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Existence and Exponential Stability of Anti-Periodic Solutions for Inertial Quaternion-Valued High-Order Hopfield Neural Networks With State-Dependent Delays

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ABSTRACT This paper deals with a class of inertial quaternion-valued high-order Hopfield neural networks with state-dependent delays. Without decomposing the considered neural networks into real-valued systems, based on a continuation theorem of coincidence degree theory and the Wirtinger inequality, the existence of anti-periodic solutions of the networks is established. By constructing a suitable Lyapunov function, the global exponential stability of anti-periodic solutions of the networks is obtained. Finally, a numerical example is given to show the feasibility of our results.

INDEX TERMS Anti-periodic solution, global exponential stability, inertial high-order Hopfield neural networks, quaternion, state-dependent delay.

I. INTRODUCTION

Since high-order Hopfield neural networks have stronger approximation characteristics, faster convergence speed, larger storage capacity and higher fault tolerance than low-order neural networks, they have become the object of intensive analysis by many scholars in recent years. Since all of these applications depend to a large extent on their dynamics, many scholars have studied various dynamic properties of high-order Hopfield neural networks [3], [1]–[5].

On the one hand, it is worth noting that the introduction of inertial terms into neural network models can easily lead to complex dynamic behaviors in networks [6], [7]. In addition, the inclusion of inertial terms in standard neural network models also has some practical backgrounds [8], [9]. Therefore, in some cases, it is more reasonable to describe neural network models using second-order differential equations. At present, there is not much literature on the study of inertial neural network dynamics [10]–[13].

On the other hand, quaternion algebra was introduced into mathematics by Hamilton in 1843. The skew field of quaternions is denoted by

$$\mathbb{H} := \{x | x = x^R + ix^I + jx^J + kx^K\},$$

where x^R, x^I, x^J, $x^K \in \mathbb{R}$, i, j, and k are the fundamental quaternion units, which obey the Hamilton's multiplication rules: $ij = -ji = k, jk = -kj = i, ki = -ik = j, i^2 = j^2 = k^2 = -1$. For every $x \in \mathbb{H}$, the conjugate of x is defined as

$$x^* = x^R - ix^I - jx^J - kx^K$$

and the norm of x is defined as

$$\|x\| = \sqrt{xx^*} = \sqrt{(x^R)^2 + (x^I)^2 + (x^J)^2 + (x^K)^2}.$$

Due to the non-commutativity of quaternion multiplication, the study of quaternions is much more difficult than for real and complex numbers. However, with the continuous development of quaternion algebra theory, modern science and technology, the application fields of quaternions are becoming more and more extensive, such as attitude control, quantum mechanics and computer graphics. In particular, since quaternion-valued neural networks can use multi-state activation functions to process multi-level information, research on them has become a hot topic.

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Although quaternion-valued neural networks contain real-valued neural networks and complex-valued neural networks as their special cases, due to the non-commutativity of quaternion multiplication, in general, methods for studying real-valued and complex-valued neural networks cannot be directly used to study quaternion-valued neural networks. In order to avoid the non-commutativity, there are two commonly feasible methods: one is to decompose a quaternion-valued neural network into four real-valued systems according to Hamilton's multiplication rules and the other is to decompose a quaternion-valued neural network into two complex-valued systems based on the complex decomposition properties of quaternions. For papers on the study of quaternion-valued neural networks using the method of decomposing a quaternion-valued neural network into four real-valued neural networks, see [14]–[16]; for papers on the study of quaternion-valued neural networks using the method of decomposing a quaternion-valued neural network into two complex-valued neural networks, see [17], [18]. However, to date, little research has been done on the dynamics of quaternion-valued neural networks by using non-decomposition methods [19].

In addition, as we know, periodicity, anti-periodicity, and almost periodicity are important dynamic properties of time-varying neural network systems, and since neural network signal transmission processes can often be described as anti-periodic processes, many authors have studied the anti-periodicity of various types of neural networks [20]–[27].

Motivated by the aforementioned discussions, in this paper, we are concerned with the following inertial quaternion-valued high-order Hopfield neural network with state-dependent delays:

$$\begin{aligned}\ddot{x}_p(t) = &-\alpha_p(t)\dot{x}_p(t) - \beta_p(t)x_p(t)\\ &+\sum_{q=1}^{n} a_{pq}(t)f_q(x_q(t-\tau_{pq}(t,x_q(t))))\\ &+\sum_{q=1}^{n}\sum_{l=1}^{n} b_{pql}(t)g_q(x_q(t-\sigma_{pql}(t,x_q(t))))\\ &\times g_l(x_l(t-\nu_{pql}(t,x_q(t)))) + I_p(t), \end{aligned} \tag{1}$$

where $p = 1, 2, \ldots, n$ and n is the number of units in the neural network, $x_p(t) \in \mathbb{H}$ corresponds to the state vector of the pth unit at time t; $\alpha_p(t) \geq 0$, $\beta_p(t) \geq 0$ are the coefficients of the first derivative term and the connected term, respectively; $a_{pq}(t), b_{pql}(t) \in \mathbb{H}$ are the first- and second-order connection weights of the neural network at time t, $\tau_{pq}(t, x_q(t)) \geq 0$, $\sigma_{pql}(t, x_q(t)) \geq 0$ and $\nu_{pql}(t, x_q(t)) \geq 0$ correspond to the transmission delays, $I_p(t) \in \mathbb{H}$ denotes the external input at time t, and $f_q, g_q : \mathbb{H} \to \mathbb{H}$ are the activation functions of signal transmission.

Remark 1: In all existing studies on inertial neural networks, the coefficients of both the first derivative term and the connected term are considered as constants. Considering in real electronic circuit systems, the parameters of the electronic components may vary with the environment. Therefore, it is more practical to assume that all the parameters in the network considered in this paper are time-varying ones.

Throughout this paper, we assume that

(H_1) For $p, q, l = 1, 2, \ldots, n, \alpha_p, \beta_p \in C(\mathbb{R}, \mathbb{R}^+)$, τ_{pq}, σ_{pql}, $\nu_{pql} \in BC(\mathbb{R}\times\mathbb{H}, \mathbb{R})$, $f_q, g_q \in C(\mathbb{H}, \mathbb{H})$, $a_{pq}, b_{pql}, I_p \in C(\mathbb{R}, \mathbb{H})$ and there exists a constant $\omega > 0$ such that

$$\begin{aligned}\alpha_p(t+\frac{\omega}{2}) &= \alpha_p(t),\ \ \beta_p(t+\frac{\omega}{2}) = \beta_p(t),\\ a_{pq}(t+\frac{\omega}{2})f_q(u) &= -a_{pq}(t)f_q(-u),\\ b_{pql}(t+\frac{\omega}{2})g_q(u)g_l(v) &= -b_{pql}(t)g_q(-u)g_l(-v),\\ \tau_{pq}(t+\frac{\omega}{2},u) &= \tau_{pq}(t,u) = \tau_{pq}(t,-u),\\ \sigma_{pql}(t+\frac{\omega}{2},u) &= \sigma_{pql}(t,u) = \sigma_{pql}(t,-u),\\ \nu_{pql}(t+\frac{\omega}{2},u) &= \nu_{pql}(t,u) = \nu_{pql}(t,-u),\\ I_p(t+\frac{\omega}{2}) &= -I_p(t),\ \ \forall t\in\mathbb{R},\ \ u,v\in\mathbb{H}.\end{aligned}$$

(H_2) For $j = 1, 2, \ldots, n$, there exist constants $L_q^f > 0$ and $L_q^g > 0$ such that for all $u, v \in \mathbb{H}$,

$$\begin{aligned}\|f_q(u)-f_q(v)\| &\leq L_q^f\|u-v\|,\\ \|g_q(u)-g_q(v)\| &\leq L_q^g\|u-v\|.\end{aligned}$$

