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Novel Generalized Distance Measure of Pythagorean Fuzzy Sets and a Compromise Approach for Multiple Criteria Decision Analysis Under Uncertainty

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ABSTRACT This paper aims at developing a novel generalized distance measure of Pythagorean fuzzy (PF) sets and constructing a distance-based compromise approach for multiple criteria decision analysis (MCDA) within PF environments. The theory of Pythagorean fuzziness provides a representative model of non-standard fuzzy sets; it is valuable for representing complex vague or imprecise information in many practical applications. The distance measure for Pythagorean membership grades is important because it can effectively quantify the separation between PF information. Based on the essential characteristics of PF sets (membership, non-membership, strength, and direction), this paper proposes several distance measures, namely, new Hamming and Euclidean distances and a generalized distance measure that is based on them for Pythagorean membership grades and for PF sets. Moreover, the useful and desirable properties of the proposed PF distance measures are investigated to evaluate their advantages and form a solid theoretical basis. In addition, to evaluate the performance of the proposed distance measures in practice, this paper establishes a PF-distance-based compromise approach for addressing MCDA problems that involve PF information. The effectiveness and practicability of the developed approach are further evaluated through a case study on bridge-superstructure construction methods. According to the application results and comparative analysis, the proposed PF distance measures are accurate and outperform other methods in handling the inherent uncertainties of evaluation information. Furthermore, the PF-distance-based compromise approach can accommodate the much higher degrees of uncertainty in real-life decision scenarios and effectively determine the priority ranking among candidate alternatives for managing complicated MCDA problems.

INDEX TERMS Generalized distance measure, Pythagorean fuzzy set, distance-based compromise approach, multiple criteria decision analysis, Pythagorean membership grade.

I. INTRODUCTION

Multiple criteria decision analysis (MCDA) involves ranking the priority orders of alternatives and selecting the optimal compromise solution among a finite set of candidate alternatives based on a finite set of evaluative criteria [1]–[3]. The foundation for the compromise approach is the establishment of an agreement via mutual concessions [2]. Decision makers are assumed to prefer alternatives that are closer to the positive-ideal solution and farther from the negative-ideal solution. Accordingly, the compromise model attempts to identify an alternative that is closest to the positive-ideal solution and farthest from the negative-ideal solution. The core concepts in classical compromising models are distance measures and/or similarity measures [4], [5]. Among the numerous available methods for conflict management and decision analysis, the most prevalent compromise approach is the technique for order preference by similarity to ideal

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solutions (TOPSIS) [1]. The compromise methodology constitutes an important branch of MCDA methods and has been applied to many real-life problems [3], [6], [7].

However, exact evaluation information is often unrealistic and inadequate for modeling practical decision scenarios because decision makers' assessments and estimations of the performances of competing alternatives are subject to inherent vagueness and imprecision [4], [8], [9]. Moreover, MCDA models and methods that are based on ordinary fuzzy sets might be insufficient for modeling practical scenarios because of the increased complexity of the decision-making environment [5]. Pythagorean fuzzy (PF) sets, which were introduced by Yager [10] and later extended by Yager [11], [12] and Yager and Abbasov [13], are useful for expressing the incomplete, inexact, and/or ambiguous information that is contained in human subjective evaluations and judgments. Since Zhang and Xu [14] provided operationally mathematical representations of PF sets, the theory of Pythagorean fuzziness has become increasingly popular in the field of MCDA [15], [16].

Yager [10]–[12] and Yager and Abbasov [13] introduced a new class of nonstandard membership grades: Pythagorean membership grades. Pythagorean membership grades are characterized by the membership degree, the non-membership degree, the indeterminacy degree, the strength of commitment, and the direction of commitment [11], [13], [17], [18]. PF sets with Pythagorean membership grades satisfy the relaxed condition that the square sum of the membership degree and the non-membership degree is equal to or less than one [10]–[13], [19], [20]. This relaxed condition provides an obvious advantage to PF sets, namely, a wider coverage of the information span [19]. Due to various information insufficiency issues in practical decision situations, PF sets have become an important tool because they can effectively model higher-order fuzziness and uncertainty in MCDA problems [16], [19], [21]–[24].

Many studies on compromise models and methods have been developed for solving MCDA problems within PF uncertain environments, such as a PF TOPSIS method that is based on the revised closeness [14]; a hybrid TOPSIS method that uses the PF ordered weighted averaging weighted average distance operator [25]; an extension of the TOPSIS model with hesitant PF sets (combinations of PF sets and hesitant fuzzy sets) [26]; a PF VIKOR (VlseKriterijumska Optimizacija I Kompromisno Resenje, which translates to multicriteria optimization and compromise solution) method [27]; a correlation-based compromise approach that is based on information energy, correlations, and correlation coefficients for PF characteristics [28]; an improved risk assessment approach that uses linguistic terms, TOPSIS, and PF sets [29]; a three-way method that uses ideal TOPSIS solutions as PF information [30]; a three-phase method that is based on the extended TOPSIS via normalized projections of PF values [6]; a TOPSIS-based Pythagorean normal cloud approach for group decision making [31]; a Pearson-like correlation-based PF compromise approach [21]; a PF TOPSIS method that uses new PF correlation-based closeness indices [3]; and a group decision-making sustainable supplier selection approach that uses an extended TOPSIS approach that is based on interval-valued PF sets [7].

From both the theoretical and practical perspectives, the compromise methodology determines the optimal compromise solution by explicitly evaluating the candidate alternatives over multiple conflicting criteria in practice. Central to most of the compromise approaches is the concept of distance measures. A distance measure measures the separation between uncertain information [9], [32]. Moreover, distance measures are important concepts when dealing with applications of mathematical theories [33], [34]. In the PF context, decision makers require a proper model for representing uncertain information; moreover, they require an adequate measure for processing such information. PF distance measures are important tools for identifying the separation between PF data and for comparing complex PF information. Several useful distance measures that are based on PF sets have been introduced by Chen [27], Li and Zeng [17], Peng and Dai [35], Ren *et al.* [36], Zeng *et al.* [18], and Zhang and Xu [14].

Zhang and Xu [14] proposed the Hamming distance between PF numbers and applied it to develop an extended TOPSIS method. Ren *et al.* [36] presented the Euclidean distance between PF numbers and constructed an extended TODIM (an acronym in Portuguese for interactive multiple criteria decision making) approach. Peng and Dai [35] proposed a distance measure of PF sets for solving stochastic MCDA problems that is based on prospect theory and regret theory. However, their developed distance measure yields a PF number, not a scalar. By extending Zhang and Xu's Hamming distance and Ren *et al.*'s Euclidean distance, Chen [27] introduced the generalized distance measure for PF information and developed a novel remoteness-index-based VIKOR method. Considering the newly developed distance measures in the PF context, Li and Zeng [17] proposed a new distance measure for PF sets that is based on four fundamental parameters of PF numbers: the membership degree, the non-membership degree, the strength of commitment about membership, and the direction of commitment. Furthermore, Zeng *et al.* [18] considered five fundamental parameters of PF numbers, namely, the membership degree, the non-membership degree, the indeterminacy degree, the strength, and the direction, to define a more comprehensive distance measure for PF sets.

However, with these PF distance measures, various limitations and difficulties may be encountered. The commonly used PF distance measures, e.g., the current Hamming and Euclidean distances, do not incorporate the unique characteristics of PF sets into the measurement specification. In contrast to other nonstandard fuzzy models such as intuitionistic fuzzy sets, the main features of PF sets are the strength of commitment and the direction of commitment within a Pythagorean membership grade [11], [17], [18]. Nevertheless, Peng and Dai [35] only employed the degrees of membership and non-membership to compute the PF distance. Zhang and Xu [14], Ren *et al.* [36], and Chen [27] utilized the degrees of membership, non-membership, and indeterminacy to define their distance measures. These PF distance measures ignore the influences of the unique features of PF sets; namely, they do not take the strength of commitment and the direction of commitment into consideration.

The unique features of PF sets have been incorporated into the newly developed distance measures by Li and Zeng [17] and Zeng *et al.* [18]. However, in contrast to the widely used Hamming and Euclidean distances, Li and Zeng's and Zeng *et al.*'s measures do not employ the squared terms of membership degrees, non-membership degrees, indeterminacy degrees, and strengths of commitment. Additionally, the normalization approaches that were employed in their measurements underestimate the maximal values of the normalized distance measures. Moreover, the degree of indeterminacy and the strength of commitment are dual concepts. Hence, Zeng *et al.*'s measurement would encounter a double weighting problem in the computation process due to the dual concepts.

The determination of PF distance measures is a significant issue in the theory of Pythagorean fuzziness, especially for the development of distance-based compromise approaches. Nevertheless, as discussed above, with the existing distance measures in the PF context, various limitations and difficulties are encountered. Therefore, it is necessary to develop more suitable distance measures that are based on PF sets to address the critical issues regarding the lack of consideration of characteristics of PS sets, the lack of square terms in Pythagorean membership degrees, the use of unsuitable normalization approaches, and the double weighting problem, which motivates the research of this paper.

The main objective of this paper is to develop a variety of novel distance measures that are based on PF sets and propose a PF-distance-based compromise approach for addressing MCDA problems under complex uncertainty regarding PF information. To avoid the double weighting problem of the dual concepts (the degree of indeterminacy and the strength of commitment), this paper incorporates four characteristics, namely, the degree of membership, the degree of non-membership, the strength of commitment, and the direction of commitment, into the determination approach of the proposed PF distance measures. To overcome the difficulties that are encountered with the current distance measures, this paper develops several suitable PF distances, namely, new Hamming and Euclidean distances and their generalized distance measures, for Pythagorean membership grades and PF sets. Various useful and desirable properties of the proposed PF distance measures are also investigated. The suitability and performance of the proposed measures are examined via numerical examples and comparisons. As an application of the proposed distance measures, this paper further establishes a PF-distance-based compromise approach for solving MCDA problems within the PF environment. Moreover, the performance and practicability of the proposed methodology is evaluated in a real-world case study of a selection problem for bridge-superstructure construction methods. To analyze the influence of the parameter settings, a sensitivity analysis is conducted to investigate the application results that are obtained using various distance measures. Furthermore, comparative studies are conducted to evaluate the proposed methodology against the TOPSIS-based compromise approach using other distance measures.

The main contributions of this paper are summarized as follows: (1) several useful PF distance measures are developed for overcoming the shortcomings of the current measures; (2) solid theoretical bases (e.g., properties that are related to semimetrics and metrics) of the proposed measures are demonstrated; (3) a novel PF-distance-based compromise approach is constructed for addressing MCDA problems in the PF context; (4) the performance and practicability of the developed methodology are evaluated in practice; and (5) the application results are evaluated via a sensitivity analysis and comparative studies.

The remainder of this paper is organized as follows: Section II introduces basic concepts regarding Pythagorean membership grades and PF sets. Section III reviews the available distance measures for Pythagorean membership grades and discusses the limitations of these distance measures. Section IV constructs a variety of new distance measures within PF environments and investigates their useful and desirable properties. The proposed PF distance measures are evaluated via numerical comparisons and the results are discussed. Section V establishes the useful concept of closeness-based precedence indices and develops a novel PF-distance-based compromise approach for addressing MCDA problems that involve PF information. Section VI applies the proposed methodology to the selection of bridge-superstructure construction methods to evaluate its feasibility and practicality. A sensitivity analysis and comparative studies are also conducted to explore the advantages of the proposed methodology. Finally, Section VII presents the conclusions of this study.

II. BASIC CONCEPT OF PF SETS

This section introduces fundamental concepts that are related to Pythagorean membership grades and PF sets [10]–[14].

Let X be a finite universe of discourse. Let $\mu_P(x)$, $\nu_P(x)$, $r_P(x)$, and $d_P(x)$, which are defined in the unit interval [0,1], denote the degree of membership, the degree of non-membership, the strength of commitment, and the direction of commitment, respectively, regarding the membership of element $x \in X$ in PF set P. Let $\theta_P(x)$ be expressed in radians in the range $[0, \pi/2]$. A Pythagorean membership grade, which is denoted by p, is expressed as follows:

$$p = (\mu_P(x), \nu_P(x); r_P(x), d_P(x)), \tag{1}$$

where the relevant parameters are defined as follows:

$$\mu_P(x) = r_P(x) \cdot \cos(\theta_P(x)), \tag{2}$$

$$\nu_P(x) = r_P(x) \cdot \sin(\theta_P(x)), \tag{3}$$

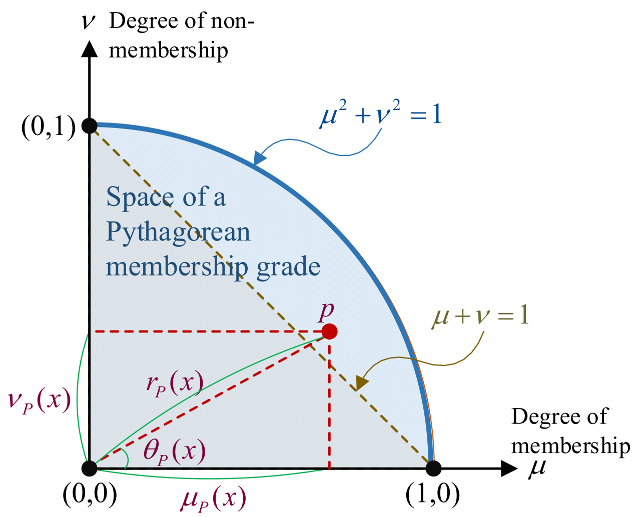


FIGURE 1. Space of a Pythagorean membership grade $\mathbf{p}=(\mu_P(x), \nu_P(x); r_P(x), d_P(x))$.

$$d_P(x)=\frac{\pi-2\cdot\theta_P(x)}{\pi}. \tag{4}$$

A PF set P is characterized by a set of ordered parameters, namely, $\mu_P(x)$, $\nu_P(x)$, $r_P(x)$, and $d_P(x)$, in X, where $\mu_P(x), \nu_P(x), r_P(x), d_P(x) \in [0, 1]$; it is defined as follows:

$$\begin{aligned} P &= \{\langle x, p\rangle | x \in X\} \\ &= \{\langle x, (\mu_P(x), \nu_P(x); r_P(x), d_P(x))\rangle | x \in X\}, \end{aligned} \tag{5}$$

subject to the following condition:

$$0 \le (\mu_P(x))^2 + (\nu_P(x))^2 \le 1. \tag{6}$$

A convenient geometrical interpretation of the space of Pythagorean membership grades is presented in Figure 1. A Pythagorean membership grade p allows lack of commitment and uncertainty in assigning the degrees of membership and non-membership. The degree of membership, which is denoted as $\mu_P(x)$, represents the support for the membership of element $x \in X$ in PF set P, whereas the degree of non-membership, which is denoted as $\nu_P(x)$, represents the support against the membership of x in P.

The degrees $\mu_P(x)$ and $\nu_P(x)$ are related via Pythagorean complements with respect to the strength of commitment $r_P(x)$. According to the Pythagorean Theorem, $\cos^2(\theta_P(x))+\sin^2(\theta_P(x))=1$. Moreover, $\mu_P(x)=r_P(x)\cdot\cos(\theta_P(x))$ and $\nu_P(x)=r_P(x)\cdot\sin(\theta_P(x))$, which yield the following result:

$$(\mu_P(x))^2 + (\nu_P(x))^2 = (r_P(x))^2. \tag{7}$$

It follows that $(\mu_P(x))^2 = (r_P(x))^2 - (\nu_P(x))^2$ and $(\nu_P(x))^2 = (r_P(x))^2 - (\mu_P(x))^2$. Therefore, $\mu_P(x)$ and $\nu_P(x)$ are Pythagorean complements with respect to $r_P(x)$. The complement p^c of p is defined as follows:

$$\begin{aligned} p^c &= (\mu_{P^c}(x), \nu_{P^c}(x); r_{P^c}(x), d_{P^c}(x)) \\ &= (\nu_P(x), \mu_P(x); r_P(x), 1-d_P(x)), \end{aligned} \tag{8}$$

where $\mu_{P^c}(x)=\nu_P(x)$, $\nu_{P^c}(x)=\mu_P(x)$, $r_{P^c}(x)=r_P(x)$, and $d_{P^c}(x)=1-d_P(x)$.

