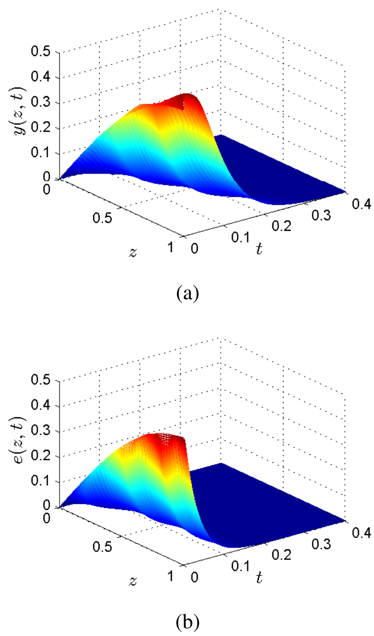

FIGURE 4. Closed-loop simulation results of the closed-loop coupled PDE systems (32)-(33) and (3)-(4) with $P(z) \neq s(z)$ and $d(t) = 0$: (a) Evolutionary profile of $y(z, t)$ (b) Profile of evolution of $e(z, t)$.

A. H_∞ CONTROL FOR COLLOCATED OBSERVATION CASE

We consider the H_∞ control for collocated observation case (i.e. $\boldsymbol{P}(z) = \boldsymbol{s}(z)$). The actuators and sensors are respectively taken action at the same region of this spatial domain as [0.2, 0.3], [0.4, 0.5] and [0.7, 0.8] . That is, $\bar{z}_1^s = \tilde{z}_1^s = 0.2$, $\bar{z}_1^h = \tilde{z}_1^h = 0.3$, $\bar{z}_2^s = \tilde{z}_2^s = 0.4$, $\bar{z}_2^h = \tilde{z}_2^h = 0.5$, $\bar{z}_3^s = \tilde{z}_3^s = 0.7$, $\bar{z}_3^h = \tilde{z}_3^h = 0.8$. We can obtain $\phi_1 = 0.09$, $\phi_2 = 0.0711$, $\phi_3 = 0.09$. Given $\tau^* = 8.1761$, by solving the LMIs (10), we get $k_1 = 13.6392$, $k_2 = 13.0998$, $k_3 = 13.6392$. The simulation results of the closed-loop PDE system (9) with $\boldsymbol{P}(z) = \boldsymbol{s}(z)$ and $\boldsymbol{d}(t) = 0$ are presented in Figure 2, which described the evolutionary profile of $y(z, t)$, the trajectory of $|y(\cdot, t)|_2$ and the trajectory of the static output feedbcak controller (8). In Figure 2, the state of the system $y(z, t)$ tends to a plane and no longer varying along with t, the values of $|y(\cdot, t)|_2$ and controller input $\boldsymbol{w}(t)$ tend to zero when $t \to \infty$. That is, the designed H_∞ static feedback controller (8) can stabilize the linear parabolic PDE system (1) with $\boldsymbol{P}(z) = \boldsymbol{s}(z)$ and $\boldsymbol{d}(t) = 0$.

Define the initial state value of the closed-loop PDE system (9) as $y_0(z) = 0$ and $\boldsymbol{d}(t) = [\exp(-0.1t) \quad \exp(-0.2t)$

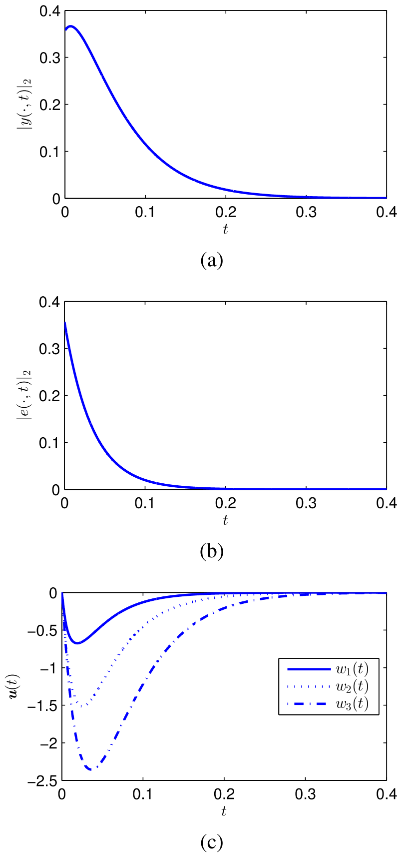

FIGURE 5. **Closed-loop simulation results of the closed-loop coupled PDE systems (32)-(33) and (3)-(4) with $P(z) \neq s(z)$ and $d(t) = 0$: (a) Trajectory of $|y(\cdot,t)|_2$ (b) Trajectory of $|e(\cdot,t)|_2$ (c)Trajectory of the H_∞ observer-based dynamic output feedback controller (30).**

$\exp(-\,0.3t)]^T, t > 0$. Define $\tau(t) \triangleq \sqrt{\frac{\int_0^t |y(\cdot,s)|_2^2 ds}{\int_0^t ||d(s)||^2 ds}}$ Figure 3 shows the trajectory of $\tau(t)$ for the linear parabolic PDE system (1) with $P(z) = s(z)$ and the initial value $y_0(z) = 0$. From Figure (3), it's easily seen that $\tau(t) < \tau^*$, the H_∞ performance (7) is guaranteed.

