IEEE Access
Multidisciplinary : Rapid Review : Open Access Journal

Received April 23, 2019, accepted May 3, 2019, date of publication May 15, 2019, date of current version June 6, 2019.
Digital Object Identifier 10.1109/ACCESS.2019.2916908

Finite Time Synchronization for Reactive Diffusion Complex Networks via Boundary Control

YIPING LUO, ZHAOMING LING, AND YUEJIE YAO
Hunan Institute of Engineering, Xiangtan 411101, China
Corresponding author: Yiping Luo (lyp@hnie.edu.cn)

This work was supported in part by the National Natural Science Foundation of China under Grant No.11372107, and in part by the Natural Science Foundation of Hunan Province under Grant 2017JJ4004.

ABSTRACT This paper investigates the finite-time synchronization problems of complex spatiotemporal networks with time delays and diffusion terms. First, a boundary controller based on Lyapunov stability theory, Wirtinger's inequality, and finite-time analysis is designed. Subsequently, sufficient conditions for finite-time synchronization are obtained, and the setting time of finite-time synchronization is estimated. Finally, a simulation example is given to demonstrate the effectiveness of the obtained result.

INDEX TERMS Finite-time synchronization, complex network, time delay, diffusion term, boundary control.

I. INTRODUCTION

Complex dynamic networks (CDNs) include sensor networks [1], multiagent systems [2], and neural networks [3]. In recent years, complex networks have been widely used in various fields, such as mathematics, finance, Internet, society, and biology, and yielded outstanding achievements. Meanwhile, synchronous control of complex networks has become a research hotspot in many disciplines because of its practical and potential implications, such as intelligent control [4] and communication encryption [5], which can be classified as synchronous control problems. In the past 10 years, many synchronization concepts have been widely proposed, such as asymptotic synchronization [6], full synchronization [7], projection synchronization [8], lag synchronization [9], phase synchronization [10], approximate synchronization [11], pinning synchronization [12], finite-time synchronization [13], hybrid synchronization [14], cluster synchronization [15], and outer synchronization [16]. Outstanding achievements continue to emerge in this research field. However, delay remains inevitable in practice. For example, information transfer in a network is often accompanied by delay, which is a primary cause of system instability. Therefore, studying the stability of time-delay systems has become an important topic in control theory, as validated by various previous studies [17]–[24].

Many phenomena occur in the real world in different fields, such as chemical engineering, neurophysiology, and biodynamics, in which state variables depend not only on time but also on spatial location. Additionally, many complex spatiotemporal dynamical networks (CDNs) with time and space characteristics are present in nature and subject areas [25]. Therefore, the study of complex spatio-temporal dynamic network synchronization has attracted considerable attention. Synchronous control of complex space–time dynamic networks (CSDNs) is difficult to study because of its infinite dimensionality, but it has attracted many researchers in the past few decades with respect to such fields as matrix proportional [26], P-sD [27], pulse [28], intermittent [29], [30], activation control [31], linear separation [32], sliding mode control [33], and adaptive control [34]. These controllers are based on state feedback and require sensors and drivers distributed throughout the space domain. These characteristics make the application of these controllers difficult in situations where the state is unknown. Boundary control can solve this problem, and in the past few years, it has been applied to the synchronization of complex networks. For example, Liming Wang studied the asymptotic synchronization problem of coupled time-delay systems with boundary control in [35]. Jinde Cao discussed the nonlinear CSDN cluster

The associate editor coordinating the review of this manuscript and approving it for publication was Weisi Guo.

synchronization control problem with cluster structure through boundary control in [36].

Complex networks must not only be synchronized but also have sufficient synchronization performance. Most of the control methods that have been proposed in the literature for complex network synchronization control issues require infinite time but ignore complex network time synchronization. In practical applications, systems often need to reach stable synchronization within finite time. Recently, researchers have proposed a control method that can achieve complex network synchronization in finite time according to real-time requirements and produced positive results. For example, Jinde Cao studied the finite- and fixed-time synchronization problems of time-varying inertial memristive neural networks in [37]. Lu solved the problem of overlapping cluster synchronization of coupled complex networks by adaptive finite-time control in [38]. Wu studied the finite-time boundary stabilization problem of reaction–diffusion neural networks without time delay in [39]. Wang X investigated the synchronization problem of a class of fully complex-valued networks with coupling delay by using linear feedback control in finite time in [40]. Espitia studied the stability problem of event-triggered boundary-controlled hyperbolic partial differential equations in [41]. However, the finite-time synchronization of complex networks based on boundary-controlled delay-diffusion delays has not been studied.

Based on the above analysis and the problems to be solved, this study considers the finite-time boundary control problem of CSDNs with diffusion terms and space–time characteristics. At present, results have not been reported for the finite-time boundary control of complex networks with reaction–diffusion time delay. The main difficulty is the design of finite-time boundary controllers under Neumann boundary conditions. On this basis, a complex network with coupled time delay and the dynamic behavior of system nodes with time delays are given, and sufficient conditions for the finite-time synchronization of the boundary controller are obtained. Finally, numerical simulations verify our theoretical results.

Notation: All the notations used in this study are standard. For $x \in R^n$, let $\|x\|$ denote the Euclidean vector norm, i.e., $\|x\| = \sqrt{(x^T x)}$. For $A \in R^{n\times n}$, let $\|P\|$ indicate the norm of P induced by the Euclidean vector norm, i.e., $\|P\| = \sqrt{\lambda_{max}(P^T P)}$, where $\lambda_{max}(P^T P)$ is the maximum eigenvalue of $P^T P$. I denotes the identity matrix with appropriate dimensions, $R^{m\times n}$ denotes the set of all $m \times n$ real matrices, and R^n is the n-dimensional Euclidean space. For symmetric matrices A and B, the notation $A > B (A \geq B)$ means that the matrix $A - B$ is positive definite (nonnegative). The superscript "T" stands for the transpose of a vector or a matrix. $diag\{\ldots\}$ is used to denote the block diagonal matrix, $y_{i,t}$ means $\frac{dy_i}{dt}$, y_t means $\frac{dy}{dt}$, $y_{i,xx}$ means $\frac{\partial^2 y_{i,xx}}{\partial x^2}$, and means $\frac{\partial^2 y_{xx}}{\partial x^2}$, and $y_x(x,t)\,|_a^b$ means $y_x(a,t) - y_x(b,t)$.

