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# Complex Synchronization of a Ring-Structured Network of FitzHugh-Nagumo Neurons With Single-and Dual-State Gap Junctions Under Ionic Gates and External Electrical Disturbance

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**ABSTRACT** Synchronization plays an essential role in processing information and decisions by neurons and their networks in the brain, and it is useful to study the synchronization of neuron networks, as a part of the process of understanding the functionality of both healthy and diseased brains. In the past, most studies had developed control schemes relating to synchronization problems which were limited to two or three neurons, which cannot depict the dynamic synchronization behavior of neuron networks. In this paper, we investigated the synchronization issues associated with a ring-structured network of FitzHugh–Nagumo (FHN) neurons, under external electrical stimulation, and with single- and dual-state gap junctions. In addition, the gap junctions (coupling)s and ionic gate disturbances were included in the dynamics of this FHN neuronal network, making our work both more realistic, and more challenging. Thus, each neuron in this network was influenced synaptically by its neighboring two neurons. A simple, robust, and adaptive control scheme, for both a single- and the dual-gap-junction network has been proposed, which will compensate for the nonlinear dynamics without direct cancelation to achieve synchronization. Sufficient conditions to guarantee synchronization of both membrane potentials and recovery variables were derived by using Lyapunov stability theory. Finally, the proposed scheme was validated, and its efficacy was comprehensively analyzed through numerical simulations.

**INDEX TERMS** Synchronization, FitzHugh–Nagumo model, neuronal network, adaptive control, Lyapunov stability theory.

## I. INTRODUCTION

Synchronization is a very interesting and essential phenomenon in both our daily activities [1]–[3], and in nonlinear science [4]. Over the past few years, synchronization has played a crucial role in different scientific fields, including biology, ecology, physics, chemistry, secure communication, electrical circuits and systems, image processing, parameters estimation, neuroscience, and mechanics [1], [4]–[12]. For instance, in the processing of biotic information in a mammalian system, synchronization plays a significant role [13]. Chaos synchronization has been studied extensively, in physical and biological systems [14], attracting attention from many researchers [3], [15]. Since the pioneering work of Pecora and Carroll on the synchronization of chaotic systems [16], chaotic synchronization has been widely used to investigate and understand brain functionality.

In neuroscience, analysis of healthy and diseased brain functionality plays an important role in understanding the complex dynamic mechanisms behind the interactions of neurons and their networks [17], [18], [19]. For example, researchers have investigated the generation of signals generated by a healthy brain in the performance of some tasks, and have analyzed the effect of brain disease on the functionality of neurons [20], [21].

The brain is a complex, nonlinear system, with its own dynamics in terms of connections with neurons in different

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parts of the brain. In the past, this interaction between networks of neurons has been investigated by many researchers, using different schemes and brain imaging systems [22]–[26], and it has been shown through experiments that in several brain areas, synchronization plays a critical role in processing information between neurons [27], [28]. In a neuronal system, synchronization might be the basis of several complex mechanisms of both normal and abnormal brain functions [27]. For instance, synchronization occurs in the olfactory and hippocampal regions of the brain [7], [29], [30], and it has been reported previously that the absence of synchronization between neurons can cause brain disorders, affecting body functionality, such as gait or heart rhythm [31]. Furthermore, brain diseases like epilepsy, Parkinson's disease, schizophrenia, and autism, are disorders caused by a lack of neuronal synchronization [7], [27], [32], [33].

In the literature, many mathematical models depicting neuronal functionality have been developed, to simulate the real neuronal system, in which there exists a chaotic phenomenon, which may be simple or complex [34]. Among different mathematical models of neurons, the FitzHugh–Nagumo (FHN) neuronal model, developed by Fitzhugh [35], and Nagumo et al. [36], has been used as a primary tool to investigate neuronal synchronization problems [3], [13]–[15], [27], [37]–[45]. Past studies have also shown that synchronization of FHN neurons became more difficult once gap junctions were introduced into their dynamics [13]–[15], [27], [43]–[45].

Recent studies have proposed different control strategies, including active control, sliding mode control, adaptive control, nonlinear control, and back stepping design, to study the synchronization phenomenon of neurons. Wang et al. [15] investigated the synchronization of two coupled neurons, by designing a nonlinear controller, without requiring consideration of the gap junction coupling strength. The response of a system consisting of two FHN neurons, coupled with a gap junction, was investigated, and the significance of the coupling coefficient was studied by Mishra et al. [44]. Wang et al. [3] studied the synchronization problem of two FHN neuronal systems by proposing nonlinear feedback linearization and adaptive control schemes. Che et al. [43] presented a fuzzy-based, robust, adaptive control scheme, to synchronize coupled FHN neurons with gap junctions. In another study, Che et al. [14] investigated the synchronization phenomena in a coupled FHN system, with gap junctions. They proposed an adaptive neural network, sliding mode controller, to guarantee the synchronization of the system. Haitao et al. [27] used hybrid sliding mode control and back-stepping techniques to guarantee the synchronization of an FHN neuron system, with gap junctions. Yang and Lin [42] studied the synchronization between two, uncoupled FHN neurons, and used a robust, adaptive, sliding mode controller, instead of active elimination, to achieve synchronization. Peterson and Alberto [41] considered the singularly perturbed limit of a periodically excited, two-dimensional FHN system, and they investigated the occurrence of chaos by varying system parameters. Mehdiabadi et al. [13] addressed the synchronization problem of two FHN neurons, coupled with weak gap junctions, under external electrical stimulation, by developing an adaptive fractional controller. Sergei [39] studied synchronization in two coupled, delayed FHN neurons, with heterogeneities. Guadalupe et al. [40] investigated noise-sustained synchronization in two coupled rings of FHN cells. Sergei and Alexander [38] studied the synchronization in two FHN systems with discrete coupling, using a linear matrix inequality method. Zhang and Liao [37] studied the synchronization and chaos in coupled, memristor-based FHN circuits with memristor synapses.

Even though researchers have successfully developed control strategies to guarantee the synchronization of two or three coupled neurons, the human brain has hundreds of thousands of neurons, which transmit information via synapses, to form complex and large neuronal networks. Neurons transmit information to other neurons in the same network, and other networks, by synchronizing their activities through chemical and electrical synapses. Therefore, the synchronization phenomenon plays a critical role in processing the brain's signals, to ensure efficient communications between neurons [46], [47]. Furthermore, coupling between neurons can greatly affect the dynamic properties of a neuronal network [31].

In the literature, most studies considered the simple problem of synchronizing two or three neurons, but cannot depict the neuronal network synchronization mechanism, as it is one of the complex dynamic processes of the brain. Consequently, it will be an important but challenging task to investigate the synchronization of neuronal networks, rather than the simple synchronization of a two or three neuron system. Accordingly, a network of FHN neurons will display more complex dynamic behavior than two or three coupled neurons and will show more complicated dynamics under external electrical stimulation through gap junctions.

These gap junctions, known as protein channels, play an important role in the transmission of information between neurons. Furthermore, inclusion of the dynamic effects of ionic gate disturbances in this network will make it a more realistic, but more challenging problem. Therefore, investigation of the synchronization process, in a coupled FHN neuronal network, under ionic and external electrical stimulation, with gap junctions, is particularly challenging, in terms of both theoretical and practical applications. Furthermore, a recent study showed that the inclusion of gap junctions in both states of FHN neuron foreshadows even more real and complex dynamics. These dynamics can help to understanding phenomena such as elliptic seizures, since it is neuronal communications between various regions which give rise to these phenomena. Therefore, investigation of such dual-gap-junctioned FHN networks is an important step forward in the understanding of many of the complex phenomena involved in brain communication [48].