Compared with other nonstandard fuzzy sets (e.g., intuitionistic fuzzy sets), the strength of commitment and the direction of commitment within Pythagorean membership grades are the unique features of PF sets [11], [17], [18]. More fundamentally, a Pythagorean membership grade $p=(\mu_P(x), \nu_P(x); r_P(x), d_P(x))$ can be considered as a point on a circle of radius $r_P(x)$ because of the condition in (7). The larger the value of $r_P(x)$, the stronger the commitment regarding membership at point x, and the lower the uncertainty. The direction of commitment $d_P(x)$ indicates on a scale from 0 to 1 how fully the strength $r_P(x)$ is pointing toward membership. The direction of $r_P(x)$ is completely toward membership if $d_P(x)=1$ and non-membership if $d_P(x)=0$, whereas intermediate values of $d_P(x)$ indicate partial support for membership and non-membership.

Referring to (2)–(4), angle $\theta_P(x)\in[0,\pi/2]$ can be derived from $d_P(x)$, which can be obtained from $\mu_P(x)$ and $r_P(x)$ or $\nu_P(x)$ and $r_P(x)$ as follows:

$$\begin{aligned} \theta_P(x) &= \frac{\pi}{2}(1-d_P(x)) \\ &= \arccos\left(\frac{\mu_P(x)}{r_P(x)}\right) = \arcsin\left(\frac{\nu_P(x)}{r_P(x)}\right). \end{aligned} \tag{9}$$

Furthermore, angle $\theta_{P^c}(x)$, which is associated with the Pythagorean complement p^c, is computed as follows:

$$\theta_{P^c}(x) = \arccos\left(\frac{\nu_P(x)}{r_P(x)}\right) = \arcsin\left(\frac{\mu_P(x)}{r_P(x)}\right). \tag{10}$$

III. DISTANCE MEASURES FOR PYTHAGOREAN MEMBERSHIP GRADES

This section presents a dual concept of the strength of commitment that is based on PF sets and reviews the existing distance measures between Pythagorean membership grades [14], [17], [18], [27], [36].

Zhang and Xu [14] introduced the degree of indeterminacy of an element $x \in X$ to a PF set P, which is denoted as $\tau_P(x)$, where $\tau_P(x)\in[0,1]$. The function $\tau_P(x)$ expresses a lack of knowledge regarding whether x belongs to P [27] and it can be regarded as a dual concept that is associated with the strength of commitment $r_P(x)$. The indeterminacy degree $\tau_P(x)$ is defined as follows:

$$\tau_P(x)=\sqrt{1-(\mu_P(x))^2-(\nu_P(x))^2}=\sqrt{1-(r_P(x))^2}. \tag{11}$$

It follows that $(\tau_P(x))^2 = 1-(r_P(x))^2$, which implies the duality of $\tau_P(x)$ and $r_P(x)$.

Let p_1 and p_2 be two Pythagorean membership grades in the universe of discourse X. Zhang and Xu's proposed Hamming distance measure (D_H^{ZX}) between p_1 and p_2 takes into account the squared differences in the membership degrees, non-membership degrees and indeterminacy degrees [14]. The distance $D_H^{ZX}(p_1, p_2)$is defined as follows:

$$\begin{aligned} D_H^{ZX}(p_1,p_2)=\frac{1}{2}\cdot\Big(&\left|(\mu_{P_1}(x))^2-(\mu_{P_2}(x))^2\right|+\Big|(\nu_{P_1}(x))^2 \\ &-(\nu_{P_2}(x))^2\Big|+\left|(\tau_{P_1}(x))^2-(\tau_{P_2}(x))^2\right|\Big). \end{aligned} \tag{12}$$

Ren *et al.* [36] presented a Euclidean distance measure (D_E^{RXG}) that is based on the membership degree, the non-membership degree, and the indeterminacy degree in Pythagorean membership grades. The distance $D_E^{\mathrm{RXG}}(p_1, p_2)$ between p_1 and p_2 is defined as follows:

$$D_E^{\mathrm{RXG}}(p_1, p_2) = \left\{ \frac{1}{2} \cdot \left[\left((\mu_{P_1}(x))^2 - (\mu_{P_2}(x))^2 \right)^2 + \left((\nu_{P_1}(x))^2 - (\nu_{P_2}(x))^2 \right)^2 + \left((\tau_{P_1}(x))^2 - (\tau_{P_2}(x))^2 \right)^2 \right] \right\}^{\frac{1}{2}}. \quad (13)$$

Let β denote a distance parameter, where $\beta \geq 1$. Chen [27] developed a generalized distance measure ($D_{G,\beta}^{\mathrm{C}}$) by extending the distance measures $D_H^{\mathrm{ZX}}$and D_E^{RXG}. Measures D_H^{ZX} and D_E^{RXG} are special cases of the generalized distance measure; namely, $D_{G,\beta}^{\mathrm{C}}$ reduces to D_H^{ZX} and D_E^{RXG} when $\beta = 1$ and 2, respectively, namely, $D_{G,1}^{\mathrm{C}}(p_1, p_2) = D_H^{\mathrm{ZX}}(p_1, p_2)$ and $D_{G,2}^{\mathrm{C}}(p_1, p_2) = D_E^{\mathrm{RXG}}(p_1, p_2)$. The distance $D_{G,\beta}^{\mathrm{C}}(p_1, p_2)$ between p_1 and p_2 is defined as follows:

$$D_{G,\beta}^{\mathrm{C}}(p_1, p_2) = \left[\frac{1}{2} \cdot \left(\left| (\mu_{P_1}(x))^2 - (\mu_{P_2}(x))^2 \right|^\beta + \left| (\nu_{P_1}(x))^2 - (\nu_{P_2}(x))^2 \right|^\beta + \left| (\tau_{P_1}(x))^2 - (\tau_{P_2}(x))^2 \right|^\beta \right) \right]^{\frac{1}{\beta}}. \quad (14)$$

Li and Zeng [17] incorporated the direction of commitment regarding membership into the specification of distance measures that are based on PF sets. By considering the differences in four parameters (the degree of membership, the degree of non-membership, the strength of commitment, and the direction of commitment), Li and Zeng [17] proposed a normalized generalized distance measure ($D_{G,\beta}^{\mathrm{LZ}}$) of Pythagorean membership grades, where the distance parameter satisfies $\beta \geq 1$. The distance $D_{G,\beta}^{\mathrm{LZ}}(p_1, p_2)$between p_1 and p_2 is defined as follows:

$$D_{G,\beta}^{\mathrm{LZ}}(p_1, p_2) = \left[\frac{1}{4} \left(|\mu_{P_1}(x) - \mu_{P_2}(x)|^\beta + |\nu_{P_1}(x) - \nu_{P_2}(x)|^\beta + |r_{P_1}(x) - r_{P_2}(x)|^\beta + |d_{P_1}(x) - d_{P_2}(x)|^\beta \right) \right]^{\frac{1}{\beta}}. \quad (15)$$

Furthermore, Zeng *et al.* [18] considered each Pythagorean membership grade to be characterized by five parameters (the membership degree, non-membership degree, indeterminacy degree, strength, and direction). In this regard, they proposed a new distance measure ($D_{G,\beta}^{\mathrm{ZLY}}$) that is a function of the five fundamental parameters of PF sets, where $\beta \geq 1$. The distance $D_{G,\beta}^{\mathrm{ZLY}}(p_1, p_2)$ between p_1 and p_2 is defined as follows:

$$D_{G,\beta}^{\mathrm{ZLY}}(p_1, p_2) = \left[\frac{1}{5} \left(|\mu_{P_1}(x) - \mu_{P_2}(x)|^\beta + |\nu_{P_1}(x) - \nu_{P_2}(x)|^\beta + |\tau_{P_1}(x) - \tau_{P_2}(x)|^\beta + |r_{P_1}(x) - r_{P_2}(x)|^\beta + |d_{P_1}(x) - d_{P_2}(x)|^\beta \right) \right]^{\frac{1}{\beta}}. \quad (16)$$

The distance measures that are discussed above for Pythagorean membership grades within the PF environment suffer from several limitations and/or identification difficulties: First, Li and Zeng [17] and Zeng *et al.* [18] addressed the lack of consideration of the unique features of PS sets. As described earlier, the features of PF sets, in contrast with other nonstandard fuzzy sets, are the strength of commitment $r_P(x)$ and the direction of commitment $d_P(x)$ within a Pythagorean membership grade. Nonetheless, Zhang and Xu [14], Ren *et al.* [36], and Chen [27] did not incorporate $r_P(x)$ and $d_P(x)$ into their distance measures (D_H^{ZX}, D_E^{RXG}, and $D_{G,\beta}^{\mathrm{C}}$).

Second, although Li and Zeng [17] and Zeng *et al.* [18] considered $r_P(x)$ and $d_P(x)$ in identifying distance measures for PF sets, their proposed distance measures $D_{G,\beta}^{\mathrm{LZ}}$ and $D_{G,\beta}^{\mathrm{ZLY}}$ do not fully utilize the squared degrees of membership, non-membership, indeterminacy, and strength of the Pythagorean membership grades. The $D_{G,\beta}^{\mathrm{LZ}}$ measure was developed based on the differences with respect to $\mu_P(x)$, $\nu_P(x)$, $r_P(x)$, and $d_P(x)$. The differences that correspond to $(\mu_P(x))^2$, $(\nu_P(x))^2$, and $(r_P(x))^2$ were not incorporated into the $D_{G,\beta}^{\mathrm{LZ}}$ measure. The computation of the $D_{G,\beta}^{\mathrm{ZLY}}$ measure is based on the respective differences in terms of $\mu_P(x)$, $\nu_P(x)$, $\tau_P(x)$, $r_P(x)$, and $d_P(x)$. Analogously, the basis of the $D_{G,\beta}^{\mathrm{ZLY}}$ measure does not depend on the respective differences with respect to $(\mu_P(x))^2$, $(\nu_P(x))^2$, $(\tau_P(x))^2$, and $(r_P(x))^2$.

Third, decision makers may be confronted by an underestimation problem when employing the $D_{G,\beta}^{\mathrm{LZ}}$ and $D_{G,\beta}^{\mathrm{ZLY}}$ measures. The normalization approaches that were employed in Li and Zeng's and Zeng *et al.*'s definitions will underestimate the maximal values of the normalized distance measures. For example, the maximal values of the $D_{G,1}^{\mathrm{LZ}}$ and $D_{G,1}^{\mathrm{ZLY}}$ measures are 0.75 and 0.6, respectively, when $\beta = 1$. Moreover, when $\beta = 2$, the maximal values of the $D_{G,2}^{\mathrm{LZ}}$ and $D_{G,2}^{\mathrm{ZLY}}$ measures are 0.866 and 0.7746, respectively.

Fourth, in Zeng *et al.*'s definitions, because of the duality of $\tau_P(x)$ and $r_P(x)$, the computation of the $D_{G,\beta}^{\mathrm{ZLY}}$ measure will result in the double weighting problem regarding the dual concepts. To overcome these difficulties, this paper attempts to develop a variety of novel distance measures for Pythagorean membership grades and for PF sets.

IV. PROPOSED PF DISTANCE MEASURES

This section constructs a variety of novel PF distance measures within the PF environment, which fully take into account the four fundamental parameters of Pythagorean membership grades. Useful and desirable properties of the proposed metrics are also investigated in this section. Moreover, this section compares various distance measures in the PF context via illustrative examples.

As discussed earlier, various limitations and difficulties are encountered when applying the specification approaches of the existing distance measures for Pythagorean membership grades. To address the issues regarding the lack of consideration of the unique features of PF sets, the lack of adoption of the square terms of Pythagorean membership degrees, the use of unsuitable normalization approaches, and the double weighting problem, this paper proposes a new approach for determining the PF distance measures of Pythagorean membership grades and the normalized PF distance measures for PF sets. Because the indeterminacy degree $\tau_P(x)$ and the strength of commitment $r_P(x)$ are dual concepts, namely, $(\tau_P(x))^2 = 1 - (r_P(x))^2$, this paper fully considers the influences of four parameters, namely, $\mu_P(x)$, $\nu_P(x)$, $r_P(x)$, and $d_P(x)$, while neglecting $\tau_P(x)$, in defining new PF distance measures. In the following, this paper presents several novel PF distances: the Hamming PF distance measure D_H, the Euclidean PF distance measure D_E, and the generalized PF distance measure $D_{G,\beta}$. Essential properties of these measures are also investigated. Measures D_H and D_E are special cases of measure $D_{G,\beta}$. Namely, $D_{G,\beta}$ can be regarded as a generalized version of D_H and D_E.

Let $p_1 = (\mu_{P_1}(x), \nu_{P_1}(x); r_{P_1}(x), d_{P_1}(x))$ and $p_2 = (\mu_{P_2}(x), \nu_{P_2}(x); r_{P_2}(x), d_{P_2}(x))$ be two Pythagorean membership grades in the universe of discourse X. The Hamming PF distance measure between p_1 and p_2 is defined as follows:

$$D_H(p_1, p_2) = \frac{1}{3}\left(\left|(\mu_{P_1}(x))^2 - (\mu_{P_2}(x))^2\right| + \left|(\nu_{P_1}(x))^2 - (\nu_{P_2}(x))^2\right| + \left|(r_{P_1}(x))^2 - (r_{P_2}(x))^2\right| + \left|d_{P_1}(x) - d_{P_2}(x)\right|\right). \tag{17}$$

The Euclidean PF distance measure between p_1 and p_2 is defined as follows:

$$D_E(p_1, p_2) = \left\{\frac{1}{3}\left[\left((\mu_{P_1}(x))^2 - (\mu_{P_2}(x))^2\right)^2 + \left((\nu_{P_1}(x))^2 - (\nu_{P_2}(x))^2\right)^2 + \left((r_{P_1}(x))^2 - (r_{P_2}(x))^2\right)^2 + \left(d_{P_1}(x) - d_{P_2}(x)\right)^2\right]\right\}^{\frac{1}{2}}. \tag{18}$$

The generalized PF distance measure between p_1 and p_2 is defined as follows:

$$D_{G,\beta}(p_1, p_2) = \left[\frac{1}{3}\left(\left|(\mu_{P_1}(x))^2 - (\mu_{P_2}(x))^2\right|^\beta + \left|(\nu_{P_1}(x))^2 - (\nu_{P_2}(x))^2\right|^\beta + \left|(r_{P_1}(x))^2 - (r_{P_2}(x))^2\right|^\beta + \left|d_{P_1}(x) - d_{P_2}(x)\right|^\beta\right)\right]^{\frac{1}{\beta}}, \tag{19}$$

where β is a distance parameter and $\beta \geq 1$. If $\beta = 1$, the generalized PF distance reduces to the Hamming PF distance, namely, $D_H(p_1, p_2) = D_{G,1}(p_1, p_2)$. If $\beta = 2$, the generalized PF distance reduces to the Euclidean PF distance, namely, $D_E(p_1, p_2) = D_{G,2}(p_1, p_2)$. Although $D_{G,\beta}$ is the generalized version of D_H and D_E ($D_H = D_{G,1}$ and $D_E = D_{G,2}$), their properties differ. Thus, this paper separately explores the properties of the three measures: D_H, D_E, and $D_{G,\beta}$.

The generalized PF distance measure $D_{G,\beta}$ is a semimetric because it satisfies the requirements of reflexivity, separability, and symmetry for all β values, according to the following theorem.

Theorem 1: Let $p_1 = (\mu_{P_1}(x), \nu_{P_1}(x); r_{P_1}(x), d_{P_1}(x))$ and $p_2 = (\mu_{P_2}(x), \nu_{P_2}(x); r_{P_2}(x), d_{P_2}(x))$ be two Pythagorean membership grades in the universe of discourse X. The generalized PF distance measure $D_{G,\beta}$ satisfies the following properties:

(T1.1) Boundedness: $0 \leq D_{G,\beta}(p_1, p_2) \leq 1$;

(T1.2) Reflexivity: $D_{G,\beta}(p_1, p_1) = 0$;

(T1.3) Separability: $D_{G,\beta}(p_1, p_2) = 0$ if and only if $p_1 = p_2$;

(T1.4) Symmetry: $D_{G,\beta}(p_1, p_2) = D_{G,\beta}(p_2, p_1)$.