II. FINITE-TIME SYNCHRONIZATION OF COMPLEX NETWORKS BY BOUNDARY CONTROL

$$\begin{cases} g_{i,t}(x,t) = \Theta_1 g_{i,xx}(x,t) + f(g_i(x,t)) \\ \qquad + \sum\limits_{i=i}^{n} a_{ij}\Gamma_1 g_j(x,t) + \sum\limits_{i=i}^{n} b_{ij}\Gamma_1 g_j(x,t-\tau) \\ g_{i,x}(x,t)\,|_{x=0} = 0,\ g_{i,x}(x,t)\,|_{x=L} = Cu_i(t) \\ g_i(x,0) = g_{i,0}(x) \\ (x,t) \in [0,L]\times[0,\infty] \end{cases} \tag{1}$$

where $g_i(x,t) \triangleq [g_{i1}(x,t), g_{i2}(x,t), \ldots, g_{in}(x,t)]^T \in R^n$ are the states. $x \in [0,L] \in R$ and $t \in [0,\infty)$ are the spatial and time variables, respectively. The subscripts x and t are the partial derivatives with respect to x and t, respectively. $u_i(t) \in R^n$ denotes the boundary control inputs. Dispersal matrices $\Theta_1 \in R^{n\times n}$ are assumed to be positive. C is a negative number, and τ is the time delay. $f(g_i(x,t))$ represents the nonlinear perturbation of time and spatial variables. $A \triangleq (a_{ij})_{N\times N}$, $B \triangleq (b_{ij})_{N\times N}$ describe the diffusive topological structure of the CSDN, defined as $a_{ij} > 0, b_{ij} > 0 (i \neq j), a_{ii} = -\sum\limits_{j=1,j\neq i}^{N} a_{ij}, b_{ii} = -\sum\limits_{j=1,j\neq i}^{N} b_{ij}$, $i \in N$.

$$\begin{cases} s_t(x,t) = \Theta_1 s_{xx}(x,t) + Bs(x,t) + f(s(x,t)) \\ s_x(x,t)\,|_{x=0} = s_x(x,t)\,|_{x=L} = 0 \\ s(x,0) = s_0(x) \\ (x,t) \in [0,L]\times[0,\infty], \end{cases} \tag{2}$$

where $s(x,t)$ may be an equilibrium point, a periodic orbit, or a chaotic orbit.

Denote the synchronization errors $e_i(x,t) \triangleq g_i(x,t) - s(x,t)$, $i \in N$. The synchronization error system of the $i-th$ node can be obtained from (1) and (2) as

$$\begin{cases} e_{i,t}(x,t) = \Theta_1 e_{i,xx}(x,t) + \tilde{f}(g_i(x,t)) \\ \qquad + \sum\limits_{i=i}^{n} a_{ij}\Gamma_1 e_j(x,t) + \sum\limits_{i=i}^{n} b_{ij}\Gamma_1 e_j(x,t-\tau) \\ e_{i,x}(x,t)\,|_{x=0} = 0,\ e_{i,x}(x,t)\,|_{x=L} = Cu_i(t) \\ e_i(x,0) = e_{i,0}(x) \\ (x,t) \in [0,L]\times[0,\infty], \end{cases} \tag{3}$$

where $\tilde{f}(g_i(x,t)) \triangleq f(g_i(x,t)) - f(s(x,t))$.

The synchronization error system (15) can be rewritten in a compact manner as

$$\begin{cases} e_t = (\Theta_1 \otimes I_n)\, e_{xx} + (A\otimes\Gamma_1)\, e + \tilde{f}(g(x,t)) \\ \qquad + B\otimes\Gamma_2 e(x,t-\tau) \\ e_x\,|_{x=0} = 0,\ e_x\,|_{x=L} = Cu(t) \\ e(x,0) = e_0(x) \\ (x,t) \in [0,L]\times[0,\infty], \end{cases} \tag{4}$$

where

$$e(x,t-\tau) \triangleq \left[e_1^T(x,t-\tau), e_2^T(x,t-\tau), \ldots, e_n^T(x,t-\tau)\right]^T,$$
$$u(t) \triangleq \left[u_1^T(t), u_2^T(t), \ldots, u_n^T(t)\right]^T,$$
$$e(x,t) \triangleq \left[e_1^T(x,t), e_2^T(x,t), \ldots, e_n^T(x,t)\right]^T.$$

We design the following boundary controller for the ith node of the system (1):

$$u(t) = \int_0^L ke(x,t)dx, + \frac{e(L,t)}{\|e(L,t)\|^2} \cdot \frac{k}{2}\left[\int_0^L e^T(x,t)e(x,t)dx + \int_0^L \int_{t-\tau}^t e^T(x,t)e(x,t)dsdx\right]^\beta. \quad (5)$$

Let $\tilde{f}(g_i(x,t)) = f(g_i(x,t)) - f(s(x,t))$, where C is a negative number, and K is the gain coefficient to be determined and satisfies $0 < \beta < 1$, $|e|^2 = e^T e$.

Remark 1: Controller (5), which is located at the boundary position, requires only one actuator at $x = L$, whereas the controllers located in the systems require an array of actuators all over the spatial domain, that is, $x \in [0, L]$.

Lemma 1 (see [42]): Assume that a continuous, positive-definite function $V(t)$ satisfies the following differential inequality:

$$\dot{V}(t) \le -\alpha V^\eta(t), \quad t \ge t_0, \; V(t_0) \ge 0,$$

where $\alpha > 0$, $0 < \eta < 1$. For any given t_0, $V(t)$ satisfies the following inequality:

$$V^{1-\eta}(t) \le V^{1-\eta}(t_0) - \alpha(1-\eta)(t-t_0), \quad t_0 \le t \le T,$$

and

$$V(t) \equiv 0 \quad \text{for all } t \ge t_0,$$

where $t_1 = t_0 + \frac{V^{1-\eta}(t_0)}{\alpha(1-\eta)}$

Lemma 2 (Wirtinger's inequality [39]): Given a square integrable vector function $z(x)$ with $z(0) = 0$ or $z(L) = 0$, for any symmetric matrix $S > 0$, the following inequality holds:

$$\int_0^L z^T(s)Sz(s)ds \le 4L^2\pi^{-2}\int_0^L (dz(s)/ds)^T S(dz(s)/ds)\, ds.$$

Lemma 3 (see [36]): For two square integrable vector functions$a(x)$, $b(x)$, $x \in [0, L]$,the following inequality holds for any scalar $\alpha > 0$:

$$2\int_0^L a^T(x)b(x)dx \le \alpha\int_0^L a^T(x)a(x)dx + \alpha^{-1}\int_0^L b^T(x)b(x)dx.$$

Assumption 1: Assume that the nonlinear function $f(g)$ satisfy the Lipschitz condition, i.e., for any ς, γ, there exist scalars $l_1 > 0$ satisfying

$$\|f(\varsigma) - f(\gamma)\| \le l_1 \|\varsigma - \gamma\|.$$

Definition 1: If a settling time $T > 0$ exists depending on the initial value, $\lim_{t\to T}\|x_i(t) - s(t)\| = 0$ for all $\forall i, j$, $t \ge T$, and $t > T$, then $\|x_i(t) - s(t)\| \equiv 0$ is established. The complex network reaches finite-time synchronization.