In the light of above, this paper reports on our investigation of two different synchronization problems, of ring-structured,

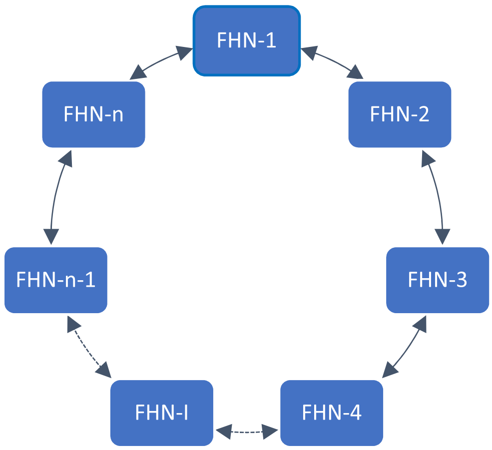


**FIGURE 1. Schematic of the ring structured network of FHN neurons.**

coupled networks of identical FHN neurons, with different gap junctions, under external electrical stimulation, and ionic gates disturbance. Each neuron of this ring-structured network of coupled FHN neurons receives signals via synapses from its nearest two neurons, as shown in Figure 1. Adaptive control theory was used to propose a simple, robust, adaptive control strategy, to investigate the synchronization problem of the FHN neuronal network. Conditions which guaranteed the synchronization of the ring-structured network of n-FHN coupled neurons were derived analytically, using Lyapunov stability theory. Furthermore, a small network of five FHN neurons was considered for validation and verification of the proposed adaptive control scheme. The validation and effectiveness of the proposed adaptive control scheme has been shown through numerical simulation. The main contributions in this study includes the following: **(i)** synchronization of $n$ FHN networks, with single-gap junctions; **(ii)** synchronization of $n$ FHN networks, with dual-gap junctions; **(iii)** development of a unique adaptive control scheme, to synchronize both the single- and dual-gap-junctioned FHN networks; **(iv)** synchronization of both membrane and recovery states has been achieved, for both single- and dual-gap junctioned, FHN neuronal networks, by the proposed adaptive control scheme.

The remaining sections of the paper are structured as follows: In Section II, we briefly describe a single-neuron FHN model. In Section III, details of the ring-structured network of coupled, n-FHN neurons under ionic and external electrical stimulation, with single- and dual-state gap-junctions, configured in a ring structure, are presented. The adaptive control scheme developed for achieving synchronization of both single- and dual-state gap-junctioned networks, their derivations and proofs of sufficient conditions using Lyapunov stability theory, are also presented in Section III. Numerical simulation with a small network of five, coupled FHN neurons, and its results, are presented in Section IV, followed by conclusions in the last section.

## II. GENERAL DYNAMICS OF SINGLE FHN NEURON

Mathematical modelling of neurons plays an important role in understanding the functionality of the brain, since the neuron is the fundamental unit of the brain. Several dynamic models, including the integrated and fire model [49], the Hodgkin and Huxley model (HH) [50], the FHN model [27], the Morris–Lecar model [51], and the Hindmarsh and Rose model [52], have been developed in the past. These models can replicate many real neuronal behaviors, such as phase spiking, tonic spiking, phasic bursting, class 1 excitability, and resonator and integrator behaviors. Among these models, FHN is relatively simple, and has been the most commonly employed as a synchronization investigation tool. The FHN model can be represented as

$$\begin{aligned} \dot{x} &= x(x-1)(1-rx) - y + I + d \\ \dot{y} &= bx - cy \end{aligned} \tag{1}$$

In (1), $x$ and $y$ are state variables for action potential and recovery, respectively, $r$, $b$, and $c$ are positive constants, $d = 0.01\sin(0.2t)$ is the ionic gate disturbance and $I = \frac{A}{\omega}\cos(\omega t)$ is the stimulus current, with $A$ and $\omega = 2\pi f$ representing amplitude and frequency, respectively. The time response of the membrane potential, recovery variable, and phase plane diagram of the FHN model, with parameter values $r = 10$, $b = 1$, $c = 0.001$, $A = 0.1, f = 0.129$, and initial conditions of $(x(0), (y(0)) = (0, 0)$, are shown in Figure 2.

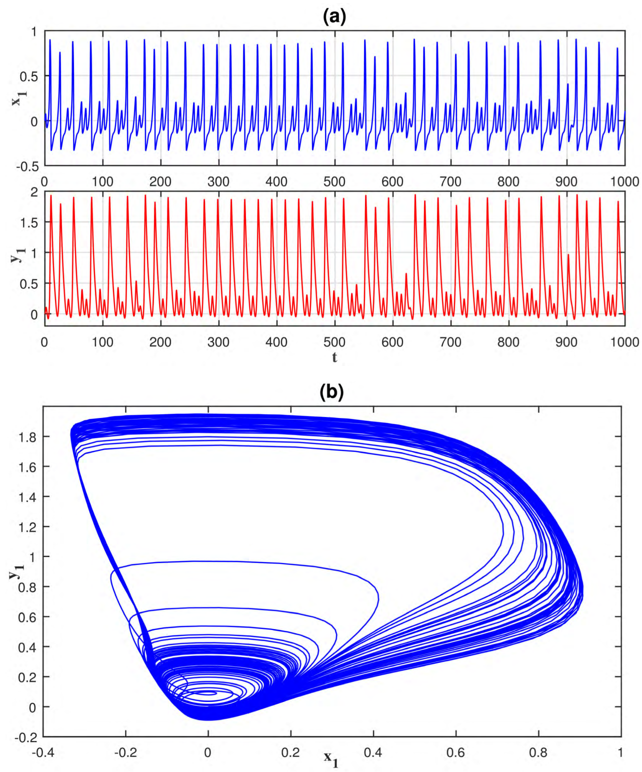


**FIGURE 2. Dynamic behavior of an FHN neuron model: (a) Time responses of the membrane potential and recovery variable, (b) phase plane diagram.**

## III. NETWORK OF *n*-FHN NEURONS AND CONTROL DESIGN

This section describes formulation of the synchronization issue for a ring-structured network of $n$ coupled FHN neurons, under ionic and external electrical stimulation, with gap junctions. This network will show dynamics that are more complicated, with gap junctions, and under external electrical stimulation. Furthermore, the inclusion of the dynamic effects of ionic gate disturbances in this network will make it more realistic, and also more challenging to analyze. In this ring-structured configuration, all neurons are attached to its neighbors, as in a chain, i.e., the first neuron is connected with second and last neurons, as shown in Figure 1.

### A. NETWORK OF FHN NEURONS WITH SINGLE-GAP JUNCTION

In this section, we will describe and discus the formation and synchronization of an $n$ FHN neuronal network, under ionic and external electrical stimulation, with single-gap junctions. Mathematically, this network can be expressed as shown in (2).

$$\begin{aligned}\dot{x}_i &= x_i\,(x_i-1)\,(1-rx_i)-y_i-g\,(x_i-x_{i-1})\\&\quad -g\,(x_i-x_{i+1})+I+d+u_i\\ \dot{y}_i &= bx_i-cy_i \end{aligned}\tag{2}$$

In (2), $i = 1, 2, 3, \cdots n$, where $x_i$ and $y_i$ are stated action potential variables and recovery variables respectively, $r$, $b$, and $c$ are positive constants, $d$ is the ionic gate disturbance and $I$ is the stimulus current, respectively, while $g$ represents the coupling strength of the gap junction between neighboring neurons and the neurons and the parameters $u_i$ represents the control.

Furthermore, in this ring structure of $n$ FHN neurons, $x_0 = x_n, x_{n+1} = x_1, y_0 = y_n$ and $y_{n+1} = y_1$.

In the control theory, the synchronization problem of a master and slave system can be formulated as a tracking problem, i.e., the output of a master neurons can be used to control the trajectory of a slave neuron, such that the output of the slave neuron follows the drive neuron [27].