B. H_∞ CONTROL FOR NON-COLLOCATED OBSERVATION CASE

We consider the H_∞ control for non-collocated observation case (i.e. $P(z) \neq s(z)$). The actuators are respectively active during the domain [0.1, 0.2], [0.4, 0.5] and [0.7, 0.8] of this spatial domain. That is, $\bar{z}_1^s = 0.1$, $\bar{z}_1^h = 0.2$, $\bar{z}_2^s = 0.4$, $\bar{z}_2^h = 0.5$, $\bar{z}_3^s = 0.7$, $\bar{z}_3^h = 0.8$. The sensors are respectively active during the domain [0.2, 0.3], [0.5, 0.6] and [0.8, 0.9] of this spatial domain. That is, $\tilde{z}_1^s = 0.2$, $\tilde{z}_1^h = 0.3$, $\tilde{z}_2^s = 0.5$, $\tilde{z}_2^h = 0.6$, $\tilde{z}_3^s = 0.8$, $\tilde{z}_3^h = 0.9$. We can obtain $\phi_1 = 0.0544$, $\phi_2 = 0.0711$, $\phi_3 = 0.09$, $\varphi_1 = 0.09$, $\varphi_2 = 0.0711$, $\varphi_3 = 0.0544$. Given $\tau^* = 55.3514$, $r_1 = 0.2$, $r_2 = 0.1$ and $r_3 = 0.1$, by solving the LMIs (34), we get $k_1 = 18.0054$, $k_2 = 13.6897$, $k_3 = 11.8222$, $l_1 = 10.7453$, $l_2 = 8.7235$, $l_3 = 8.1029$. The simulation results of the closed-loop coupled PDE systems (32)-(33) and (3)-(4) with $P(z) \neq s(z)$ and $d(t) = 0$ are present in Figure 4 and Figure 5, which described

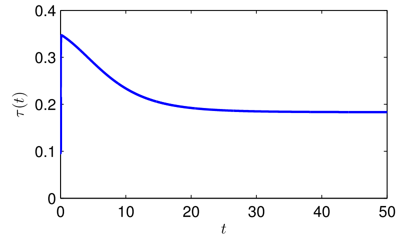

FIGURE 6. **Closed-loop coupled PDE systems (32)-(33) and (3)-(4) with $e_0(z) = 0$: the trajectory of $\tau(t)$.**

the profile of evolution of $y(z,t)$ and $y(z,t)$, the trajectory of $|y(\cdot,t)|_2$, $|e(\cdot,t)|_2$ and the H_∞ observer-based dynamic output feedback controller (30). In Figure 3, the states of the system $y(z,t)$ and estimate error $e(z,t)$ tend to planes and no longer vary along with t. In Figure 4, the values of $|y(\cdot,t)|_2$, $|e(\cdot,t)|_2$ and the controller input $w(t)$ tend to zero when $t \to \infty$. That is, the designed H_∞ observer-based dynamic output feedback controller (30) can stabilize the linear parabolic PDE system (1) with $P(z) \neq s(z)$ and $d(t) = 0$.

Define the initial value of the closed-loop coupled PDE systems (32)-(33) and (3)-(4) as $y_0(z) = 0$. Set $d(t) = [\exp(-0.2t) \quad \exp(-0.4t) \quad \exp(-0.3t)]^T, t > 0$. Define $\tau(t) \triangleq \sqrt{\frac{\int_0^t |y(\cdot,s)|_2^2 ds}{\int_0^t ||d(s)||^2 ds}}$ Figure 6 shows the trajectory of disturbance attenuation level $\tau(t)$ for the closed-loop coupled PDE systems (32)-(33) and (3)-(4) with $P(z) \neq s(z)$ and the initial value $e_0(z) = 0$. From Figure (6), it's easily seen that $\tau(t) < \tau^*$, the performance (7) is guaranteed for the closed-loop coupled PDE systems (32)-(33) and (3)-(4).

VI. CONCLUSION

This paper has presented an H_∞ piecewise control problem for linear parabolic DPSs with piecewise observation in space. The method of H_∞ control is proposed to reduce the effect of the measurement disturbance. The closed-loop systems are constructed via the static feedback controller design for the collocated observation case and the observer-based dynamic feedback controller design for the non-collocated observation case, respectively. Based on the definitions of H_∞ stability and exponentially stability in the sense of $|\cdot|_2$, the exponentially stability of the closed-loop system is guaranteed under a prescribed H_∞ performance constraint. Theorem 1 and Theorem 2 present the sufficient conditions on the existence of such controllers in the form of LMI. Finally, numerical simulation results of the closed-loop system with free disturbance and the disturbance attenuation level are shown, which illustrate the proposed H_∞ control method is effective.

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