Theorem 1: Consider dynamics (4). When positive numbers L and l_1, negative number C, and positive definite matrix Θ exist, we have

$$\begin{pmatrix} \Pi & [(\Theta\otimes I_n)Ck]^T \\ (\Theta\otimes I_n)Ck & -\frac{\pi^2}{2L^2}(\Theta\otimes I_n)\end{pmatrix} < 0, \quad (6)$$

where

$$\Pi = \left(2(A\otimes\Gamma_1) + (B\otimes\Gamma_2)^2 + (2l_1+1)I_n - 2(\Theta\otimes I_n)Ck\right).$$

Then, estimation error $e(x,t)$ converges in finite time to zero, and the settling time for FTS can be given by

$$t_1 = t_0 + \frac{V^{1-\beta}(t_0)}{2(\Theta\otimes I_n)Ck(1-\beta)},$$

where k is the gain coefficient to be determined and satisfies $0 < \beta < 1$.

Proof: Consider the following Lyapunov function for the system (1):

$$V(t) = \int_0^L \sum_{i=1}^n e_i^T(x,t)e_i(x,t)dx + \int_0^L \int_{t-\tau}^t \sum_{i=1}^n e_i^T(x,t)e_i(x,t)dtds \quad (7)$$

let $\tilde{f}(g_i(x,t)) = f(g_i(x,t)) - f(s(x,t))$, and computing the derivative of $V(t)$ along the solution trajectories of the error system (4), we obtain

$$\begin{aligned}\dot{V}(t) &= 2\int_0^L e^T(x,t)e_t(x,t)dx + \int_0^L e^T(x,t)e(x,t)dx \\ &\quad - \int_0^L e^T(x,t-\tau)e(x,t-\tau)dx \\ &= 2\left[\int_0^L l_1 e^T(x,t)e(x,t)dx + \int_0^L e^T(x,t)(A\otimes\Gamma_1)e(x,t)dx\right] \\ &\quad + 2\left[\int_0^L e^T(x,t)(B\otimes\Gamma_2)e(x,t-\tau)dx + \int_0^L e^T(x,t)(\Theta_1\otimes I_n)e_{xx}(x,t)dx\right] \\ &\quad + \int_0^L e^T(x,t)e(x,t)dx - \int_0^L e^T(x,t-\tau)e(x,t-\tau)dx \\ &= \int_0^L e^T(x,t)\left(2(A\otimes\Gamma_1) + (B\otimes\Gamma_2)^2 + (2l_1+1)I_n\right)e(x,t)dx \\ &\quad + 2\int_0^L e^T(x,t)(\Theta_1\otimes I_n)e_{xx}(x,t)dx \quad (8)\end{aligned}$$

Given that $\Theta > 0$,we can obtain $(\Theta\otimes I_n) > 0$.

By integrating parts and considering the boundary conditions, Lemma 1, Lemma 2, Assumption 1, and taking to $\alpha \to 1^+$ we can find that

$$
\begin{aligned}
2 & \int_0^L e^T(x,t)\,(\Theta \otimes I_n)\,e_{xx}(x,t)dx \\
& = 2e^T(x,t)\,(\Theta \otimes I_n)\,e_x(x,t)\Big|_0^L \\
& \quad - 2\int_0^L {e_x}^T(x,t)\,(\Theta \otimes I_n)\,e_x(x,t)dx \\
& = -2e(L,t)(\Theta \otimes I_n)Cu(t) \\
& \quad - \frac{\pi^2}{2L^2}\int_0^L \bar{e}^T(x,t)\,(\Theta \otimes I_n)\,\bar{e}(x,t)dx \\
& = 2\int_0^L (\bar{e}(x,t) - e(x,t))(\Theta \otimes I_n)Cke(x,t)dx \\
& \quad - 2(\Theta \otimes I_n)Ck\left[\int_0^L e^T(x,t)e(x,t)dx \right.\\
& \quad \left. + \int_0^L \int_{t-\tau}^t e^T(x,t)e(x,t)dsdx\right]^\beta \\
& \quad - \frac{\pi^2}{2L^2}\int_0^L \bar{e}(x,t)\,(\Theta \otimes I_n)\,\bar{e}(x,t)dx
\end{aligned} \tag{9}
$$

where $\bar{e}(x,t) \triangleq e(x,t) - e(L,t)$ and then $\bar{e}(L,t) = 0$,

$$
\begin{aligned}
\dot{V} \le & \int_0^L e^T(x,t)\,(\Pi + 2(\Theta \otimes I_n)Ck)\,e(x,t)dx \\
& - 2\int_0^L e^T(0,t)(\Theta \otimes I_n)Cke(x,t)dx - 2(\Theta \otimes I_n)CkV^\beta(t) \\
& - \frac{\pi^2}{2L^2}\int_0^L \bar{e}^T(x,t)\,(\Theta \otimes I_n)\,\bar{e}(x,t)dx \\
= & \, 2\int_0^L \bar{e}^T(x,t)(\Theta \otimes I_n)Cke(x,t)dx \\
& - \frac{\pi^2}{2L^2}\int_0^L \bar{e}^T(x,t)\,(\Theta \otimes I_n)\,\bar{e}(x,t)dx \\
& + \int_0^L e^T(x,t)\Pi e(x,t)dx - 2(\Theta \otimes I_n)CkV^\beta(t) \\
= & \int_0^1 \begin{pmatrix} e \\ \bar{e} \end{pmatrix}^T \begin{pmatrix} \Pi & [(\Theta \otimes I_n)Ck]^T \\ (\Theta \otimes I_n)Ck & -\frac{\pi^2}{2L^2}(\Theta \otimes I_n) \end{pmatrix} \begin{pmatrix} e \\ \bar{e} \end{pmatrix} dx \\
& - 2(\Theta \otimes I_n)CkV^\beta(t) \\
\le & -2(\Theta \otimes I_n)CkV^\beta(t)
\end{aligned} \tag{10}
$$

where

$$
\Pi = \left(2\,(A \otimes \Gamma_1) + (B \otimes \Gamma_2)^2 + (2l_1+1)I_n - 2(\Theta \otimes I_n)Ck\right).
$$

The settling time is:

$$
t_1 = t_0 + \frac{V^{1-\beta}(t_0)}{2(\Theta \otimes I_n)Ck(1-\beta)}. \tag{11}
$$

The proof is complete.