Proof: $D_{G,\beta}$ satisfies properties (T1.2) and (T1.4) directly. Thus, it is only necessary to prove that (T1.1) and (T1.3) are satisfied. $D_{G,\beta}(p_1, p_2) \geq 0$ according to the definition in (19). From (7), one obtains:

$$\begin{aligned}
&\left|(\mu_{P_1}(x))^2 - (\mu_{P_2}(x))^2\right| + \left|(\nu_{P_1}(x))^2 - (\nu_{P_2}(x))^2\right| + \left|(r_{P_1}(x))^2 - (r_{P_2}(x))^2\right| \\
&= \left|(\mu_{P_1}(x))^2 - (\mu_{P_2}(x))^2\right| + \left|(\nu_{P_1}(x))^2 - (\nu_{P_2}(x))^2\right| + \left|(\mu_{P_1}(x))^2 - (\mu_{P_2}(x))^2 + (\nu_{P_1}(x))^2 - (\nu_{P_2}(x))^2\right| \\
&\leq (\mu_{P_1}(x))^2 + (\nu_{P_1}(x))^2 + (\mu_{P_2}(x))^2 + (\nu_{P_2}(x))^2 \\
&\leq 1 + 1 = 2.
\end{aligned}$$

Moreover, $|d_{P_1}(x) - d_{P_2}(x)| \leq 1$ because $0 \leq d_{P_1}(x), d_{P_2}(x) \leq 1$. Thus, $|(\mu_{P_1}(x))^2 - (\mu_{P_2}(x))^2| + |(\nu_{P_1}(x))^2 - (\nu_{P_2}(x))^2| + |(r_{P_1}(x))^2 - (r_{P_2}(x))^2| + |d_{P_1}(x) - d_{P_2}(x)| \leq 3$. Recall that $\beta \geq 1$. The following is derived:

$$\begin{aligned}
&D_{G,\beta}(p_1, p_2) \\
&= \left[\frac{1}{3}\left(\left|(\mu_{P_1}(x))^2 - (\mu_{P_2}(x))^2\right|^\beta + \left|(\nu_{P_1}(x))^2 - (\nu_{P_2}(x))^2\right|^\beta + \left|(r_{P_1}(x))^2 - (r_{P_2}(x))^2\right|^\beta + \left|d_{P_1}(x) - d_{P_2}(x)\right|^\beta\right)\right]^{\frac{1}{\beta}} \\
&\leq \left[\frac{1}{3}\left(\left|(\mu_{P_1}(x))^2 - (\mu_{P_2}(x))^2\right| + \left|(\nu_{P_1}(x))^2 - (\nu_{P_2}(x))^2\right| + \left|(r_{P_1}(x))^2 - (r_{P_2}(x))^2\right| + \left|d_{P_1}(x) - d_{P_2}(x)\right|\right)\right]^{\frac{1}{\beta}} \\
&\leq \left(\frac{1}{3} \cdot 3\right)^{\frac{1}{\beta}} = 1.
\end{aligned}$$

It follows that $0 \leq D_{G,\beta}(p_1, p_2) \leq 1$, namely, (T1.1) is valid. For the necessity of separability in (T1.3), if $D_{G,\beta}(p_1, p_2) = 0$, then the absolute values $|(\mu_{P_1}(x))^2 - (\mu_{P_2}(x))^2|$, $|(\nu_{P_1}(x))^2 - (\nu_{P_2}(x))^2|$, $|(r_{P_1}(x))^2 - (r_{P_2}(x))^2|$, and $|d_{P_1}(x) - d_{P_2}(x)|$ must be equal to 0; hence, $p_1 = p_2$. For the sufficiency of separability in (T1.3), if $p_1 = p_2$, then $D_{G,\beta}(p_1, p_2) = 0$. Therefore, the property of separability is satisfied, namely, (T1.3) is valid. This completes the proof.

According to the following theorem, the proposed Hamming PF distance measure D_H is a metric because it satisfies the requirements of reflexivity, separability, symmetry, and the triangle inequality.

Theorem 2: Let $p_i = (\mu_{P_i}(x), \nu_{P_i}(x); r_{P_i}(x), d_{P_i}(x))$ (i =1, 2, 3) be three Pythagorean membership grades in the universe of discourse X. The Hamming PF distance measure D_H satisfies the following properties:

(T2.1) Boundedness: $0 \leq D_H(p_1, p_2) \leq 1$;

(T2.2) Reflexivity: $D_H(p_1, p_1) = 0$;

(T2.3) Separability: $D_H(p_1, p_2) = 0$ if and only if $p_1 = p_2$;

(T2.4) Symmetry: $D_H(p_1, p_2) = D_H(p_2, p_1)$;

(T2.5) Triangle inequality: $D_H(p_1, p_3) \leq D_H(p_1, p_2) + D_H(p_2, p_3)$.

Proof: The Hamming PF distance measure D_H can be considered a special case of the generalized PF distance measure $D_{G,\beta}$ ($D_{G,\beta}$ reduces to D_H if β =1). Accordingly, D_H satisfies properties (T2.1)–(T2.4). Thus, one needs to prove only (T2.5). Consider the component that is related to the membership degree. This proof investigates the validity of inequality $|(\mu_{P_1}(x))^2 - (\mu_{P_3}(x))^2| \leq |(\mu_{P_1}(x))^2 - (\mu_{P_2}(x))^2| + |(\mu_{P_2}(x))^2 - (\mu_{P_3}(x))^2|$. Based on the relationships among $(\mu_{P_1}(x))^2$, $(\mu_{P_2}(x))^2$, and $(\mu_{P_3}(x))^2$, the following four assumptions are investigated: (i) $(\mu_{P_1}(x))^2 \leq (\mu_{P_2}(x))^2 \leq (\mu_{P_3}(x))^2$; (ii) $(\mu_{P_1}(x))^2 \geq (\mu_{P_2}(x))^2 \geq (\mu_{P_3}(x))^2$; (iii) $(\mu_{P_2}(x))^2 \leq \min\{(\mu_{P_1}(x))^2, (\mu_{P_3}(x))^2\}$; and (iv) $(\mu_{P_2}(x))^2 \geq \max\{(\mu_{P_1}(x))^2, (\mu_{P_3}(x))^2\}$. From assumptions (i) and (ii), it follows that inequality $|(\mu_{P_1}(x))^2 - (\mu_{P_3}(x))^2| \leq |(\mu_{P_1}(x))^2 - (\mu_{P_2}(x))^2| + |(\mu_{P_2}(x))^2 - (\mu_{P_3}(x))^2|$ is satisfied. From assumption (iii), it directly follows that $(\mu_{P_1}(x))^2 - (\mu_{P_2}(x))^2 \geq 0$ and $(\mu_{P_3}(x))^2 - (\mu_{P_2}(x))^2 \geq 0$. When $(\mu_{P_1}(x))^2 \geq (\mu_{P_3}(x))^2$, the following holds:

$$\begin{aligned}&\left|(\mu_{P_1}(x))^2 - (\mu_{P_2}(x))^2\right| + \left|(\mu_{P_2}(x))^2 - (\mu_{P_3}(x))^2\right| \\ &\quad - \left|(\mu_{P_1}(x))^2 - (\mu_{P_3}(x))^2\right| \\ &= (\mu_{P_1}(x))^2 - (\mu_{P_2}(x))^2 + (\mu_{P_3}(x))^2 - (\mu_{P_2}(x))^2 \\ &\quad - (\mu_{P_1}(x))^2 + (\mu_{P_3}(x))^2 \geq 0.\end{aligned}$$

When $(\mu_{P_1}(x))^2 \leq (\mu_{P_3}(x))^2$, the following holds:

$$\begin{aligned}&\left|(\mu_{P_1}(x))^2 - (\mu_{P_2}(x))^2\right| + \left|(\mu_{P_2}(x))^2 - (\mu_{P_3}(x))^2\right| \\ &\quad - \left|(\mu_{P_1}(x))^2 - (\mu_{P_3}(x))^2\right| \\ &= (\mu_{P_1}(x))^2 - (\mu_{P_2}(x))^2 + (\mu_{P_3}(x))^2 - (\mu_{P_2}(x))^2 \\ &\quad + (\mu_{P_1}(x))^2 - (\mu_{P_3}(x))^2 \geq 0.\end{aligned}$$

Accordingly, $|(\mu_{P_1}(x))^2 - (\mu_{P_3}(x))^2| \leq |(\mu_{P_1}(x))^2 - (\mu_{P_2}(x))^2| + |(\mu_{P_2}(x))^2 - (\mu_{P_3}(x))^2|$ is satisfied under assumption (iii). Assumption (iv) directly implies that $(\mu_{P_2}(x))^2 - (\mu_{P_1}(x))^2 \geq 0$ and $(\mu_{P_2}(x))^2 - (\mu_{P_3}(x))^2 \geq 0$. When $(\mu_{P_1}(x))^2 \geq (\mu_{P_3}(x))^2$, one obtains:

$$\begin{aligned}&\left|(\mu_{P_1}(x))^2 - (\mu_{P_2}(x))^2\right| + \left|(\mu_{P_2}(x))^2 - (\mu_{P_3}(x))^2\right| \\ &\quad - \left|(\mu_{P_1}(x))^2 - (\mu_{P_3}(x))^2\right| \\ &= (\mu_{P_2}(x))^2 - (\mu_{P_1}(x))^2 + (\mu_{P_2}(x))^2 - (\mu_{P_3}(x))^2 \\ &\quad - (\mu_{P_1}(x))^2 + (\mu_{P_3}(x))^2 \geq 0.\end{aligned}$$

In contrast, when $(\mu_{P_1}(x))^2 \leq (\mu_{P_3}(x))^2$, it follows that:

$$\begin{aligned}&\left|(\mu_{P_1}(x))^2 - (\mu_{P_2}(x))^2\right| + \left|(\mu_{P_2}(x))^2 - (\mu_{P_3}(x))^2\right| \\ &\quad - \left|(\mu_{P_1}(x))^2 - (\mu_{P_3}(x))^2\right| \\ &= (\mu_{P_2}(x))^2 - (\mu_{P_1}(x))^2 + (\mu_{P_2}(x))^2 - (\mu_{P_3}(x))^2 \\ &\quad + (\mu_{P_1}(x))^2 - (\mu_{P_3}(x))^2 \geq 0.\end{aligned}$$

Therefore, $|(\mu_{P_1}(x))^2 - (\mu_{P_3}(x))^2| \leq |(\mu_{P_1}(x))^2 - (\mu_{P_2}(x))^2| + |(\mu_{P_2}(x))^2 - (\mu_{P_3}(x))^2|$ is satisfied under assumption (iv). In a similar way, for the components that are related to the non-membership degree and the strength of commitment, the following relationships can be proven: $|(\nu_{P_1}(x))^2 - (\nu_{P_3}(x))^2| \leq |(\nu_{P_1}(x))^2 - (\nu_{P_2}(x))^2| + |(\nu_{P_2}(x))^2 - (\nu_{P_3}(x))^2|$ and $|(r_{P_1}(x))^2 - (r_{P_3}(x))^2| \leq |(r_{P_1}(x))^2 - (r_{P_2}(x))^2| + |(r_{P_2}(x))^2 - (r_{P_3}(x))^2|$. Next, consider the component that is related to the direction of commitment. Inequality $|d_{P_1}(x) - d_{P_3}(x)| \leq |d_{P_1}(x) - d_{P_2}(x)| + |d_{P_2}(x) - d_{P_3}(x)|$ is satisfied under assumptions $d_{P_1}(x) \leq d_{P_2}(x) \leq d_{P_3}(x)$ and $d_{P_1}(x) \geq d_{P_2}(x) \geq d_{P_3}(x)$. From assumption $d_{P_2}(x) \leq \min\{d_{P_1}(x), d_{P_3}(x)\}$, the following result is obtained:

$$\begin{aligned}&\left|d_{P_1}(x) - d_{P_2}(x)\right| + \left|d_{P_2}(x) - d_{P_3}(x)\right| - \left|d_{P_1}(x) - d_{P_3}(x)\right| \\ &= \begin{cases} d_{P_1}(x) - d_{P_2}(x) + d_{P_3}(x) - d_{P_2}(x) - d_{P_1}(x) + d_{P_3}(x) \\ \text{if } d_{P_1}(x) \geq d_{P_3}(x) \\ d_{P_1}(x) - d_{P_2}(x) + d_{P_3}(x) - d_{P_2}(x) + d_{P_1}(x) - d_{P_3}(x) \\ \text{if } d_{P_1}(x) \leq d_{P_3}(x) \end{cases} \\ &= 2\left(\min\left\{d_{P_1}(x), d_{P_3}(x)\right\} - d_{P_2}(x)\right) \geq 0.\end{aligned}$$

Hence, $|d_{P_1}(x) - d_{P_3}(x)| \leq |d_{P_1}(x) - d_{P_2}(x)| + |d_{P_2}(x) - d_{P_3}(x)|$ is satisfied. From these results, one concludes that the triangle inequality, namely, $D_H(p_1, p_3) \leq D_H(p_1, p_2) + D_H(p_2, p_3)$, holds. Therefore, (T2.5) is valid. This completes the proof.

Theorem 3: Let p_1 and p_2 be two Pythagorean membership grades in the universe of discourse X. The Euclidean PF distance measure D_E satisfies the following properties:

(T3.1) Boundedness: $0 \leq D_E(p_1, p_2) \leq 1$;

(T3.2) Reflexivity: $D_E(p_1, p_1) = 0$;

(T3.3) Separability: $D_E(p_1, p_2) = 0$ if and only if $p_1 = p_2$;

(T3.4) Symmetry: $D_E(p_1, p_2) = D_E(p_2, p_1)$.

Proof: The D_E measure is the special case of the generalized PF distance measure $D_{G,\beta}$ ($D_{G,\beta}$ reduces to D_E when β =2). Accordingly, properties (T3.1)–(T3.4) are satisfied based on the proofs of (T1.1)–(T1.4), respectively. Moreover, D_E is a semimetric because it satisfies the requirements of reflexivity, separability, and symmetry. This completes the proof.

The angle θ and the direction of commitment d have the following desirable properties:

Theorem 4: Let $p_1^c = (\mu_{P_1^c}(x), \nu_{P_1^c}(x); r_{P_1^c}(x), d_{P_1^c}(x))$ and $p_2^c = (\mu_{P_2^c}(x), \nu_{P_2^c}(x); r_{P_2^c}(x), d_{P_2^c}(x))$ be the complements of Pythagorean membership grades p_1 and p_2, respectively, in the universe of discourse X. The degree θ that is associated with p_1, p_2, p_1^c, and p_2^c satisfies the following properties:

(T4.1) $|\theta_{P_1}(x) - \theta_{P_2}(x)| = |\theta_{P_1^c}(x) - \theta_{P_2^c}(x)|$;

(T4.2) $|\theta_{P_1}(x) - \theta_{P_2^c}(x)| = |\theta_{P_1^c}(x) - \theta_{P_2}(x)|$.