The error states of the system (2) can be defined as shown in (3).

$$e_{x_i} = x_i - x_{i+1}, \quad e_{y_i} = y_i - y_{i+1} \tag{3}$$

The time derivative of the error system (3), yields Eq (4)

$$\begin{aligned}\dot{e}_{x_i} &= -re_{x_i}(x_i^2+x_ix_{i+1}+x_{i+1}^2)+(re_{x_i}+e_{x_i})(x_i+x_{i+1})\\&\quad -e_{x_i}-e_{y_i}+ge_{x_{i-1}}-2ge_{x_i}+ge_{x_{i+1}}+u_i-u_{i+1}\\ \dot{e}_{y_i} &= be_{x_i}-ce_{y_i}\end{aligned}\tag{4}$$

In (4) above,

$$u_0 = u_n, u_{n+1} = u_1, \quad e_{x_0} = e_{x_n},\ e_{x_{n+1}} = e_{x_1},\ e_{y_0} = e_{y_n}$$

and $e_{y_{n+1}} = e_{y_1}$. The synchronization problem of the ring structured system can be converted into a problem of how to achieve asymptotic stabilization of the error system (4) at the origin. This can be achieved by designing a control scheme independent of initial conditions, such that $\lim_{t\to\infty} e_{x_i}$ and $\lim_{t\to\infty} e_{y_i}(i = 1, 2, \ldots, n)$, and for any value of gap junction $g$.

Next, we will use the adaptive control theory, and Lyapunov stability theory, to propose a simple adaptive control law.

*Theorem 1:* Consider a ring-structured network of identical $n$-FHN coupled neurons, as in system (2), with synchronization error dynamics as in system (4). If the controllers, $u_i$, in system (4) are designed as $u_1 = e_{x_n}$, $u_2 = e_{x_1} \cdots$, $u_n = e_{x_{n-1}}$, then it will guarantee the synchronization of the ring structured network of $n$-FHN neurons, by converging the error of the synchronized system to zero.

*Proof:* Using Lyapunov stability theory, it is enough to prove that for the Lyapunov function $V$ which is chosen as

$$V(t) = \frac{1}{2}\sum_{i=1}^{n}(e_{x_i}^2+e_{y_i}^2) \tag{5}$$

$\frac{dv}{dt} < 0$ for all $t$.

It is apparent that the function $V$ is a positive definite function. The time derivative of (5) yields the theorem described by

$$\begin{aligned}\dot{V}(t) &= \sum_{i=1}^{n} e_{x_i}^2\begin{bmatrix}(x_i+x_{i+1})(1+r)\\ -r(x_i^2+x_ix_{i+1}+x_{i+1}^2)-2(1+g)\end{bmatrix}\\&\quad -(1-b)\sum_{i=1}^{n}e_{x_i}e_{y_i}+g\sum_{i=}^{n}e_{x_i}e_{x_{i+1}}\\&\quad +(1+g)\sum_{i=1}^{n}e_{x_i}e_{x_{i-1}}-c\sum_{i=1}^{n}e_{y_i}^2\end{aligned}\tag{6}$$

Since the FHN system (2) has bounded trajectories, there exists a sufficiently small positive constant M>0, such that $|x_i| \le M$ $(i = 1, 2, \cdots, n)$, Thus we get

$$\begin{aligned}\dot{V}(t) &\le -(-3rM^2-2rM-2M+2(1+g))\sum_{i=1}^{n}e_{x_i}^2\\&\quad -(1-b)\sum_{i=1}^{n}|e_{x_i}|\,|e_{y_i}|+g\sum_{i=1}^{n}|e_{x_i}|\,|e_{x_{i+1}}|\\&\quad +(1+g)\sum_{i=1}^{n}|e_{x_i}|\,|e_{x_{i-1}}|-c\sum_{i=1}^{5}e_{y_i}^2\end{aligned}\tag{7}$$

Using

$$2(1+g) \ge 2 > 3rM^2+2rM+2M,\ 1-b \ge 0,\quad 0 \le g < 0.1$$

and $0 \le c < 0.01$, we obtain (8)

$$\dot{V}(t) \le -EPE^T, \tag{8}$$

where

$$E = \begin{pmatrix}|e_{x_1}| & |e_{y_1}| & |e_{x_2}| & |e_{y_2}| & \cdots & |e_{x_n}| & |e_{y_n}|\end{pmatrix}, \tag{9}$$

$$P = \begin{bmatrix} A & B & 0 & \cdots & 0 & C \\ C & A & B & \cdots & 0 & 0 \\ 0 & C & A & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A & B \\ B & 0 & 0 & \cdots & C & A \end{bmatrix} \tag{10}$$

Here $A$, $B$, and $C$ are given by

$$A = \begin{bmatrix} (-3rM^2 - 2rM - 2M + 2(1+g)) & \frac{1}{2}(1-b) \\ \frac{1}{2}(1-b) & c \end{bmatrix} \tag{11}$$

$$B = \begin{bmatrix} -g & 0 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} -(1+g) & 0 \\ 0 & 0 \end{bmatrix} \tag{12}$$

To guarantee that the origin of synchronized error system (4) is asymptotically stable, matrix $P$ should be a positive definite. The synchronization conditions which ensures the positive definiteness of matrix P(for a network of five neurons) are described in Appendix. According to Lyapunov stability theory, the error system (4) is asymptotically stable about the origin, for any value of $0 \le g < 0.1$. That is, a ring-structured network of $n$-identical, coupled FHN neurons, with different gap junctions, under ionic gates disturbance and external electrical stimulation, will achieve synchronization. This completes the proof.

### B. A NETWORK OF FHN NEURONS WITH DUAL-GAP JUNCTIONS

In this section we will describe and discus the formation and synchronization of an $n$ FHN neuronal network, under ionic and external electrical stimulation, with dual-gap junctions. Mathematically, this network can be expressed as shown in (13).

$$\begin{aligned} \dot{x}_i &= x_i(x_i - 1)(1 - rx_i) - y_i - g_1(x_i - x_{i-1}) \\ &\quad - g_1(x_i - x_{i+1}) + I + d + u_i \\ \dot{y}_i &= bx_i - cy_i - g_2(y_i - y_{i-1}) - g_2(y_i - y_{i+1}) \end{aligned} \tag{13}$$

In (13), g1 and g2 represent the coupling strength of the gap junctions between neighbor neurons.

The error states of system (13) can be defined as shown in (14).

$$e_{x_i} = x_i - x_{i+1}, \quad e_{y_i} = y_i - y_{i+1} \tag{14}$$

Time derivative of the error state (15), yields

$$\begin{aligned} \dot{e}_{x_i} &= -re_{x_i}(x_i^2 + x_i x_{i+1} + x_{i+1}^2) \\ &\quad + (re_{x_i} + e_{x_i})(x_i + x_{i+1}) - e_{x_i} - e_{y_i} \\ &\quad + g_1 e_{x_{i-1}} - 2g_1 e_{x_i} + g_1 e_{x_{i+1}} + u_i - u_{i+1} \\ \dot{e}_{y_i} &= be_{x_i} - ce_{y_i} - 2g_2 e_{y_i} + g_2 e_{y_{i-1}} + g_2 e_{y_{i+1}} \end{aligned} \tag{15}$$

The synchronization problem of the ring structured system can be converted into a problem of how to achieve the asymptotical stabilization of the error system (15) at the origin. This can be achieved by designing a control scheme independent of initial conditions, such that $\lim_{t\to\infty} e_{x_i}$ and $\lim_{t\to\infty} e_{y_i}(i = 1, 2, \ldots, n)$ and for any value of gap junctions $g_1$ and $g_2$.

Next, we will use adaptive control theory, and Lyapunov stability theory, to propose a simple adaptive control law.

*Theorem 2:* Consider a ring-structured network of identical $n$-FHN coupled neurons, as in system (13) with synchronization error dynamics as in system (15). If the controllers $u_i$ in the system (15) are designed as $u_1 = e_{x_n}, u_2 = e_{x_1} \cdots, u_n = e_{x_{n-1}}$, then it will guarantee the synchronization of the ring structured network of $n$-FHN neurons, by converging the error of the synchronized system to zero.