Remark 2: The boundary conditions given in many complex network synchronization problems with diffusion terms are often Dirichlet boundary conditions, that is, directly giving values at the boundary, such as [44]. By contrast, the present work discusses complex boundary conditions. Thus, we adopt the Neumann boundary condition, which is the partial derivative value given at the boundary. The study of Dirichlet boundary conditions in this type of study is more complex.

Remark 3: O. M. studied the finite-time stability problem with diffusion-term neural networks in [45]. M. Syed indetermined Markov complex dynamic networks with mixed time-varying delays and reaction–diffusion terms in [46] for the finite-time robust random synchronization problem. The controller designed in the above article is a global controller, which means that it is placed at every point in the spatial domain. Placing every point of the controller in the spatial domain is difficult in engineering practice. For reaction–diffusion complex network systems, the use of boundary control to achieve finite-time synchronization is an effective way to proceed to engineering applications.

Remark 4: Wu studied the boundary finite-time stabilization problem of a reaction–diffusion neural network in [39] and found important results. However, the author did not consider the effect of time delay on the studied system. In fact, many complex network systems have time lags, which often negatively affect such systems. On the contrary, the current study considers the coupled time delays and the dynamic behavior of system nodes with time delays and their effect on the synchronization of the system.

In Theorem 1, we consider a complex network model with time delays. We consider a complex network model without time delays as a simplified case.

$$
\begin{cases}
y_{i,t}(x,t) = \Theta y_{i,xx}(x,t) + By_i(x,t) + f(y_i(x,t)) \\
\qquad\qquad + \sum_{i=i}^{n} a_{ij}\Gamma_1 y_j(x,t) \\
y_{i,x}(x,t)\,|_{x=0} = 0,\ y_{i,x}(x,t)\,|_{x=L} = Cu_i(t) \\
y_i(x,0) = y_{i,0}(x) \\
(x,t) \in [0,L] \times [0,\infty]
\end{cases}. \tag{12}
$$

We design the following boundary controller for the ith node of the system (4):

$$
u(t) = \int_0^L K_1\bar{e}(x,t)dx + \int_0^L K_2e(x,t)dx + \int_0^L e^{-\eta}(x,t)dx, \tag{13}
$$

where K_1 and K_2 are the gain coefficients to be determined and satisfy $K_2 \le K_1$.

Let $\tilde{f}(y_i(x,t)) = f(y_i(x,t)) - f(s(x,t))$.

Corollary 1: Consider dynamics (1). When positive numbers c_1 and l_1,negative number C, and positive definite matrix Θ exist, we have

$$
\begin{aligned}
B \otimes I_n + c_1A \otimes \Gamma_1 + l_1I_n - (\Theta \otimes I_n)\,CK_2 & \\
- \frac{(\Theta \otimes I_n)^T\,(\Theta \otimes I_n)}{2} < 0,\ (\Theta \otimes I_n)\,CK_1 & \\
+ \frac{C}{2}\,(\Theta \otimes I_n)^T\,(\Theta \otimes I_n) < 0. &
\end{aligned} \tag{14}
$$

Then, estimation error $e(x,t)$ converges in finite time to zero, and the settling time for FTS can be given by

$$t_1 = t_0 - \frac{2V^{1-\eta}}{C(1-\eta)},$$

where K_1 and K_2 are the gain coefficients to be determined and satisfy $K_2 \leq K_1, 0 < \eta < 1$.

Remark 5: Wu investigated the synchronization problem of coupled linear partial differential systems with boundary control in [35]. Wang evaluated the asymptotic synchronization problem of coupled time-delay partial differential systems with boundary control in [43]. Jinde Cao discussed the cluster synchronization control problem of a nonlinear CSDN with cluster structure in [36]. These works presented important results in CSDN boundary control but studied the synchronization of complex network systems at infinity instead of considering synchronization in finite time. In actual engineering, systems are often expected to synchronize in limited time. Unlike the aforementioned related literature, we achieve system stability by designing a finite-time boundary controller, which has better practicability than those in the above studies.

In Corollary 1, we studied the finite-time synchronization problem of complex networks with time-delay and diffusion terms. At the same time, delays usually occur in the dynamic behavior between nodes. So, we will discuss the finite-time synchronization problem of complex networks with node dynamics with time-delay in Theorem 2. Consider the following complex network model:

$$\begin{cases} e_t = (\Theta_1 \otimes I_n)\, e_{xx} + (A \otimes \Gamma_1)\, e + \tilde{f}(g(x,t-\tau)) \\ \qquad + B \otimes \Gamma_2 e(x,t-\tau) \\ e_x\,|_{x=0} = 0,\; e_x\,|_{x=L} = Cu(t) \\ e(x,0) = e_0(x) \\ (x,t) \in [0,L] \times [0,\infty] \end{cases} \tag{15}$$

We design the following boundary controller for the ith node of the system (15):

$$\begin{aligned} u(t) = & \int_0^L ke(x,t)dx + \int_0^L ge(x,t-\tau)dx, \\ & + \frac{e(L,t)}{|e(L,t)^2|} \cdot \frac{k}{2} \left[\int_0^L e^T(x,t)e(x,t)dx \right. \\ & \left. + \int_0^L \int_{t-\tau}^t e^T(x,t)e(x,t)dsdx \right]. \end{aligned} \tag{16}$$

Let $\tilde{f}(g_i(x,t-\tau)) = f(g_i(x,t-\tau)) - f(s(x,t-\tau))$, where is a negative number, and $k > 0$, $g > 0$ are the gain coefficients to be determined and should satisfy $0 < \beta < 1$, $|e|^2 = e^T e$.

Theorem 2: Consider dynamics (15). If there exist positive numbers L and l_1,negative number C, and positive definite matrix Θ such that

$$\begin{pmatrix} \Pi_1 & [(\Theta \otimes I_n)Ck]^T & 0 \\ (\Theta \otimes I_n)Ck & -\frac{\pi^2}{2L^2}(\Theta \otimes I_n) & 0 \\ 0 & 0 & 2Cg + 2l_1 \end{pmatrix} < 0, \tag{17}$$

where

$$\Pi_1 = 2\,(A \otimes \Gamma_1) + (B \otimes \Gamma_2)^2 + I_n - 2(\Theta \otimes I_n)Ck,$$

then, estimation error $e(x,t)$ will converge in finite time to zero, and the settling time for FTS can be given by

$$t_1 = t_0 + \frac{V^{1-\beta}(t_0)}{2(\Theta \otimes I_n)Ck(1-\beta)},$$

where $k > 0$, $g < 0$ are the gain coefficients to be determined and satisfy $0 < \beta < 1$.