Proof: According to the definition in (9), $\theta_{P_1}(x) = \arccos(\mu_{P_1}(x)/r_{P_1}(x))$ and $\theta_{P_2}(x) = \arccos(\mu_{P_2}(x)/r_{P_2}(x))$

are associated withp_1 and p_2, respectively. From (8) and (10), it follows directly that:

$$\theta_{P_1^c}(x) = \arccos\left(\frac{\mu_{P_1^c}(x)}{r_{P_1^c}(x)}\right) = \arccos\left(\frac{\nu_{P_1}(x)}{r_{P_1}(x)}\right),$$
$$\theta_{P_2^c}(x) = \arccos\left(\frac{\mu_{P_2^c}(x)}{r_{P_2^c}(x)}\right) = \arccos\left(\frac{\nu_{P_2}(x)}{r_{P_2}(x)}\right),$$

where $\theta_{P_1^c}(x), \theta_{P_2^c}(x) \in [0, \pi/2]$. It follows that:

$$\cos^2\left(\theta_{P_1}(x)\right) + \cos^2\left(\theta_{P_1^c}(x)\right) = \frac{\left(\mu_{P_1}(x)\right)^2}{\left(r_{P_1}(x)\right)^2} + \frac{\left(\nu_{P_1}(x)\right)^2}{\left(r_{P_1}(x)\right)^2} = 1.$$

According to the Pythagorean Theorem, $\cos(\theta_{P_1^c}(x)) = \sin(\theta_{P_1}(x))$ because $\cos^2(\theta_{P_1}(x)) + \sin^2(\theta_{P_1}(x)) = 1$. Thus, $\theta_{P_1^c}(x) = \pi/2 - \theta_{P_1}(x)$, namely, $\theta_{P_1}(x)$ and $\theta_{P_1^c}(x)$ are angle pairs whose measures sum to a right angle, namely, $\pi/2$ radians. Analogously, $\theta_{P_2^c}(x)$ is the complementary angle of $\theta_{P_2}(x)$, namely, $\theta_{P_2^c}(x) = \pi/2 - \theta_{P_2}(x)$. It is proven that:

$$\begin{aligned}\left|\theta_{P_1^c}(x) - \theta_{P_2^c}(x)\right| &= \left|\left(\frac{\pi}{2} - \theta_{P_1}(x)\right) - \left(\frac{\pi}{2} - \theta_{P_2}(x)\right)\right| \\ &= \left|\theta_{P_1}(x) - \theta_{P_2}(x)\right|, \\ \left|\theta_{P_1^c}(x) - \theta_{P_2}(x)\right| &= \left|\left(\frac{\pi}{2} - \theta_{P_1}(x)\right) - \theta_{P_2}(x)\right| \\ &= \left|-\theta_{P_1}(x) + \left(\frac{\pi}{2} - \theta_{P_2}(x)\right)\right| \\ &= \left|\theta_{P_1}(x) - \theta_{P_2^c}(x)\right|.\end{aligned}$$

Therefore, (T4.1) and (T4.2) are satisfied, which completes the proof.

Theorem 5: Consider two Pythagorean membership grades, namely, p_1 and p_2, and their complements, namely,p_1^c and p_2^c, respectively, in the universe of discourse X. The directions of commitment d that are associated with p_1, p_2, p_1^c, and p_2^c satisfy the following properties:

(T5.1) $|d_{P_1}(x) - d_{P_2}(x)| = |d_{P_1^c}(x) - d_{P_2^c}(x)|$;

(T5.2) $|d_{P_1}(x) - d_{P_2^c}(x)| = |d_{P_1^c}(x) - d_{P_2}(x)|$.

Proof: From (4), it follows that:

$$\begin{aligned}\left|d_{P_1}(x) - d_{P_2}(x)\right| &= \left|\frac{\pi - 2\cdot\theta_{P_1}(x)}{\pi} - \frac{\pi - 2\cdot\theta_{P_2}(x)}{\pi}\right| \\ &= \frac{2}{\pi}\left|\theta_{P_1}(x) - \theta_{P_2}(x)\right|, \\ \left|d_{P_1^c}(x) - d_{P_2^c}(x)\right| &= \left|\frac{\pi - 2\cdot\theta_{P_1^c}(x)}{\pi} - \frac{\pi - 2\cdot\theta_{P_2^c}(x)}{\pi}\right| \\ &= \frac{2}{\pi}\left|\theta_{P_1^c}(x) - \theta_{P_2^c}(x)\right|.\end{aligned}$$

According to (T4.1), $|d_{P_1}(x) - d_{P_2}(x)| = |d_{P_1^c}(x) - d_{P_2^c}(x)|$. Analogously, $|d_{P_1}(x) - d_{P_2^c}(x)| = |d_{P_1^c}(x) - d_{P_2}(x)|$ according to (T4.2). This completes the proof.

Theorem 6: Let p_1 and p_2 be two Pythagorean membership grades in the universe of discourse X. The Hamming PF distance measure D_H, the Euclidean PF distance measure D_E, and the generalized PF distance measure $D_{G,\beta}$satisfy the following properties:

(T6.1) $D_H(p_1, p_2) = D_H(p_1^c, p_2^c)$, $D_E(p_1, p_2) = D_E(p_1^c, p_2^c)$, and $D_{G,\beta}(p_1, p_2) = D_{G,\beta}(p_1^c, p_2^c)$ for all $\beta \geq 1$;

(T6.2) $D_H(p_1, p_2^c) = D_H(p_1^c, p_2)$, $D_E(p_1, p_2^c) = D_E(p_1^c, p_2)$, and $D_{G,\beta}(p_1, p_2^c) = D_{G,\beta}(p_1^c, p_2)$ for all $\beta \geq 1$.

Proof: This proof only validates the cases of the Hamming PF distance measure. The other cases can be proven via a similar approach. From (8) and (17), it follows that:

$$\begin{aligned}&D_H(p_1^c, p_2^c) \\ &= \frac{1}{3}\left(\left|\left(\mu_{P_1^c}(x)\right)^2 - \left(\mu_{P_2^c}(x)\right)^2\right| + \left|\left(\nu_{P_1^c}(x)\right)^2 \right.\right. \\ &\quad \left. - \left(\nu_{P_2^c}(x)\right)^2\right| + \left|\left(r_{P_1^c}(x)\right)^2 - \left(r_{P_2^c}(x)\right)^2\right| \\ &\quad \left. + \left|d_{P_1^c}(x) - d_{P_2^c}(x)\right|\right) \\ &= \frac{1}{3}\left(\left(\left|\left(\nu_{P_1}(x)\right)^2 - \left(\nu_{P_2}(x)\right)^2\right| + \left|\left(\mu_{P_1}(x)\right)^2 - \left(\mu_{P_2}(x)\right)^2\right|\right.\right. \\ &\quad \left. + \left|\left(r_{P_1}(x)\right)^2 - \left(r_{P_2}(x)\right)^2\right| + \left|d_{P_1^c}(x) - d_{P_2^c}(x)\right|\right).\end{aligned}$$

Because $|d_{P_1}(x) - d_{P_2}(x)| = |d_{P_1^c}(x) - d_{P_2^c}(x)|$ according to (T5.1), $D_H(p_1, p_2) = D_H(p_1^c, p_2^c)$ holds. Equations $D_E(p_1, p_2) = D_E(p_1^c, p_2^c)$ and $D_{G,\beta}(p_1, p_2) = D_{G,\beta}(p_1^c, p_2^c)$ for all $\beta \geq 1$ can be proven analogously. Thus, (T6.1) is satisfied. Next, Hamming PF distances $D_H(p_1, p_2^c)$ and $D_H(p_1^c, p_2)$ are computed as follows:

$$\begin{aligned}&D_H(p_1, p_2^c) \\ &= \frac{1}{3}\left(\left|\left(\mu_{P_1}(x)\right)^2 - \left(\mu_{P_2^c}(x)\right)^2\right| + \left|\left(\nu_{P_1}(x)\right)^2 \right.\right. \\ &\quad \left. - \left(\nu_{P_2^c}(x)\right)^2\right| + \left|\left(r_{P_1}(x)\right)^2 - \left(r_{P_2^c}(x)\right)^2\right| \\ &\quad \left. + \left|d_{P_1}(x) - d_{P_2^c}(x)\right|\right) \\ &= \frac{1}{3}\left(\left|\left(\mu_{P_1}(x)\right)^2 - \left(\nu_{P_2}(x)\right)^2\right| + \left|\left(\nu_{P_1}(x)\right)^2 - \left(\mu_{P_2}(x)\right)^2\right|\right. \\ &\quad \left. + \left|\left(r_{P_1}(x)\right)^2 - \left(r_{P_2}(x)\right)^2\right| + \left|d_{P_1}(x) - d_{P_2^c}(x)\right|\right),\end{aligned}$$

$$\begin{aligned}&D_H(p_1^c, p_2) \\ &= \frac{1}{3}\left(\left|\left(\mu_{P_1^c}(x)\right)^2 - \left(\mu_{P_2}(x)\right)^2\right| + \left|\left(\nu_{P_1^c}(x)\right)^2 \right.\right. \\ &\quad \left. - \left(\nu_{P_2}(x)\right)^2\right| + \left|\left(r_{P_1^c}(x)\right)^2 - \left(r_{P_2}(x)\right)^2\right| \\ &\quad \left. + \left|d_{P_1^c}(x) - d_{P_2}(x)\right|\right) \\ &= \frac{1}{3}\left(\left(\left|\left(\nu_{P_1}(x)\right)^2 - \left(\mu_{P_2}(x)\right)^2\right| + \left|\left(\mu_{P_1}(x)\right)^2 - \left(\nu_{P_2}(x)\right)^2\right|\right.\right. \\ &\quad \left. + \left|\left(r_{P_1}(x)\right)^2 - \left(r_{P_2}(x)\right)^2\right| + \left|d_{P_1^c}(x) - d_{P_2}(x)\right|\right).\end{aligned}$$

According to (T5.2), $|d_{P_1}(x) - d_{P_2^c}(x)| = |d_{P_1^c}(x) - d_{P_2}(x)|$. Therefore, equation $D_H(p_1, p_2^c) = D_H(p_1^c, p_2)$ holds. Analogously, one can prove that $D_E(p_1, p_2^c) = D_E(p_1^c, p_2)$ and $D_{G,\beta}(p_1, p_2^c) = D_{G,\beta}(p_1^c, p_2)$ for all $\beta \geq 1$, namely, (T6.2) is satisfied. This completes the proof.

Furthermore, the suitability and effectiveness of the proposed PF distance measures (D_H, D_E, and $D_{G,\beta}$) are examined via the following numerical examples and comparisons. The compared distance measures are D_H^{ZX}, D_E^{RXG}, $D_{G,\beta}^{\mathrm{C}}$, $D_{G,\beta}^{\mathrm{LZ}}$, and $D_{G,\beta}^{\mathrm{ZLY}}$, which were developed by Zhang and Xu [14], Ren *et al.* [36], Chen [27], Li and Zeng [17], and Zeng et al. [18], respectively.

Example 1 (Maximal Normalized Distance): Let $p_1 = (1, 0; 1, 1)$ and $p_2 = (0, 1; 1, 0)$ be two Pythagorean membership grades in the universe of discourse X. Consider the Hamming distance model. The following results were obtained: $D_H^{\mathrm{ZX}}(p_1, p_2) = D_{G,1}^{\mathrm{C}}(p_1, p_2) = D_H(p_1, p_2) = D_{G,1}(p_1, p_2) = 1$, $D_{G,1}^{\mathrm{LZ}}(p_1, p_2) = 0.75$, and $D_{G,1}^{\mathrm{ZLY}}(p_1, p_2) = 0.6$. Next, considering the Euclidean distance model, the following results were obtained: $D_E^{\mathrm{RXG}}(p_1, p_2) = D_{G,2}^{\mathrm{C}}(p_1, p_2) = D_E(p_1, p_2) = D_{G,2}(p_1, p_2) = 1$, $D_{G,2}^{\mathrm{LZ}}(p_1, p_2) = 0.8660$, and $D_{G,2}^{\mathrm{ZLY}}(p_1, p_2) = 0.7746$. Chen [27] presented the fundamental concept of a lattice within the PF environment. PF numbers p_1 and p_2 are the largest and smallest PF numbers, respectively, because they are the top and bottom elements, respectively, of the lattice. Accordingly, the distance between p_1 and p_2 should be 1 in the lattice because it is based on PF sets. However, the results that were computed using Li and Zeng's and Zeng *et al.*'s distance measures are unreasonable and unacceptable because $D_{G,1}^{\mathrm{LZ}}(p_1, p_2)$, $D_{G,1}^{\mathrm{ZLY}}(p_1, p_2)$, $D_{G,2}^{\mathrm{LZ}}(p_1, p_2)$, and $D_{G,2}^{\mathrm{ZLY}}(p_1, p_2)$ are not equal to 1. As discussed previously, the normalization approaches in $D_{G,\beta}^{\mathrm{LZ}}$ and $D_{G,\beta}^{\mathrm{ZLY}}$ are unsuitable because they underestimate the maximal values of the normalized distance measures.

Example 2 (Failure to Consider the Direction of Commitment): Let $p_1 = (0.35, 0.42; 0.5467, 0.4423)$, $p_2 = (0.42, 0.35; 0.5467, 0.5577)$, and $p_3 = (0.62, 0.73; 0.9578, 0.4482)$ be three Pythagorean membership grades in X. Consider the Hamming distance model as an example. Based on the distance measures of Zhang and Xu [14] and Chen [27], one has $D_H^{\mathrm{ZX}}(p_1, p_3) = D_H^{\mathrm{ZX}}(p_2, p_3) = 0.6184$ and $D_{G,1}^{\mathrm{C}}(p_1, p_3) = D_{G,1}^{\mathrm{C}}(p_2, p_3) = 0.6184$. Namely, the Hamming distances between p_1 andp_3 and between p_2 andp_3 are the same. From $\mu_{P_1}(x) = \nu_{P_2}(x)(= 0.35)$ and $\nu_{P_1}(x) = \mu_{P_2}(x)(= 0.42)$, it is inferred that $\tau_{P_1}(x) = \tau_{P_2}(x)(= 0.8373)$. Thus, using Zhang and Xu's and Chen's measures will yield the same results because D_H^{ZX} and $D_{G,\beta}^{\mathrm{C}}$ only involve the following components: the membership degree μ_p, the non-membership degree ν_p, and the indeterminacy degree τ_p. However, the results that are discussed above are unreasonable and unacceptable. Although p_1 andp_2 have the same strength of 0.5467, they have different directions of 0.4423 and 0.5577. The degrees μ_p, ν_p, and τ_p in a Pythagorean membership grade p do not decompose the interval of [0,1] linearly. Hence, the direction of commitment d_p plays a key role in determining PF distance measures. Because $d_{P_1}(x) \neq d_{P_2}(x)$, it is reasonable to anticipate that $D_H^{\mathrm{ZX}}(p_1, p_3) \neq D_H^{\mathrm{ZX}}(p_2, p_3)$ and $D_{G,1}^{\mathrm{C}}(p_1, p_3) \neq D_{G,1}^{\mathrm{C}}(p_2, p_3)$. Nevertheless, Zhang and Xu's and Chen's measures lead to conflicting results ($D_H^{\mathrm{ZX}}(p_1, p_3) = D_H^{\mathrm{ZX}}(p_2, p_3)$ and $D_{G,1}^{\mathrm{C}}(p_1, p_3) = D_{G,1}^{\mathrm{C}}(p_2, p_3)$), which are unconvincing.

Example 3 (Consideration of the Direction of Commitment): This example is continued from Example 2. Based on the distance measures of Li and Zeng [17], Zeng *et al.* [18], and this paper, the Hamming distances between p_1 andp_3 were calculated as follows: $D_{G,1}^{\mathrm{LZ}}(p_1, p_3) = 0.2493$, $D_{G,1}^{\mathrm{ZLY}}(p_1, p_3) = 0.3093$, and $D_H(p_1, p_3) = D_{G,1}(p_1, p_3) = 0.4143$. The results between p_2 andp_3 were calculated as follows: $D_{G,1}^{\mathrm{LZ}}(p_2, p_3) = 0.2752$, $D_{G,1}^{\mathrm{ZLY}}(p_2, p_3) = 0.3301$, and $D_H(p_2, p_3) = D_{G,1}(p_2, p_3) = 0.4488$. It is observed that $D_{G,1}^{\mathrm{LZ}}(p_1, p_3) \neq D_{G,1}^{\mathrm{LZ}}(p_2, p_3)$, $D_{G,1}^{\mathrm{ZLY}}(p_1, p_3) \neq D_{G,1}^{\mathrm{ZLY}}(p_2, p_3)$, $D_H(p_1, p_3) \neq D_H(p_2, p_3)$, and $D_{G,1}(p_1, p_3) \neq D_{G,1}(p_2, p_3)$. Therefore, in contrast to D_H^{ZX} and $D_{G,1}^{\mathrm{C}}$, measures $D_{G,1}^{\mathrm{LZ}}$, $D_{G,1}^{\mathrm{ZLY}}$, D_H, and $D_{G,1}$ can differentiate between p_1 andp_2 even though the two Pythagorean membership grades have the same strength of 0.5467.