*Proof:* Lyapunov stability theory can be used as a valuable tool to prove this theorem. Using Lyapunov stability theory, the Lyapunov function candidate $V$ is chosen as

$$V(t) = \frac{1}{2}\sum_{i=1}^{n}(e_{x_i}^2 + e_{y_i}^2) \tag{16}$$

We note that the candidate function is positive definite. The time derivative of (16) yields as follows.

$$\begin{aligned} \dot{V}(t) &= \sum_{i=1}^{n} e_{x_i}^2 \begin{bmatrix} (x_i + x_{i+1})(1+r) \\ -r(x_i^2 + x_i x_{i+1} + x_{i+1}^2) - 2(1+g_1) \end{bmatrix} \\ &\quad - (1-b)\sum_{i=1}^{n} e_{x_i}e_{y_i} + g_1 \sum_{i=}^{n} e_{x_i}e_{x_{i+1}} \\ &\quad + (1+g_1)\sum_{i=1}^{n} e_{x_i}e_{x_{i-1}} - (c+2g_2)\sum_{i=1}^{n} e_{y_i}^2 \\ &\quad + g_2\sum_{i=1}^{n} e_{y_i}e_{y_{i-1}} + g_2\sum_{i=1}^{n} e_{y_i}e_{y_{i+1}} \end{aligned} \tag{17}$$

Since system (13) has bounded trajectories, there exists a sufficiently small positive constant $M$, such that $|x_i| \le M$ $(i = 1, 2, \cdots, n)$, thus we achieve

$$\begin{aligned} \dot{V}(t) &= (3rM^2 + 2rM + 2M - 2(1+g_1))\sum_{i=1}^{n} e_{x_i}^2 \\ &\quad - (1-b)\sum_{i=1}^{n} |e_{x_i}|\,|e_{y_i}| + g_1\sum_{i=1}^{n} |e_{x_i}|\,|e_{x_{i+1}}| \\ &\quad + (1+g_1)\sum_{i=1}^{n} |e_{x_i}|\,|e_{x_{i-1}}| - (c+2g_2)\sum_{i=1}^{n} e_{y_i}^2 \\ &\quad + g_2\sum_{i=1}^{n} |e_{y_i}|\,|e_{y_{i-1}}| + g_2\sum_{i=1}^{n} |e_{y_i}|\,|e_{y_{i+1}}| \end{aligned} \tag{18}$$

Using $2(1+g_1) \ge 2 > 3rM^2 + 2rM + 2M$, $1 - b \ge 0$, $0 \le g_1 < 0.1$, $0 \le c < 0.01$, $(c + 2g_2) \ge 0$, we obtain

$$\dot{V}(t) \le -E_1 P E_1^T, \tag{19}$$

where

$$E_1 = \left(|e_{x_1}| \quad |e_{y_1}| \quad |e_{x_2}| \quad |e_{y_2}| \quad \cdots \quad |e_{x_n}| \quad |e_{y_n}|\right) \tag{20}$$

$$P_1 = \begin{bmatrix} A_1 & B_1 & 0 & \cdots & 0 & C_1 \\ C_1 & A_1 & B_1 & \cdots & 0 & 0 \\ 0 & C_1 & A_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_1 & B_1 \\ B_1 & 0 & 0 & \cdots & C_1 & A_1 \end{bmatrix} \quad (21)$$

Here $A_1$, $B_1$, and $C_1$ are given by

$$A_1 = \begin{bmatrix} (-3rM^2 - 2rM - 2M + 2(1+g_1)) & \frac{1}{2}(1-b) \\ \frac{1}{2}(1-b) & c \end{bmatrix} \quad (22)$$

$$B_1 = \begin{bmatrix} -g_1 & 0 \\ 0 & -g_2 \end{bmatrix}, \quad C_1 = \begin{bmatrix} -(1+g_1) & 0 \\ 0 & -g_2 \end{bmatrix} \quad (23)$$

It is apparent that, to guarantee that the origin of synchronized error system (15) is asymptotically stable, matrix $P_1$ should be positive definite. The synchronization conditions which ensures the positive definiteness of matrix $P_1$(for a network of five neurons) are described in Appendix. According to the Lyapunov stability theory, the error system (15) is asymptotically stable about the origin, for any positive values of $0 \leq g_1 < 0.1$ and $g_2$. That is, a ring-structured network, of $n$-identical coupled FHN neurons, with different gap junctions, under ionic gates disturbance and external electrical stimulation, will achieve synchronization. This completes the proof.

## IV. NUMERICAL SIMULATION

For the purpose of verification, a ring structured network of five, coupled FHN neurons, with single- and dual-gap junctions, external disturbance, ionic gate disturbance, and control parameters can be expressed as shown in Case I and Case II.

### A. CASE I

$$\begin{aligned} \dot{x}_i &= x_i(x_i - 1)(1 - rx_i) - y_i - g(x_i - x_{i-1}) \\ &\quad - g(x_i - x_{i+1}) + I + d + u_i \\ \dot{y}_i &= bx_i - cy_i, \end{aligned} \quad (24)$$

In (24), $(i = 1, 2, \cdots, 5)$. The error state of system (24) are defined as shown in (25).

$$e_{x_i} = x_i - x_{i+1}, \quad e_{y_i} = y_i - y_{i+1} \quad (25)$$

Taking the derivative with respect to time, for system (25), the error system can be expressed as shown in (26).

$$\begin{aligned} \dot{e}_{x_i} &= -re_{x_i}(x_i^2 + x_i x_{i+1} + x_{i+1}^2) \\ &\quad + (re_{x_i} + e_{x_i})(x_i + x_{i+1}) - e_{x_i} - e_{y_i} \\ &\quad + g_1 e_{x_{i-1}} - 2g_1 e_{x_i} + g e_{x_{i+1}} + u_i - u_{i+1} \\ \dot{e}_{y_i} &= be_{x_i} - ce_{y_i} \end{aligned} \quad (26)$$

The synchronization problem of the ring-structured system can be converted into a problem of how to achieve the asymptotical stabilization of the error system (28), at the origin. This can be achieved by designing a control scheme independent of initial conditions, such that $\lim_{t\to\infty} e_{x_i}$ and $\lim_{t\to\infty} e_{y_i}(i = 1, 2, \ldots, 5)$, for any value of gap junction $g$.

Next, we will use adaptive control theory, and Lyapunov stability theory, to propose a simple adaptive control law.

Consider a ring-structured network of five FHN neurons, as in system (24), with synchronization error dynamics, as in system (26). If the controllers $u_i$ in system (26) are designed as $u_1 = e_{x_5}, u_2 = e_{x_1} \cdots, u_5 = e_{x_4}$, then it will guarantee the synchronization of the ring structured network of five-FHN neurons, by converging the error of the synchronized system to zero. The synchronization conditions which ensures the positive definiteness of matrix P are described in Appendix.

### B. CASE II

$$\begin{aligned} \dot{x}_i &= x_i(x_i - 1)(1 - rx_i) - y_i - g_1(x_i - x_{i-1}) \\ &\quad - g_1(x_i - x_{i+1}) + I + d + u_i \\ \dot{y}_i &= bx_i - cy_i - g_2(y_i - y_{i-1}) - g_2(y_i - y_{i+1}) \end{aligned} \quad (27)$$

The error state of the system (27) are defined as shown in (28).

$$e_{x_i} = x_i - x_{i+1}, \quad e_{y_i} = y_i - y_{i+1} \quad (28)$$

Taking the derivative with respect to time, for the error state (28), the error system can be expressed as shown in (29).

$$\begin{aligned} \dot{e}_{x_i} &= -re_{x_i}(x_i^2 + x_i x_{i+1} + x_{i+1}^2) \\ &\quad + (re_{x_i} + e_{x_i})(x_i + x_{i+1}) - e_{x_i} - e_{y_i} \\ &\quad + g_1 e_{x_{i-1}} - 2g_1 e_{x_i} + g_1 e_{x_{i+1}} + u_i - u_{i+1} \\ \dot{e}_{y_i} &= be_{x_i} - ce_{y_i} - 2g_2 e_{y_i} + g_2 e_{y_{i-1}} + g_2 e_{y_{i+1}} \end{aligned} \quad (29)$$

The synchronization problem of the ring structured system can be converted into a problem of how to achieve the asymptotical stabilization of the error system (29), at the origin. This can be achieved by designing a control scheme independent of initial conditions, such that $\lim_{t\to\infty} e_{x_i}$ and $\lim_{t\to\infty} e_{y_i}(i = 1, 2, \cdots, 5)$, for any value of the gap junctions $g_1$ and $g_2$.