Proof: Consider the following Lyapunov function for the system (15)

$$\begin{aligned} V(t) = & \int_0^L \sum_{i=1}^n e_i^T(x,t)e_i(x,t)dx \\ & + \int_0^L \int_{t-\tau}^t \sum_{i=1}^n e_i^T(x,t)e_i(x,t)dtdx \end{aligned} \tag{18}$$

Let $\tilde{f}(g_i(x,t)) = f(g_i(x,t)) - f(s(x,t))$, and computing the derivative of $V(t)$ along the solution trajectories of the error system (15), we obtain

$$\begin{aligned} \dot{V}(t) = & 2\int_0^L e^T(x,t)e_t(x,t)dx \\ & + \int_0^L e^T(x,t)e(x,t)dx - \int_0^L e^T(x,t-\tau)e(x,t-\tau)dx \\ = & \int_0^L e^T(x,t)\left(2\,(A \otimes \Gamma_1) + (B \otimes \Gamma_2)^2 + I_n\right) e(x,t)dx \\ & + 2\int_0^L e^T(x,t)\,(\Theta_1 \otimes I_n)\, e_{xx}(x,t)dx \\ & + \int_0^L e^T(x,t-\tau)\,(2l_1)\, e(x,t-\tau)dx \end{aligned} \tag{19}$$

Given that $\Theta > 0$,we can obtain $(\Theta \otimes I_n) > 0$.

By integrating by parts and considering the boundary conditions, Lemma 1, Lemma 2, Assumption 1, and taking to $\alpha \to 1^+$ we can find that

$$\begin{aligned} & 2\int_0^L e^T(x,t)\,(\Theta \otimes I_n)\, e_{xx}(x,t)dx \\ & = 2e^T(x,t)\,(\Theta \otimes I_n)\, e_x(x,t)\Big|_0^L \\ & \quad - 2\int_0^L e_x^T(x,t)\,(\Theta \otimes I_n)\, e_x(x,t)dx \\ & = 2e(L,t)(\Theta \otimes I_n)Cu(t) \\ & \quad - \frac{\pi^2}{2L^2}\int_0^L \bar{e}^T(x,t)\,(\Theta \otimes I_n)\,\bar{e}(x,t)dx \\ & = 2\int_0^L (\bar{e}^T(x,t) - e^T(x,t))(\Theta \otimes I_n)Cke(x,t)dx \\ & \quad - 2(\Theta \otimes I_n)Ck\left[\int_0^L e^T(x,t)e(x,t)dx \right. \\ & \quad \left. + \int_0^L \int_{t-\tau}^t e^T(x,t)e(x,t)dsdx\right]^\beta \end{aligned}$$

$$
\begin{aligned}
&-\frac{\pi^2}{2L^2}\int_0^L \bar{e}^T(x,t)\,(\Theta\otimes I_n)\,\bar{e}(x,t)dx \\
&+2\int_0^L (\bar{e}(x,t)-e(x,t))(\Theta\otimes I_n)Cge(x,t-\tau)dx
\end{aligned} \tag{20}
$$

where $\bar{e}(x,t)\overset{\Delta}{=}e(x,t)-e(L,t)$ then $\bar{e}(L,t)=0$,

$$
\begin{aligned}
&2\int_0^L (\bar{e}^T(x,t)-e^T(x,t))(\Theta\otimes I_n)Cge(x,t-\tau)dx \\
&\quad = 2\int_0^L \bar{e}^T(x,t)(\Theta\otimes I_n)Cge(x,t-\tau)dx \\
&\qquad -2\int_0^L e^T(x,t)(\Theta\otimes I_n)Cge(x,t-\tau)dx \\
&\quad \le Cg\int_0^L \bar{e}^T(x,t)(\Theta\otimes I_n)^2\bar{e}(x,t)dx \\
&\qquad +2Cg\int_0^L e^T(x,t-\tau)e(x,t-\tau)dx \\
&\qquad +Cg\int_0^L e^T(x,t)(\Theta\otimes I_n)^2e(x,t)dx \\
&\quad \le 2Cg\int_0^L e^T(x,t-\tau)e(x,t-\tau)dx
\end{aligned} \tag{21}
$$

We have

$$
\begin{aligned}
\dot{V} \le& \int_0^L e^T(x,t)\left(2\,(A\otimes\Gamma_1)+(B\otimes\Gamma_2)^2+I_n\right)e(x,t)dx \\
&+(2Cg+2l_1)\int_0^L e^T(x,t-\tau)e(x,t-\tau)dx \\
&-2\int_0^L e(0,t)(\Theta\otimes I_n)Cke(x,t)dx \\
&-2(\Theta\otimes I_n)CkV^\beta(t)-\frac{\pi^2}{2L^2}\int_0^L \bar{e}^T(x,t)(\Theta\otimes I_n)\,\bar{e}(x,t)dx \\
=&\,2\int_0^L \bar{e}^T(x,t)(\Theta\otimes I_n)Cke(x,t)dx \\
&-\frac{\pi^2}{2L^2}\int_0^L \bar{e}^T(x,t)\,(\Theta\otimes I_n)\,\bar{e}(x,t)dx \\
&+(2Cg+2l_1)\int_0^L e^T(x,t-\tau)(x,t-\tau)dx \\
&+\int_0^L e^T(x,t)\left(2\,(A\otimes\Gamma_1)+(B\otimes\Gamma_2)^2\right. \\
&\left.-2(\Theta\otimes I_n)Ck\right)e(x,t)dx-2(\Theta\otimes I_n)CkV^\beta(t) \\
=&\int_0^1\begin{pmatrix} e\\ \bar{e}\\ e_\tau\end{pmatrix}^T\begin{pmatrix}\Pi_1 & [(\Theta\otimes I_n)Ck]^T & 0\\ (\Theta\otimes I_n)Ck & -\frac{\pi^2}{2L^2}(\Theta\otimes I_n) & 0\\ 0 & 0 & 2Cg+2l_1\end{pmatrix} \\
&\times\begin{pmatrix} e\\ \bar{e}\\ e_\tau\end{pmatrix}dx-2(\Theta\otimes I_n)CkV^\beta(t) \\
\le& -2(\Theta\otimes I_n)CkV^\beta(t)
\end{aligned} \tag{22}
$$

The settling time is:

$$
t_1=t_0+\frac{V^{1-\beta}(t_0)}{2(\Theta\otimes I_n)Ck(1-\beta)} \tag{23}
$$

The proof is complete.