(H_3) There exist positive constants M_q^f and M_q^g such that

$$\|f_q(u)\| \leq M_q^f,\quad \|g_q(u)\| \leq M_q^g,\quad \text{for all } u\in\mathbb{H},\quad q=1,2,\ldots,n.$$

We will adopt the following notation:

$$\begin{aligned}\alpha_p^- &= \inf_{t\in[0,\omega]}\alpha_p(t),\quad \beta_p^- = \inf_{t\in[0,\omega]}\beta_p(t),\\ a_{pq}^+ &= \sup_{t\in[0,\omega]}\|a_{pq}(t)\|,\quad b_{pql}^+ = \sup_{t\in[0,\omega]}\|b_{pql}(t)\|,\\ \tau^+ &= \max_{1\leq p,q\leq n}\Big\{\sup_{t\in[0,\omega],u\in\mathbb{H}}\tau_{pq}(t,u)\Big\},\\ \sigma^+ &= \max_{1\leq p,q,l\leq n}\Big\{\sup_{t\in[0,\omega],u\in\mathbb{H}}\sigma_{pql}(t,u)\Big\},\\ \nu^+ &= \max_{1\leq p,q,l\leq n}\Big\{\sup_{t\in[0,\omega],u\in\mathbb{H}}\nu_{pql}(t,u)\Big\},\\ \rho &= \max\{\tau^+,\sigma^+,\nu^+\},\quad I_p^+ = \sup_{t\in[0,\omega]}\|I_p(t)\|.\end{aligned}$$

The initial value of system (1) is given by

$$\begin{cases} x_p(s) = \varphi_p(s),\\ \dot{x}_p(s) = \psi_p(s), \quad s\in[-\rho,0],\ \ p=1,2,\ldots,n,\end{cases} \tag{2}$$

where $\varphi_p, \psi_p \in C([-\rho, 0], \mathbb{H})$.

The main purpose of this paper is to study the existence and global exponential stability of anti-periodic solutions of system (1). Without decomposing system (1) into real or complex value systems, by applying a continuation theorem of coincidence degree theory and the Wirtinger inequality, we establish the existence of anti-periodic solutions of system (1) and by constructing a suitable Lyapunov function, we derive a set of sufficient conditions guaranteeing the exponential stability of anti-periodic solutions of system (1). The results and methods of this paper are brand new.

Remark 2: Even when (1) degenerates into a real-valued system or a complex-valued system, the existence and stability of anti-periodic solutions of system (1) has not been studied.

The rest of this paper is organized as follows. In Section 2, we transform system (1) into a system of first-order differential equations and introduce some lemmas. In Section 3, we study the existence of anti-periodic solutions of system (1). In Section 4, we investigate the global exponential stability of anti-periodic solutions of system (1). In Section 5, we give a numerical example to illustrate the feasibility of the obtained results. We give a brief conclusion in Section 6.

II. PRELIMINARIES

Let $(x_1, x_2, \ldots, x_n)^T$ be a solution of system (1) with initial value (2). Set variable transformation:

$$y_p(t) = \dot{x}_p(t) + x_p(t), \quad p = 1, 2, \ldots, n, \tag{3}$$

then system (1) becomes

$$\begin{cases} \dot{x}_p(t) = -x_p(t) + y_p(t) \triangleq \Pi_p(x, y, t), \\ \dot{y}_p(t) = -(1 + \beta_p(t) - \alpha_p(t))x_p(t) - (\alpha_p(t) - 1)y_p(t) \\ + \sum_{q=1}^{n} a_{pq}(t)f_q(x_q(t - \tau_{pq}(t, x_q(t)))) \\ + \sum_{q=1}^{n}\sum_{l=1}^{n} b_{pql}(t)g_q(x_q(t - \sigma_{pql}(t, x_q(t)))) \\ \times g_l(x_l(t - \nu_{pql}(t, x_q(t)))) \\ + I_p(t) \triangleq \Gamma_p(x, y, t), \quad p = 1, 2, \ldots, n \end{cases} \tag{4}$$

with the initial value:

$$\begin{cases} x_p(s) = \varphi_p(s), \quad \dot{x}_p(s) = \psi_p(s), \\ y_p(s) = \varphi_p(s) + \psi_p(s) = \phi_p(s), \quad s \in [-\rho, 0], \end{cases}$$

where $p = 1, 2, \ldots, n$.

Remark 3: If $u = (x_1, \ldots, x_n, y_1, \ldots, y_n)^T$ is a solution of system (4), then $x = (x_1, \ldots, x_n)^T$ is a solution of (1). Conversely, if $x = (x_1, \ldots, x_n)^T$ is a solution of (1), then $u = (x_1, \ldots, x_n, y_1, \ldots, y_n)^T$ is a solution of system (4) via the transformation (3).

Lemma 1: [28] (Wirtinger Inequality) If u is a C^1 function such that $u(0) = u(T)$, then

$$\|u - \bar{u}\|_{L_2} \le \frac{T}{2\pi}\|u'\|_{L_2},$$

where $\|u\|_{L_2} := \left(\int_0^T |u(t)|^2 \mathrm{d}t\right)^{\frac{1}{2}}$ and $\bar{u} = \frac{1}{T}\int_0^T u(t)\mathrm{d}t$.

It is easy to draw the following conclusion through a direct calculation:

Lemma 2: For all $a, b \in \mathbb{H}$, $a^*b + b^*a \le a^*a + b^*b$.

Lemma 3: [28] Let X and Y be Banach spaces, and let $L : \operatorname{Dom} L \subset X \to Y$ be linear, $N : X \to Y$ be continuous. Assume that L is one-to-one and $K := L^{-1}N$ is compact. Furthermore, assume that there exists a bounded and open subset $\Omega \subset X$ with $0 \in \Omega$ such that equation $Lu = \lambda Nu$ has no solutions in $\partial\Omega \cap \operatorname{Dom} L$ for any $\lambda \in (0, 1)$. Then the problem $Lu = Nu$ has at least one solution in $\overline{\Omega}$.

Definition 1: Let $u = (x_1, \ldots, x_n, y_1, \ldots, y_n)^T$ and $\tilde{u} = (\tilde{x}_1, \ldots, \tilde{x}_n, \tilde{y}_1, \ldots, \tilde{y}_n)^T$ be two arbitrary solutions of system (4) with initial values $\Phi = (\varphi_1, \ldots, \varphi_n, \phi_1, \ldots, \phi_n)^T$ and $\Psi = (\tilde{\varphi}_1, \ldots, \tilde{\varphi}_n, \tilde{\phi}_1, \ldots, \tilde{\phi}_n)^T$, respectively. If there exist constants $\varepsilon > 0$ and $M > 0$ such that

$$\|u(t) - \tilde{u}(t)\|_{\mathbb{H}^{2n}} \le M\|\Phi - \Psi\|_0 e^{-\varepsilon t}, \quad t > 0,$$

where

$$\|u(t) - \tilde{u}(t)\|_{\mathbb{H}^{2n}} = \left[\sum_{p=1}^{n}\left(\|x_p(t) - \tilde{x}_p(t)\|^2 + \|y_p(t) - \tilde{y}_p(t))\|^2\right)\right]^{\frac{1}{2}},$$

$$\|\Phi - \Psi\|_0 = \left[\sum_{p=1}^{n}\left(\sup_{s\in[-\rho,0]}\|\varphi_p(s) - \tilde{\varphi}_p(s)\|^2 + \sup_{s\in[-\rho,0]}\|\phi_p(s) - \tilde{\phi}_p(s))\|^2\right)\right]^{\frac{1}{2}},$$

then every solution of system (4) is said to be globally exponentially stable.

III. THE EXISTENCE OF ANTI-PERIODIC SOLUTIONS

In this section, based on Lemma 3, we shall study the existence of anti-periodic solution of (1).