Next, we will use adaptive control theory, and Lyapunov stability theory, to propose a simple adaptive control law.

Consider a ring-structured network of five-FHN neurons, as in system (27), with synchronization error dynamics as in system (29). If the controllers $u_i$, in system (29), are designed as $u_1 = e_{x_n}, u_2 = e_{x_1} \cdots, u_5 = e_{x_4}$, then it will guarantee the synchronization of the ring-structured network of five FHN neurons, by converging the error of the synchronized system to zero. The synchronization conditions which ensures the positive definiteness of matrix $P_1$ are described in Appendix.

Numerical simulations were carried out, to validate and analyze the effect of the adaptive control laws proposed for the synchronization of the ring-structured, coupled network of five FHN neurons. Parameter values selected were: $r = 10, b = 1, A = 0.1, f = 0.129, g = g_1 = 0.05, g_2 = 0.001, c = 0.001$, with initial conditions $(x_1(0), y_1(0), x_2(0), y_2(0), x_3(0), y_3(0), x_4(0), y_4(0), x_5(0), y_5(0)) = (0, 0, 0.1, 0.1, 0.5, 0.5, 0.2, 0.2, 0.3, 0.3)$. For Case I, it can be seen in Figures 3 and 4 that the phase plane dynamics of

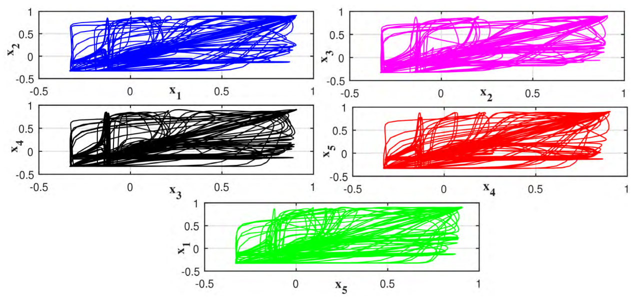


**FIGURE 3.** Synchronization analysis of x states of five FHN neurons with single gap-junction coupled in a ring structure without controls.

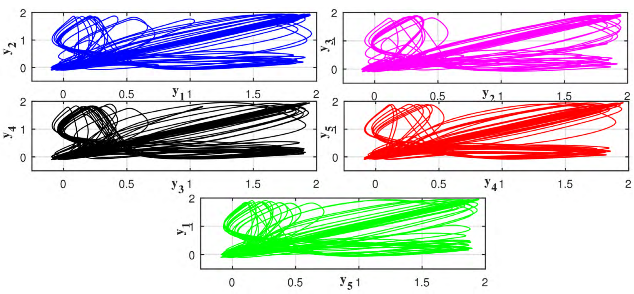


**FIGURE 4.** Synchronization analysis of y states of five FHN neurons with single gap-junction coupled in a ring structure without controls.

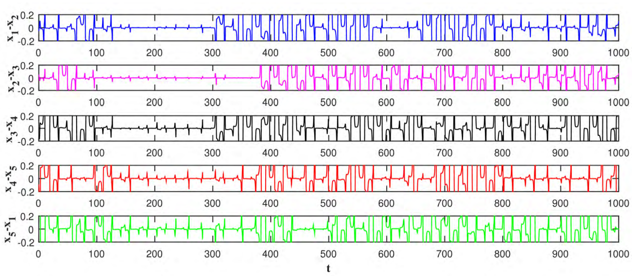


**FIGURE 5.** Dynamics of synchronization errors between x states of five FHN neurons with single gap-junction coupled in a ring structure without controls.

membrane potential and the recovery variable showed abrupt behavior, indicating that the original network of FHN neurons was not synchronized. Furthermore, Figures 5 and 6 show the time dynamics of the errors for membrane potential, and for the recovery variable, respectively. These results revealed that both states of each neuron in the network showed spiked errors, with non-converging behavior. It can be concluded that, at this point, there was no synchronization between the firing of the five neurons of the network of the ring-structure FHN system.

Now, by using the adaptive control scheme proposed in Case I, we have simulated the ring-structured network of the five neurons. Figures 7 and 8 show the phase plane diagrams of the membrane action potentials, and recovery variables, between each neuron of the network, respectively.

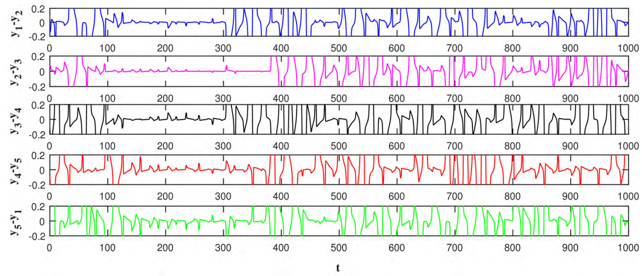


**FIGURE 6.** Dynamics of synchronization errors between y states of five FHN neurons with single gap-junction coupled in a ring structure without controls.

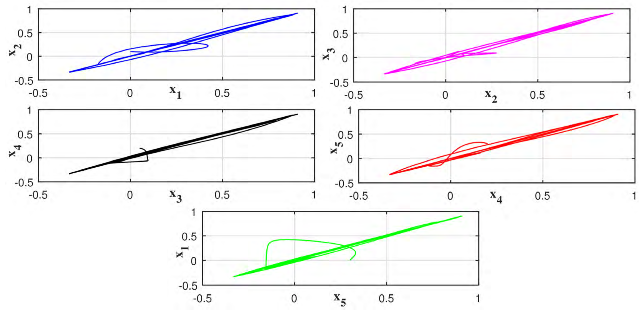


**FIGURE 7.** Synchronization analysis of x states of five FHN neurons with single gap-junction coupled in a ring structure with proposed control scheme.

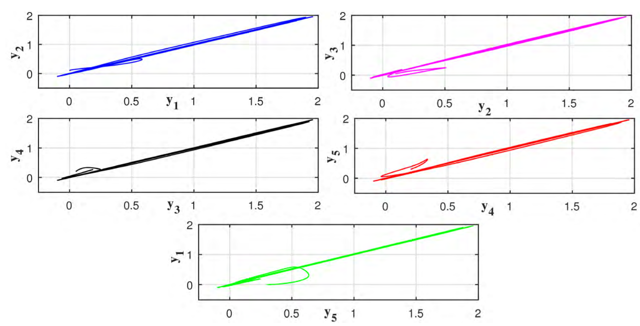


**FIGURE 8.** Synchronization analysis of y states of five FHN neurons with single gap-junction coupled in a ring structure with proposed control scheme.

It can be seen that each state of the neurons is synchronized with the respective state of the slave neuron, since the resultant plot shows a straight line passing through the origin with a slope of 1. In addition, we validated the proposed scheme by analyzing the synchronization dynamics of the ring-structured network of FHN neurons, with and without the controller in the Case I. The results of this analysis are shown in Figures 9 and 10. In these simulations, the proposed adaptive controllers were switched on at $t = 400$. It can be seen that the error dynamics, for both the membrane potential and the recovery variable, of the network, showed non-synchronous and oscillatory behavior, but as soon as the proposed adaptive controls scheme was applied, the errors

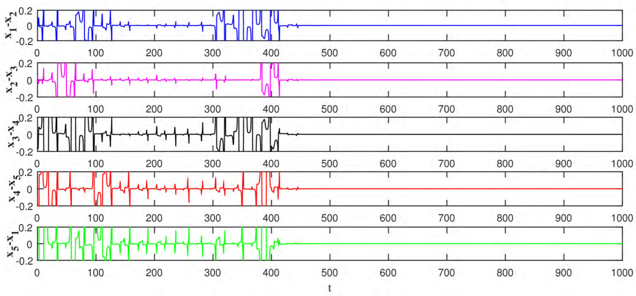

**FIGURE 9.** Dynamics of synchronization errors between x states of five FHN neurons with single gap-junction coupled in a ring structure with proposed control scheme.