Example 4 (Distance from the Complement): Let $p = (0.77, 0.43; 0.8819, 0.6758)$ be a Pythagorean membership grade in X. The complement of p is $p^c = (0.43, 0.77; 0.8819, 0.3242)$. Regarding the distance between p and p^c, the following results were obtained: $D_H^{\mathrm{ZX}}(p, p^c) = D_{G,1}^{\mathrm{C}}(p, p^c) = 0.4080$, $D_{G,1}^{\mathrm{LZ}}(p, p^c) = 0.2579$, $D_{G,1}^{\mathrm{ZLY}}(p, p^c) = 0.2063$, and $D_H(p, p^c) = D_{G,1}(p, p^c) = 0.3892$ based on the Hamming distance model; and $D_E^{\mathrm{RXG}}(p, p^c) = D_{G,2}^{\mathrm{C}}(p, p^c) = 0.4080$, $D_{G,2}^{\mathrm{LZ}}(p, p^c) = 0.2978$, $D_{G,2}^{\mathrm{ZLY}}(p, p^c) = 0.2664$, and $D_E(p, p^c) = D_{G,2}(p, p^c) = 0.3901$ based on the Euclidean distance model. $D_{G,1}^{\mathrm{C}}(p, p^c) = D_{G,2}^{\mathrm{C}}(p, p^c) = 0.4080$ (or, equivalently, $D_H^{\mathrm{ZX}}(p, p^c) = D_E^{\mathrm{RXG}}(p, p^c) = 0.4080$). Moreover, in this example, $D_{G,\beta}^{\mathrm{C}}(p, p^c) = 0.4080$ if $1 \leq \beta \leq 790$; hence, the parameter β does not affect the distance between p and p^c. Because Chen [27] merely considered the degrees of membership, non-membership, and indeterminacy in defining the generalized distance measures, $D_{G,\beta}^{\mathrm{C}}$ cannot properly capture the separation between a Pythagorean membership grade and its complement. Such results are unsuitable for use by decision makers. In contrast, the parameter β can exert its effect sufficiently in generalized distance measures $D_{G,\beta}^{\mathrm{LZ}}$, $D_{G,\beta}^{\mathrm{ZLY}}$, and $D_{G,\beta}$ (e.g., $D_{G,1}^{\mathrm{LZ}}(p, p^c) \neq D_{G,2}^{\mathrm{LZ}}(p, p^c)$, $D_{G,1}^{\mathrm{ZLY}}(p, p^c) \neq D_{G,2}^{\mathrm{ZLY}}(p, p^c)$, and $D_{G,1}(p, p^c) \neq D_{G,2}(p, p^c)$). Nevertheless, the normalized distances that use the measures $D_{G,\beta}^{\mathrm{LZ}}$ and $D_{G,\beta}^{\mathrm{ZLY}}$ are smaller compared to other measures (e.g., 0.2063 and 0.2664 based on $D_{G,1}^{\mathrm{ZLY}}$ and $D_{G,2}^{\mathrm{ZLY}}$, respectively). As discussed previously, the unsuitable normalization approaches in $D_{G,\beta}^{\mathrm{LZ}}$ and $D_{G,\beta}^{\mathrm{ZLY}}$ will lead to invalid results because they underestimate the PF distances. In contrast to $D_{G,\beta}^{\mathrm{LZ}}$ and $D_{G,\beta}^{\mathrm{ZLY}}$, $D_{G,1}(p, p^c) = 0.3892$ and $D_{G,2}(p, p^c) = 0.3901$ (or, equivalently, $D_H(p, p^c) = 0.3892$ and $D_E(p, p^c) = 0.3901$). Therefore, in contrast to the compared distance measures, the proposed measures, namely, D_H, D_E, and $D_{G,\beta}$, yield reasonable and acceptable results.

The useful and desirable properties of the developed PF distance measures, namely, D_H, D_E, and $D_{G,\beta}$, have been specified in Theorems 1−6. The numerical comparisons and discussions in Examples 1−4 have further demonstrated that the proposed measures outperform D_H^{ZX}, D_E^{RXG}, $D_{G,\beta}^{\mathrm{C}}$, $D_{G,\beta}^{\mathrm{LZ}}$, and $D_{G,\beta}^{\mathrm{ZLY}}$.

V. PF-DISTANCE-BASED COMPROMISE APPROACH

This section attempts to utilize the proposed distance measures to develop a simple and effective MCDA method that is based on PF sets. This section formulates an MCDA problem within the PF decision environment and presents the characteristics of the ideal PF solutions. Based on the proposed measures, namely, D_H, D_E, and $D_{G,\beta}$, the weighted PF distance measures towards the ideal PF solutions are determined to establish a closeness-based precedence index. A novel PF-distance-based compromise approach is proposed for addressing MCDA problems that involve PF information.

An MCDA problem can be expressed as a decision matrix whose entries are the evaluative ratings of candidate alternatives with respect to each criterion. Consider the following MCDA problem in the PF context: Let $Z = \{z_1, z_2, \cdots, z_m\}$ be a discrete set of m candidate alternatives and let $C = \{c_1, c_2, \cdots, c_n\}$ be a finite set of n evaluative criteria, where integers $m, n \geq 2$. Based on each criterion, the set C can be divided into C_{I} (the set of benefit criteria) and C_{II} (the set of cost criteria), where $C_{\mathrm{I}} \cap C_{\mathrm{II}} = \emptyset$ and $C_{\mathrm{I}} \cup C_{\mathrm{II}} = C$. Let w $= (w_1, w_2, \cdots, w_n)$ be the weight vector of the criteria, which satisfies $0 \leq w_j \leq 1$ $(j \in \{1, 2, \cdots, n\})$ and $\sum_{j=1}^{n} w_j = 1$. Let a Pythagorean membership grade $p_{ij} = (\mu_{ij}, \nu_{ij}; r_{ij}, d_{ij})$ denote the evaluative rating of an alternative $z_i \in Z$ with respect to a criterion $c_j \in C$. The MCDA problem within the PF environment can be represented in matrix form as follows:

$$\mathrm{p} = [p_{ij}]_{m\times n} = \left[(\mu_{ij}, \nu_{ij}; r_{ij}, d_{ij})\right]_{m\times n} = \begin{bmatrix} (\mu_{11}, \nu_{11}; r_{11}, d_{11}) & (\mu_{12}, \nu_{12}; r_{12}, d_{12}) & \cdots & (\mu_{1n}, \nu_{1n}; r_{1n}, d_{1n}) \\ (\mu_{21}, \nu_{21}; r_{21}, d_{21}) & (\mu_{22}, \nu_{22}; r_{22}, d_{22}) & \cdots & (\mu_{2n}, \nu_{2n}; r_{2n}, d_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (\mu_{m1}, \nu_{m1}; r_{m1}, d_{m1}) & (\mu_{m2}, \nu_{m2}; r_{m2}, d_{m2}) & \cdots & (\mu_{mn}, \nu_{mn}; r_{mn}, d_{mn}) \end{bmatrix}. \tag{20}$$

The characteristic of an alternative $z_i \in Z$ can be expressed as a PF set P_i, which is characterized by a set $(\mu_{ij}, \nu_{ij}; r_{ij}, d_{ij})$ for all $c_j \in C$ as follows:

$$P_i = \left\{\left\langle c_j, p_{ij}\right\rangle \middle| \, c_j \in C\right\} = \left\{\left\langle c_j, (\mu_{ij}, \nu_{ij}; r_{ij}, d_{ij})\right\rangle \middle| \, c_j \in C\right\}, \tag{21}$$

which satisfies the condition $0 \leq (\mu_{ij})^2 + (\nu_{ij})^2 \leq 1$. Moreover, $r_{ij} = \sqrt{(\mu_{ij})^2 + (\nu_{ij})^2}$, $\theta_{ij} = \arccos(\mu_{ij}/r_{ij}) = \arcsin(\nu_{ij}/r_{ij})$, and $d_{ij} = (\pi - 2 \cdot \theta_{ij})/\pi$.

For the PF decision matrix p in an MCDA problem, let z_* denote the positive-ideal PF solution. The characteristic P_* of z_* is defined as follows:

$$\begin{aligned} P_* &= \left\{\left\langle c_j, p_{*j}\right\rangle \middle| \, c_j \in C\right\} \\ &= \left\{\left\langle c_j, (\mu_{*j}, \nu_{*j}; r_{*j}, d_{*j})\right\rangle \middle| \, c_j \in C\right\}, \end{aligned} \tag{22}$$

where $p_{*j} = (\mu_{*j}, \nu_{*j}; r_{*j}, d_{*j})$ denotes the evaluative rating of z_* regarding a criterion c_j. The parameters that correspond to p_{*j} are computed as follows:

$$\mu_{*j} = \begin{cases} \max_{i=1}^{m} \mu_{ij} & \text{if} c_j \in C_{\mathrm{I}}, \\ \min_{i=1}^{m} \mu_{ij} & \text{if} c_j \in C_{\mathrm{II}}, \end{cases} \tag{23}$$

$$\nu_{*j} = \begin{cases} \min_{i=1}^{m} \nu_{ij} & \text{if} c_j \in C_{\mathrm{I}}, \\ \max_{i=1}^{m} \nu_{ij} & \text{if} c_j \in C_{\mathrm{II}}, \end{cases} \tag{24}$$

$r_{*j} = \sqrt{(\mu_{*j})^2 + (\nu_{*j})^2}$, and $d_{*j} = (\pi - 2 \cdot \theta_{*j})/\pi$, where $\theta_{*j} = \arccos(\mu_{*j}/r_{*j}) = \arcsin(\nu_{*j}/r_{*j})$. The condition $0 \leq (\mu_{*j})^2 + (\nu_{*j})^2 \leq 1$ holds for all $c_j \in C$.

Let $z_\#$ denote the negative-ideal PF solution in p. The characteristic $P_\#$ of $z_\#$ is defined as follows:

$$\begin{aligned} P_\# &= \left\{\left\langle c_j, p_{\#j}\right\rangle \middle| \, c_j \in C\right\} \\ &= \left\{\left\langle c_j, (\mu_{\#j}, \nu_{\#j}; r_{\#j}, d_{\#j})\right\rangle \middle| \, c_j \in C\right\}, \end{aligned} \tag{25}$$

where $p_{\#j} = (\mu_{\#j}, \nu_{\#j}; r_{\#j}, d_{\#j})$ denotes the evaluative rating of $z_\#$ with respect to a criterion c_j. The parameters that correspond to $p_{\#j}$ are computed as follows:

$$\mu_{\#j} = \begin{cases} \min_{i=1}^{m} \mu_{ij} & \text{if} c_j \in C_{\mathrm{I}}, \\ \max_{i=1}^{m} \mu_{ij} & \text{if} c_j \in C_{\mathrm{II}}, \end{cases} \tag{26}$$

$$\nu_{\#j} = \begin{cases} \max_{i=1}^{m} \nu_{ij} & \text{if} c_j \in C_{\mathrm{I}}, \\ \min_{i=1}^{m} \nu_{ij} & \text{if} c_j \in C_{\mathrm{II}}, \end{cases} \tag{27}$$

$r_{\#j} = \sqrt{(\mu_{\#j})^2 + (\nu_{\#j})^2}$, and $d_{\#j} = (\pi - 2 \cdot \theta_{\#j})/\pi$, where $\theta_{\#j} = \arccos(\mu_{\#j}/r_{\#j}) = \arcsin(\nu_{\#j}/r_{\#j})$. Moreover, the condition $0 \leq (\mu_{\#j})^2 + (\nu_{\#j})^2 \leq 1$ is satisfied for all $c_j \in C$.

By utilizing the proposed measure $D_{G,\beta}$, the weighted generalized PF distance between the characteristics P_i and P_* is calculated as follows:

$$D^{W}_{G,\beta}(P_i, P_*) = \left[\frac{1}{3}\sum_{j=1}^{n} w_j \left(\left|(\mu_{ij})^2 - (\mu_{*j})^2\right|^{\beta} + \left|(\nu_{ij})^2 - (\nu_{*j})^2\right|^{\beta} + \left|(r_{ij})^2 - (r_{*j})^2\right|^{\beta} + \left|d_{ij} - d_{*j}\right|^{\beta}\right)\right]^{\frac{1}{\beta}}. \tag{28}$$

When the positive-ideal PF solution z_* is employed to facilitate anchored judgments, the smaller $D^{W}_{G,\beta}(P_i, P_*)$ is, the higher the performance of alternative z_i. For $\beta = 1$ and $\beta = 2$, the weighted measure $D^{W}_{G,\beta}$ reduces to the weighted Hamming and Euclidean PF distances, respectively, where $D^{W}_{H}(P_i, P_*)$ $(= D^{W}_{G,1}(P_i, P_*))$ and $D^{W}_{E}(P_i, P_*)$ $(= D^{W}_{G,2}(P_i, P_*))$ are expressed as follows:

$$D^{W}_{H}(P_i, P_*) = \frac{1}{3}\sum_{j=1}^{n} w_j \left(\left|(\mu_{ij})^2 - (\mu_{*j})^2\right| + \left|(\nu_{ij})^2 - (\nu_{*j})^2\right|\right.$$

$$+\left|(r_{ij})^2-(r_{*j})^2\right|+\left|d_{ij}-d_{*j}\right|\Big), \tag{29}$$

$$D_E^W(P_i,P_*)=\left\{\frac{1}{3}\sum_{j=1}^{n}w_j\left[\left((\mu_{ij})^2-(\mu_{*j})^2\right)^2+\left((\nu_{ij})^2-(\nu_{*j})^2\right)^2+\left((r_{ij})^2-(r_{*j})^2\right)^2+\left(d_{ij}-d_{*j}\right)^2\right]\right\}^{\frac{1}{2}}. \tag{30}$$

Similarly, the weighted generalized PF distance between the characteristics P_i and $P_\#$ is calculated as follows:

$$D_{G,\beta}^W(P_i,P_\#)=\left[\frac{1}{3}\sum_{j=1}^{n}w_j\left(\left|(\mu_{ij})^2-(\mu_{\#j})^2\right|^\beta+\left|(\nu_{ij})^2-(\nu_{\#j})^2\right|^\beta+\left|(r_{ij})^2-(r_{\#j})^2\right|^\beta+\left|d_{ij}-d_{\#j}\right|^\beta\right)\right]^{\frac{1}{\beta}}. \tag{31}$$

When the negative-ideal PF solution $z_\#$ is employed to facilitate anchored judgments, the larger $D_{G,\beta}^W(P_i,P_\#)$ is, the higher the performance of alternative z_i. Moreover, the weighted Hamming PF distance $D_H^W(P_i,P_\#)$ ($=D_{G,1}^W(P_i,P_\#)$) and the weighted Euclidean PF distance $D_E^W(P_i,P_\#)$ ($=D_{G,2}^W(P_i,P_\#)$) are expressed as follows:

$$D_H^W(P_i,P_\#)=\frac{1}{3}\sum_{j=1}^{n}w_j\Big(\left|(\mu_{ij})^2-(\mu_{\#j})^2\right|+\left|(\nu_{ij})^2-(\nu_{\#j})^2\right|+\left|(r_{ij})^2-(r_{\#j})^2\right|+\left|d_{ij}-d_{\#j}\right|\Big), \tag{32}$$

$$D_E^W(P_i,P_\#)=\left\{\frac{1}{3}\sum_{j=1}^{n}w_j\left[\left((\mu_{ij})^2-(\mu_{\#j})^2\right)^2+\left((\nu_{ij})^2-(\nu_{\#j})^2\right)^2+\left((r_{ij})^2-(r_{\#j})^2\right)^2+\left(d_{ij}-d_{\#j}\right)^2\right]\right\}^{\frac{1}{2}}. \tag{33}$$

The characteristic P_i that is closest to P_* does not accord with the characteristic that is farthest from $P_\#$. To address this issue, this paper establishes a closeness-based precedence index that is similar to the closeness coefficient (the relative closeness to the ideal solutions) in TOPSIS methods. Based on the generalized distance model, the closeness-based precedence index, which is denoted as $CI_{G,\beta}$, for each alternative $z_i\in Z$ is defined as follows:

$$CI_{G,\beta}(P_i)=\frac{D_{G,\beta}^W(P_i,P_\#)}{D_{G,\beta}^W(P_i,P_*)+D_{G,\beta}^W(P_i,P_\#)}. \tag{34}$$

It follows that $0\le CI_{G,\beta}(P_i)\le 1$. If $P_i=P_*$, then $CI_{G,\beta}(P_i)=1$; if $P_i=P_\#$, then $CI_{G,\beta}(P_i)=0$. Moreover, the larger $CI_{G,\beta}(P_i)$ is, the better the characteristic P_i performs and the greater the preference is for alternative z_i. Accordingly, by setting a suitable value of the distance parameter β, the precedence ranks among candidate alternatives can be obtained based on the descending order of the $CI_{G,\beta}(P_i)$ values in the PF-distance-based compromise approach.