Let

$$X = \left\{u : u = (x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)^T \in C(\mathbb{R}, \mathbb{H}^{2n}), u\left(t + \frac{\omega}{2}\right) = -u(t), \forall t \in \mathbb{R}\right\},$$

$$\|u\|_X = \sum_{p=1}^{n}(|x_p|_0 + |y_p|_0),$$

where for $p = 1, 2, \ldots, n$,

$$|x_p|_0 = \sup_{t\in[0,\omega]}\sqrt{x_p(t)x_p^*(t)},$$

$$|y_p|_0 = \sup_{t\in[0,\omega]}\sqrt{y_p(t)y_p^*(t)},$$

then X is a Banach space with the norm $\|\cdot\|_X$.

Theorem 1: Assume that (H_1)-(H_3) hold. Furthermore, suppose that

(A_1) $2\pi - \omega > 0$ and $2\pi(2\pi - \omega) - C > 0$, where $C = (B_1 - B_2)\omega^2 + 2\pi\omega B_2$, $B_1 = \max_{t\in[0,\omega]}|1 + \beta_p(t) - \alpha_p(t)|$, $B_2 = \max_{t\in[0,\omega]}|\alpha_p(t) - 1|$, then system (1) has at least one $\frac{\omega}{2}$-anti-periodic solution, which lies in the region:

$$X_0 = \left\{x \in X : ||x||_X \le \sum_{p=1}^{n}\sqrt{\omega}\left(\frac{\omega A_p}{2\pi - \omega} + A_p\right) + 1\right\},$$

where

$$A_p = \sqrt{\omega}\left(1 + \frac{C}{2\pi(2\pi-\omega)-C}\right)\left(\sum_{q=1}^{n} a_{pq}^{+}M_q^f + \sum_{q=1}^{n}\sum_{l=1}^{n} b_{pql}^{+}(M_q^g)^2 + I_p^+\right),\ p = 1, 2, \ldots, n.$$

Proof: Define a linear operator $L : \mathrm{Dom}\, L \subset X \to X$ by $Lu = \dot{u}$, where $\mathrm{Dom}\, L = \{u : u \in X, \dot{u} \in X\}$ and a continuous operator $N : X \to X$ by

$$(Nu)(t) = \big(\Pi_1(x, y, t), \Pi_2(x, y, t), \ldots, \Pi_n(x, y, t), \Gamma_1(x, y, t), \Gamma_2(x, y, t), \ldots, \Gamma_n(x, y, t)\big)^T.$$

It is easy to see that $\mathrm{Ker}\, L = \{\mathbf{0}\}$ and $\mathrm{Im}\, L = \left\{y \in X, \int_0^{2T} y(t)dt = 0\right\} = X$. Hence, $L : \mathrm{Dom}\, L \to X$ is one-to-one. Denote by L^{-1} the inverse of L and take $K := L^{-1}N$, then by using Arzela-Ascoli theorem, we can verify that K is compact.

Let $u \in X$ be an arbitrary solution of $Lu = \lambda Nu$ for a certain $\lambda \in (0, 1)$, then we have

$$\begin{cases} \dot{x}_p(t) = \lambda \Pi_p(x, y, t), \\ \dot{y}_p(t) = \lambda \Gamma_p(x, y, t), \quad p = 1, 2, \ldots, n. \end{cases} \tag{5}$$

Multiplying by $\dot{x}_p^*(t)$ on both sides of the first equation of (5) and then integrating it over the interval $[0, \omega]$, for $p = 1, 2, \ldots, n$, we have

$$\begin{aligned}
\int_0^{\omega} \|\dot{x}_p(t)\|^2 dt
&= \lambda \int_0^{\omega} (-x_p(t) + y_p(t))\dot{x}_p^*(t) dt \\
&\le \int_0^{\omega} \|x_p(t)\dot{x}_p^*(t)\| dt + \int_0^{\omega} \|y_p(t)\dot{x}_p^*(t)\| dt \\
&\le \left(\int_0^{\omega} \|x_p(t)\|^2 dt\right)^{\frac{1}{2}} \left(\int_0^{\omega} \|\dot{x}_p(t)\|^2 dt\right)^{\frac{1}{2}} \\
&\quad + \left(\int_0^{\omega} \|y_p(t)\|^2 dt\right)^{\frac{1}{2}} \left(\int_0^{\omega} \|\dot{x}_p(t)\|^2 dt\right)^{\frac{1}{2}},
\end{aligned}$$

that is,

$$\left(\int_0^{\omega} \|\dot{x}_p(t)\|^2 dt\right)^{\frac{1}{2}} \le \left(\int_0^{\omega} \|x_p(t)\|^2 dt\right)^{\frac{1}{2}} + \left(\int_0^{\omega} \|y_p(t)\|^2 dt\right)^{\frac{1}{2}}. \tag{6}$$

Multiplying by $\dot{y}_p^*(t)$ on both sides of the second equation of (5) and then integrating it over the interval $[0, \omega]$, we have

$$\begin{aligned}
\int_0^{\omega} \|\dot{y}_p(t)\|^2 dt
&\le B_1 \int_0^{\omega} \|x_p(t)\dot{y}_p^*(t)\| dt + B_2 \int_0^{\omega} \|y_p(t)\dot{y}_p^*(t)\| dt \\
&\quad + \left(\sum_{q=1}^{n} a_{pq}^{+}M_q^f + \sum_{q=1}^{n}\sum_{l=1}^{n} b_{pql}^{+}(M_q^g)^2 + I_p^+\right) \times \int_0^{\omega} \|\dot{y}_p^*(t)\| dt \\
&\le B_1 \left(\int_0^{\omega} \|x_p(t)\|^2 dt\right)^{\frac{1}{2}} \left(\int_0^{\omega} \|\dot{y}_p(t)\|^2 dt\right)^{\frac{1}{2}} \\
&\quad + B_2 \left(\int_0^{\omega} \|y_p(t)\|^2 dt\right)^{\frac{1}{2}} \left(\int_0^{\omega} \|\dot{y}_p(t)\|^2 dt\right)^{\frac{1}{2}} \\
&\quad + \sqrt{\omega}\left(\sum_{q=1}^{n} a_{pq}^{+}M_q^f + \sum_{q=1}^{n}\sum_{l=1}^{n} b_{pql}^{+}(M_q^g)^2 + I_p^+\right) \\
&\quad \times \left(\int_0^{\omega} \|\dot{y}_p(t)\|^2 dt\right)^{\frac{1}{2}}, \quad p = 1, 2, \ldots, n,
\end{aligned}$$

that is, for $p = 1, 2, \ldots, n$,

$$\begin{aligned}
\left(\int_0^{\omega} \|\dot{y}_p(t)\|^2 dt\right)^{\frac{1}{2}}
&\le B_1 \left(\int_0^{\omega} \|x_p(t)\|^2 dt\right)^{\frac{1}{2}} + B_2 \left(\int_0^{\omega} \|y_p(t)\|^2 dt\right)^{\frac{1}{2}} \\
&\quad + \sqrt{\omega}\left(\sum_{q=1}^{n} a_{pq}^{+}M_q^f + \sum_{q=1}^{n}\sum_{l=1}^{n} b_{pql}^{+}(M_q^g)^2 + I_p^+\right).
\end{aligned} \tag{7}$$

Since x_p, y_p are $\frac{\omega}{2}$-anti-periodic and $x_p, y_p \in C^1, p = 1, 2, \ldots, n$, then by Lemma 1, we have

$$\left(\int_0^{\omega} \|x_p(t)\|^2 dt\right)^{\frac{1}{2}} \le \frac{\omega}{2\pi}\left(\int_0^{\omega} \|\dot{x}_p(t)\|^2 dt\right)^{\frac{1}{2}} \tag{8}$$

and

$$\left(\int_0^{\omega} \|y_p(t)\|^2 dt\right)^{\frac{1}{2}} \le \frac{\omega}{2\pi}\left(\int_0^{\omega} \|\dot{y}_p(t)\|^2 dt\right)^{\frac{1}{2}}. \tag{9}$$