Remark 6: In [31], Yang studied the synchronization problem of a class of nonlinear multidelay complex spatiotemporal networks modeled by semilinear parabolic partial differential equations. Wang X investigated the synchronization problem of uncertain complex networks with time-varying node delays and multiple time-varying coupled delays in [49]. In the present study, we assume that all delays in the system are the same. Given that different delays can be handled by their maximum bounds, time-varying delays can also be handled in this way. Although such a treatment will increase the conservativeness of the system, the computational complexity of determining the limited synchronization time and the amount of calculation of the system will be greatly reduced.

Remark 7: For time-delay partial differential systems with spatiotemporal features, Bq Yang proposed a boundary controller to achieve finite-time stability in [32]. However, in the system discussed in this study, only the transmission time lag was included. By contrast, we study the finite-time synchronization of complex network systems with time delay. The system we adopt includes coupled time delays and the dynamic behavior of system nodes with time delays.

Remark 8: According to **Theorems 1 and 2,** the constants k and β are contained in Controllers (5) and (16), respectively. These constants affect the settling time. Therefore, we can choose the right k and β to obtain the required settling time.

III. NUMERICAL SIMULATION

In this section, the exponential synchronization conditions obtained in this paper is illustrated with an example.

Consider a general complex dynamical network consisting of dynamical nodes with linear couplings system in Theorem3.

$$
\begin{cases}
e_t=(\Theta_1\otimes I_n)\,e_{xx}+(A\otimes\Gamma_1)\,e+\tilde{f}(g(x,t-\tau)) \\
\qquad\quad +B\otimes\Gamma_2 e(x,t-\tau) \\
e_x\mid_{x=0}=0,\ e_x\mid_{x=L}=Cu(t) \\
e(x,0)=e_0(x) \\
(x,t)\in[0,L]\times[0,\infty]
\end{cases}
$$

and consider a complex network system with 4 nodes, each of which is a 4-dimensional linear system Where β=0.5; τ=0.1; L=3; $C=-1$;l_1=0.4; $I=\mathrm{diag}\{1,1,1,1\}$;

$$
\Gamma_1=\Gamma_2=\mathrm{diag}\{2.8,3.1,1.5,2.9\};
$$

$$
A=\begin{bmatrix}-0.5 & 0.2 & 0 & 0.3\\ 0.2 & -0.6 & 0.2 & 0.2\\ 0 & 0.2 & -0.2 & 0\\ 0.3 & 0.2 & 0 & -0.5\end{bmatrix};
$$

$$
B=\begin{bmatrix}-0.3 & 0.1 & 0 & 0.2\\ 0.1 & -0.4 & 0.2 & 0.1\\ 0 & 0.2 & -0.2 & 0\\ 0.2 & 0.1 & 0 & -0.3\end{bmatrix}.
$$

The initial conditions be $e_1(3,:)=2\sin(x)+\cos(x)$, $e_2(3,:)=\exp(x+\cos(x))$ and $e_3(3,:)=2\sin(x)-\cos(x)$, $e_4(3,:)=5\sin(x)+3\cos(x)$, $x\in[0,3]$, respectively.

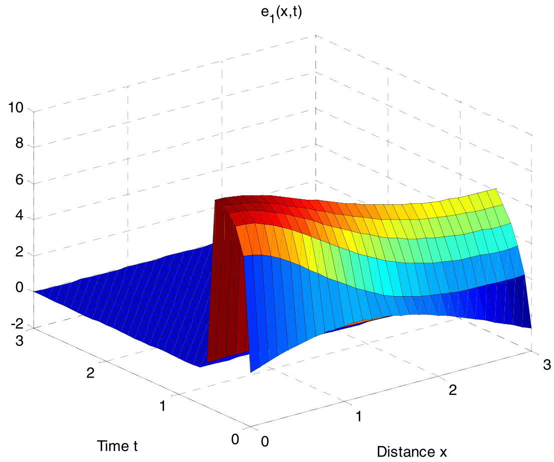

FIGURE 1. The state of the node e_1 with controller.

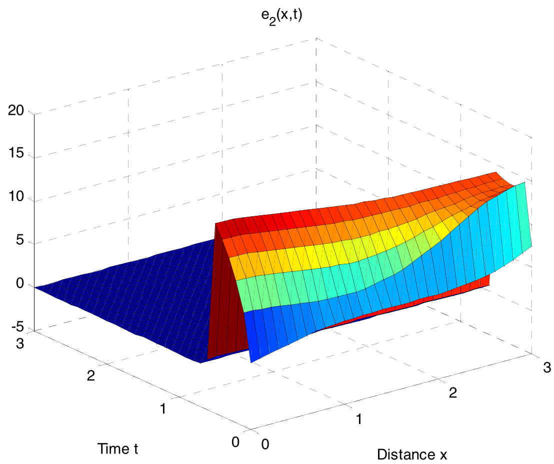

FIGURE 2. The state of the node e_2 with controller.

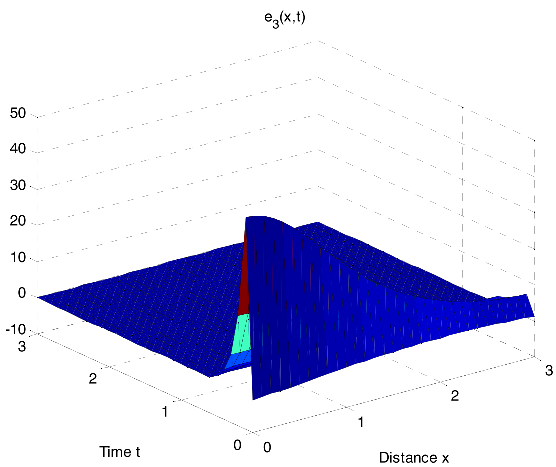

FIGURE 3. The state of the node e_3 with controller.

Figure 1, 2, 3, 4 is the node 1, 2, 3, 4 status curves with controller.

As can be seen from the figure, after a period of time, where time unit is s, the system state is synchronous, and

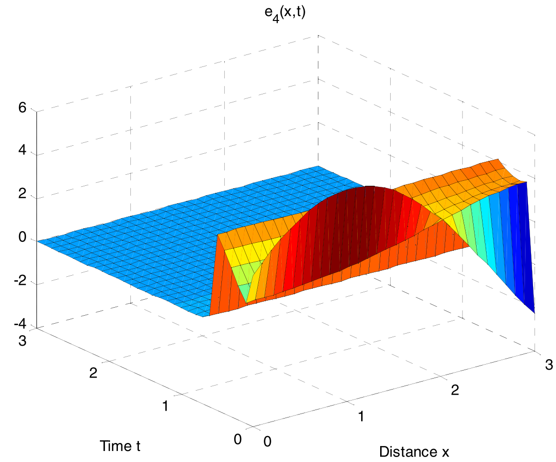

FIGURE 4. The state of the node e_4 with controller.

the feasibility and effectiveness of the control method are demonstrated

IV. CONCLUSIONS AND DISCUSSIONS

This study presents a problem of the finite-time synchronization of complex spatiotemporal networks with diffusion terms and proposes a method of boundary control that can be used to achieve finite-time synchronization for complex network systems. We provide a basis for future research directions, such as achieving synchronization through lenient conditions and generalizing existing methods to the synchronization problems of complex networks with event-triggered controllers. Thus, we will consider these goals in the future.