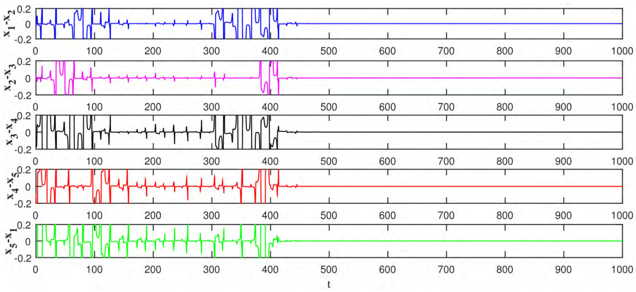

**FIGURE 10.** Dynamics of synchronization errors between y states of five FHN neurons with single gap-junction coupled in a ring structure with proposed control scheme.

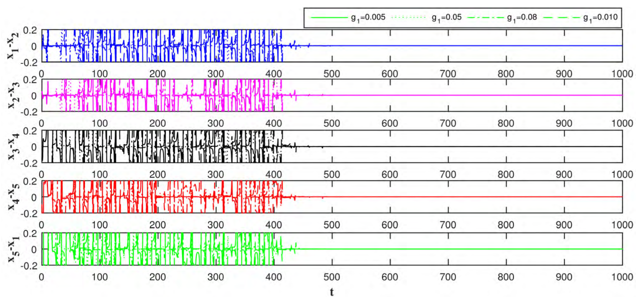


**FIGURE 11.** Synchronization analysis of x states of five FHN neurons with different values of single gap-junction coupled in a ring structure with proposed control scheme.

between the membrane potential and recovery variable states converged to zero, indicating synchronization. Moreover, Figures 11 and 12 show the dynamics of the network of neurons for different values of gap-junction $g$. It can be seen that the proposed control scheme ensures the synchronization for all values of $g$, indicating that the proposed control scheme can be used for any value of gap-junction parameter.

In Case II, even though the dynamics were more complex in comparison with single-state, gap-junctioned FHN network, neurons, it can be seen that the proposed control scheme successfully achieved the synchronization. It can be seen from Figures 13–16 that this network also showed unsynchronized behavior before application of the proposed control scheme. As for Case I, the results illustrated in Figures 17–20 showed that synchronization could

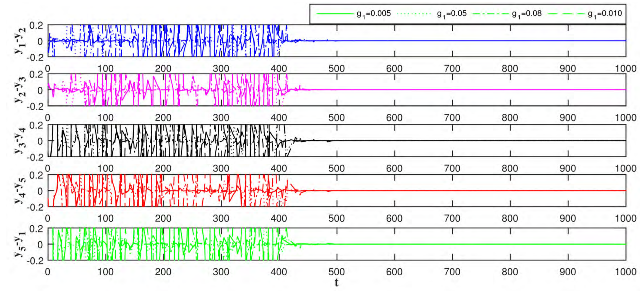


**FIGURE 12.** Synchronization analysis of y states of five FHN neurons with different values of single gap-junction coupled in a ring structure with proposed control scheme.

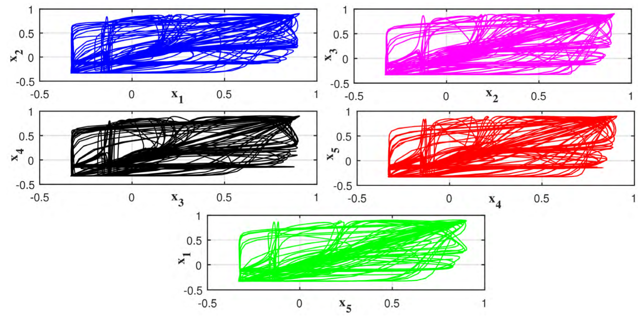

**FIGURE 13.** Synchronization analysis of x states of five FHN neurons with dual gap-junction coupled in a ring structure without controls.

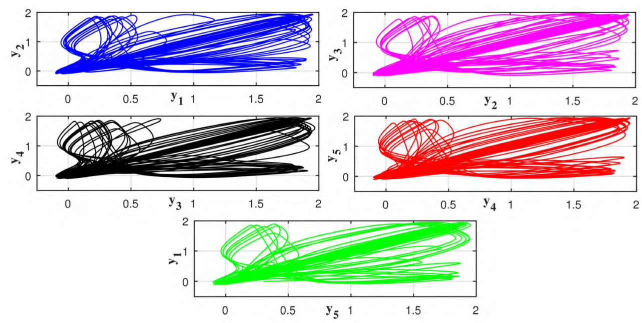

**FIGURE 14.** Synchronization analysis of y states of five FHN neurons with dual gap-junction coupled in a ring structure without controls.

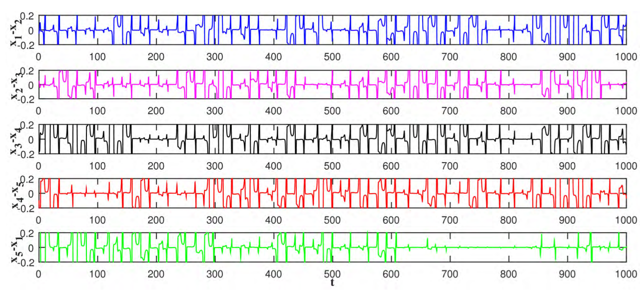

**FIGURE 15.** Dynamics of synchronization errors between x states of five FHN neurons with dual gap-junction coupled in a ring structure without controls.

be achieved by using the proposed control scheme. The convergence of error dynamics to zero guaranteed the synchronization of the ring-structured network of five FHN

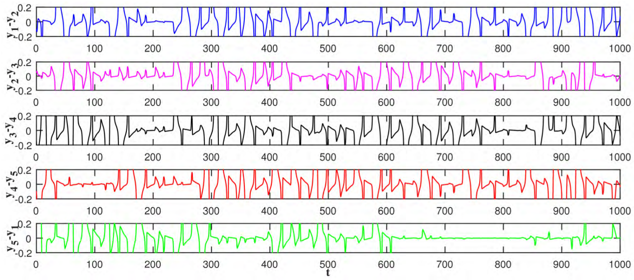

**FIGURE 16.** Dynamics of synchronization errors between y states of five FHN neurons with dual gap-junction coupled in a ring structure without controls.

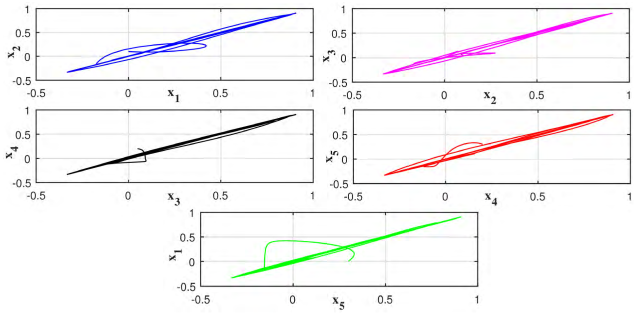

**FIGURE 17.** Synchronization analysis of x states of five FHN neurons with dual gap-junction coupled in a ring structure with proposed control scheme.