From (6)-(9), for $p = 1, 2, \ldots, n$, we can get

$$\left(\int_0^{\omega} \|\dot{x}_p(t)\|^2 dt\right)^{\frac{1}{2}} \le \frac{\omega}{2\pi}\left(\int_0^{\omega} \|\dot{x}_p(t)\|^2 dt\right)^{\frac{1}{2}} + \frac{\omega}{2\pi}\left(\int_0^{\omega} \|\dot{y}_p(t)\|^2 dt\right)^{\frac{1}{2}}$$

and

$$\begin{aligned}
\left(\int_0^{\omega} \|\dot{y}_p(t)\|^2 dt\right)^{\frac{1}{2}}
&\le \frac{\omega B_1}{2\pi}\left(\int_0^{\omega} \|\dot{x}_p(t)\|^2 dt\right)^{\frac{1}{2}} + \frac{\omega B_2}{2\pi}\left(\int_0^{\omega} \|\dot{y}_p(t)\|^2 dt\right)^{\frac{1}{2}} \\
&\quad + \sqrt{\omega}\left(\sum_{q=1}^{n} a_{pq}^{+}M_q^f + \sum_{q=1}^{n}\sum_{l=1}^{n} b_{pql}^{+}(M_q^g)^2 + I_p^+\right).
\end{aligned}$$

Hence, for $p = 1, 2, \ldots, n$, we find

$$\left(\int_0^{\omega} \|\dot{x}_p(t)\|^2 dt\right)^{\frac{1}{2}} \le \frac{\omega}{2\pi-\omega}\left(\int_0^{\omega} \|\dot{y}_p(t)\|^2 dt\right)^{\frac{1}{2}}$$

and

$$\left(\int_0^{\omega} \|\dot{y}_p(t)\|^2 dt\right)^{\frac{1}{2}}$$

$$\leq \sqrt{\omega}\left(1+\frac{C}{2\pi(2\pi-\omega)-C}\right)\left(\sum_{q=1}^{n} a_{pq}^{+}M_q^f + \sum_{q=1}^{n}\sum_{l=1}^{n} b_{pql}^{+}(M_q^g)^2 + I_p^{+}\right) \triangleq A_p,$$

that is,

$$\left(\int_0^{\omega}\|\dot{x}_p(t)\|^2\mathrm{d}t\right)^{\frac{1}{2}} \leq \frac{\omega A_p}{2\pi-\omega},\ \ p=1,2,\ldots,n \qquad (10)$$

and

$$\left(\int_0^{\omega}\|\dot{y}_p(t)\|^2\mathrm{d}t\right)^{\frac{1}{2}} \leq A_p,\ \ p=1,2,\ldots,n. \qquad (11)$$

For $u=(x_1,x_2,\ldots,x_n,y_1,y_2,\ldots,y_n)\in X$, we can assume that $x_p(t)=x_p^R(t)+ix_p^I(t)+jx_p^J(t)+kx_p^K(t)$, $y_p(t)=y_p^R(t)+iy_p^I(t)+jy_p^J(t)+ky_p^K(t)$, $x_p^l(t+\frac{\omega}{2})=x_p^l(t)$, $y_p^l(t+\frac{\omega}{2})=y_p^l(t)$, where $x_p^l, y_p^l\in C(\mathbb{R},\mathbb{R})$, $l=R,I,J,K$, $p=1,2,\ldots,n$.

Since x_p^l, y_p^l are $\frac{\omega}{2}$-anti-periodic real-valued functions, there exist $\xi_p^l, \eta_p^l\in[0,\omega]$ such that

$$x_p^l(\xi_p^l)=y_p^l(\eta_p^l)=0,\quad l=R,I,J,K,\ \ p=1,2,\ldots,n.$$

Hence, we have

$$\begin{aligned}|x_p^l|_\infty :&= \sup_{t\in[0,\omega]}|x_p^l(t)|\\ &= \sup_{t\in[0,\omega]}\left|x_p^l(\xi_p^l)+\int_{\xi_p^l}^{t}\dot{x}_p^l(s)\mathrm{d}s\right|\\ &\leq \sqrt{\omega}\left(\int_0^{\omega}|\dot{x}_p^l(t)|^2\mathrm{d}t\right)^{\frac{1}{2}},\end{aligned}$$

$$|y_p^l|_\infty := \sup_{t\in[0,\omega]}|y_p^l(t)| \leq \sqrt{\omega}\left(\int_0^{\omega}|\dot{y}_p^l(t)|^2\mathrm{d}t\right)^{\frac{1}{2}},$$

that is, for $p=1,2,\ldots,n$,

$$|x_p^l|_\infty^2 \leq \omega\int_0^{\omega}|\dot{x}_p^l(t)|^2\mathrm{d}t,\ |y_p^l|_\infty^2 \leq \omega\int_0^{\omega}|\dot{y}_p^l(t)|^2\mathrm{d}t. \quad (12)$$

Therefore, it follows from (10)-(12) that

$$\begin{aligned}&\|u\|_X\\ &\leq \sum_{p=1}^{n}\left(\sqrt{|x_p^R|_\infty^2+|x_p^I|_\infty^2+|x_p^J|_\infty^2+|x_p^K|_\infty^2}\right.\\ &\quad\left.+\sqrt{|y_p^R|_\infty^2+|y_p^I|_\infty^2+|y_p^J|_\infty^2+|y_p^K|_\infty^2}\right)\\ &\leq \sum_{p=1}^{n}\sqrt{\omega}\left[\left(\int_0^{\omega}\|\dot{x}_p(t)\|^2\mathrm{d}t\right)^{\frac{1}{2}}+\left(\int_0^{\omega}\|\dot{y}_p(t)\|^2\mathrm{d}t\right)^{\frac{1}{2}}\right]\\ &\leq \sum_{p=1}^{n}\sqrt{\omega}\left(\frac{\omega A_p}{2\pi-\omega}+A_p\right)\triangleq D.\end{aligned}$$

Take $\Omega=\{u\in X:\|u\|_X<D+1\}$, then $\Omega\subset X$ with $0\in\Omega$ such that equation $Lu=\lambda Nu$ has no solutions in $\partial\Omega\cap\mathrm{Dom}\,L$ for any $\lambda\in(0,1)$. Thus, by Lemma 3, we obtain that system (4) has at least one $\frac{\omega}{2}$-anti-periodic solution in Ω. In view of Remark 3, we see that system (1) has at least one $\frac{\omega}{2}$-anti-periodic solution in X_0. The proof is complete. ■

IV. THE GLOBAL EXPONENTIAL STABILITY OF ANTI-PERIODIC SOLUTION

In this section, we construct Lyapunov functionals and use some inequality techniques to study the global exponential stability of anti-periodic solutions of (1).