For each alternative z_i, the closeness-based precedence indices CI_H and CI_E, which are based on the Hamming and Euclidean distance models, respectively, are defined as follows:

$$CI_H(P_i)=CI_{G,1}(P_i)=\frac{D_H^W(P_i,P_\#)}{D_H^W(P_i,P_*)+D_H^W(P_i,P_\#)}, \tag{35}$$

$$CI_E(P_i)=CI_{G,2}(P_i)=\frac{D_E^W(P_i,P_\#)}{D_E^W(P_i,P_*)+D_E^W(P_i,P_\#)}. \tag{36}$$

Analogously, $0\le CI_H(P_i), CI_E(P_i)\le 1$. If $P_i=P_*$, then $CI_H(P_i)=1$ (or $CI_E(P_i)=1$); if $P_i=P_\#$, then $CI_H(P_i)=0$ (or $CI_E(P_i)=0$). The larger $CI_H(P_i)$ (or $CI_E(P_i)$) is, the better the characteristic P_i performs and the greater the preference is for alternative z_i. Therefore, the precedence relationships among competing alternatives can be effectively determined according to the descending order of the $CI_H(P_i)$ (or $CI_E(P_i)$) values.

The proposed PF-distance-based compromise approach for addressing MCDA problems under complex uncertainty based on PF sets can be implemented using a simple and effective algorithm. This algorithm is comprised of six phases: formulating an MCDA problem, collecting relevant decision information, identifying the characteristics of ideals, calculating the separation between the characteristics, obtaining the closeness-based precedence indices, and identifying the precedence relationships. The six phases are implemented via the following steps:

Step 1 (Formulate an MCDA problem): Specify the set of candidate alternatives $Z=\{z_1,z_2,\cdots,z_m\}$. Identify the set of evaluative criteria $C=\{c_1,c_2,\cdots,c_n\}$, which is divided into a set of benefit criteriaC_{I} and a set of cost criteriaC_{II}.

Step 2 (Collect relevant decision information): Establish the weight vector $\mathrm{w}=(w_1,w_2,\cdots,w_n)$, where $\sum_{j=1}^n w_j=1$. Construct the PF evaluative rating $p_{ij}=(\mu_{ij},\nu_{ij};r_{ij},d_{ij})$ of $z_i\in Z$ with respect to $c_j\in C$. Form a PF decision matrix $\mathrm{p}=[p_{ij}]_{m\times n}$ and calculate the characteristic P_i of each z_i.

Step 3 (Identify the characteristics of the ideals): Determine the evaluative ratings $p_{*j}=(\mu_{*j},\nu_{*j};r_{*j},d_{*j})$ and $p_{\#j}=(\mu_{\#j},\nu_{\#j};r_{\#j},d_{\#j})$ for all $c_j\in C$ to identify the characteristics P_* and $P_\#$ of the positive-ideal PF solution z_* and the negative-ideal PF solution $z_\#$, respectively.

Step 4 (Calculate the separation between the characteristics): Employ the weighted measures D_H^W ($D_{G,1}^W$), D_E^W ($D_{G,2}^W$), and $D_{G,\beta}^W$ to compute the weighted PF distances between P_i and P_* and between P_i and $P_\#$.

Step 5 (Obtain the closeness-based precedence indices): Compute the closeness-based precedence indices $CI_H(P_i)$ (based on D_H^W), $CI_E(P_i)$ (based on D_E^W), and $CI_{G,\beta}(P_i)$ (based on $D_{G,\beta}^W$) of the characteristic P_i for each $z_i\in Z$.

Step 6 (Identify the precedence relationships): Rank the m alternatives according to the descending order of the $CI_H(P_i)$,

$CI_E(P_i)$, or $CI_{G,\beta}(P_i)$ values to identify the precedence relationships among the alternatives in Z. The alternative that has the largest closeness-based precedence index is the optimal solution.

VI. APPLICATION AND COMPARITIVE STUDIES

This section applies the new PF-distance-based compromise approach to a real-world case of bridge-superstructure construction methods to evaluate the usefulness and practicability of the proposed methodology. Moreover, a sensitivity analysis is implemented to investigate the results that are obtained using several distance measures under various parameter settings. Finally, comparative studies are conducted to compare the proposed methodology to other relevant approaches.

A. PRACTICAL APPLICATION

The real-world case, which was adopted from Chen [15], involves the construction of a concrete-based bridge superstructure for the Suhua Highway Alternative Road Project in Taiwan. Because construction methods for building bridge superstructures differ in terms of their construction characteristics, applicable environments, construction costs, and construction durations, the assessment of bridge-superstructure construction methods has a decisive effect on successful bridge construction [37]. This case study aims at addressing the selection problem of bridge-superstructure construction methods under complex uncertainty based on PF sets.

In Step 1, the MCDA problem of bridge-superstructure construction was formulated based on a set of candidate alternatives, namely, $Z = \{z_1, z_2, z_3, z_4\}$, and a set of evaluative criteria, namely, $C = \{c_1, c_2, \cdots, c_8\}$. Set Z consists of four commonly used bridge-superstructure construction methods: the advanced shoring method (z_1), the incremental launching method (z_2), the balanced cantilever method (z_3), and the precast segmental method (z_4). Set C is comprised of the durability (c_1), damage cost (c_2), construction cost (c_3), traffic effect (c_4), site condition (c_5), climatic condition (c_6), landscape (c_7), and environmental impact (c_8). Set C is divided into $C_{\mathrm{I}} = \{c_1, c_5\}$ and $C_{\mathrm{II}} = \{c_2, c_3, c_4, c_6, c_7, c_8\}$, namely, the durability and site condition are benefit criteria, while the remaining six criteria are cost criteria.

In Step 2, based on the bridge-superstructure construction case that was presented by Chen [15], the weight vector for the eight criteria was defined as follows: w=(0.1404, 0.1252, 0.1090, 0.0839, 0.1361, 0.1252, 0.1408, 0.1394). Moreover, the data of the PF evaluative rating $p_{ij} = (\mu_{ij}, \nu_{ij}; r_{ij}, d_{ij})$ for all $z_i \in Z$ and $c_j \in C$ are listed in Table 1. Additionally, the values of τ_{ij} and θ_{ij} that are associated with each p_{ij} are listed in the last two columns. The PF decision matrix, namely, $\mathrm{p} = [p_{ij}]_{4\times 8}$, and the characteristic P_i of each alternative can be constructed based on all the PF evaluative ratings. For example, according to (21), the characteristic of the advanced shoring method was as follows: $P_1 = \{\langle c_1, (0.5407, 0.6781; 0.8673, 0.4285)\rangle, \langle c_2, (0.8075, 0.2638; 0.8495, 0.7990)\rangle, \langle c_3, (0.1654, 0.9508; 0.9651, 0.1096)\rangle, \langle c_4, (0.4802, 0.6774; 0.8303, 0.3926)\rangle, \langle c_5, (0.9285, 0.3402; 0.9889, 0.7764)\rangle, \langle c_6, (0.1443, 0.9485; 0.9594, 0.0961)\rangle, \langle c_7, (0.2848, 0.5778; 0.6442, 0.2915)\rangle, \langle c_8, (0.0727, 0.9961; 0.9987, 0.0464)\rangle\}$.

TABLE 1. PF evaluative ratings in PF decision matrix p.

z_i	c_j	$p_{ij}=(\mu_{ij}, \nu_{ij}; r_{ij}, d_{ij})$	τ_{ij}	θ_{ij}
z_1	c_1	(0.5407, 0.6781; 0.8673, 0.4285)	0.4978	0.8977
	c_2	(0.8075, 0.2638; 0.8495, 0.7990)	0.5276	0.3158
	c_3	(0.1654, 0.9508; 0.9651, 0.1096)	0.2618	1.3986
	c_4	(0.4802, 0.6774; 0.8303, 0.3926)	0.5573	0.9541
	c_5	(0.9285, 0.3402; 0.9889, 0.7764)	0.1488	0.3512
	c_6	(0.1443, 0.9485; 0.9594, 0.0961)	0.2818	1.4198
	c_7	(0.2848, 0.5778; 0.6442, 0.2915)	0.7649	1.1128
	c_8	(0.0727, 0.9961; 0.9987, 0.0464)	0.0504	1.4979
z_2	c_1	(0.9550, 0.2198; 0.9800, 0.8560)	0.1992	0.2262
	c_2	(0.1050, 0.9644; 0.9701, 0.0690)	0.2427	1.4623
	c_3	(0.2871, 0.6480; 0.7088, 0.2655)	0.7054	1.1537
	c_4	(0.2504, 0.9371; 0.9700, 0.1662)	0.2432	1.3097
	c_5	(0.6526, 0.4723; 0.8056, 0.6012)	0.5925	0.6265
	c_6	(0.2359, 0.9692; 0.9975, 0.1520)	0.0701	1.3320
	c_7	(0.8861, 0.3390; 0.9487, 0.7674)	0.3159	0.3654
	c_8	(0.4863, 0.3709; 0.6116, 0.5852)	0.7912	0.6516
z_3	c_1	(0.9665, 0.2400; 0.9959, 0.8450)	0.0908	0.2434
	c_2	(0.0666, 0.9452; 0.9475, 0.0448)	0.3195	1.5005
	c_3	(0.1795, 0.7727; 0.7933, 0.1453)	0.6088	1.3425
	c_4	(0.6608, 0.6856; 0.9522, 0.4883)	0.3056	0.8038
	c_5	(0.9285, 0.3402; 0.9889, 0.7764)	0.1488	0.3512
	c_6	(0.4213, 0.5224; 0.6711, 0.4321)	0.7414	0.8921
	c_7	(0.2496, 0.9579; 0.9899, 0.1623)	0.1417	1.3159
	c_8	(0.5887, 0.7902; 0.9854, 0.4076)	0.1704	0.9305
z_4	c_1	(0.7324, 0.6090; 0.9525, 0.5584)	0.3045	0.6937
	c_2	(0.4794, 0.8505; 0.9763, 0.3268)	0.2165	1.0575
	c_3	(0.5830, 0.7978; 0.9881, 0.4018)	0.1534	0.9397
	c_4	(0.4900, 0.6619; 0.8235, 0.4057)	0.5673	0.9335
	c_5	(0.3881, 0.7003; 0.8007, 0.3222)	0.5991	1.0647
	c_6	(0.8540, 0.2077; 0.8789, 0.8481)	0.4770	0.2386
	c_7	(0.1127, 0.9902; 0.9966, 0.0721)	0.0830	1.4575
	c_8	(0.2174, 0.9720; 0.9960, 0.1401)	0.0895	1.3508

In Step 3, the membership degree, namely, μ_{*j}, and the non-membership degree, namely, ν_{*j}, within p_{*j} can be obtained via (23) and (24), respectively. Moreover, the degrees $\mu_{\#j}$ and $\nu_{\#j}$ within $p_{\#j}$ can be calculated via (26) and (27), respectively. Based on these results, the characteristic of the positive-ideal PF solution z_* was determined via (22) as follows: $P_* = \{\langle c_1, (0.9665, 0.2198; 0.9912, 0.8576)\rangle, \langle c_2, (0.0666, 0.9644; 0.9667, 0.0439)\rangle, \langle c_3, (0.1654, 0.9508; 0.9651, 0.1096)\rangle, \langle c_4, (0.2504, 0.9371; 0.9700, 0.1662)\rangle, \langle c_5, (0.9285, 0.3402; 0.9889, 0.7764)\rangle, \langle c_6, (0.1443, 0.9692; 0.9799, 0.0941)\rangle, \langle c_7, (0.1127, 0.9902; 0.9966, 0.0721)\rangle, \langle c_8, (0.0727, 0.9961; 0.9987, 0.0464)\rangle\}$. Moreover, via (25), the characteristic of the negative-ideal PF solution $z_\#$ was determined as follows: $P_\# = \{\langle c_1, (0.5407, 0.6781; 0.8673, 0.4285)\rangle, \langle c_2, (0.8075, 0.2638; 0.8495, 0.7990)\rangle, \langle c_3, (0.5830, 0.6480; 0.8717, 0.4664)\rangle, \langle c_4, (0.6608, 0.6619; 0.9353, 0.4995)\rangle, \langle c_5, (0.3881, 0.7003; 0.8007, 0.3222)\rangle, \langle c_6, (0.8540, 0.2077; 0.8789, 0.8481)\rangle, \langle c_7, (0.8861, 0.3390; 0.9487, 0.7674)\rangle, \langle c_8, (0.5887, 0.3709; 0.6958, 0.6421)\rangle\}$.

In Step 4, considering the universality and practicability of the Hamming and Euclidean distance models, the distance

TABLE 2. Separation measures and closeness-based precedence indices.

z_i	$D_H^W(P_i,P_*)$	Rank	$D_H^W(P_i,P_\#)$	Rank	$CI_H(P_i)$	Rank
z_1	0.2877	2	0.4462	2	0.6080	2
z_2	0.3158	3	0.3946	3	0.5555	3
z_3	0.1908	1	0.5244	1	0.7332	1
z_4	0.3467	4	0.3471	4	0.5003	4
	$D_E^W(P_i,P_*)$	Rank	$D_E^W(P_i,P_\#)$	Rank	$CI_E(P_i)$	Rank
z_1	0.2740	2	0.4122	2	0.6007	2
z_2	0.3054	3	0.3727	3	0.5496	3
z_3	0.1895	1	0.4946	1	0.7230	1
z_4	0.3254	4	0.3287	4	0.5025	4

parameter in the separation measures was set to $\beta = 1$ and $\beta = 2$. Based on the weighted measure D_H^W ($D_{G,1}^W$), the calculation results of the weighted Hamming PF distances $D_H^W(P_i, P_*)$ and $D_H^W(P_i, P_\#)$ were obtained via (29) and (32), respectively, and are listed in the top part of Table 2. Based on the weighted measure D_E^W ($D_{G,2}^W$), the results of the weighted Euclidean PF distances $D_E^W(P_i, P_*)$ and $D_E^W(P_i, P_\#)$ were calculated via (30) and (33), respectively, and are listed in the bottom part of Table 2.

In Step 5, the closeness-based precedence indices, namely, $CI_H(P_i)$ and $CI_E(P_i)$, were derived by employing (35) and (36), respectively. The computed results are listed in Table 2. In Step 6, according to the descending order of the $CI_H(P_i)$ values, the precedence relationships among the four bridge-superstructure construction methods were determined: $z_3 \succ z_1 \succ z_2 \succ z_4$. The same final ranking result was obtained based on the $CI_E(P_i)$ values. Furthermore, Table 2 lists the ranking orders of the four alternatives according to the ascending order of the $D_H^W(P_i, P_*)$ (or $D_E^W(P_i, P_*)$) values. In addition, this table presents the ranking results that are based on the descending order of the $D_H^W(P_i, P_\#)$ (or $D_E^W(P_i, P_\#)$) values. Using D_H^W or D_E^W yielded the same ranking result: $z_3 \succ z_1 \succ z_2 \succ z_4$. Therefore, the balanced cantilever method (z_3) is indeed the optimal choice for the Suhua Highway Alternative Road Project.

B. SENSITIVITY ANALYSIS

Using the proposed PF-distance-based compromise approach that is based on the generalized PF distance measure $D_{G,\beta}$, this subsection conducts a sensitivity analysis to investigate the influence of the distance parameter β on the solution results.