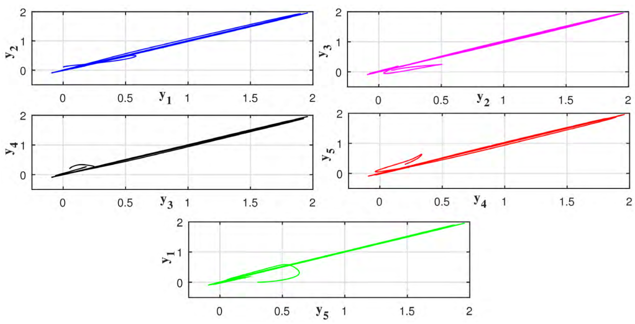

**FIGURE 18.** Synchronization analysis of y states of five FHN neurons with dual gap-junction coupled in a ring structure with proposed control scheme.

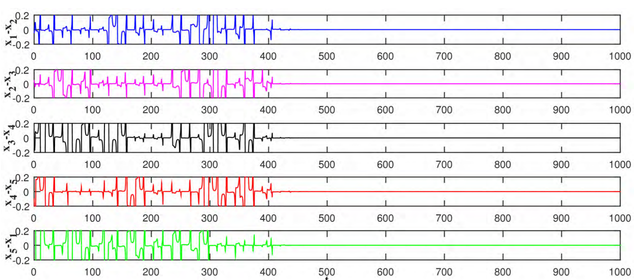

**FIGURE 19.** Dynamics of synchronization errors between x states of five FHN neurons with dual gap-junction coupled in a ring structure with proposed control scheme.

neurons, under conditions of external electrical stimulation, ionic gates disturbance and gap junctions. It can also be observed that the errors between neurons, for both single- and

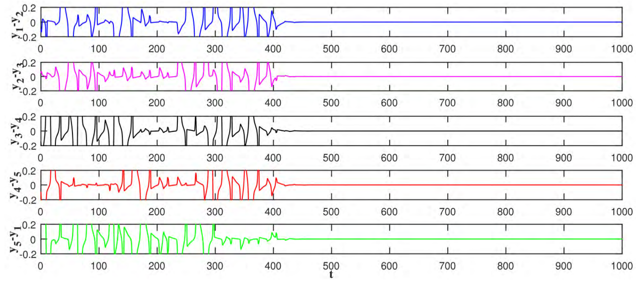

**FIGURE 20.** Dynamics of synchronization errors between y states of five FHN neurons with dual gap-junction coupled in a ring structure with proposed control scheme.

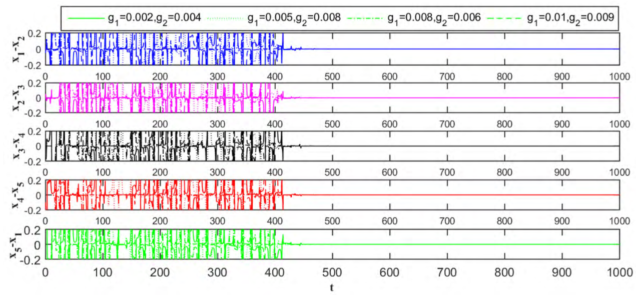


**FIGURE 21.** Synchronization analysis of x states of five FHN neurons with different values of dual gap-junctions coupled in a ring structure with proposed control scheme.

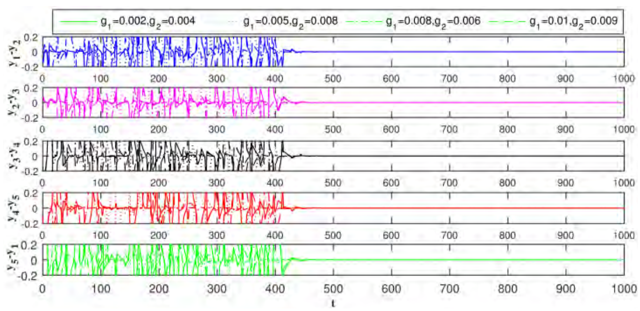


**FIGURE 22.** Synchronization analysis of x states of five FHN neurons with different values of dual gap-junctions coupled in a ring structure with proposed control scheme.

dual-state gap-junctioned networks converged to zero very rapidly, showing the design efficiency of the proposed control scheme. Moreover, Figures 21 and 22 show the dynamics of the network of neurons for different values of the dual gap-junctions $g_1$ and $g_2$. It can be seen that the proposed control scheme ensures the synchronization for all values of dual gap-junctions, indicating that the proposed control scheme can be used for any value of gap-junction parameters.

## V. CONCLUSION

In a brain, neuronal activity is synchronized, to process and transfer information. Synchronization plays an important role in understanding brain function. Consequently, networks of FHN neurons show more complex behavior and dynamics. In the work reported here, synchronization of a network of

$n$ identical FHN neurons, with gap junction, ion gate disturbance, and external electrical stimulation, was addressed. Two different networks, one with gap junctions in the membrane state, and the other with gap junctions in both states, were considered for the first time. A simple adaptive control strategy was proposed, to achieve synchronization in the neuronal networks, which compensated for the nonlinear dynamics without direct cancelation. Using Lyapunov stability theory to derive sufficient conditions guaranteed synchronization of the FHN neuronal networks.

Numerical simulations were performed, which demonstrated the efficiency of the proposed control scheme.

## APPENDIX

To show that matrix P in Theorem 1 is positive definite, we will show that determinants of all its leading principal minors have positive values. We will consider a network of five neurons as described in Case I of section IV. Matrix P for this network can be written see (P), as shown at the top of the next page.
where $Z = (-3rM^2 - 2rM - 2M + 2(1 + g))$. After substituting values of the model parameters used in numerical simulation, the main sub-determinants of the above matrix are as follows

$$M_1 = 2 + 2g \tag{A1}$$
$$M_2 = 0.0002 + 0.0002g \tag{A2}$$
$$M_3 = 0.0004 + 0.0007g + 0.0003g^2 \tag{A3}$$
$$M_4 = 4. \times 10^{-8} + 7.0 \times 10^{-8}g + 3. \times 10^{-8}g^2 \tag{A4}$$
$$M_5 = 8. \times 10^{-8} + 2. \times 10^{-7}g + 1.6 \times 10^{-7}g^2 + 4. \times 10^{-8}g^3 \tag{A5}$$
$$M_6 = 8. \times 10^{-12} + 2. \times 10^{-11}g + 1.6 \times 10^{-11}g^2 + 4. \times 10^{-12}g^3 \tag{A6}$$
$$M_7 = 1.6 \times 10^{-11} + 5.2 \times 10^{-11}g + 6.1 \times 10^{-11}g^2 + 3. \times 10^{-11}g^3 + 5. \times 10^{-12}g^4 \tag{A7}$$
$$M_8 = 1.6 \times 10^{-15} + 5.2 \times 10^{-15}g + 6.1 \times 10^{-15}g^2 + 3. \times 10^{-15}g^3 + 5. \times 10^{-16}g^4 \tag{A8}$$
$$M_9 = 3.1 \times 10^{-15} + 1.15 \times 10^{-14}g + 1.6 \times 10^{-14}g^2 + 1. \times 10^{-14}g^3 + 2.5 \times 10^{-15}g^4 \tag{A9}$$
$$M_{10} = 3.1 \times 10^{-19} + 1.15 \times 10^{-18}g + 1.6 \times 10^{-18}g^2 + 1.0 \times 10^{-18}g^3 + 2.5 \times 10^{-19}g^4 \tag{A10}$$

where $(A_1)$ - $(A_{10})$ are ten main sub-determinants of matrix P. It is easy to verify that these sub-determinants are positive for all values of coupling strength gap-junction $g$, which ensure that matrix P positive definite.