Theorem 2: Let (H_1)-(H_3) and (A_1) hold. Furthermore, assume that

(A_2) $\tau_{pq}(t,u)\equiv\tau_{pq}(t)$, $\sigma_{pql}(t,u)\equiv\sigma_{pql}(t)$, $\nu_{pql}(t,u)\equiv\nu_{pql}(t)$, $\tau_{pq},\sigma_{pql},\nu_{pql}\in C^1(\mathbb{R},\mathbb{R}^+)$, and

$$\mu = \max_{1\leq p,q,l\leq n}\left\{\sup_{t\in\mathbb{R}}\dot{\tau}_{pq}(t),\ \sup_{t\in\mathbb{R}}\dot{\sigma}_{pql}(t),\ \sup_{t\in\mathbb{R}}\dot{\nu}_{pql}(t)\right\} < 1.$$

(A_3) There exists a positive constant $\varepsilon>0$ satisfying

$$\begin{aligned}\Lambda = \max_{1\leq p\leq n}\Bigg\{&2\varepsilon+4-\alpha_p^{-}-\beta_p^{-},\\ &\alpha_p^{+}-1-\beta_p^{-}+\sum_{q=1}^{n}(a_{qp}^{+})^2(L_q^f)^2\frac{e^{2\varepsilon\tau^{+}}}{1-\mu}\\ &+\sum_{q=1}^{n}\sum_{l=1}^{n}(b_{lqp}^{+})^2(M_q^g)^2(L_q^g)^2\frac{e^{2\varepsilon\nu^{+}}}{1-\mu}\\ &+\sum_{q=1}^{n}\sum_{l=1}^{n}(b_{qpl}^{+})^2(M_q^g)^2(L_q^g)^2\frac{e^{2\varepsilon\sigma^{+}}}{1-\mu}\Bigg\}<0.\end{aligned}$$

Then system (1) has a unique $\frac{\omega}{2}$-anti-periodic solution and it is globally exponentially stable.

Proof: By Theorem 1, we know that system (4) has an $\frac{\omega}{2}$-periodic solution $\tilde{u}(t)$ and suppose that its initial value is $\Psi(t)$. Let $u(t)$ be an arbitrary solution of system (4) with the initial value $\Phi(t)$. Set $v_p(t)=x_p(t)-\tilde{x}_p(t)$, $w_p(t)=y_p(t)-\tilde{y}_p(t)$, by (4), we have

$$\begin{cases}\dot{v}_p(t)=-v_p(t)+w_p(t),\\ \dot{w}_p(t)=-(1+\beta_p(t)-\alpha_p(t))v_p(t)-(\alpha_p(t)-1)w_p(t)\\ +\sum\limits_{q=1}^{n}a_{pq}(t)\tilde{f}_q(v_q(t-\tau_{pq}(t)))+\sum\limits_{q=1}^{n}\sum\limits_{l=1}^{n}b_{pql}(t)\\ \times\big[g_q(x_q(t-\sigma_{pql}(t)))\tilde{g}_l(v_l(t-\nu_{pql}(t)))\\ +g_l(\tilde{x}_l(t-\nu_{pql}(t)))\tilde{g}_q(v_q(t-\sigma_{pql}(t)))\big],\end{cases} \quad (13)$$

where $p=1,2,\ldots,n$,

$$\begin{aligned}\tilde{f}_q(v_q(t-\tau_{pq}(t)))&=f_q(x_q(t-\tau_{pq}(t)))-f_q(\tilde{x}_q(t-\tau_{pq}(t))),\\ \tilde{g}_l(v_l(t-\nu_{pql}(t)))&=g_l(x_l(t-\nu_{pql}(t)))-g_l(\tilde{x}_l(t-\nu_{pql}(t))),\\ \tilde{g}_q(v_q(t-\sigma_{pql}(t)))&=g_q(x_q(t-\sigma_{pql}(t)))-g_q(\tilde{x}_q(t-\sigma_{pql}(t))).\end{aligned}$$

Define a Lyapunov functional as follows:

$$V(t)=\sum_{p=1}^{n}e^{2\varepsilon t}v_p^*(t)v_p(t)+\sum_{p=1}^{n}e^{2\varepsilon t}w_p^*(t)w_p(t)$$

$$
\begin{aligned}
&+\sum_{p=1}^{n}\sum_{q=1}^{n}(a_{pq}^{+})^{2}(L_{q}^{f})^{2}\frac{e^{2\varepsilon\tau^{+}}}{1-\mu}\\
&\times\int_{t-\tau_{pq}(t)}^{t}e^{2\varepsilon s}v_{q}^{*}(s)v_{q}(s)\mathrm{d}s\\
&+\sum_{p=1}^{n}\sum_{q=1}^{n}\sum_{l=1}^{n}(b_{pql}^{+})^{2}(M_{q}^{g})^{2}(L_{q}^{g})^{2}\frac{e^{2\varepsilon\nu^{+}}}{1-\mu}\\
&\times\int_{t-\nu_{pql}(t)}^{t}e^{2\varepsilon s}v_{l}^{*}(s)v_{l}(s)\mathrm{d}s\\
&+\sum_{p=1}^{n}\sum_{q=1}^{n}\sum_{l=1}^{n}(b_{pql}^{+})^{2}(M_{q}^{g})^{2}(L_{q}^{g})^{2}\frac{e^{2\varepsilon\sigma^{+}}}{1-\mu}\\
&\times\int_{t-\sigma_{pql}(t)}^{t}e^{2\varepsilon s}v_{q}^{*}(s)v_{q}(s)\mathrm{d}s.
\end{aligned}
$$

By (13) and Lemma 2, we can obtain

$$
\begin{aligned}
&D^{+}\left(\sum_{p=1}^{n}e^{2\varepsilon t}v_{p}^{*}(t)v_{p}(t)\right)\\
&=\sum_{p=1}^{n}2\varepsilon e^{2\varepsilon t}v_{p}^{*}(t)v_{p}(t)+\sum_{p=1}^{n}e^{2\varepsilon t}(-v_{p}^{*}(t)+w_{p}^{*}(t))v_{p}(t)\\
&\quad+\sum_{p=1}^{n}e^{2\varepsilon t}v_{p}^{*}(t)(-v_{p}(t)+w_{p}(t))\\
&=(2\varepsilon-2)\sum_{p=1}^{n}e^{2\varepsilon t}v_{p}^{*}(t)v_{p}(t)\\
&\quad+\sum_{p=1}^{n}e^{2\varepsilon t}(v_{p}^{*}(t)w_{p}(t)+w_{p}^{*}(t)v_{p}(t))\\
&\le(2\varepsilon-1)\sum_{p=1}^{n}e^{2\varepsilon t}v_{p}^{*}(t)v_{p}(t)+\sum_{p=1}^{n}e^{2\varepsilon t}w_{p}^{*}(t)w_{p}(t)\qquad(14)
\end{aligned}
$$