To explore the application results that are yielded by measure $D_{G,\beta}$ under various parameter settings, the following values of the distance parameter were considered in the sensitivity analysis: $\beta = 1, 2, 5, 10, 20, 50, 100, 200, 500, 1000$, and 2000. For the bridge-superstructure construction case, the results of the sensitivity analysis for the various β values on the separation measures between P_i and P_* and between P_i and $P_\#$ are presented graphically in Figures 2 and 3, respectively. The patterns among the four alternatives are moderately stable when $\beta \leq 500$. However, the distributions of the separation measures show irregular patterns when

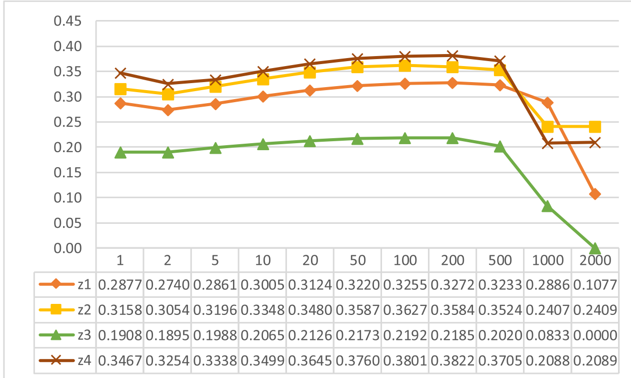


FIGURE 2. Sensitivity analysis of the separation between P_i and P_*.

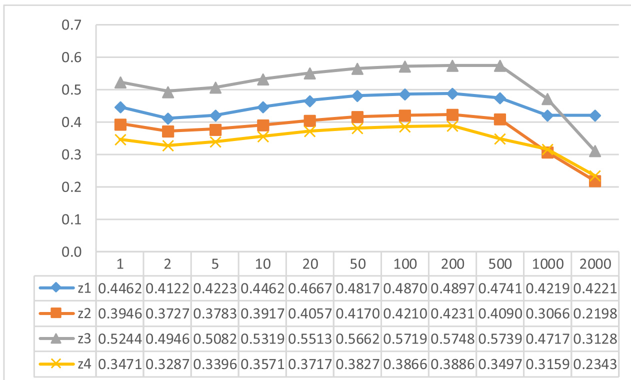
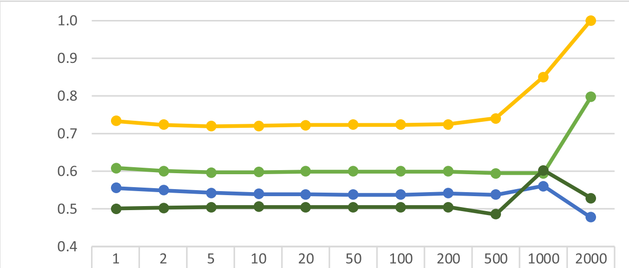


FIGURE 3. Sensitivity analysis of the separation between P_i and $P_\#$.

$\beta > 500$. The change in the distribution patterns affects the priority orders of the alternatives. As shown in Figure 2, the consistent ranking result of $z_3 \succ z_1 \succ z_2 \succ z_4$ was obtained based on the ascending order of the $D_{G,\beta}^W(P_i, P_*)$ values for $\beta \leq 500$. However, different ranking results were acquired for $\beta > 500$, e.g., $z_3 \succ z_4 \succ z_2 \succ z_1$ in the case of $\beta = 1000$ and $z_3 \succ z_1 \succ z_4 \succ z_2$ in the case of $\beta = 2000$. According to Figure 3, the identical ranking result of $z_3 \succ z_1 \succ z_2 \succ z_4$ was obtained according to the descending order of the $D_{G,\beta}^W(P_i, P_\#)$ values for $\beta \leq 500$. For $\beta > 500$, distinct ranking results were obtained, e.g., $z_3 \succ z_1 \succ z_4 \succ z_2$ and $z_1 \succ z_3 \succ z_4 \succ z_2$ in the cases of $\beta = 1000$ and $\beta = 2000$, respectively. To acquire more stable and reliable results for facilitating decision making, it is suggested to set the β value within the range of $[1, 500]$ in this case.

The results of the sensitivity analysis on the closeness-based precedence indices are graphically presented in Figure 4. The common final ranking of $z_3 \succ z_1 \succ z_2 \succ z_4$ was derived for $\beta \leq 500$, which indicates the stable and credible precedence relationships among four candidate alternatives. Similar to Figures 2 and 3, the distribution of the closeness-based precedence indices exhibits irregular patterns when $\beta > 500$. For example, two final ranking results of $z_3 \succ z_4 \succ z_1 \succ z_2$ and $z_3 \succ z_1 \succ z_4 \succ z_2$ were obtained in the cases of $\beta = 1000$ and $\beta = 2000$,



	1	2	5	10	20	50	100	200	500	1000	2000
z1	0.6080	0.6007	0.5961	0.5976	0.5990	0.5994	0.5994	0.5994	0.5945	0.5938	0.7968
z2	0.5555	0.5496	0.5420	0.5392	0.5383	0.5375	0.5372	0.5414	0.5372	0.5602	0.4771
z3	0.7332	0.7230	0.7188	0.7203	0.7217	0.7227	0.7229	0.7246	0.7397	0.8498	1.0000
z4	0.5003	0.5025	0.5043	0.5051	0.5049	0.5044	0.5042	0.5041	0.4856	0.6020	0.5286

FIGURE 4. Sensitivity analysis of closeness-based precedence indices.

respectively. Based on the sensitivity analysis of the parameter β on the $CI_{G,\beta}(P_i)$ values for all $z_i \in Z$, it is concluded that alternative z_3 has the highest priority in all final ranking results. Therefore, the balanced cantilever method (z_3) is the most suitable bridge-superstructure construction method for the Suhua Highway Alternative Road Project.

Based on the comparison results of the sensitivity analysis, the separation measures between P_i and P_* and between P_i and $P_\#$ and the closeness-based precedence indices of alternatives correspond to moderately steady and reliable patterns when $\beta \leq 500$. Accordingly, the specification of the distance parameter β within the range of [1, 500] is suitable for facilitating decision support because of the reasonable and creditable results that are obtained via the sensitivity analysis. The Hamming PF distance measure D_H ($D_{G,1}$) and the Euclidean PF distance measure D_E ($D_{G,2}$) belong to the most widely used distance models. It is suggested to set the β value to 1 or 2 for simple computation and convenient application. According to the sensitivity analysis and the comparative study, the effectiveness and usefulness of the proposed methodology in practice are satisfactory because the developed PF-distance-based compromise approach can yield reasonable results and present desirable outcomes to support the authority in decision making. Moreover, the robustness and validity of the proposed methodology have been supported by the results of a comparative study under various parameter settings.

C. COMPARATIVE ANALYSIS AND DISCUSSION

This subsection conducts two comparative studies to evaluate the effectiveness and advantages of the proposed methodology and discusses the results.

To facilitate a consistent comparison, the comparative studies focus on two commonly used distance models: the Hamming and Euclidean distance models. The first comparative study incorporated the Hamming distance model into the core structure of TOPSIS. Developed measure $D_{G,1}$ (Hamming distance measure D_H) is compared with current distance measures D_H^{ZX} ($D_{G,1}^{\mathrm{C}}$) [14], [27], $D_{G,1}^{\mathrm{LZ}}$ [17], and $D_{G,1}^{\mathrm{ZLY}}$ [18]. To determine the separation between P_i and P_*, weighted measures $D_H^{\mathrm{ZX},W}$ ($D_{G,1}^{\mathrm{C},W}$), $D_{G,1}^{\mathrm{LZ},W}$, and $D_{G,1}^{\mathrm{ZLY},W}$ are calculated as follows:

$$D_H^{\mathrm{ZX},W}(P_i,P_*) = \frac{1}{2}\sum_{j=1}^{n} w_j \left(\left|(\mu_{ij})^2-(\mu_{*j})^2\right| + \left|(\nu_{ij})^2-(\nu_{*j})^2\right| + \left|(\tau_{ij})^2-(\tau_{*j})^2\right|\right), \tag{37}$$

$$D_{G,1}^{\mathrm{LZ},W}(P_i,P_*) = \frac{1}{4}\sum_{j=1}^{n} w_j \left(\left|\mu_{ij}-\mu_{*j}\right| + \left|\nu_{ij}-\nu_{*j}\right| + \left|r_{ij}-r_{*j}\right| + \left|d_{ij}-d_{*j}\right|\right), \tag{38}$$

$$D_{G,1}^{\mathrm{ZLY},W}(P_i,P_*) = \frac{1}{5}\sum_{j=1}^{n} w_j \left(\left|\mu_{ij}-\mu_{*j}\right| + \left|\nu_{ij}-\nu_{*j}\right| + \left|\tau_{ij}-\tau_{*j}\right| + \left|r_{ij}-r_{*j}\right| + \left|d_{ij}-d_{*j}\right|\right). \tag{39}$$

The separation measures between P_i and $P_\#$ that are based on $D_H^{\mathrm{ZX},W}$, $D_{G,1}^{\mathrm{LZ},W}$, and $D_{G,1}^{\mathrm{ZLY},W}$ are calculated as follows:

$$D_H^{\mathrm{ZX},W}(P_i,P_\#) = \frac{1}{2}\sum_{j=1}^{n} w_j \left(\left|(\mu_{ij})^2-(\mu_{\#j})^2\right| + \left|(\nu_{ij})^2-(\nu_{\#j})^2\right| + \left|(\tau_{ij})^2-(\tau_{\#j})^2\right|\right), \tag{40}$$

$$D_{G,1}^{\mathrm{LZ},W}(P_i,P_\#) = \frac{1}{4}\sum_{j=1}^{n} w_j \left(\left|\mu_{ij}-\mu_{\#j}\right| + \left|\nu_{ij}-\nu_{\#j}\right| + \left|r_{ij}-r_{\#j}\right| + \left|d_{ij}-d_{\#j}\right|\right), \tag{41}$$

$$D_{G,1}^{\mathrm{ZLY},W}(P_i,P_\#) = \frac{1}{5}\sum_{j=1}^{n} w_j \left(\left|\mu_{ij}-\mu_{\#j}\right| + \left|\nu_{ij}-\nu_{\#j}\right| + \left|\tau_{ij}-\tau_{\#j}\right| + \left|r_{ij}-r_{\#j}\right| + \left|d_{ij}-d_{\#j}\right|\right). \tag{42}$$

Consider the practical application to the bridge-superstructure construction case. The first comparative analysis focuses on the results that are yielded by the proposed methodology based on $D_{G,1}$ and by the TOPSIS-based compromise approaches based on D_H^{ZX}, $D_{G,1}^{\mathrm{LZ}}$, and $D_{G,1}^{\mathrm{ZLY}}$. Figures 5 and 6 present the comparison results of the separation measures between P_i and P_* and between P_i and $P_\#$, respectively, that are based on the Hamming distance model.

Figure 5 compares the weighted Hamming PF distances $D_H^{\mathrm{ZX},W}(P_i,P_*)$ ($D_{G,1}^{\mathrm{C},W}(P_i,P_*)$), $D_{G,1}^{\mathrm{LZ},W}(P_i,P_*)$, $D_{G,1}^{\mathrm{ZLY},W}(P_i,P_*)$ and $D_H^{W}(P_i,P_*)$ ($D_{G,1}^{W}(P_i,P_*)$), which are based on measures D_H^{ZX} ($D_{G,1}^{\mathrm{C}}$) [14], [27], $D_{G,1}^{\mathrm{LZ}}$ [17], $D_{G,1}^{\mathrm{ZLY}}$ [18], and the proposed D_H ($D_{G,1}$), respectively. The smaller the separation measure between P_i and P_*, the closer the alternative z_i is to the positive-ideal PF solution z_*. Then, the four alternatives can be ranked based on the ascending order of the obtained weighted Hamming PF distances. According to Figure 5, $D_H^{\mathrm{ZX},W}(P_i,P_*)$, $D_{G,1}^{\mathrm{LZ},W}(P_i,P_*)$, $D_{G,1}^{\mathrm{ZLY},W}(P_i,P_*)$, and $D_H^{W}(P_i,P_*)$ yield the consistent ranking result of $z_3 \succ z_1 \succ z_2 \succ z_4$.

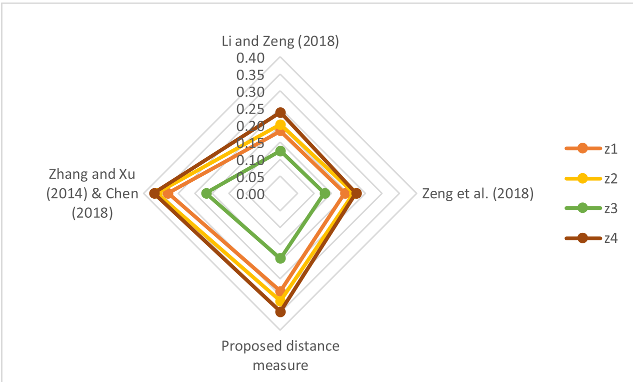


FIGURE 5. Comparison results of the separation between P_i and P_* that are based on the Hamming distance model.

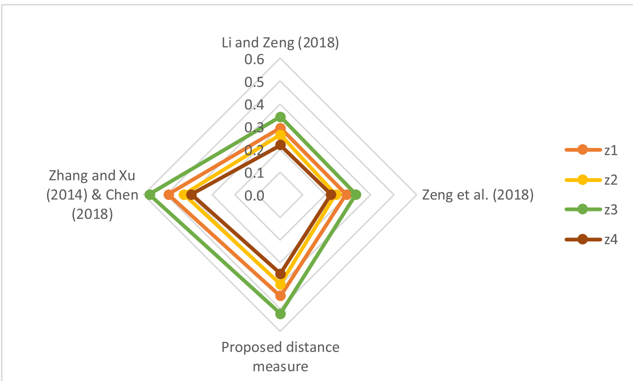


FIGURE 6. Comparison results of the separation between P_i and $P_\#$ that are based on the Hamming distance model.

Figure 6 presents the comparison results of $D_H^{\mathrm{ZX},W}(P_i,P_\#)$ $(D_{G,1}^{\mathrm{C},W}(P_i,P_\#))$, $D_{G,1}^{\mathrm{LZ},W}(P_i,P_\#)$, $D_{G,1}^{\mathrm{ZLY},W}(P_i,P_\#)$, and $D_H^{W}(P_i,P_\#)$ $(D_{G,1}^{W}(P_i,P_\#))$. The larger the separation measure between P_i and $P_\#$, the farther the alternative z_i is from the negative-ideal PF solution $z_\#$. The four alternatives can also be ranked according to the descending order of the obtained weighted Hamming PF distances. $D_H^{\mathrm{ZX},W}(P_i,P_\#)$, $D_{G,1}^{\mathrm{LZ},W}(P_i,P_\#)$, $D_{G,1}^{\mathrm{ZLY},W}(P_i,P_\#)$, and $D_H^{W}(P_i,P_\#)$ all lead to the identical ranking result of $z_3 \succ z_1 \succ z_2 \succ z_4$, which accords with the ranking order in Figure 5.