A. AUTHORS' CONTRIBUTIONS

Both authors read and carefully approved the final manuscript.

B. COMPETING INTERESTS

The authors declare that they have no competing interests.

ACKNOWLEDGEMENT

The authors would like to thank X. Zou for her great help in many aspects. *(Zhaoming Ling and Yuejie Yao contributed equally to this work.)*

REFERENCES

[1] E. Garone, A. Gasparri, and F. Lamonaca, "Clock synchronization protocol for wireless sensor networks with bounded communication delays," *Automatica*, vol. 59, pp. 60–72, Sep. 2015.

[2] H. Li, X. Liao, T. Huang, and W. Zhu, "Event-triggering sampling based leader-following consensus in second-order multi-agent systems," *IEEE Trans. Autom. Control*, vol. 60, no. 7, pp. 1998–2003, Jul. 2015.

[3] J. D. Cao, R. Rakkiyappan, K. Maheswari, and A. Chandrasekar, "Exponential H_∞ filtering analysis for discrete-time switched neural networks with random delays using sojourn probabilities," *Sci. China Technol. Sci.*, vol. 59, pp. 387–402, Mar. 2016.

[4] H. Li, G. Chen, T. Huang, and Z. Dong, "Event-triggered distributed average consensus over directed digital networks with limited communication bandwidth," *IEEE Trans. Cybern.*, vol. 46, no. 12, pp. 3098–3110, Dec. 2016.

[5] X. Wu, C. J. Zhu, and H. B. Kan, "An improved secure communication scheme based passive synchronization of hyperchaotic complex nonlinear system," *Appl. Math. Comput.*, vol. 252, pp. 201–214, Feb. 2015.

[6] X. Li, R. Rakkiyappan, and N. Sakthivel, "Non-fragile synchronization control for Markovian jumping complex dynamical networks with probabilistic time-varying coupling delays," *Asian J. Control*, vol. 17, pp. 1678–1695, Sep. 2015.

[7] C. Yao, Q. Zhao, and J. Yu, "Complete synchronization induced by disorder in coupled chaotic lattices," *Phys. Lett. A*, vol. 377, no. 5, pp. 370–377, 2013.

[8] G. M. Mahmoud and E. E. Mahmoud, "Complex modified projective synchronization of two chaotic complex nonlinear systems," *Nonlinear Dyn.*, vol. 73, pp. 2231–2240, Sep. 2013.

[9] K. Rajagopal and S. Vaidyanathan, "Adaptive lag synchronization of a modified Rucklidge chaotic system with unknown parameters and its LabVIEW implementation," *Sensor Transducers*, vol. 200, pp. 37–44, Apr. 2016.

[10] F. A. S. Ferrari, R. L. Viana, S. R. Lopes, and R. Stoop, "Phase synchronization of coupled bursting neurons and the generalized kuramoto model," *Neural Netw.*, vol. 66, pp. 107–118, Jun. 2015.

[11] T. Li and B. P. Rao, "Criteria of Kalman's type to the approximate controllability and the approximate synchronization for a coupled system of wave equations with Dirichlet boundary controls," *SIAM J. Control Optim.*, vol. 54, no. 1, pp. 49–72, 2016.

[12] W. Yu, G. Chen, and J. Lü, "On pinning synchronization of complex dynamical networks," *Automatica*, vol. 45, no. 2, pp. 429–435, 2009.

[13] J. Cao and R. Li, "Fixed-time synchronization of delayed memristor-based recurrent neural networks," *Sci. China Inf. Sci.*, vol. 60, Mar. 2017, Art. no. 032201.

[14] R. Nagornov, G. Osipov, M. Komarov, A. Pikovsky, and A. Shilnikov, "Mixed-mode synchronization between two inhibitory neurons with post-inhibitory rebound," *Commun. Nonlinear Sci. Numer. Simul.*, vol. 36, pp. 175–191, Jul. 2016.

[15] F. Sorrentino, L. M. Pecora, A. M. Hagerstrom, T. E. Murphy, and R. Roy, "Complete characterization of the stability of cluster synchronization in complex dynamical networks," *Sci. Adv.*, vol. 2, no. 4, 2016, Art. no. e1501737.

[16] C. Zhang, X. Wang, C. Luo, J. Li, and C. Wang, "Robust outer synchronization between two nonlinear complex networks with parametric disturbances and mixed time-varying delays," *Phys. A, Stat. Mech. Appl.*, vol. 494, pp. 251–264, Mar. 2018.

[17] N. Vu Phat and V. Le Hien, "An application of Razumikhin theorem to exponential stability for linear non-autonomous systems with time-varying delay," *Appl. Math. Lett.*, vol. 22, no. 9, pp. 1412–1417, 2009.

[18] X. Zhang, D. Li, and X. Zhang, "Adaptive fuzzy impulsive synchronization of chaotic systems with random parameters," *Chaos, Solitons Fractals*, vol. 104, pp. 77–83, Nov. 2017.

[19] M. A. A. Ahmed, Y. Liu, W. Zhang, A. Alsaedi, and T. Hayat, "Exponential synchronization for a class of complex networks of networks with directed topology and time delay," *Neurocomputing*, vol. 266, pp. 274–283, Nov. 2017.

[20] S. H. Lee, M. J. Park, O. M. Kwon, and R. Sakthivel, "Advanced sampled-data synchronization control for complex dynamical networks with coupling time-varying delays," *Inf. Sci.*, vol. 420, pp. 454–465, Dec. 2017.

[21] Q. Xie, G. Si, Y. Zhang, Y. Yuan, and R. Yao, "Finite-time synchronization and identification of complex delayed networks with Markovian jumping parameters and stochastic perturbations," *Chaos, Solitons Fractals*, vol. 86, pp. 35–49, May 2016.

[22] L. Shi, H. Zhu, S. Zhong, K. Shi, and J. Cheng, "Function projective synchronization of complex networks with asymmetric coupling via adaptive and pinning feedback control," *ISA Trans.*, vol. 65, pp. 81–87, Nov. 2016.