and

$$
\begin{aligned}
&D^{+}\left(\sum_{p=1}^{n}e^{2\varepsilon t}w_{p}^{*}(t)w_{p}(t)\right)\\
&=\sum_{p=1}^{n}e^{2\varepsilon t}\Big\{(2\varepsilon+2-2\alpha_{p}(t))w_{p}^{*}(t)w_{p}(t)\\
&\quad-(1+\beta_{p}(t)-\alpha_{p}(t))\big(v_{p}^{*}(t)w_{p}(t)+w_{p}^{*}(t)v_{p}(t)\big)\\
&\quad+\sum_{q=1}^{n}\Big[\big(a_{pq}(t)\tilde{f}_{q}(v_{q}(t-\tau_{pq}(t)))\big)^{*}w_{p}(t)\\
&\quad+w_{p}^{*}(t)\big(a_{pq}(t)\tilde{f}_{q}(v_{q}(t-\tau_{pq}(t)))\big)\Big]\\
&\quad+\sum_{q=1}^{n}\sum_{l=1}^{n}\Big[\big(b_{pql}(t)g_{q}(x_{q}(t-\sigma_{pql}(t)))\\
&\quad\times\tilde{g}_{l}(v_{l}(t-\nu_{pql}(t)))\big)^{*}w_{p}(t)\\
&\quad+w_{p}^{*}(t)\big(b_{pql}(t)g_{q}(x_{q}(t-\sigma_{pql}(t)))\tilde{g}_{l}(v_{l}(t-\nu_{pql}(t)))\big)\Big]\\
&\quad+\sum_{q=1}^{n}\sum_{l=1}^{n}\Big[\big(b_{pql}(t)g_{l}(\tilde{x}_{l}(t-\nu_{pql}(t)))\\
&\quad\times\tilde{g}_{q}(v_{q}(t-\sigma_{pql}(t)))\big)^{*}w_{p}(t)\\
&\quad+w_{p}^{*}(t)\big(b_{pql}(t)g_{l}(\tilde{x}_{l}(t-\nu_{pql}(t)))\\
&\quad\times\tilde{g}_{q}(v_{q}(t-\sigma_{pql}(t)))\big)\Big]\Big\}\\
&\le\sum_{p=1}^{n}e^{2\varepsilon t}\Big\{(2\varepsilon+2-2\alpha_{p}(t))w_{p}^{*}(t)w_{p}(t)\\
&\quad-(1+\beta_{p}(t)-\alpha_{p}(t))\big(v_{p}^{*}(t)v_{p}(t)+w_{p}^{*}(t)w_{p}(t)\big)\\
&\quad+\sum_{q=1}^{n}\big(a_{pq}(t)\tilde{f}_{q}(v_{q}(t-\tau_{pq}(t)))\big)^{*}\\
&\quad\times\big(a_{pq}(t)\tilde{f}_{q}(v_{q}(t-\tau_{pq}(t)))\big)+w_{p}^{*}(t)w_{p}(t)\\
&\quad+\sum_{q=1}^{n}\sum_{l=1}^{n}\big(b_{pql}(t)g_{q}(x_{q}(t-\sigma_{pql}(t)))\\
&\quad\times\tilde{g}_{l}(v_{l}(t-\nu_{pql}(t)))\big)^{*}\\
&\quad\times\big(b_{pql}(t)g_{q}(x_{q}(t-\sigma_{pql}(t)))\tilde{g}_{l}(v_{l}(t-\nu_{pql}(t)))\big)\\
&\quad+w_{p}^{*}(t)w_{p}(t)+\sum_{q=1}^{n}\sum_{l=1}^{n}\big(b_{pql}(t)g_{l}(\tilde{x}_{l}(t-\nu_{pql}(t)))\\
&\quad\times\tilde{g}_{q}(v_{q}(t-\sigma_{pql}(t)))\big)^{*}\\
&\quad\times\big(b_{pql}(t)g_{l}(\tilde{x}_{l}(t-\nu_{pql}(t)))\tilde{g}_{q}(v_{q}(t-\sigma_{pql}(t)))\big)\\
&\quad+w_{p}^{*}(t)w_{p}(t)\Big\}\\
&\le\sum_{p=1}^{n}e^{2\varepsilon t}\Big\{(2\varepsilon+4-\alpha_{p}(t)-\beta_{p}(t))w_{p}^{*}(t)w_{p}(t)\\
&\quad-(1+\beta_{p}(t)-\alpha_{p}(t))v_{p}^{*}(t)v_{p}(t)\\
&\quad+\sum_{q=1}^{n}(a_{pq}^{+})^{2}(L_{q}^{f})^{2}v_{q}^{*}(t-\tau_{pq}(t))v_{q}(t-\tau_{pq}(t))\\
&\quad+\sum_{q=1}^{n}\sum_{l=1}^{n}(b_{pql}^{+})^{2}(M_{q}^{g})^{2}(L_{q}^{g})^{2}\\
&\quad\times v_{l}^{*}(t-\nu_{pql}(t))v_{l}(t-\nu_{pql}(t))\\
&\quad+\sum_{q=1}^{n}\sum_{l=1}^{n}(b_{pql}^{+})^{2}(M_{q}^{g})^{2}(L_{q}^{g})^{2}\\
&\quad\times v_{q}^{*}(t-\sigma_{pql}(t))v_{q}(t-\sigma_{pql}(t))\Big\}.\qquad(15)
\end{aligned}
$$

Further, based on (14) and (15), we have

$$
\begin{aligned}
&D^{+}V(t)\\
&\le\sum_{p=1}^{n}e^{2\varepsilon t}\Big\{(2\varepsilon+4-\alpha_{p}(t)-\beta_{p}(t))w_{p}^{*}(t)w_{p}(t)\\
&\quad-(1+\beta_{p}(t)-\alpha_{p}(t))v_{p}^{*}(t)v_{p}(t)\\
&\quad+\sum_{q=1}^{n}(a_{pq}^{+})^{2}(L_{q}^{f})^{2}v_{q}^{*}(t-\tau_{pq}(t))v_{q}(t-\tau_{pq}(t))\\
&\quad+\sum_{q=1}^{n}\sum_{l=1}^{n}(b_{pql}^{+})^{2}(M_{q}^{g})^{2}(L_{q}^{g})^{2}
\end{aligned}
$$

$$\times v_l^*(t-\nu_{pql}(t))v_l(t-\nu_{pql}(t))$$
$$+\sum_{q=1}^n\sum_{l=1}^n(b_{pql}^+)^2(M_q^g)^2(L_q^g)^2$$
$$\times v_q^*(t-\sigma_{pql}(t))v_q(t-\sigma_{pql}(t))$$
$$+\sum_{q=1}^n(a_{pq}^+)^2(L_q^f)^2\frac{e^{2\varepsilon\tau^+}}{1-\mu}v_q^*(t)v_q(t)$$
$$-\sum_{q=1}^n(a_{pq}^+)^2(L_q^f)^2\frac{e^{2\varepsilon\tau^+}}{1-\mu}(1-\dot{\tau}_{pq}(t))e^{-2\varepsilon\tau_{pq}(t)}$$
$$\times v_q^*(t-\tau_{pq}(t))v_q(t-\tau_{pq}(t))$$
$$+\sum_{q=1}^n\sum_{l=1}^n(b_{pql}^+)^2(M_q^g)^2(L_q^g)^2\frac{e^{2\varepsilon\nu^+}}{1-\mu}v_l^*(t)v_l(t)$$
$$-\sum_{q=1}^n\sum_{l=1}^n(b_{pql}^+)^2(M_q^g)^2(L_q^g)^2\frac{e^{2\varepsilon\nu^+}}{1-\mu}(1-\dot{\nu}_{pql}(t))$$
$$\times e^{-2\varepsilon\nu_{pql}(t)}v_l^*(t-\nu_{pql}(t))v_l(t-\nu_{pql}(t))$$
$$+\sum_{q=1}^n\sum_{l=1}^n(b_{pql}^+)^2(M_q^g)^2(L_q^g)^2\frac{e^{2\varepsilon\sigma^+}}{1-\mu}v_q^*(t)v_q(t)$$
$$-\sum_{q=1}^n\sum_{l=1}^n(b_{pql}^+)^2(M_q^g)^2(L_q^g)^2$$
$$\times\frac{e^{2\varepsilon\sigma^+}}{1-\mu}(1-\dot{\sigma}_{pql}(t))e^{-2\varepsilon\sigma_{pql}(t)}$$
$$\times v_q^*(t-\sigma_{pql}(t))v_q(t-\sigma_{pql}(t))\Bigg\}$$
$$\le\sum_{p=1}^n e^{2\varepsilon t}\Bigg\{(2\varepsilon+4-\alpha_p^--\beta_p^-)w_p^*(t)w_p(t)$$
$$-(1+\beta_p^--\alpha_p^+)v_p^*(t)v_p(t)$$
$$+\sum_{q=1}^n(a_{pq}^+)^2(L_q^f)^2\frac{e^{2\varepsilon\tau^+}}{1-\mu}v_q^*(t)v_q(t)$$
$$+\sum_{q=1}^n\sum_{l=1}^n(b_{pql}^+)^2(M_q^g)^2(L_q^g)^2\frac{e^{2\varepsilon\nu^+}}{1-\mu}v_l^*(t)v_l(t)$$
$$+\sum_{q=1}^n\sum_{l=1}^n(b_{pql}^+)^2(M_q^g)^2(L_q^g)^2\frac{e^{2\varepsilon\sigma^+}}{1-\mu}v_q^*(t)v_q(t)\Bigg\}$$
$$=\sum_{p=1}^n e^{2\varepsilon t}\Bigg\{(2\varepsilon+4-\alpha_p^--\beta_p^-)w_p^*(t)w_p(t)$$
$$-(1+\beta_p^--\alpha_p^+)v_p^*(t)v_p(t)$$
$$+\sum_{q=1}^n(a_{qp}^+)^2(L_q^f)^2\frac{e^{2\varepsilon\tau^+}}{1-\mu}v_p^*(t)v_p(t)$$
$$+\sum_{q=1}^n\sum_{l=1}^n(b_{lqp}^+)^2(M_q^g)^2(L_q^g)^2\frac{e^{2\varepsilon\nu^+}}{1-\mu}v_p^*(t)v_p(t)$$
$$+\sum_{q=1}^n\sum_{l=1}^n(b_{qpl}^+)^2(M_q^g)^2(L_q^g)^2\frac{e^{2\varepsilon\sigma^+}}{1-\mu}v_p^*(t)v_p(t)\Bigg\}$$
$$=\sum_{p=1}^n e^{2\varepsilon t}\Bigg\{(2\varepsilon+4-\alpha_p^--\beta_p^-)w_p^*(t)w_p(t)$$
$$+\Bigg(\alpha_p^+-1-\beta_p^-+\sum_{q=1}^n(a_{qp}^+)^2(L_q^f)^2\frac{e^{2\varepsilon\tau^+}}{1-\mu}$$
$$+\sum_{q=1}^n\sum_{l=1}^n(b_{lqp}^+)^2(M_q^g)^2(L_q^g)^2\frac{e^{2\varepsilon\nu^+}}{1-\mu}$$
$$+\sum_{q=1}^n\sum_{l=1}^n(b_{qpl}^+)^2(M_q^g)^2(L_q^g)^2\frac{e^{2\varepsilon\sigma^+}}{1-\mu}\Bigg)v_p^*(t)v_p(t)\Bigg\}$$
$$\le\Lambda e^{2\varepsilon t}\sum_{p=1}^n(w_p^*(t)w_p(t)+v_p^*(t)v_p(t))\le 0,$$