The second comparative study employed the Euclidean distance model in the core structure of TOPSIS. Proposed measure $D_{G,2}$ (the Euclidean distance measure D_E) is compared with existing distance measures D_E^{RXG} $(D_{G,2}^{\mathrm{C}})$ [27], [36], $D_{G,2}^{\mathrm{LZ}}$ [17], and $D_{G,2}^{\mathrm{ZLY}}$ [18]. The separation measures between P_i and P_* that are based on weighted measures $D_E^{\mathrm{RXG},W}$ $(D_{G,2}^{\mathrm{C},W})$, $D_{G,2}^{\mathrm{LZ},W}$, and $D_{G,2}^{\mathrm{ZLY},W}$ are calculated as follows:

$$D_E^{\mathrm{RXG},W}(P_i,P_*) = \left\{\frac{1}{2}\sum_{j=1}^{n} w_j\left[\left((\mu_{ij})^2-(\mu_{*j})^2\right)^2 + \left((\nu_{ij})^2-(\nu_{*j})^2\right)^2 + \left((\tau_{ij})^2-(\tau_{*j})^2\right)^2\right]\right\}^{\frac{1}{2}}, \quad (43)$$

$$D_{G,2}^{\mathrm{LZ},W}(P_i,P_*) = \left\{\frac{1}{4}\sum_{j=1}^{n} w_j\left[(\mu_{ij}-\mu_{*j})^2 + (\nu_{ij}-\nu_{*j})^2 + (r_{ij}-r_{*j})^2 + (d_{ij}-d_{*j})^2\right]\right\}^{\frac{1}{2}}, \quad (44)$$

$$D_{G,2}^{\mathrm{ZLY},W}(P_i,P_*) = \left\{\frac{1}{5}\sum_{j=1}^{n} w_j\left[(\mu_{ij}-\mu_{*j})^2 + (\nu_{ij}-\nu_{*j})^2 + (\tau_{ij}-\tau_{*j})^2 + (r_{ij}-r_{*j})^2 + (d_{ij}-d_{*j})^2\right]\right\}^{\frac{1}{2}}. \quad (45)$$

The separation measures between P_i and $P_\#$ that are based on $D_E^{\mathrm{RXG},W}$, $D_{G,2}^{\mathrm{LZ},W}$, and $D_{G,2}^{\mathrm{ZLY},W}$ are calculated as follows:

$$D_E^{\mathrm{RXG},W}(P_i,P_\#) = \left\{\frac{1}{2}\sum_{j=1}^{n} w_j\left[\left((\mu_{ij})^2-(\mu_{\#j})^2\right)^2 + \left((\nu_{ij})^2-(\nu_{\#j})^2\right)^2 + \left((\tau_{ij})^2-(\tau_{\#j})^2\right)^2\right]\right\}^{\frac{1}{2}}, \quad (46)$$

$$D_{G,2}^{\mathrm{LZ},W}(P_i,P_\#) = \left\{\frac{1}{4}\sum_{j=1}^{n} w_j\left[(\mu_{ij}-\mu_{\#j})^2 + (\nu_{ij}-\nu_{\#j})^2 + (r_{ij}-r_{\#j})^2 + (d_{ij}-d_{\#j})^2\right]\right\}^{\frac{1}{2}}, \quad (47)$$

$$D_{G,2}^{\mathrm{ZLY},W}(P_i,P_\#) = \left\{\frac{1}{5}\sum_{j=1}^{n} w_j\left[(\mu_{ij}-\mu_{\#j})^2 + (\nu_{ij}-\nu_{\#j})^2 + (\tau_{ij}-\tau_{\#j})^2 + (r_{ij}-r_{\#j})^2 + (d_{ij}-d_{\#j})^2\right]\right\}^{\frac{1}{2}}. \quad (48)$$

For the bridge-superstructure construction case, Figures 7 and 8 present the comparison results of the separation measures between P_i and P_* and between P_i and $P_\#$, respectively, that are based on the Euclidean distance model. Figure 7 compares weighted Euclidean PF distances $D_E^{\mathrm{RXG},W}(P_i,P_*)$ $(D_{G,2}^{\mathrm{C},W}(P_i,P_*))$, $D_{G,2}^{\mathrm{LZ},W}(P_i,P_*)$, $D_{G,2}^{\mathrm{ZLY},W}(P_i,P_*)$, and $D_E^{W}(P_i,P_*)$ $(D_{G,2}^{W}(P_i,P_*))$, which are based on measures D_E^{RXG} $(D_{G,2}^{\mathrm{C}})$ [27], [36], $D_{G,2}^{\mathrm{LZ}}$ [17], $D_{G,2}^{\mathrm{ZLY}}$ [18], and the proposed D_E $(D_{G,2})$, respectively. $D_E^{\mathrm{RXG},W}(P_i,P_*)$, $D_{G,2}^{\mathrm{LZ},W}(P_i,P_*)$, $D_{G,2}^{\mathrm{ZLY},W}(P_i,P_*)$, and $D_E^{W}(P_i,P_*)$ all yield the same ranking: $z_3 \succ z_1 \succ$

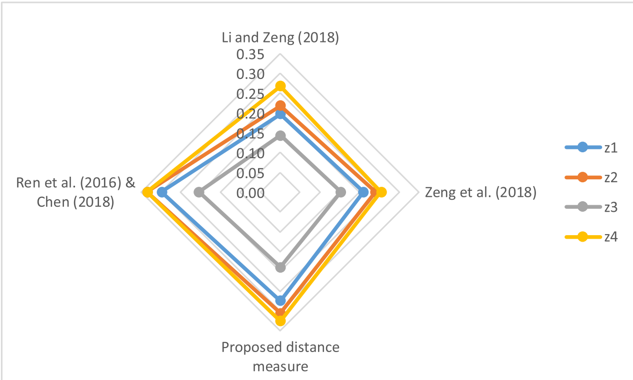


FIGURE 7. Comparison results of the separation between P_i and P_* that are based on the Euclidean distance model.

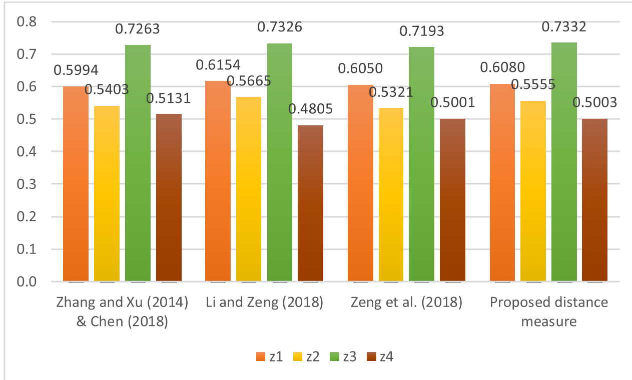


FIGURE 9. Comparison results of closeness-based precedence indices that are based on the Hamming distance model.

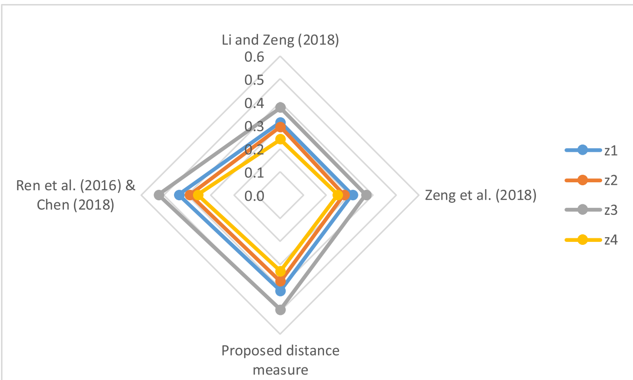


FIGURE 8. Comparison results of the separation between P_i and $P_\#$ that are based on the Euclidean distance model.

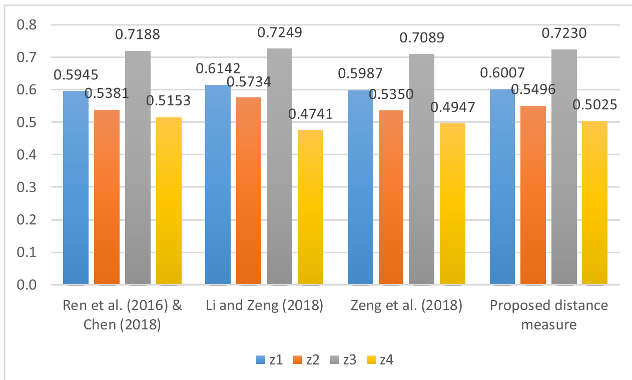


FIGURE 10. Comparison results of closeness-based precedence indices that are based on the Euclidean distance model.

$z_2 \succ z_4$. Additionally, Figure 8 presents the comparison results of $D_E^{\mathrm{RXG},W}(P_i, P_\#)$ ($D_{G,2}^{\mathrm{C},W}(P_i, P_\#)$), $D_{G,2}^{\mathrm{LZ},W}(P_i, P_\#)$, $D_{G,2}^{\mathrm{ZLY},W}(P_i, P_\#)$, and $D_E^{W}(P_i, P_\#)$ ($D_{G,2}^{W}(P_i, P_\#)$); these results render the same ranking of $z_3 \succ z_1 \succ z_2 \succ z_4$.

Consider the first comparative study, which is based on the Hamming distance model. The closeness-based precedence indices, namely, CI_H^{ZX} (using (37) and (40)), $CI_{G,1}^{\mathrm{LZ}}$ (using (38) and (41)), and $CI_{G,1}^{\mathrm{ZLY}}$ (using (39) and (42)), for each alternative z_i are computed as follows:

$$CI_H^{\mathrm{ZX}}(P_i) = \frac{D_H^{\mathrm{ZX},W}(P_i, P_\#)}{D_H^{\mathrm{ZX},W}(P_i, P_*) + D_H^{\mathrm{ZX},W}(P_i, P_\#)}, \tag{49}$$

$$CI_{G,1}^{\mathrm{LZ}}(P_i) = \frac{D_{G,1}^{\mathrm{LZ},W}(P_i, P_\#)}{D_{G,1}^{\mathrm{LZ},W}(P_i, P_*) + D_{G,1}^{\mathrm{LZ},W}(P_i, P_\#)}, \tag{50}$$

$$CI_{G,1}^{\mathrm{ZLY}}(P_i) = \frac{D_{G,1}^{\mathrm{ZLY},W}(P_i, P_\#)}{D_{G,1}^{\mathrm{ZLY},W}(P_i, P_*) + D_{G,1}^{\mathrm{ZLY},W}(P_i, P_\#)}. \tag{51}$$

In the second comparative study, which is based on the Euclidean distance model, the closeness-based precedence indices, namely, CI_E^{RXG} (using (43) and (46)), $CI_{G,2}^{\mathrm{LZ}}$ (using (44) and (47)), and $CI_{G,2}^{\mathrm{ZLY}}$ (using (45) and (48)), for each alternative z_i are computed as follows:

$$CI_E^{\mathrm{RXG}}(P_i) = \frac{D_E^{\mathrm{RXG},W}(P_i, P_\#)}{D_E^{\mathrm{RXG},W}(P_i, P_*) + D_E^{\mathrm{RXG},W}(P_i, P_\#)}, \tag{52}$$

$$CI_{G,2}^{\mathrm{LZ}}(P_i) = \frac{D_{G,2}^{\mathrm{LZ},W}(P_i, P_\#)}{D_{G,2}^{\mathrm{LZ},W}(P_i, P_*) + D_{G,2}^{\mathrm{LZ},W}(P_i, P_\#)}, \tag{53}$$

$$CI_{G,2}^{\mathrm{ZLY}}(P_i) = \frac{D_{G,2}^{\mathrm{ZLY},W}(P_i, P_\#)}{D_{G,2}^{\mathrm{ZLY},W}(P_i, P_*) + D_{G,2}^{\mathrm{ZLY},W}(P_i, P_\#)}. \tag{54}$$

The comparison results of the closeness-based precedence indices that are based on the Hamming and Euclidean distance models are presented in Figures 9 and 10, respectively.

Consider the comparison results in Figures 5–8. As discussed previously, the smaller the separation measure between P_i and P_* is, the higher the performance of alternative z_i. According to Figures 5 and 7, the common ranking result of $z_3 \succ z_1 \succ z_2 \succ z_4$ was determined according to the weighted Hamming and Euclidean PF distances between P_i and P_*. In contrast, the larger the separation measure between P_i and $P_\#$ is, the higher the performance of alternative z_i. As shown in Figures 6 and 8, the same ranking result, namely, $z_3 \succ z_1 \succ z_2 \succ z_4$, was obtained based on the weighted PF distances between P_i and

$P_{\#}$. These ranking results accord with the solution results that are yielded by the proposed methodology. Nevertheless, the separation measures that were proposed by Li and Zeng [17] and Zeng *et al.* [18] yield substantially lower values than those by Zhang and Xu [14], Ren *et al.* [36], Chen [27], and the proposed distance measures according to Figures 5–8. Weighted measures $D_{G,\beta}^{\mathrm{LZ},W}$ and $D_{G,\beta}^{\mathrm{ZLY},W}$ will underestimate the separation measures between P_i and P_* and between P_i and $P_{\#}$. Additionally, in contrast to weighted measure $D_{G,\beta}^{\mathrm{C},W}$, the proposed $D_{G,\beta}^{W}$ fully considers the influences of four characteristics of PF sets. Based on the comparison results with the weighted generalized distance measures, namely, $D_{G,\beta}^{\mathrm{C},W}$, $D_{G,\beta}^{\mathrm{LZ},W}$, and $D_{G,\beta}^{\mathrm{ZLY},W}$, the proposed $D_{G,\beta}^{W}$ can yield more reasonable separation measure results.

According to Figures 9 and 10, a clear consensus on the precedence relationships among four competing alternatives is obtained: $z_3 \succ z_1 \succ z_2 \succ z_4$. Therefore, the optimal alternative is the balanced cantilever method (z_3). In addition to the proposed measure, using the other distance measures ($D_{G,\beta}^{\mathrm{C}}$, $D_{G,\beta}^{\mathrm{LZ}}$, and $D_{G,\beta}^{\mathrm{ZLY}}$), the PF-distance-based compromise approach yields the same final ranking result in the bridge-superstructure construction case. Therefore, the robustness of the proposed methodology is further demonstrated by the results of the comparative analysis.

VII. CONCLUSIONS

In the theory of Pythagorean fuzziness, suitable distance measures for Pythagorean membership grades must be identified for measuring the separation between PF sets. The determination of PF distance measures is essential for the development of distance-based compromise approaches within the PF environment. However, with the current PF distance measures, various limitations and difficulties are encountered. In response, this paper has introduced new PF distance measures as a core technique for handling sophisticated PF information in which Pythagorean membership grades are utilized to accommodate the complicated uncertainty of MCDA problems.

This paper has investigated the limitations and difficulties of the current distance measures in PF contexts. To address these issues, a variety of novel distance measures that are based on PF sets has been developed by employing the following four essential characteristics: the degree of membership, the degree of non-membership, the strength of commitment, and the direction of commitment. The proposed PF distance measures have been evaluated via theoretical analysis and comparative studies on critical topics (the maximal normalized distance, failure to consider the direction of commitment, and the distance from the complement). The proposed Hamming, Euclidean, and generalized PF distances are effective measures for distinguishing the separation between the characteristics of alternatives. Based on the specification of suitable distance measures, this paper has established a PF-distance-based compromise approach for addressing uncertain MCDA problems within the PF environment. This approach has been examined using a real-world case study of bridge-superstructure construction methods and the influence of the distance parameter has been further explored via a sensitivity analysis. To evaluate the methodological effectiveness, the proposed approach has been compared with the TOPSIS-based compromise methods that are based on the other distance measures. According to the results of the sensitivity analysis and comparative studies, the feasibility and reliability of the proposed methodology have been demonstrated through comprehensive comparisons with other relevant distance-based techniques.

In summary, compared to other relevant approaches, the proposed methodology has the advantages of overcoming the difficulties of the existing PF distance measures, providing a flexible and practical approach for separation measurement that is based on PF sets, conducting MCDA calculations in PF contexts more conveniently and effectively, and manipulating uncertain information via a simple PF-distance-based compromise approach. The application results, along with a comprehensive comparative analysis, have demonstrated that the proposed methodology can produce more convincing and reasonable outcomes than other distance-based techniques.

Future research can focus on applying the PF-distance-based compromise approach to more complicated real-world problems, such as supplier evaluation and selection, investment portfolio and financial planning, land-use planning, water resource management, transportation investment and planning, econometric development and planning, public policy, and environmental issues. Moreover, the proposed PF distance measures can be combined with other methods such as TOPSIS, TODIM, VIKOR, and PROMETHEE to construct more useful decision-making models within the PF environment.

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DATA AVAILABILITY STATEMENT

The datasets that were generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

CONFLICT OF INTEREST STATEMENT

The author declares that there is no conflict of interest regarding the publication of this paper.

ETHICAL APPROVAL

This article does not contain any studies with human participants or animals that were performed by the author.

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