To show that matrix $P_1$ in Theorem 2 is positive definite, we will show that determinants of all its leading principal minors have positive values. We will consider a network of five neurons as described in Case I of section IV. Matrix $P_1$ for this network can be written see $(P_1)$, as shown at the top of the next page.
where $Z = (-3rM^2 - 2rM - 2M + 2(1 + g_1))$. After substituting values of the model parameters used in numerical simulation, the main sub-determinants of the above matrix are as follows

$$M_1 = 2 + 2g_1 \tag{A11}$$
$$M_2 = 0.0002 + 0.0002g_1 + 4g_2 + 4g_1g_2 \tag{A12}$$
$$M_3 = 0.0004 + 0.0007g_1 + 0.0003g_1^2 + 8g_2 + 14g_1g_2 + 6g_1^2g_2 \tag{A13}$$
$$M_4 = 4. \times 10^{-8} + 7. \times 10^{-8}g_1 + 3. \times 10^{-8}g_1^2 + \left(0.0016 + 0.0028g_1 + 0.0012g_1^2\right)g_2 + \left(12 + 21g_1 + 9g_1^2\right)g_2^2 \tag{A14}$$
$$M_5 = 8. \times 10^{-8} + 2. \times 10^{-7}g_1 + 1.6 \times 10^{-7}g_1^2 + 4. \times 10^{-8}g_1^3 + \left(0.0032 + 0.008g_1 + 0.0064g_1^2 + 0.0016g_1^3\right)g_2 + \left(24 + 60g_1 + 48g_1^2 + 12g_1^3\right)g_2^2 \tag{A15}$$
$$M_6 = (1+g_1)\begin{pmatrix} 8. \times 10^{-12} + 1.2 \times 10^{-11}g_1 \\ +4. \times 10^{-12}g_1^2 \end{pmatrix} + (1+g_1)\begin{pmatrix} 4.8 \times 10^{-7} + 7.2 \times 10^{-7}g_1 + \\ 2.4 \times 10^{-7}g_1^2 \end{pmatrix}g_2 + (1+g_1)\left(0.008 + 0.012g_1 + 0.004g_1^2\right)g_2^2 + (1+g_1)\left(32 + 48g_1 + 16g_1^2\right)g_2^3 \tag{A16}$$
$$M_7 = (1+g_1)\begin{pmatrix} 1.6 \times 10^{-11} + 3.6 \times 10^{-11}g_1 \\ +2.5 \times 10^{-11}g_1^2 + 5. \times 10^{-12}g_1^3 \end{pmatrix} + (1+g_1)\begin{pmatrix} 9.6 \times 10^{-7} + 0.00000216g_1 + \\ 0.0000015g_1^2 + 3. \times 10^{-7}g_1^3 \end{pmatrix}g_2 + (1+g_1)\begin{pmatrix} 0.016 + 0.036g_1 + 0.025g_1^2 + \\ 0.005g_1^3 \end{pmatrix}g_2^2 + (1+g_1)\left(64 + 144g_1 + 100g_1^2 + 20g_1^3\right)g_2^3 \tag{A17}$$
$$M_8 = 1.6 \times 10^{-15} + 5.2 \times 10^{-15}g_1 + 6.1 \times 10^{-15}g_1^2 + 3. \times 10^{-15}g_1^3 + 5. \times 10^{-16}g_1^4 + \begin{pmatrix} 1.28 \times 10^{-10} + 4.16 \times 10^{-10}g_1 + \\ 4.88 \times 10^{-10}g_1^2 + 2.4 \times 10^{-10}g_1^3 + \\ 4. \times 10^{-11}g_1^4 \end{pmatrix}g_2 + \begin{pmatrix} 0.00000336 + 0.00001092g_1 + \\ 0.00001281g_1^2 + 0.0000063g_1^3 + \\ 0.00000105g_1^4 \end{pmatrix}g_2^2 + \begin{pmatrix} 0.032 + 0.104g_1 + 0.122g_1^2 + 0.06g_1^3 \\ +0.01g_1^4 \end{pmatrix}g_2^3 + \left(80 + 260g_1 + 305g_1^2 + 150g_1^3 + 25g_1^4\right)g_2^4 \tag{A18}$$

$$P=\begin{bmatrix} Z & \frac{1}{2}(1-b) & -g & 0 & 0 & 0 & 0 & 0 & -(1+g) & 0\\ \frac{1}{2}(1-b) & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ -(1+g) & 0 & Z & \frac{1}{2}(1-b) & -g & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & \frac{1}{2}(1-b) & c & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -(1+g) & 0 & Z & \frac{1}{2}(1-b) & -g & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{2}(1-b) & c & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -(1+g) & 0 & Z & \frac{1}{2}(1-b) & -g & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(1-b) & c & 0 & 0\\ -g_1 & 0 & 0 & 0 & 0 & 0 & -(1+g) & 0 & Z & \frac{1}{2}(1-b)\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(1-b) & c \end{bmatrix}$$

$$P_1=\begin{bmatrix} Z & \frac{1}{2}(1-b) & -g_1 & 0 & 0 & 0 & 0 & 0 & -(1+g_1) & 0\\ \frac{1}{2}(1-b) & c+2g_2 & 0 & -g_2 & 0 & 0 & 0 & 0 & 0 & -g_2\\ -(1+g_1) & 0 & Z & \frac{1}{2}(1-b) & -g_1 & 0 & 0 & 0 & 0 & 0\\ 0 & -g_2 & \frac{1}{2}(1-b) & c+2g_2 & 0 & -g_2 & 0 & 0 & 0 & 0\\ 0 & 0 & -(1+g_1) & 0 & Z & \frac{1}{2}(1-b) & -g_1 & 0 & 0 & 0\\ 0 & 0 & 0 & -g_2 & \frac{1}{2}(1-b) & c+2g_2 & 0 & -g_2 & 0 & 0\\ 0 & 0 & 0 & 0 & -(1+g_1) & 0 & Z & \frac{1}{2}(1-b) & -g_1 & 0\\ 0 & 0 & 0 & 0 & 0 & -g_2 & \frac{1}{2}(1-b) & c+2g_2 & 0 & -g_2\\ -g_1 & 0 & 0 & 0 & 0 & 0 & -(1+g_1) & 0 & Z & \frac{1}{2}(1-b)\\ 0 & -g_2 & 0 & 0 & 0 & 0 & 0 & -g_2 & \frac{1}{2}(1-b) & c+2g_2 \end{bmatrix}$$

$$\begin{aligned} M_9 =\ & 3.1\times 10^{-15}+1.15\times 10^{-14}g_1+1.6\times 10^{-14}g_1^2\\ & +1.\times 10^{-14}g_1^3+2.5\times 10^{-15}g_1^4\\ & +\begin{pmatrix} 2.48\times 10^{-10}+9.2\times 10^{-10}g_1+\\ 1.28\times 10^{-9}g_1^2 8.\times 10^{-10}g_1^3+\\ 2.\times 10^{-10}g_1^4 \end{pmatrix} g_2\\ & +\begin{pmatrix} 0.00000651+0.00002415g_1+\\ 0.0000336g_1^2 0.000021g_1^3+\\ 0.00000525g_1^4 \end{pmatrix} g_2^2\\ & +\begin{pmatrix} 0.062+0.23g_1+0.32g_1^2+0.2g_1^3\\ +0.05g_1^4 \end{pmatrix} g_2^3\\ & +\begin{pmatrix} 155+575g_1+800g_1^2+500g_1^3\\ +125g_1^4 \end{pmatrix} g_2^4 \end{aligned} \tag{A$_{19}$}$$

$$\begin{aligned} M_{10} =\ & \left(31+115g_1+160g_1^2+100g_1^3+25g_1^4\right) *\\ & \begin{pmatrix} 1.\times 10^{-20}+1.\times 10^{-15}g_2+3.5\times 10^{-11}g_2^2+\\ 5.\times 10^{-7}g_2^3+0.0025g_2^4 \end{pmatrix} \end{aligned} \tag{A$_{20}$}$$

where ($A_{11}$) - ($A_{20}$) are ten main sub-determinants of matrix $P_1$. It is easy to verify that these sub-determinants are positive for all values of coupling strength gap-junction $g_1$ and $g_2$, which ensure that matrix $P_1$ positive definite.

## DATA AVAILABILITY

No data has been used.

## CONFLICT OF INTEREST

The authors declare that they have no conflict of interest.

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