which implies that $V(t)\le V(0)$ for $t\ge 0$. From the definition of $V(t)$, we have

$$V(t)\ge e^{2\varepsilon t}\sum_{p=1}^n(v_p^*(t)v_p(t)+w_p^*(t)w_p(t))$$
$$=(e^{\varepsilon t}\|u(t)-\tilde{u}(t)\|_{\mathbb{H}^{2n}})^2$$

and

$$V(0)\le\sum_{p=1}^n\Bigg(\sup_{s\in[-\rho,0]}\|v_p(s)\|^2+\sup_{s\in[-\rho,0]}\|w_p(s)\|^2\Bigg)$$
$$+\sum_{p=1}^n\sum_{q=1}^n(a_{qp}^+)^2(L_q^f)^2\frac{e^{2\varepsilon\tau^+}}{1-\mu}\tau^+\sup_{s\in[-\rho,0]}\|v_p(s)\|^2$$
$$+\sum_{p=1}^n\sum_{q=1}^n\sum_{l=1}^n(b_{lqp}^+)^2(M_q^g)^2(L_q^g)^2$$
$$\times\frac{e^{2\varepsilon\nu^+}}{1-\mu}\nu^+\sup_{s\in[-\rho,0]}\|v_p(s)\|^2$$
$$+\sum_{p=1}^n\sum_{q=1}^n\sum_{l=1}^n(b_{qpl}^+)^2(M_q^g)^2(L_q^g)^2$$
$$\times\frac{e^{2\varepsilon\sigma^+}}{1-\mu}\sigma^+\sup_{s\in[-\rho,0]}\|v_p(s)\|^2$$
$$\le\sum_{p=1}^n\Bigg\{\Bigg(1+\sum_{q=1}^n(a_{qp}^+)^2(L_q^f)^2\frac{e^{2\varepsilon\tau^+}}{1-\mu}\tau^+$$
$$+\sum_{q=1}^n\sum_{l=1}^n(b_{lqp}^+)^2(M_q^g)^2(L_q^g)^2\frac{e^{2\varepsilon\nu^+}}{1-\mu}\nu^+$$
$$+\sum_{q=1}^n\sum_{l=1}^n(b_{qpl}^+)^2(M_q^g)^2(L_q^g)^2\frac{e^{2\varepsilon\sigma^+}}{1-\mu}\sigma^+\Bigg)$$
$$\times\sup_{s\in[-\rho,0]}\|\varphi_p(s)-\tilde{\varphi}_p(s)\|^2$$
$$+\sup_{s\in[-\rho,0]}\|\phi_p(s)-\tilde{\phi}_p(s)\|^2\Bigg\}$$
$$\le\Theta\|\Phi-\Psi\|_0^2,$$

where

$$\Theta = \max_{1\le p\le n}\Bigg\{1+\sum_{q=1}^{n}(a_{qp}^{+})^2(L_q^f)^2\frac{e^{2\varepsilon\tau^+}}{1-\mu}\tau^+ + \sum_{q=1}^{n}\sum_{l=1}^{n}(b_{lqp}^{+})^2(M_q^g)^2(L_q^g)^2\frac{e^{2\varepsilon\nu^+}}{1-\mu}\nu^+ + \sum_{q=1}^{n}\sum_{l=1}^{n}(b_{qpl}^{+})^2(M_q^g)^2(L_q^g)^2\frac{e^{2\varepsilon\sigma^+}}{1-\mu}\sigma^+\Bigg\} > 1.$$

Hence, we can obtain

$$(e^{\varepsilon t}\|u(t)-\tilde{u}(t)\|_{\mathbb{H}^{2n}})^2 \le \Theta\|\Phi-\Psi\|_0^2,$$

that is,

$$\|u(t)-\tilde{u}(t)\|_{\mathbb{H}^{2n}} \le M\|\Phi-\Psi\|_0 e^{-\varepsilon t}, \quad t>0,$$

where $M=\sqrt{\Theta}$.

Therefore, the $\frac{\omega}{2}$-anti-periodic solution of system (4) is globally exponentially stable. In view of Remark 3, the $\frac{\omega}{2}$-anti-periodic solution of system (1) is also globally exponentially stable. The proof is complete. ■

V. NUMERICAL EXAMPLE

In this section, we give an example to show the feasibility and effectiveness of the results obtained in this paper.

Example 1: In system (1), let $n=2$ and take the coefficients as follows: for $p, q, l = 1, 2$,

$$\alpha_p(t) = \cos 12t + 2, \quad \beta_p(t) = \cos 12t + 5,$$
$$\tau_{pq}(t,u) = \frac{1}{10}|\sin t|, \quad \sigma_{pql}(t,u) = \frac{1}{20}|\sin t|,$$
$$\nu_{pql}(t,u) = \frac{1}{20}|\cos t|,$$
$$a_{11}(t) = a_{21}(t) = 0.3e^{-0.01} - i0.4e^{-0.01}\sin 12t + j0.5e^{-0.01}\cos 12t + k0.63e^{-0.01}\cos 12t,$$
$$a_{12}(t) = a_{22}(t) = 0.225e^{-0.01} - i0.05e^{-0.01}\cos 12t + j0.31e^{-0.01}\sin 12t + k0.225e^{-0.01}\cos 12t,$$
$$b_{pq1}(t) = 0.5e^{-0.005}\sin 6t + ie^{-0.005}\cos 6t + j1.32e^{-0.005}\sin 6t - ke^{-0.005}\cos 6t,$$
$$b_{pq2}(t) = 1.2e^{-0.005}\cos 6t - i0.5e^{-0.005}\sin 6t - je^{-0.005}\cos 6t - ke^{-0.005}\sin 6t,$$
$$I_p(t) = 0.6\sin 6t + i\cos 6t + j\sin 6t + k2\cos 6t,$$

and for any $x\in\mathbb{H}$, $x = x^R + ix^I + jx^J + kx^K$, take the activation functions as

$$f_1(x) = f_2(x) = \frac{1}{2}\sin x^R + i\frac{1}{2}\sin x^I + j\frac{1}{2}\sin x^J + k\frac{1}{2}\sin x^K,$$
$$g_1(x) = g_2(x) = \frac{1}{3}|\sin x^R| + i\frac{1}{4}|\sin x^I| + j\frac{1}{5}|\sin x^J| + k\frac{1}{6}|\sin x^K|.$$