[23] J. Wang, J. Feng, C. Xu, Y. Zhao, and J. Feng, "Pinning synchronization of nonlinearly coupled complex networks with time-varying delays using M-matrix strategies," *Neurocomputing*, vol. 177, pp. 89–97, Feb. 2016.

[24] W.-H. Chen, Z. Jiang, X. Lu, and S. Luo, "H_∞ synchronization for complex dynamical networks with coupling delays using distributed impulsive control," *Nonlinear Anal., Hybrid Syst.*, vol. 17, pp. 111–127, Aug. 2015.

[25] L. Sheng, H. Yang, and X. Lou, "Adaptive exponential synchronization of delayed neural networks with reaction-diffusion terms," *Chaos Solitons Fractals*, vol. 40, no. 2, pp. 930–939, 2009.

[26] F. Yu and H. J. Jiang, "Global exponential synchronization of fuzzy cellular neural networks with delays and reaction–diffusion terms," *Neurocomputing*, vol. 74, pp. 509–515, Jan. 2011.

[27] C. Yang, J. Qiu, and H. He, "Exponential synchronization for a class of complex spatio-temporal networks with space-varying coefficients," *Neurocomputing*, vol. 151, pp. 401–407, Mar. 2015.

[28] C. Hu, H. Jiang, and Z. Teng, "Impulsive control and synchronization for delayed neural networks with reaction–diffusion terms," *IEEE Trans. Neural Netw.*, vol. 21, no. 1, pp. 67–81, Jan. 2010.

[29] C. Hu, J. Yu, H. Jiang, and Z. Teng, "Exponential synchronization for reaction–diffusion networks with mixed delays in terms of p-norm via intermittent driving," *Neural Netw.*, vol. 31, pp. 1–11, Jul. 2012.

[30] C. Zhang, X. Wang, C. Wang, and Z. Xia, "Outer synchronization of complex networks with internal delay and coupling delay via aperiodically intermittent pinning control," *Int. J. Modern Phys. C*, vol. 28, no. 8, 2017, Art. no. 1750108.

[31] Y. Chengdong *et al.*, "Synchronization for nonlinear complex Spatio-temporal networks with multiple time-invariant delays and multiple time-varying delays," *Neural Process. Lett.*, 2018.

[32] K. Wu and B. Yang, "Finite-time boundary control for delay reaction–diffusion systems," *Appl. Math. Comput.*, vol. 329, pp. 52–63, Jul. 2018.

[33] W. Qiao, "Synchronization of the fractional order hyperchaos ressler systems with activation feedback control," Tech. Rep.,

[34] Q. Gan, "Adaptive synchronization of stochastic neural networks with mixed time delays and reaction–diffusion terms," *Nonlinear Dyn.*, vol. 69, pp. 2207–2219, Sep. 2012.

[35] K.-N. Wu, T. Tian, and L. Wang, "Synchronization for a class of coupled linear partial differential systems via boundary control," *J. Franklin Inst.*, vol. 353, no. 16, pp. 4062–4073, 2016.

[36] C. Yang, J. Cao, T. Huang, J. Zhang, and J. Qiu, "Guaranteed cost boundary control for cluster synchronization of complex spatio-temporal dynamical networks with community structure," *Sci. China Inf. Sci.*, vol. 61, no. 5, 2018, Art. no. 052203.

[37] R. Wei, J. Cao, and A. Alsaedi, "Finite-time and fixed-time synchronization analysis of inertial memristive neural networks with time-varying delays," *Cogn. Neurodyn.*, vol. 12, no. 3, pp. 121–134, 2017.

[38] X. Liu and Y. Wei, "Finite-time cluster synchronization of nonlinearly coupled reaction-diffusion neural networks via spatial coupling and control," in *Proc. Int. Conf. Syst. Inform.*, 2016, pp. 24–29.

[39] K.-N. Wu, H.-X. Sun, P. Shi, and C.-C. Lim, "Finite-time boundary stabilization of reaction-diffusion systems," *Int. J. Robust Nonlinear Control*, vol. 28, no. 5, pp. 1641–1652, 2017.

[40] C. Zhang, X. Wang, S. Wang, W. Zhou, and Z. Xia, "Finite-time synchronization for a class of fully complex-valued networks with coupling delay," *IEEE Access*, vol. 6, pp. 17923–17932, 2018.

[41] N. Espitia, A. Girard, N. Marchand, and C. Prieur, "Event-based boundary control of a linear 2×2 hyperbolic system via backstepping approach," *IEEE Trans. Autom. Control*, vol. 63, no. 8, pp. 2686–2693, Aug. 2017.

[42] S. P. Bhat and D. S. Bernstein, "Finite-time stability of continuous autonomous systems," *SIAM J. Control Optim.*, vol. 38, no. 3, pp. 751–766, 2000.

[43] K. N. Wu, T. Tian, L. Wang, and W.-W. Wang, "Asymptotical synchronization for a class of coupled time-delay partial differential systems via boundary control," *Neurocomputing*, vol. 197, pp. 113–118, Jul. 2016.

[44] J. G. Lu, "Global exponential stability and periodicity of reaction-diffusion delayed recurrent neural networks with Dirichlet boundary conditions," *Chaos, Solitons Fractals*, vol. 35, no. 1, pp. 116–125, 2008.

[45] M. S. Ali, J. Yogambigai, and O. M. Kwon, "Finite-time robust passive control for a class of switched reaction-diffusion stochastic complex dynamical networks with coupling delays and impulsive control," *Int. J. Syst. Sci.*, vol. 49, no. 4, pp. 718–735, 2018.

[46] M. S. Ali and J. Yogambigai, "Finite-time robust stochastic synchronization of uncertain Markovian complex dynamical networks with mixed time-varying delays and reaction–diffusion terms via impulsive control," *J. Franklin Inst.*, vol. 354, no. 5, pp. 2415–2436, 2017.

[47] W. Xingyuan and H. Yijie, "Projective synchronization of fractional order chaotic system based on linear separation," *Phys. Lett. A*, vol. 372, no. 4, pp. 435–441, 2007.

[48] D. Lin and X. Wang, "Observer-based decentralized fuzzy neural sliding mode control for interconnected unknown chaotic systems via network structure adaptation," *Fuzzy Sets Syst.*, vol. 161, no. 15, pp. 2066–2080, Aug. 2010.

[49] C. Zhang, X. Wang, C. Wang, and W. Zhou, "Synchronization of uncertain complex networks with time-varying node delay and multiple time-varying coupling delays," *Asian J. Control*, vol. 20, no. 6, pp. 186–195, 2017.