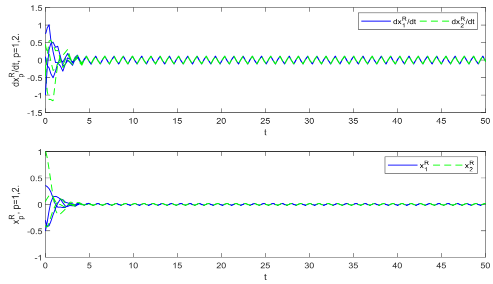

FIGURE 1. The states of $dx_p^R(t)/dt$ and $x_p^R(t)$, $p = 1, 2$.

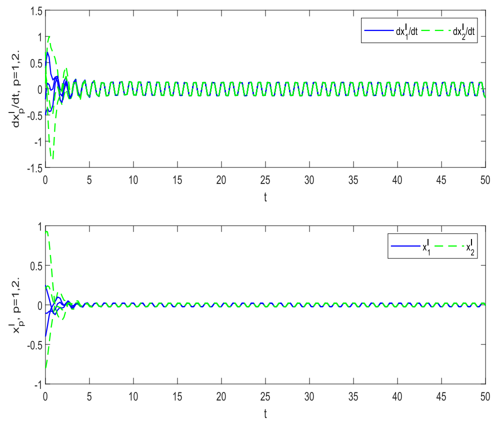

FIGURE 2. The states of $dx_p^I(t)/dt$ and $x_p^I(t)$, $p = 1, 2$.

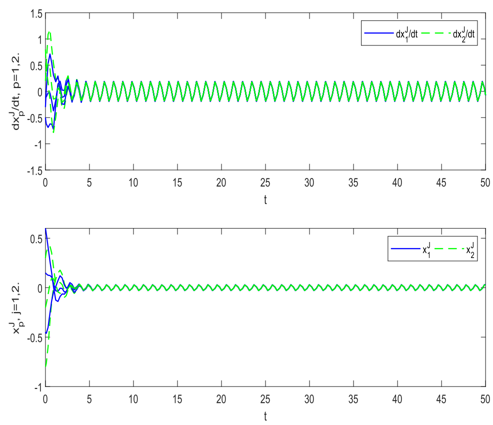

FIGURE 3. The states of $dx_p^J(t)/dt$ and $x_p^J(t)$, $p = 1, 2$.

It is easy to get that

$$L_1^f = L_2^f = \frac{1}{2}, \quad L_1^g = L_2^g = \frac{1}{3}, \quad M_1^f = M_2^f = 1,$$
$$M_1^g = M_2^g = \frac{1}{2}, \quad \omega = \frac{\pi}{3}, \ B_1 = 4, \ B_2 = 2, \ C = 2\pi^2,$$
$$\alpha_p^+ = 3, \quad \alpha_p^- = 1, \ \beta_p^+ = 6, \ \beta_p^- = 4,$$
$$\tau^+ = \frac{1}{10}, \quad \sigma^+ = \nu^+ = \frac{1}{20}, \quad \mu = \frac{1}{10},$$

then

$$2\pi(2\pi-\omega) - C \approx 13.16 > 0.$$

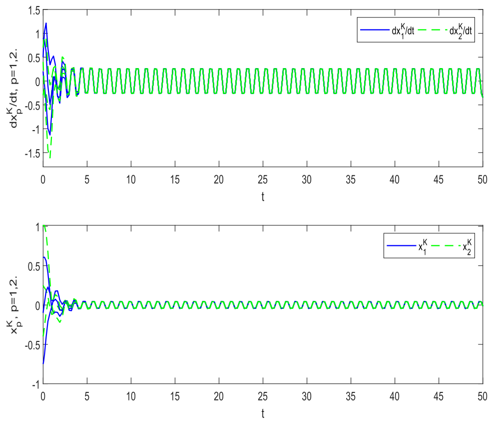

FIGURE 4. The states of $dx_p^K(t)/dt$ and $x_p^K(t)$, $p = 1, 2$.

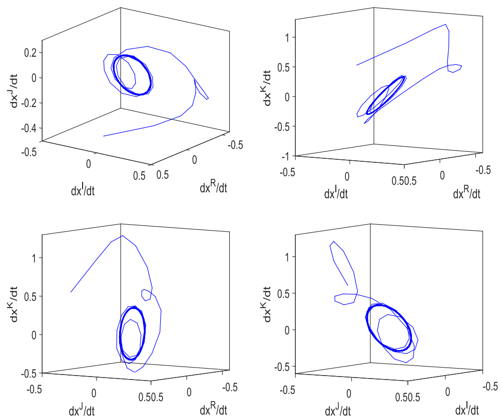

FIGURE 5. Curves of dx^R/dt, dx^I/dt, dx^J/dt and dx^K/dt in 3-dimensional space for stable case.

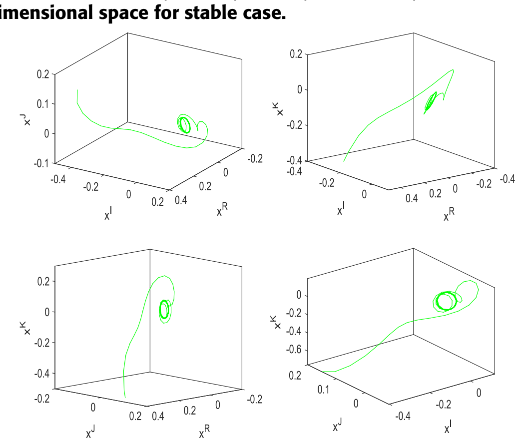

FIGURE 6. Curves of x^R, x^I, x^J and x^K in 3-dimensional space for stable case.

Taking $\lambda = 0.1$, we have

$$\Lambda = \max_{1\le p\le 2}\Big\{2\varepsilon + 4 - \alpha_p^- - \beta_p^-, \alpha_p^+ - 1 - \beta_p^- + \sum_{q=1}^{2}(a_{qp}^+)^2(L_q^f)^2\frac{e^{2\varepsilon\tau^+}}{1-\mu} + \sum_{q=1}^{2}\sum_{l=1}^{2}(b_{lqp}^+)^2(M_q^g)^2(L_q^g)^2\frac{e^{2\varepsilon\nu^+}}{1-\mu} + \sum_{q=1}^{2}\sum_{l=1}^{2}(b_{qpl}^+)^2(M_q^g)^2(L_q^g)^2\frac{e^{2\varepsilon\sigma^+}}{1-\mu}\Big\} \approx -0.53 < 0.$$

Thus conditions (H_1)-(H_3) and (A_1)-(A_3) hold. Therefore, according to Theorem 2, system (1) has at least one $\frac{\pi}{6}$-anti-periodic solution, which is globally exponentially stable (see Figures 1-6).

VI. CONCLUSION

In this paper, the existence and global exponential stability of anti-periodic solutions of inertial quaternion-valued high-order Hopfield neural networks with delays is investigated via direct methods. Our results of this paper are new even when the considered neural networks degenerate into real-valued or complex-valued ones. In addition, our method of this paper is new and can be applied to study other types of the quaternion-valued neural networks with or without inertial terms.

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